A Category of Topological Predomains

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Chapter 0

Introduction

Classical Domain Theory (CDT) started in the late 1960s, when Dana Scott discovered that *directed complete partially ordered sets* (dcpos) give rise to a denotational semantics of functional programming languages. The order on a dcpo gives rise to a topology, making a dcpo to a topological space, called *semantic domain*. From the early 1970s onward, various research groups investigated the use of semantic domains in computer science. Important results of this were that Kleene's fixed point theorem, which is used to model recursion, can be applied in domains, and the fact that in algebraic domains, the behaviour of an element is completely determined by the behaviour of certain ideal elements. A necessary requirement for domains in semantics is the existence of function spaces. But although the category **DCPO** of all dcpos and continuous maps is cartesian-closed, the category **ALG** of algebraic domains lacks this property. However, there do exists cartesian-closed subcategories of **ALG**, but is not known if these provide free algebras, which can be used to model computational effects.
In another approach, known as Synthetic Domain Theory (SDT), one looks for categories of domains within models of intuitionistic set theory, i.e. elementary topoi. Domains should be sets satisfying certain axioms, providing structural requirements to model a variety of computational phenomena. Usually one works within the realizability topos over a partial (or total) combinatory algebra (pca/tca). In this context, one was able to find categories of domains, which provide models for polymorphism and a variety of computational effects.
In recent work, Alex Simpson has discovered an interesting link between CDT and SDT (see [Sim03]). In particular, he found out that there exists a category of topological spaces (the category $\mathbf{QCB}_0$ of all quotients of countably-based T_0-spaces) which is equivalent to a subcategory of the category $\mathbf{Asm}(\mathcal{P}\omega)$ of assemblies over the tca $\mathcal{P}\omega$. He then succeeded to show that the category of complete extensional pers over $\mathcal{P}\omega$, a category considered in SDT, in fact appears as a category of topological (pre)domains, **TP**, whose objects have a nice topological description.
In this thesis, the category **TP** will be reconstructed in all details.

Throughout the thesis, the reader is assumed to have a certain familiarity with the concepts of Category Theory (see e.g. [Str03]) and Domain Theory (see e.g.

[AJ94]). The reader interested in more information on the history of Domain Theory is referred to [Hist96].
In chapter 1, some basic concepts will be introduced. We will start by considering $\mathcal{P}\omega$, which can be equipped with a binary operation, and thus becomes a tca. Then we will introduce the categories $\mathbf{Asm}(\mathcal{P}\omega)$ and $\mathbf{Mod}(\mathcal{P}\omega)$ of *assemblies* and *modest sets* over this tca, which provide a platform for SDT. We will show that the category of assemblies is in fact equivalent to the category of ω-*equiloguical spaces*, a category related to topology, which was introduced by Dana Scott to overcome the lack of cartesian-closure of **Sp**, the category of topological spaces and continuous maps. Finally in the first chapter, the categories of *sequential* and *limit spaces* will be defined.
In the second chapter, we will introduce the category **QCB** via the concept of ω-projecting quotient maps. This concept will allow us to view **QCB** as a full subcategory of the category of limit spaces and as a full reflective subcategory of the category of assemblies over $\mathcal{P}\omega$. We will also shortly mention, why **QCB** arises as a full subcategory of many cartesian-closed subcategories of **Sp**.
In the third chapter, we will define the categories of *extensional pers*, *complete extensional pers* and *replete objects* over $\mathcal{P}\omega$. We will show that they are all cartesian-closed subcategories of $\mathbf{Asm}(\mathcal{P}\omega)$
In the last chapter, we will show the equivalence of $\mathbf{QCB}_0$ and the category of extensional pers, and from this we will construct the category **TP**. In the last part, we will have a closer look at the complete extensional pers. In particular, we will show that those pers (up to isomorphism) are *admissible*.

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Chapter 1

Basic Concepts

1.1 Scott's Graph Model $\mathcal{P}\omega$

The power set of the natural numbers, $\mathcal{P}\omega$, ordered by inclusion, is a complete lattice. Moreover it is ω-algebraic, since the finite subsets of ω, denoted by $\mathcal{P}_{\mathrm{f}}\omega$, are the compact elements of the lattice and there are only countably many of these. Another fact about $\mathcal{P}\omega$ is that any countably-based T_0-space can be embedded into it. A proof of this can be found e.g. in II.3 of [Comp80].
In this section we will show that the space of Scott-continuous functions $[\mathcal{P}\omega \to \mathcal{P}\omega]$ can be obtained as a retract of $\mathcal{P}\omega$, which goes back to [Sco76]. Thus $\mathcal{P}\omega$ provides a model for untyped λ-calculus and is a partial (actually even total) combinatory algebra (pca) in the sense of [Bee85].
We will start by defining codings of pairs and finite subsets of natural numbers as natural numbers. This will be used to define functions $\mathsf{fun} : [\mathcal{P}\omega \to \mathcal{P}\omega] \to \mathcal{P}\omega$ and $\mathsf{app} : \mathcal{P}\omega \to [\mathcal{P}\omega \to \mathcal{P}\omega]$ such that $\mathsf{app} \circ \mathsf{fun} = id_{[\mathcal{P}\omega \to \mathcal{P}\omega]}$.

Definition 1.1.1. A coding of pairs of natural numbers is a primitive recursive bijection $f : \omega \times \omega \to \omega$.
A coding of finite subsets of natural numbers is given by the inverse of a primitive recursive bijection $e : \omega \to \mathcal{P}_{\mathrm{f}}\omega$.

We may assume the following codings to be given, but the results below do not depend on that particular choice.

A coding of pairs is given by $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$,
$\langle n, m \rangle = \frac{1}{2}(n + m + 1)(n + m) + m$.
For a coding of finite subsets, let $e : \omega \to \mathcal{P}_{\mathrm{f}}\omega$ be the map given by $e_n = A$ if and only if $n = \sum_{k \in A} 2^k$. The inverse of e is given by $\{n_0, n_1, \ldots, n_k\} \mapsto \sum_{i=0}^{k} 2^{n_i}$. So any $u \in \mathcal{P}_{\mathrm{f}}\omega$ has a unique representation e_n.
From these codings we derive another coding function $[\cdot, \cdot] : \mathcal{P}_{\mathrm{f}}\omega \times \omega \to \omega$ given by $[e_n, k] = \langle n, k \rangle$.
Via this function we can describe the graph of a Scott-continuous function $f : \mathcal{P}\omega \to \mathcal{P}\omega$ by $\mathsf{fun} : [\mathcal{P}\omega \to \mathcal{P}\omega] \to \mathcal{P}\omega,\ f \mapsto \{[e_n, k] |\ k \in f(e_n)\}$.
This construction can be inverted by setting

$\mathsf{app} : \mathcal{P}\omega \to [\mathcal{P}\omega \to \mathcal{P}\omega]$ as $\mathsf{app}(u)(v) = \{k \in \omega |\ \exists e_n \subseteq v.\ [e_n, k] \in u\}$.
In order to show that app really is an inverse to fun and both functions are continuous, we prove the following lemma.

Lemma 1.1.2. *For* fun *and* app*, defined as above, the following hold:*

(i) fun *is continuous,*

(ii) $\mathsf{app}(u) : \mathcal{P}\omega \to \mathcal{P}\omega$ *is continuous for each* $u \in \mathcal{P}\omega$*,*

(iii) app *is continuous,*

(iv) $\mathsf{app} \circ \mathsf{fun} = id_{[\mathcal{P}\omega \to \mathcal{P}\omega]}$.

