Categorical Intuitions Underlying Semantic Normalisation Proofs

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There have been several attempts to rationally(?) reconstruct Berger and Schwichtenberg's method of "Normalisation by Evaluation" employing ideas from categorical logic.

The key observation was the well-known fact that for a cartesian closed category $\mathcal{C}$ the Yoneda embedding $\mathsf{Y}_{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ preserves the cartesian closed structure. Specialising to the free cartesian closed category $\mathcal{C}$ generated by one base type ι one gets a natural isomorphism between $\mathsf{Y}_{\mathcal{C}}$ and the interpretation functor $\mathcal{I} : \mathcal{C} \to \widehat{\mathcal{C}}$ where the latter is the free ccc-preserving functor with $\mathcal{I}(\iota) = \mathsf{Y}_{\mathcal{C}}(\iota)$. Explicitating this natural isomorphism between $\mathsf{Y}_{\mathcal{C}}$ and $\mathcal{I}$ gives rise to "functional programs"

$$\mathsf{quote}_\sigma : \mathcal{I}(\sigma) \to \mathsf{Y}_{\mathcal{C}}(\sigma)$$

$$\mathsf{unquote}_\sigma : \mathsf{Y}_{\mathcal{C}}(\sigma) \to \mathcal{I}(\sigma)$$

defined by *mutual* structural recursion over σ.

Alas, this doesn't help too much as $\mathcal{C}$ is a *quotient* of syntax and, instead, one would like to have that quote_σ (at the empty context) maps $\mathcal{I}_\sigma$ (at the empty context) to (closed) long $\beta\eta$-normal forms of type σ.

There are two ways out of this dilemma. One, taken by Cubrić, Dybjer and P.J. Scott, is to consider *setoids* instead of sets as the category in which presheaves (over $\mathcal{C}$) take their values and to show directly that the canonical realising map underlying quote maps denotations to long $\beta\eta$-normal forms. It turns out that this map underlying quote is similar to the one used by Berger and Schwichtenberg.

Another way as taken by Altenkirch, Hofmann and Streicher is to consider $\widehat{\mathcal{V}}$, the category of presheaves over the category $\mathcal{V}$ of *variable substitutions*, instead of $\widehat{\mathcal{C}}$. As $\mathcal{V}$ is a subcategory of $\mathcal{C}$ (full on objects but *not* on morphisms) there is an obvious restriction functor $\mathsf{R} : \widehat{\mathcal{C}} \to \widehat{\mathcal{V}}$. We write $\mathsf{Y}_{\mathcal{V}}$ for the composition $\mathsf{R} \circ \mathsf{Y}_{\mathcal{C}}$. Alas, this functor $\mathsf{Y}_{\mathcal{V}}$ doesn't preserve the ccc-structure (since R doesn't). But this can be repaired by interpreting λ-calculus not in $\widehat{\mathcal{V}}$ but in the *glueing* of $\mathsf{Y}_{\mathcal{V}}$, i.e. the comma category $\mathsf{Gl}(\mathsf{Y}_{\mathcal{V}}) = \widehat{\mathcal{V}}{\downarrow}\mathsf{Y}_{\mathcal{V}}$, which is known to be cartesian closed and for which the codomain functor $\mathsf{cod} : \mathsf{Gl}(\mathsf{Y}_{\mathcal{V}}) \to \mathcal{C}$ is known to be ccc-preserving. This glueing category $\mathsf{Gl}(\mathsf{Y}_{\mathcal{C}})$ can be understood as the category of *logical predicates* on $\widehat{\mathcal{V}}$. Let NE_σ and NF_σ be the $\mathcal{V}$-presheaves of neutral terms and normal terms of type σ, respectively. Note that they can be organised into presheaves over $\mathcal{V}$ just because neutral terms and normal terms are stable under variable substitutions. As they aren't stable under arbitrary substitutions this explains why we have to take $\mathcal{V}$ instead of $\mathcal{C}$. Now one can interpret λ-calculus in the glued category where base type ι is interpreted as $\mathsf{NF}_\iota = \mathsf{NE}_\iota \hookrightarrow \mathsf{Y}_{\mathcal{V}}(\iota)$. By induction on the structure of type σ one can show that the interpretation $\mathsf{P}_\sigma \hookrightarrow \mathcal{Y}_{\mathcal{V}}(\sigma)$ satifies the following **invariant**

$$\mathsf{NE}_\sigma \subseteq \mathsf{P}_\sigma \subseteq \mathsf{NF}_\sigma \quad .$$

As the glued category is a model of λ-calculus one gets for every term t of type σ as its interpretation a (global) element in P_σ which is a normal form by the invariant above and which is contained in the equivalence class of t modulo conversion because the functor $\mathsf{cod} : \mathsf{Gl}(\mathsf{Y}_{\mathcal{V}}) \to \mathcal{C}$ preserves the ccc-structure.

Thus, *a posteriori* the predicates P_σ on the interpretation of σ in $\widehat{\mathcal{V}}$ appear as a naive implementation of Taits's idea of *computability predicates* albeit in an appropriate *presheaf topos*, namely $\widehat{\mathcal{V}}$.