Proof: It is clear that both, fun and app, are monotone, so in (i) to (iii) it remains to show that directed sups are preserved:

(i) Suppose $(f_k)_{k\in K}$ is a directed family of functions in $[\mathcal{P}\omega \to \mathcal{P}\omega]$. We have to show $\mathsf{fun}(\bigvee^{\uparrow}_{k\in K} f_k) = \bigcup^{\uparrow}_{k\in K} \mathsf{fun}(f_k)$. We get:

$$\begin{array}{rcl} \mathsf{fun}(\bigvee^{\uparrow}_{k\in K} f_k) & = & \{[e_m, n] |\ n \in (\bigvee^{\uparrow}_{k\in K} f_k)(e_m)\} \\ & = & \{[e_m, n] |\ n \in \bigcup^{\uparrow}_{k\in K} f_k(e_m)\} \\ & = & \bigcup^{\uparrow}_{k\in K}\{[e_m, n] |\ n \in f_k(e_m)\} \\ & = & \bigcup^{\uparrow}_{k\in K} \mathsf{fun}(f_k). \end{array}$$

(ii) Here we have to show that for any $u \in \mathcal{P}\omega$ and any directed subset $(v_k)_{k\in K}$ of $\mathcal{P}\omega$ we get $\mathsf{app}(u)(\bigcup^{\uparrow}_{k\in K} v_k) = \bigcup^{\uparrow}_{k\in K} \mathsf{app}(u)(v_k)$. This follows from:

$$\begin{array}{rcl} \mathsf{app}(u)(\bigcup^{\uparrow}_{k\in K} v_k) & = & \{n \in \omega |\ \exists e_m \subseteq \bigcup^{\uparrow}_{k\in K} v_k.\ [e_m, n] \in u\} \\ & =^* & \bigcup^{\uparrow}_{k\in K}\{n \in \omega |\ \exists e_m \subseteq v_k.\ [e_m, n] \in u\} \\ & = & \bigcup^{\uparrow}_{k\in K} \mathsf{app}(u)(v_k). \end{array}$$

The equality denoted by $=^*$ holds, because if $e_m \subseteq \bigcup^{\uparrow}_{k\in K} v_k$, then by finiteness of e_m, there are finitely many elements $v_{k_0}, \ldots, v_{k_z}$ such that $e_m \subseteq \bigcup^{z}_{i=0} v_{k_i}$, so since $(v_k)_{k\in K}$ is directed, there exists $j \in K$, such that $\bigcup^{z}_{i=0} v_{k_i} \subseteq v_j$, hence $e_m \subseteq v_j$.

(iii) For app being continuous, we have to show that for any directed subset $(u_k)_{k\in K}$ of $\mathcal{P}\omega$, $\mathsf{app}(\bigcup^{\uparrow}_{k\in K} u_k) = \bigvee^{\uparrow}_{k\in K} \mathsf{app}(u_k)$. It suffices to show that $\mathsf{app}(\bigcup^{\uparrow}_{k\in K} u_k)(v) = (\bigvee^{\uparrow}_{k\in K} \mathsf{app}(u_k))(v) = \bigcup^{\uparrow}_{k\in K}(\mathsf{app}(u_k)(v))$ for all $v \in \mathcal{P}\omega$, which we get from:

$$\begin{array}{rcl} \mathsf{app}(\bigcup^{\uparrow}_{k\in K} u_k)(v) & = & \{n \in \omega |\ \exists e_m \subseteq v.\ [e_m, n] \in \bigcup^{\uparrow}_{k\in K} u_k\} \\ & = & \bigcup^{\uparrow}_{k\in K}\{n \in \omega |\ \exists e_m \subseteq v.\ [e_m, n] \in u_k\} \\ & = & \bigcup^{\uparrow}_{k\in K}(\mathsf{app}(u_k)(v)). \end{array}$$

(iv) It remains to show that $\mathsf{app} \circ \mathsf{fun} = id_{[\mathcal{P}\omega \to \mathcal{P}\omega]}$, i.e. $(\mathsf{app} \circ \mathsf{fun}(f))(v) = f(v)$ for all $v \in \mathcal{P}\omega$. We get this, using injectivity of $[\cdot,\cdot]$ and continuity of f, from:

$$\begin{array}{rcl} (\mathsf{app} \circ \mathsf{fun}(f))(v) & = & \{n \in \omega \mid \exists e_m \subseteq v.\ [e_m, n] \in \mathsf{fun}(f)\} \\ & = & \{n \in \omega \mid \exists e_m \subseteq v.\ [e_m, n] \in \{[e_s, t] \mid t \in f(e_s)\}\} \\ & = & \{n \in \omega \mid \exists e_m \subseteq v.\ n \in f(e_m)\} \\ & = & \bigcup^{\uparrow}_{e_m \subseteq v} f(e_m) \\ & = & f(v). \end{array}$$

□

As a direct consequence we get:

Theorem 1.1.3. *$[\mathcal{P}\omega \to \mathcal{P}\omega]$ is a retract of $\mathcal{P}\omega$.*

Now $\mathcal{P}\omega$ provides a model for untyped λ-calculus, because via app one can define application, by setting $u \cdot v = \mathsf{app}(u)(v)$, and via fun one can define λ-abstraction, by interpreting a continuous function $\lambda x.f(x)$ as $\mathsf{fun}(f)$ in $\mathcal{P}\omega$. Throughout the thesis, we will identify continuous endomaps of $\mathcal{P}\omega$ as elements of $\mathcal{P}\omega$.
Since the application is not associative, we conveniently write $u \cdot v \cdot w$ for $(u \cdot v) \cdot w$.

Proposition 1.1.4. *In $\mathcal{P}\omega$ there exist elements k and s such that:*

(i) $k \cdot u \cdot v = u$ *for all* $u, v \in \mathcal{P}\omega$,

(ii) $s \cdot u \cdot v \cdot w = u \cdot v \cdot (u \cdot w)$ *for all* $u, v, w \in \mathcal{P}\omega$.

Proof:

(i) k is given by the λ-term $\lambda xy.x$,

(ii) s by $\lambda xyz.xy(xz)$. □

Thus $\mathcal{P}\omega$ is a pca in the sense of [Bee85].
For the sections below, note that the following combinators can be implemented in our pca $\mathcal{P}\omega$:

- i, given by $\lambda x.x$,
- pair, given by $\lambda xyz.zxy$,
- fst, given by $\lambda z.z(\lambda xy.x)$,
- snd, given by $\lambda z.z(\lambda xy.y)$

such that for all $u, v \in \mathcal{P}\omega$, $\mathsf{i} \cdot u = u$, $\mathsf{fst} \cdot (\mathsf{pair} \cdot u \cdot v) = u$ and $\mathsf{snd} \cdot (\mathsf{pair} \cdot u \cdot v) = v$ hold. Instead of $\mathsf{pair} \cdot u \cdot v$, we will also write $\langle u, v \rangle$.

1.2 Assemblies and Modest Sets

In the first section, we have shown that $\mathcal{P}\omega$ has a surprisingly rich structure. Now we will define the categories $\mathbf{Asm}(\mathcal{P}\omega)$ and $\mathbf{Mod}(\mathcal{P}\omega)$, which arise from this structure, and show that they are cartesian-closed. We will also introduce the category $\mathbf{PER}(\mathcal{P}\omega)$ of partial equivalence relations on $\mathcal{P}\omega$ and show that it is equivalent to $\mathbf{Mod}(\mathcal{P}\omega)$. In chapters 3 and 4, we will construct the categories $\mathbf{ExPER_\Sigma}(\mathcal{P}\omega)$ and $\mathbf{CE_\Sigma}(\mathcal{P}\omega)$ from these. But first we will motivate where the ideas below come from.

We have seen that $\mathcal{P}\omega$ provides a model for untyped λ-calculus. Now we will introduce datatypes into our model. A datatype is given by a set X and we want to represent elements of X in our λ-calculus model $\mathcal{P}\omega$. This is done by a function $\|\cdot\|_X : X \to \mathcal{P}(\mathcal{P}\omega) \setminus \emptyset$, so for each element $x \in X$ we have a nonempty subset $\|x\|_X \subseteq \mathcal{P}\omega$ of *realizers*, representing x. Given datatypes X, Y, we also want to be able to express functions $f : X \to Y$ in terms of realizers. For given $f : X \to Y$, we say that f is *tracked* (or *realized*) by $a \in \mathcal{P}\omega$, if:

$$(\forall x \in X)\ u \in \|x\|_X \Rightarrow a \cdot u \in \|f(x)\|_Y,$$

i.e. there is a continuous function on $\mathcal{P}\omega$, that acts on the realizers like f on the data. f is then called a *realizable map*.
Thus we can introduce the following category:

Definition 1.2.1. The category $\mathbf{Asm}(\mathcal{P}\omega)$ is given by:

- Objects are tuples $(X, \|\cdot\|_X)$, such that X is a set and $\|\cdot\|_X : X \to \mathcal{P}(\mathcal{P}\omega)\setminus\emptyset$. These objects are called *assemblies*.
- Morphisms between assemblies $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ are realizable maps $f : X \to Y$.

It also makes sense to distinguish those datatypes X, for which $u \in \mathcal{P}\omega$ realizes at most one object. These assemblies are called *modest sets*.

Definition 1.2.2. The full subcategory $\mathbf{Mod}(\mathcal{P}\omega)$ of $\mathbf{Asm}(\mathcal{P}\omega)$ is given by those objects $(X, \|\cdot\|_X)$, for which $u \in \|x\|_X \cap \|y\|_X$ implies $x = y$.

Recall that for a category $\mathbf{C}$ to be bi-cartesian-closed means that $\mathbf{C}$ is cartesian closed (has finite products and exponentials) and has finite sums. We will allow ourselves to write X instead of $(X, \|\cdot\|_X)$ for assemblies and modest sets, when no confusion can arise.

Theorem 1.2.3. *The categories* $\mathbf{Mod}(\mathcal{P}\omega)$ *and* $\mathbf{Asm}(\mathcal{P}\omega)$ *are bi-cartesian-closed and have equalizers.*

Proof: We show that $\mathbf{Asm}(\mathcal{P}\omega)$ and $\mathbf{Mod}(\mathcal{P}\omega)$ have initial and terminal objects, binary products and sums, exponentials and equalizers. All constructions, but exponentials, arise from the constructions in **SET**. Thus uniqueness of mediating arrows, in the cases of binary products and sums and equalizers, follows directly from this observation. We only have to make sure that all morphisms used here can be tracked, and that in the case of modest sets the constructions remain modest.

- Initial object: Set $\mathbf{0} := (\emptyset, \|\cdot\|_{\mathbf{0}})$, where $\|\cdot\|_{\mathbf{0}} : \emptyset \to \mathcal{P}(\mathcal{P}\omega) \setminus \emptyset$ is the empty function.

- Terminal object: Set $\mathbf{1} := (\{*\}, \|\cdot\|_{\mathbf{1}})$, where $\| * \|_{\mathbf{1}} = \mathcal{P}\omega$. Clearly this is modest.

- Binary products: Let X, Y be assemblies. Consider the product in $\mathbf{SET}$, $X \times Y$, and $\|\cdot\|_{X\times Y}$ given by $\|(x,y)\|_{X\times Y} := \{\langle u, v\rangle|\ u \in \|x\|_X \textit{ and } v \in \|y\|_Y\}$. Then the projections are tracked by fst and snd, and if Z is an assembly so that $f : Z \to X$ is tracked by a and $g : Z \to Y$ is tracked by b, then the mediating arrow is tracked by $\lambda x.\langle ax, bx\rangle$. So $(X \times Y, \|\cdot\|_{X\times Y})$ is a product of X and Y in $\mathbf{Asm}(\mathcal{P}\omega)$.
  If X and Y are modest sets, it is clear that $u \in \|(x,y)\|_{X\times Y} \cap \|(x',y')\|_{X\times Y}$ if and only if $x = x'$ and $y = y'$, thus $(X \times Y, \|\cdot\|_{X\times Y})$ is a modest set as well.

- Binary sums: For assemblies X, Y and the set-theoretic sum $X + Y$, let $\|\cdot\|_{X+Y}$ be given by $\|(0,x)\|_{X+Y} := \{\langle \mathsf{fst}, u\rangle|\ u \in \|x\|_X\}$ and $\|(1,y)\|_{X+Y} := \{\langle \mathsf{snd}, v\rangle|\ v \in \|y\|_Y\}$. Then the inclusion $s_1 : X \to X + Y$ is tracked by $\lambda x.\langle \mathsf{fst}, x\rangle$ and $s_2 : Y \to X + Y$ by $\lambda y.\langle \mathsf{snd}, y\rangle$. If Z is another assembly and there are morphisms $f : X \to Z$ tracked by a and $g : Y \to Z$ tracked by b, then the mediating arrow is tracked by $\lambda x.\mathsf{fst}\, x\, \langle a, b\rangle(\mathsf{snd}\, x)$.
  If X and Y are modest sets, consider $u \in \|(i,t)\|_{X+Y} \cap \|(i',t')\|_{X+Y}$. Then u is either of the form $\langle \mathsf{fst}, v\rangle$ or $\langle \mathsf{snd}, v\rangle$. It then follows that $i = i'$ and, by X and Y being modest, that $t = t'$. Hence $X + Y$ is a modest set.

- Exponentials: Let X, Y be assemblies. Set $Y^X := \{f : X \to Y|\ \exists a \in \mathcal{P}\omega.\ a \textit{ tracks } f\}$, $\|f\|_{Y^X} := \{a \in \mathcal{P}\omega|\ a \textit{ tracks } f\}$. Then $(Y^X, \|\cdot\|_{Y^X})$ clearly is an assembly, and it is straightforward to check that Y^X meets the categorical requirements for an exponential.
  For a modest set Y, suppose a tracks $f, g \in Y^X$. Then for all $x \in X$, we get for $u \in \|x\|_X$, that $a \cdot u \in \|f(x)\|_Y \cap \|g(x)\|_Y$. Hence, by Y being modest, it follows that $f(x) = g(x)$. Thus $\mathbf{Mod}(\mathcal{P}\omega)$ forms an exponential ideal in $\mathbf{Asm}(\mathcal{P}\omega)$ and $(Y^X, \|\cdot\|_{Y^X})$ becomes a modest set.

- Equalizers: Let $f, g : X \to Y$ be realizable. Then we can construct the equalizer $e : Z \to X$, by setting $Z := \{x \in X|\ f(x) = g(x)\}$, $\|x\|_Z = \|x\|_X$ for all $x \in Z$, and e to be the inclusion map, which is tracked by i. It is easy to see that $(Z, \|\cdot\|_Z)$ is an assembly and a modest set, if X is. $\square$

We see that it is fairly easy to construct exponentials in $\mathbf{Mod}(\mathcal{P}\omega)$ and $\mathbf{Asm}(\mathcal{P}\omega)$. It is clear that the inclusion functor $I : \mathbf{Mod}(\mathcal{P}\omega) \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$ preserves all the constructions mentioned above.

To conclude this section, we will show that $\mathbf{Mod}(\mathcal{P}\omega)$ is equivalent to $\mathbf{PER}(\mathcal{P}\omega)$, the category of partial equivalence relations over $\mathcal{P}\omega$.

A partial equivalence relation (per) R on $\mathcal{P}\omega$ is a symmetric and transitive binary relation on $\mathcal{P}\omega$. If $u \mathsf{R} u$, we write $[u]_{\mathsf{R}}$ for the equivalence class $\{v \in \mathcal{P}\omega | v \mathsf{R} u\}$. For pers R, S we say a continuous map $a \in \mathcal{P}\omega$ is R, S-preserving if $u \mathsf{R} v$ implies $a \cdot u \mathsf{S} a \cdot v$. Thus the R, S-preserving maps form another per S^{R}, given by $a \mathsf{S}^{\mathsf{R}} b$ if and only if $a \cdot u \mathsf{S} b \cdot v$ for all $u \mathsf{R} v$. Notice that all elements of $[a]_{\mathsf{R}^{\mathsf{S}}}$ represent the same function between the pers R and S. This leads to the following definition:

Definition 1.2.4. The category $\mathbf{PER}(\mathcal{P}\omega)$ is given as follows:

- Objects are pers on $\mathcal{P}\omega$.
- Morphisms between pers R and S are equivalence classes of continuous R, S-preserving maps.

Proposition 1.2.5. *$\mathbf{Mod}(\mathcal{P}\omega)$ and $\mathbf{PER}(\mathcal{P}\omega)$ are equivalent.*

Proof: Consider the functor $F : \mathbf{Mod}(\mathcal{P}\omega) \to \mathbf{PER}(\mathcal{P}\omega)$ given by:

- for $X \in \mathbf{Mod}(\mathcal{P}\omega)$, set $(u, v) \in F(X)$, whenever there exists $x \in X$ such that $u, v \in \|x\|_X$,
- for $f \in \mathbf{Mod}(\mathcal{P}\omega)(X, Y)$, there is $a \in \mathcal{P}\omega$ tracking f. Set $F(f) = [a]_{F(Y)^{F(X)}}$.

We show that F is the required equivalence.
We have to prove that F is full and faithful and for each $\mathsf{R} \in \mathbf{per}(\mathcal{P}\omega)$ we can find $X \in \mathbf{Mod}(\mathcal{P}\omega)$ such that $F(X) \cong \mathsf{R}$.
Full: Assume $[a]_{F(Y)^{F(X)}} \in \mathbf{PER}(\mathcal{P}\omega)(F(X), F(Y))$. Then $[a]_{F(Y)^{F(X)}}$ is nonempty. For $x \in X$, set $f(x) = y$ whenever $a \cdot u \in \|y\|_Y$ for all $u \in \|x\|_X$. Note that this y is unique, since Y was a modest set. Clearly a tracks f and hence $[a]_{F(Y)^{F(X)}} = [F(f)]_{F(Y)^{F(X)}}$.
Faithful: In the proof of theorem 1.2.3, we have seen that an element $a \in \mathcal{P}\omega$ tracks at most one $f : X \to Y$, if Y is a modest set. Thus for $f, g \in \mathbf{Mod}(\mathcal{P}\omega)(X, Y)$, $F(f) = F(g)$ implies $f = g$ and so F is faithful.
For the last part of the equivalence let $\mathsf{R} \in \mathbf{PER}(\mathcal{P}\omega)$ be given. Define $\mathsf{carrier}(\mathsf{R}) := \{u \in \mathcal{P}\omega | u \mathsf{R} u\}$ and set $X := \mathsf{carrier}(\mathsf{R})/\mathsf{R} = \{[u]_{\mathsf{R}} | u \in \mathsf{carrier}(\mathsf{R})\}$. For $[u]_{\mathsf{R}} \in X$, set $\|[u]_{\mathsf{R}}\| = [u]_{\mathsf{R}} \subseteq \mathcal{P}\omega$. Then $(X, \| \cdot \|)$ clearly is a modest set over $\mathcal{P}\omega$ and it follows directly that $F(X) \cong \mathsf{R}$. □

1.3 Equilogical Spaces

It is a fact that the category **Sp** of topological spaces and continuous maps is not cartesian-closed. Therefore, D. Scott introduced the category **Equ** of equilogical spaces and equivariant maps, which contains **Sp** as a subcategory, but is cartesian-closed. Scott then showed that **Equ** is equivalent to the category **Asm(ALat)** of assemblies over algebraic lattices, where exponentials are fairly easy to construct. For details see [BBS04].
In our case, we are interested in the category ω**Equ** of countably-based equilogical spaces, which turns out to be equivalent to $\mathbf{Asm}(\mathcal{P}\omega)$. In chapter 2 we will construct the category **QCB** and show that it can be viewed as a full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$. For this we will use the category ω**Equ**, since **QCB** lives very naturally in it.

Definition 1.3.1. The category **Equ** is given by:

- Objects are *equilogical spaces*, i.e. tupels $(X, \sim)$, where X is a topological space and $\sim$ is a (total) equivalence relation on X.
- Morphisms between equilogical spaces $(X, \sim)$ and $(Y, \approx)$ are functions between the quotients $f : (X/\sim) \to (Y/\approx)$, that are realized by some continuous $\overline{f} : X \to Y$, i.e. $f \circ q_X = q_Y \circ \overline{f}$, where q_X and q_Y are the quotient maps. These morphisms are called *equivariant maps*.

Note that there is no restriction on the equivalence relation in the definition of an equilogical space. However, it is clear that each equivariant $f : (X/\sim) \to (Y/\approx)$ is continuous, when $(X/\sim)$ and $(Y/\approx)$ are equipped with the quotient topology.
The category ω**Equ** is the full subcategory of **Equ**, whose objects are constructed from countably-based topological spaces (from now on *cb-spaces*), and $\omega\mathbf{Equ}_0$ is the full subcategory of ω**Equ** whose objects are given by $(X, \sim)$ and X is a T_0-space.

In order to show the desired equivalence of ω**Equ** and $\mathbf{Asm}(\mathcal{P}\omega)$, we have to take a closer look at the properties of $\mathcal{P}\omega$. In II.3 of [Comp80], it is shown that if $i : X \hookrightarrow Y$ is a topological embedding and $f : X \to \mathcal{P}\omega$ a continuous map, then f extends along i, i.e. there exists a (not necessarily unique) continuous $\overline{f}$ making the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "\exists \overline{f}"] & \mathcal{P}\omega \\ X \arrow[u, hook, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

With this property, $\mathcal{P}\omega$ is called *injective* with respect to topological embeddings. This can be generalized via the following definition:

Definition 1.3.2. A *topological pre-embedding* is a morphism $i : X \to Y$ in **Sp**, such that for each open $U \subseteq X$ there exists an open $V \subseteq Y$ so that $U = i^{-1}(V)$.

Lemma 1.3.3. *In the above, $\overline{f}$ exists already, if i is a topological pre-embedding.*

Proof: This follows from the fact that $\mathcal{P}\omega$ is a T_0-space. Let $\mathbf{Sp}_0$ be the category of topological spaces satisfying T_0, then there exists a reflection functor $T_0 : \mathbf{Sp} \to \mathbf{Sp}_0$, called the T_0-fication. Since any map $f : X \to Z$, whose codomain is a T_0-space, extends uniquely along the reflection map $r_X : X \to T_0(X) =: X_0$, we get the following diagram:

$$\begin{tikzcd} Y \arrow[r, "r_Y"] & Y_0 \arrow[r, "\exists \overline{f}"] & \mathcal{P}\omega \\ X \arrow[u, hook, "i"] \arrow[r, "r_X"'] & X_0 \arrow[u, hook, "i_0"] \arrow[ur, "f"'] & \end{tikzcd}$$

where i_0 is a topological pre-embedding. But a topological pre-embedding between T_0-spaces already is a topological embedding and thus the claim follows. $\square$

Since any cb-space satisfying T_0 can be embedded into $\mathcal{P}\omega$, each cb-space can be topologically pre-embedded into $\mathcal{P}\omega$, via the T_0-ficaton. Moreover, whenever $f : X \to Y$ is a continuous map between cb-spaces, then, by lemma 1.3.3, there exists a continuous $a \in \mathcal{P}\omega$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{P}\omega \arrow[r, "a"] & \mathcal{P}\omega \\ X \arrow[u, "i_X"] \arrow[r, "f"'] & Y \arrow[u, "i_Y"'] \end{tikzcd}$$

where i_X, i_Y are topological pre-embeddings. Thus we get:

Theorem 1.3.4. *The categories* $\mathbf{Asm}(\mathcal{P}\omega)$ *and* $\omega\mathbf{Equ}$ *are equivalent.*

Proof: Define a functor $F : \omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$ as follows.
For any ω-equilogical space $(X, \sim)$ choose a topological pre-embedding $i_X : X \hookrightarrow \mathcal{P}\omega$ and let $F((X, \sim))$ be given by $(X/\sim, \|\cdot\|)$ such that $u \in \|[x]_\sim\|$ if and only if there is a $y \in [x]_\sim$ with $u = i_X(y)$.
On morphisms F acts as identity.
By the above considerations, it is clear that F is well-defined, full and faithful.
For the last part of the equivalence, let $(Y, \|\cdot\|) \in \mathbf{Asm}(\mathcal{P}\omega)$. Consider the set $X := \{(y, u) \in Y \times \mathcal{P}\omega |\ u \in \|y\|\}$ equipped with the topology whose base is $\{\pi_2^{-1}(O) |\ O \text{ open in } \mathcal{P}\omega\}$ (π_2 being the projection onto the second component). Define on X a relation, by setting $(x, u) \sim (x', u')$ whenever $x = x'$. Clearly there exists a bijection $X/\sim\ \leftrightarrow Y$ and it is straightforward to verify that $F(X, \sim) \cong (Y, \|\cdot\|)$. $\square$

Corollary 1.3.5. *The categories* $\omega\mathbf{Equ}_0$ *and* $\mathbf{Mod}(\mathcal{P}\omega)$ *are equivalent.*

Proof: Just repeat the above proof and start with objects of $\omega\mathbf{Equ}_0$. Then the image of F is just $\mathbf{Mod}(\mathcal{P}\omega)$. $\square$

This theorem yields that $\omega\mathbf{Equ}$ and $\omega\mathbf{Equ}_0$ are bi-cartesian-closed and provides an easy way to construct exponentials in them.

1.4 Limit and Sequential Spaces

Finally in this chapter, we will introduce the categories of sequential and limit spaces, **Seq** and **Lim**. We will show that they are cartesian closed and how to construct exponentials in them. In the next chapter, we will then construct **QCB** as a subcategory of **Seq**.
Let $\mathbb{N}$ be the set of natural numbers, equipped with its natural (discrete) topology.

By $\mathcal{N}$ we denote the one-point compactification of $\mathbb{N}$, i.e. the topological space, whose underlying set is $\mathbb{N}\cup\{\infty\}$ and whose topology has a base given by $\{\{n\}|\ n \in \mathbb{N}\} \cup \{\{k \in \mathbb{N}|\ k \leq n\}|\ n \in \mathbb{N}\}$.
We use $\mathcal{N}$ here instead of $\overline{\omega}$ to distinguish the topological space $\mathcal{N}$ from the poset $\overline{\omega}$, whose topology consists only of subsets of the form $\{\{k \in \mathbb{N}|\ k \leq n\}|\ n \in \mathbb{N}\}$.
If X is a topological space, any continuous function $f : \mathbb{N} \to X$ that extends (not necessarily uniquely, if X does not satisfy T_2) along the embedding $i : \mathbb{N} \to \mathcal{N}$ to a continuous function $\overline{f} : \mathcal{N} \to X$, as illustrated in the following diagram,

$$\begin{tikzcd}
\mathcal{N} \arrow[r, "\exists \overline{f}"] & X \\
\mathbb{N} \arrow[u, hook, "i"] \arrow[ur, "f"'] &
\end{tikzcd}$$

defines a converging sequence $(x_n)_{n\in\mathbb{N}}$ (shortly (x_n)) in X. (x_n) has limit x (shortly $(x_n) \to x$), if $\overline{f}$ can be chosen so that $\overline{f}(\infty) = x$. We will denote the set of sequences with limit points on X by $\mathrm{CS}(X)$. In particular, we will use the following notations:

- $((x_n), x) \in \mathrm{CS}(X)$, whenever $(x_n) \to x$,
- $(x_k) \preceq (x_n)$ to denote that (x_k) is a subsequence of (x_n), accordingly $(x_k) \prec (x_n)$ for proper subsequences.

In a topological space X, each open set U satisfies the following condition:

(*) $(\forall x \in U)(\forall((x_n), x) \in \mathrm{CS}(X))(\exists k \in \mathbb{N}).\ (\forall n \geq k)\ x_n \in U.$

However, if (*) holds for $U \subseteq X$, this in general does not imply U to be open. One calls the subsets of X satisfying (*) the *sequentially open* subsets of X. If each sequentially open subset of X is open, then X is called a *sequential space*. This leads to the following category:

Definition 1.4.1. The category **Seq** is given by:

- Objects are sequential spaces.
- Morphisms are continuous maps.

Thus the category **Seq** is a full subcategory of **Sp** including all cb-spaces, i.e. ω**Sp** is a subcategory of **Seq**. Notice that in **Seq**, for a function $f : X \to Y$ to be continuous, is equivalent to $(f(x_n)) \to f(x)$ in Y for all $(x_n) \to x$ in X, i.e. *sequential continuity*. Note also that we allow a sequence to converge to more than one point.

In fact the above ideas can be generalized to non-topological spaces. Let X be a set and denote by $\mathrm{CS}(X)$ a family of sequences with limit points on X. In order to give our intuition of limits a mathematical meaning, we demand the following on each $((x_n), x) \in \mathrm{CS}(X)$:

- the constant sequence (x) has limit x: $(((x), x) \in \mathrm{CS}(X))$,
- if (x_n) is a sequence with limit x, then any subsequence has limit x: $(((x_n), x) \in \mathrm{CS}(X) \text{ and } (x_k) \preceq (x_n)) \Rightarrow (((x_k), x) \in \mathrm{CS}(X))$,
- if each subsequence of (x_n) contains a subsequence with limit x, then (x_n) has limit x: $((\forall (x_k) \preceq (x_n))(\exists (x_i) \preceq (x_k))\, ((x_i), x) \in \mathrm{CS}(X)) \Rightarrow (((x_n), x) \in \mathrm{CS}(X))$.

If X is a set and $\mathrm{CS}(X)$ is a distinguished family of sequences with limits on X satisfying the properties above, then $(X, \mathrm{CS}(X))$ is called a *limit space*, leading to the following:

Definition 1.4.2. The category **Lim** is given by:

- Objects are limit spaces $(X, \mathrm{CS}(X))$.
- Morphisms between limit spaces $(X, \mathrm{CS}(X))$ and $(Y, \mathrm{CS}(Y))$ are sequentially continuous maps $f : X \to Y$.

We say a sequence (x_n) on X is *convergent*, if there exists an $x \in X$ such that $((x_n), x) \in \mathrm{CS}(X)$.

We immediately see that **Seq** is a full subcategory of **Lim**. We also can construct a topological space from a limit space $(X, \mathrm{CS}(X))$ by taking the topology of sequentially open sets of X. This yields a reflection functor $R : \mathbf{Lim} \to \mathbf{Seq}$. It follows immediately that together with the embedding functor $I : \mathbf{Seq} \hookrightarrow \mathbf{Lim}$, we get an adjunction $R \dashv I$. The sequential spaces are just those limit spaces for which sequential convergence on X and topological convergence on $R(X)$ coincide. Notice that all sequentially convergent sequences on X converge topologically on $R(X)$, i.e. $\mathrm{CS}(X) \subseteq \mathrm{CS}(R(X))$. A rather surprising fact is that the reflection functor R preserves finite products. For a basic but rather technical proof of this see [MS02]. Also note that the elements of $\mathrm{CS}(X)$ are in one-to-one correspondence to **Lim**-morphisms $\mathcal{N} \to X$.

Theorem 1.4.3. **Seq** *and* **Lim** *are bi-cartesian-closes categories, have equalizers, and the inclusion functor I preserves this structure.*

Proof: We show that both categories have initial and terminal object, binary products and sums, exponentials and equalizers and that in the case of **Seq**, constructions remain in **Seq**. Again uniqueness of mediating arrows in the constructions follows from the situation in **SET**.

- Initial object: $\mathbf{0} = \emptyset$ and there are no sequences.
- Terminal object: $\mathbf{1} = \{*\}$ and the only convergent sequence is the constant sequence.
- Binary products: Let $(X, \mathrm{CS}(X)), (Y, \mathrm{CS}(Y))$ be limit spaces, then the product is given by $(X \times Y, \mathrm{CS}(X \times Y))$, where $((x_n, y_n)) \to (x, y)$, whenever

$(x_n) \to x$ in X and $(y_n) \to y$ in Y.
That **Seq** is closed under binary products follows from the fact that the reflection functor $R : \mathbf{Lim} \to \mathbf{Seq}$ preserves binary products.

- Binary sums: For $(X, \mathrm{CS}(X)), (Y, \mathrm{CS}(Y)) \in \mathbf{Lim}$ the underlying set of the sum is given by the disjoint union, $X + Y$, and $(i_n, t_n) \to (i, t)$, whenever $(\exists k \in \mathbb{N}).\ (\forall n \geq k)\ i_n = i$ and then for $i = 0$, $(t_m) \to t$ in X, where (t_m) is the sequence $(t_k, t_{k+1}, t_{k+2}, \ldots)$, similar for $i = 1$.
If X, Y are sequential spaces, then we remain in **Seq** since the topology on $X + Y$, with basis given by $\{(0, x) |\ x \in U,\ U \subseteqq_{open} X\} \cup \{(1, y) |\ y \in V,\ V \subseteqq_{open} Y\}$, contains all sequentially open subsets.

- Exponentials: For limit spaces $(X, \mathrm{CS}(X))$ and $(Y, \mathrm{CS}(Y))$, let the underlying set of $(Y^X, \mathrm{CS}(Y^X))$ be given by $\{f : X \to Y |\ f$ sequentially continuous$\}$ and $(f_n) \to f$, whenever $(f_n(x_n)) \to f(x)$ in Y for all convergent sequences $(x_n) \to x$ in X. It is a straightforward verification, that $(Y^X, \mathrm{CS}(Y^X))$ meets the categorical requirements for an exponential.
Recall that the reflection functor R preserves finite products. Thus, by 1.857 of [FS90], **Seq** forms an exponential ideal in **Lim**, hence for $Y \in \mathbf{Seq}$, $(Y^X, \mathrm{CS}(Y^X))$ is a sequential space.

- Equalizers: Let $(X, \mathrm{CS}(X))$ and $(Y, \mathrm{CS}(Y))$ be limit spaces and $f, g : X \to Y$ be **Lim**-morphisms, then the equalizer is given by the inclusion $e : Z \to X$, where $Z = \{x \in X |\ f(x) = g(x)\}$ and $(z_n) \to z$ in Z whenever $(e(z_n)) \to e(z)$ in X. Trivially e is sequentially continuous.
For $X, Y \in \mathbf{Seq}$, we get that $(z_n) \to z$ in Z if and only if for all open subsets $U \subseteq X$ containing $e(z)$, there exists $i \in \mathbb{N}$ such that $e(z_n) \in U$ for all $n \geq i$. So (z_n) converges in Z if and only if it converges with respect to the induced topology from X. However, in general this topology will not contain all sequentially open subsets, so we have to equip Z with the sequentialization of the induced topology. Then $e : Z \to X$ is a morphism in **Seq**. $\square$

This theorem yields that **Lim** and **Seq** are "nicer" categories than **Sp**, because they are cartesian-closed and exponentials are fairly easy to construct. Notice that most spaces used in general mathematics, for example all metric spaces and all cb-spaces, are sequential. However there are topological spaces which are not sequential, for example topological dual spaces X' equipped with the weak-*-topology, where X is an infinite-dimensional $\mathbb{R}$-vector-space.

From now on, we allow ourselves to simply write X instead of $(X, \mathrm{CS}(X))$ for a limit space. We conclude this chapter with an observation that will come in very handy in the next sections.

Lemma 1.4.4. *A morphism $f : X \to Y$ in* **Lim** *is a regular mono if and only if it is one-to-one and $(x_n) \to x$ in X whenever $f(x_n) \to f(x)$ in Y.*

Proof: The "only if" direction is clear, by the description of equalizers in **Lim**. For the "if" direction, observe that f is an equalizer of $\mathbf{1}, \chi_f : Y \to \nabla\mathbf{2}$, where

$\nabla\mathbf{2}$ is the limit space, consisting of two elements $\{0,1\}$, and each sequence in $\nabla\mathbf{2}$ converges to each element, $\mathbf{1}$ is the constant function with value 1 and χ_f maps the image of f to 1 and all other elements of Y to 0. □

Proposition 1.4.5. **Seq** *contains all its regular* **Lim**-*subobjects.*

Proof: Assume $f : X \to Y$ is a regular mono in **Lim** and Y a sequential space. Then, by lemma 1.4.4, f is equalizer of $\mathbf{1}, \chi_f : Y \to \nabla\mathbf{2}$. But $\nabla\mathbf{2}$ is a sequential space (it is the two-element space equipped with the indiscrete topology) and **Seq** has equalizers, so it follows that f is a regular mono in **Seq**. □

Chapter 2

The category QCB

2.1 ω-projecting Quotients

In chapter 1, we have seen that the category **Seq** has better closure properties than **Sp**. In this section, we will introduce the category **PQ**, which will turn out to be the category of all topological quotients of cb-spaces (from now on *qcb-spaces*) and continuous maps. Then we will show that it is a full subcategory of **Seq**.

Definition 2.1.1. An *ω-projecting map* in a category, containing ω**Sp** as a subcategory, is a morphism $f : X \to Y$, such that for every cb-space A, every map $g : A \to Y$ factors through f, i.e. there exists $\overline{g} : A \to X$ with $g = f \circ \overline{g}$.
PQ is the full subcategory of **Sp**, consisting of those spaces Q for which there exists a cb-space X, together with an ω-projecting $q : X \to Q$, which moreover is a topological quotient map.

PQ stands for ω-projecting quotients. We will call the objects of **PQ** simply PQ-spaces. By $\mathbf{PQ}_0$ we denote the full subcategory of **PQ**, whose objects are T_0-spaces.

Lemma 2.1.2. *Topological quotients of sequential spaces are sequential spaces.*

Proof: Let X be a sequential space, $q : X \twoheadrightarrow Q$ a topological quotient map. For $U \subseteq Q$ sequentially open, $V = q^{-1}(U)$ is sequentially open in X, hence open, and therefore U is open in Q. □

Corollary 2.1.3. **PQ** *is a subcategory of* **Seq**.

Proof: Immediate from the previous lemma. □

2.2 PQ as subcategory of Seq

In this section, we will show that **PQ** is bi-cartesian-closed, has equalizers and that the inclusion functor $I : \mathbf{PQ} \hookrightarrow \mathbf{Seq}$ preserves this structure. The proof mainly follows the one in [MS02].

Proposition 2.2.1. **PQ** *has initial and terminal objects and the inclusion functor* **PQ** $\hookrightarrow$ **Seq** *preserves them.*

Proof: Clear, since $\mathbf{0_{Seq}}$ and $\mathbf{1_{Seq}}$ are countably-based spaces. $\square$

Lemma 2.2.2. **Lim** *has coequalizers.*

Proof: Let $f, g : X \to Y$ be morphisms in **Lim**. Define $\sim$ to be the least equivalence relation relation on Y, containing $\{(fx, gx) |\ x \in X\}$. Put $Z = Y/\sim$ and $c : Y \to Z$, $c(y) = [y]_\sim$. Furthermore, define converging sequences on Z by $(z_n) \to z$ whenever each subsequence $(z_k) \preceq (z_n)$ contains a subsequence $(z_i) \preceq (z_k)$ such that there is a convergent sequence $(y_i) \to y$ in Y, with $c(y_i) = z_i$ and $c(y) = z$. It is straightforward to check that Z is a limit space, c is continuous and coequalizer of f and g. $\square$

Notice that each surjective map $c : Y \to Z$ between limit spaces, acting on the convergent sequences of Y and Z as described above, arises as a coequalizer. For if ΔY is the limit space, whose underlying set is the one of Y and where a sequence converges if and only if it is eventually constant, and $s : Z \to Y$ is a section of c (i.e. $c \circ s = \mathrm{id}_Z$), then c becomes a coequalizer of $\mathrm{id}, s \circ c : \Delta Y \to Y$. Notice that s may not be a **Lim**-morphism, but by the construction of ΔY, $s \circ c : \Delta Y \to Y$ always is.

Lemma 2.2.3. *ω-projecting maps in* **Lim** *are regular epis.*

Proof: Let $f : X \to Y$ be an ω-projecting map in **Lim**. Clearly f must be surjective. Recall that by definition, converging sequences with limits in Y are in one-to-one correspondence to continuous maps $g : \mathcal{N} \to Y$. $\mathcal{N}$ is a cb-space and by considering the following diagram,

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\overline{g}"] \arrow[dr, "g"'] & X \arrow[d, two heads, "f"] \\ & Y \end{tikzcd}$$

it is easy to see that f satisfies the requirements for a coequalizer described above. Hence f is a regular epi. $\square$

Consider the functor $F : \mathbf{Lim} \to \mathbf{Sp}$, that maps a limit space $(X, \mathrm{CS}(X))$ to the topological space $(X, \mathcal{O}(X))$, where $\mathcal{O}(X)$ is given by the sequentially open sets of $(X, \mathrm{CS}(X))$. Let $G : \mathbf{Sp} \to \mathbf{Lim}$ be the functor that maps a topological space $(X, \mathcal{O}(X))$ to the limit space $(X, \mathrm{CS}(X))$, where $\mathrm{CS}(X)$ is given by those sequences, that converge with respect to the topology $\mathcal{O}(X)$. It is straightforward to verify that these functors form an adjunction $F \dashv G$, and moreover are equivalences, when restricted to **Seq**.

Lemma 2.2.4. *Let $f : X \to Y$ be an ω-projecting map in* **Lim** *between sequential spaces. Then f is an ω-projecting topological quotient map.*

Proof: As ω-projecting map in **Lim**, by the previous lemma, f is a regular epi. Since $F : \mathbf{Lim} \to \mathbf{Sp}$ is a left adjoint, $F(f)$ is a regular epi in **Sp**, hence a topological quotient map. Moreover, we get that $F(f) = f$ and it is clear that f is ω-projecting in **Sp**. $\square$

Lemma 2.2.5. *In* **Lim** *the following hold:*

(i) products of ω-projecting maps are ω-projecting,

(ii) compositions of ω-projecting maps are ω-projecting,

(iii) pullbacks of ω-projecting maps along arbitrary morphisms are ω-projecting,

(iv) if $f : X \to Y$ is ω-projecting and A is a cb-space, then $f^A : X^A \to Y^A$ is ω-projecting.

Proof:

(i) Let $q : X \twoheadrightarrow Q$, $r : Y \twoheadrightarrow R$ be ω-projecting maps and Z be a cb-space. The claim follows directly from the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "\langle \overline{f_1}, \overline{f_2} \rangle"] \arrow[dr, "\langle f_1, f_2 \rangle"'] & X \times Y \arrow[d, two heads, "q \times r"] \\
& Q \times R
\end{tikzcd}$$

(ii) Clear.

(iii) Let $r : Y \twoheadrightarrow R$ be ω-projecting and $f : Q \to R$ be an arbitrary morphism. Furthermore suppose Z is a cb-space and $g : Z \to Q$ a morphism. Consider the following pullback diagram:

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, "\overline{f \circ g}"] \arrow[ddr, "g"'] & & \\
& X \arrow[r] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, "r"] \\
& Q \arrow[r, "f"'] & R
\end{tikzcd}$$

The mediating arrow ensures that q is ω-projecting.

(iv) Let Z be a cb-space and $h : Z \to Y^A$ be any morphism. Then $Z \times A$ is countably-based and we get for the exponential transpose $g : Z \times A \to Y$ of h a morphism $\overline{g} : Z \times A \to X$ such that $g = f \circ \overline{g}$. Taking the exponential transpose again yields a map $\overline{h} : Z \to Y^A$ such that $h = f^A \circ \overline{h}$, and thus f^A is ω-projecting. $\square$

Lemma 2.2.6. *Binary sums in* **Lim** *are pullback-stable, i.e. for any morphism* $f : Z \to X + Y$ *in Lim and the following pullback squares:*

$$\begin{tikzcd}
Z_X \arrow[r, tail, "j_{Z_X}"] \arrow[d, "f_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "f"] & Z_Y \arrow[l, tail, "j_{Z_Y}"'] \arrow[d, "f_Y"] \arrow[dl, phantom, "\llcorner", very near start] \\
X \arrow[r, tail, "i_X"'] & X + Y & Y \arrow[l, tail, "i_X"]
\end{tikzcd}$$

it holds that $Z \cong Z_X + Z_Y$, *and* j_{Z_X}, j_{Z_Y} *form a coproduct cone.*

Proof: By the properties of a sum, there is a morphism $h : Z_X + Z_Y \to Z$, which clearly is a bijection on the underlying sets. If $(z_n) \to z$ is a converging sequence in Z, then $(f(z_n)) \to f(z)$ in $X + Y$ and hence $(f(z_n))$ is eventually the image of a convergent sequence in X or Y. By the squares being pullbacks, (z_n) is eventually the image of a convergent sequence in Z_X or Z_Y. So h is an isomorphism. $\square$

Recall that, by the proof of theorem 1.4.3, the topology on regular subobjects of sequential spaces, taken in **Seq**, is given by the sequentialization of the subspace topology. Similarly, the topology on products of sequential spaces in **Seq** is the sequentialization of the product topology. Thus we have that regular subobjects and finite products of cb-spaces, taken in **Lim**, are cb-spaces.

Proposition 2.2.7. *Let* $I : \mathbf{PQ} \hookrightarrow \mathbf{Seq}$ *be the inclusion functor. Then:*

(i) **PQ** *has binary products and* I *preserves them,*

(ii) **PQ** *contains all its regular* **Lim**-*subobjects,*

(iii) **PQ** *has equalizers and* I *preserves them,*

(iv) **PQ** *has binary sums and* I *preserves them.*

Proof:

(i) Let Q, R be PQ-spaces, then there exist cb-spaces X, Y and ω-projecting topological quotient maps $q : X \twoheadrightarrow Q$ and $r : Y \twoheadrightarrow R$. By lemma 2.2.5, $q \times r : X \times Y \to Q \times R$ is ω-projecting and, by lemma 2.2.4, ω-projecting topological quotient map. Since $X \times Y$ is cb-space, this yields that $Q \times R$ is in **PQ**.

(ii) Let R be a PQ-space, Y be countably-based, $r : Y \twoheadrightarrow R$ be an ω-projecting topological quotient map and $m : Q \rightarrowtail R$ be a regular mono in **Lim**. Consider the following pullback:

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, "r"] \\ Q \arrow[r, tail, "m"'] & R \end{tikzcd}$$

By proposition 1.4.5, X and Q are objects in **Seq**. Furthermore, by lemma 2.2.5, q is ω-projecting, hence, by lemma 2.2.4, ω-projecting topological quotient map. As regular subobject of a cb-space, X is a cb-space and so $q : X \twoheadrightarrow Q$ is in **PQ**.

(iii) Follows from (ii).

(iv) Let Q, R be PQ-spaces. Then there are cb-spaces X, Y and ω-projecting topological quotient maps $q : X \twoheadrightarrow Q$, $r : Y \twoheadrightarrow R$. We show that $q + r : X + Y \twoheadrightarrow Q + R$ is ω-projecting. Let Z be a cb-space and $f : Z \to Q + R$ be a morphism in **Lim**. Consider the following pullbacks:

$$\begin{tikzcd} Z_Q \arrow[r, tail, "j_{Z_Q}"] \arrow[d, "f_Q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "f"] & Z_R \arrow[l, tail, "j_{Z_R}"'] \arrow[d, "f_R"] \arrow[dl, phantom, "\llcorner", very near start] \\ Q \arrow[r, tail, "i_Q"'] & Q + R & R \arrow[l, tail, "i_R"] \end{tikzcd}$$

Then, by lemma 2.2.6, $Z \cong Z_Q + Z_R$ and $f \cong f_Q + f_R$. Clearly j_{Z_Q} and j_{Z_R} are regular monos, so Z_Q, Z_R are cb-spaces. As f_Q factors through q and f_R factors through r, it follows that f factors through $q + r$. Thus $q + r : X + Y \twoheadrightarrow Q + R$ is ω-projecting and, since $X + Y$ is a cb-space, $Q + R$ an object of **PQ**. $\square$

It remains to show that PQ has exponentials, which are preserved by I. For this we have to transfer the definition of pre-embeddings, given in section 1.3, from **Sp** to **Lim** and relate these pre-embeddings to w-projecting maps.

Definition 2.2.8. A **Lim**-*pre-embedding* is a morphism $f : X \to Y$ in **Lim**, such that whenever $(f(x_n)) \to f(x)$ in Y, then $(x_n) \to x$ in X.

Notice that from lemma 1.4.4, it follows that an injective **Lim**-pre-embedding is a regular mono in **Lim**.
Since converging sequences on Y are in one-to-one correspondence with **Lim**-morphisms $\mathcal{N} \to Y$, $f : X \to Y$ is a **Lim**-pre-embedding if and only if each $g : \mathcal{N} \to Y$, for which there exist $x_n \in X$ such that $g(n) = f(x_n)$, factors uniquely through f.

Lemma 2.2.9. *For* **Lim**-*pre-embeddings the following hold:*

(i) products of **Lim**-*pre-embeddings are* **Lim**-*pre-embeddings,*

(ii) compositions of **Lim**-*pre-embeddings are* **Lim**-*pre-embeddings,*

(iii) pullbacks of **Lim**-*pre-embeddings along arbitrary morphisms in* **Lim** *are* **Lim**-*pre-embeddings,*

(iv) if $i : X \to Y$ *is a* **Lim**-*pre-embedding, then so is* $i^A : X^A \to Y^A$ *for any limit space* A.

Proof:

(i) Let $i : A \to X$ and $j : B \to Y$ be **Lim**-pre-embeddings. Suppose that $\langle f_1, f_2 \rangle : \mathcal{N} \to X \times Y$ is a **Lim**-morphism, for which there exist $\langle a_n, b_n \rangle$ in $A \times B$ such that $\langle i(a_n), j(b_n) \rangle = \langle f_1(n), f_2(n) \rangle$. Since i and j are **Lim**-pre-embeddings, there exist $g_1 : \mathcal{N} \to A$ and $g_2 : \mathcal{N} \to B$ such that $i \circ g_1 = f_1$ and $j \circ g_2 = f_2$, from which it follows that $(i \times j) \circ \langle g_1, g_2 \rangle = \langle f_1, f_2 \rangle$, and thus $i \times j$ is a **Lim**-pre-embedding.

(ii) Clear.

(iii) Let $i : B \to Y$ be a **Lim**-pre-embedding and $f : X \to Y$ be any morphism. Consider the following pullback diagram:

$$\begin{tikzcd}
\mathcal{N} \arrow[dr, dashed] \arrow[drr, "g"] \arrow[ddr] & & \\
& A \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& B \arrow[r, "i"'] & Y
\end{tikzcd}$$

Suppose $g(n) = j(a_n)$. Then $f \circ g(n) = i(b_n)$ for some $b_n \in B$. Thus $f \circ g$ factors uniquely through i, as i is a **Lim**-pre-embedding. Now the mediating arrow ensures that g factors uniquely through j, making j a **Lim**-pre-embedding.

(iv) Suppose $(i^A(g_n)) \to i^A(g)$ in Y^A. Then $(i \circ g_n) \to i \circ g$ in Y^A. Hence for all converging sequences $(a_n) \to a$ in A, $(i(g_n(a_n))) \to i(g(a))$ in Y. Since i is a **Lim**-pre-embedding, this yields $(g_n(a_n)) \to g(a)$ in X and so $(g_n) \to g$ in X^A. Thus i^A is a **Lim**-pre-embedding. □

Recall that for a topological space X, a subset $B \subseteq X$ is sequentially closed if and only if all limits of converging sequences in B are contained in B. In a sequential space, a subset is closed if and only if it is sequentially closed.

Lemma 2.2.10. *If $i : X \to Y$ is a map from a sequential space into a cb-space, then i is a* **Lim***-pre-embedding if and only if it is a topological pre-embedding.*

Proof: The "if" part clearly holds for all maps between sequential spaces.
For the "only if" part, consider $B \subseteq X$ to be (sequentially) closed. We have to show, that there exists a (sequentially) closed $C \subseteq Y$ such that $i^{-1}(C) = B$. Set

$$C = \{y \in Y | \; (\exists((y_n), y) \in \mathrm{CS}(Y)). \; (\forall n \in \omega) \; y_n \in i(B)\}$$

Then, by basic properties of cb-spaces, C is closed and $i^{-1}(C) = B$, since i is a **Lim**-pre-embedding and B (sequentially) closed in A. ☐

Lemma 2.2.11. *For ω-algebraic lattices A and B, the (Scott-)continuous function space $[A \to B]$, equipped with the Scott-Topology, is isomorphic to the exponential B^A, taken in* **Seq***.*

Proof: Clearly the underlying sets of B^A and $[A \to B]$ are isomorphic, so it remains to show that convergence on both spaces coincide. A step function $(a \searrow b) : A \to B$ is a function defined by

$$(a \searrow b)(x) = \begin{cases} b & \text{if } a \sqsubseteq x \\ \perp & \text{otherwise,} \end{cases}$$

where a and b are compact elements of A and B, respectively. It is known that the compact elements of $[A \to B]$ arise as finite suprema of step functions (see e.g. chapter 4 of [AJ94])
Suppose $f_n \to f$ in B^A, i.e. $f_n(x_n) \to f(x)$ in B for all $(x_n) \to x$ in A. We have to show that $f_n \to f$ in $[A \to B]$, i.e. that for each compact element $e \sqsubseteq f$, (f_n) is eventually above e. By the above considerations, we may assume that e is a step function $(a \searrow b)$. By convergence of constant sequences on A, we get $f_n(x) \to f(x)$ in B and $(a \searrow b)(x) \sqsubseteq f(x)$. In particular, we have $b = (a \searrow b)(a) \sqsubseteq f(a)$. Thus $(f_n(a))$ is eventually above b, hence there exists $k \in \mathbb{N}$ such that for all $n \geq k$, $(a \searrow b) \sqsubseteq f_n$, and it follows that $f_n \to f$ in $[A \to B]$.
Conversely, suppose $f_n \to f$ in $[A \to B]$ and let $(x_n) \to x$ in A. We must show that $f_n(x_n) \to f(x)$ in B, i.e. that for each compact $b \sqsubseteq f(x)$, $(f_n(x_n))$ is eventually above b. There exists a compact $a \sqsubseteq x$ in A with $b \sqsubseteq f(a)$, hence $(a \searrow b) \sqsubseteq f$. But (x_n) is eventually above a, and (f_n) is eventually above $(a \searrow b)$. Thus there exists $k \in \mathbb{N}$ such that for all $n \geq k$, $b = (a \searrow b)(a) \sqsubseteq f_n(a) \sqsubseteq f_n(x_n)$, hence $f_n \to f$ in B^A. Now $B^A \cong [A \to B]$ in **Seq** follows immediately. ☐

Notice that the above lemma holds in particular for $\mathcal{P}\omega^{\mathcal{P}\omega}$. Now we can relate ω-projecting maps to **Lim**-pre-embeddings, and after that we are able to show that **PQ** has exponentials and the inclusion into **Seq** preserves them.

Lemma 2.2.12. *If $i : X \hookrightarrow Y$ is a* **Lim***-pre-embedding between cb-spaces, then $\mathcal{P}\omega^i : \mathcal{P}\omega^Y \to \mathcal{P}\omega^X$ is an ω-projecting topological quotient map.*

Proof: Suppose Z is a cb-space and $f : Z \to \mathcal{P}\omega^X$ any continuous map. Let $g : Z \times X \to \mathcal{P}\omega$ be its exponential transpose. By lemma 2.2.9, $\mathrm{id}_Z \times i : Z \times X \hookrightarrow$

$Z \times Y$ is a **Lim**-pre-embedding between cb-spaces and thus, by lemma 2.2.10, a topological pre-embedding. Hence, by lemma 1.3.3, there exists $\overline{g} : Z \times Y \to \mathcal{P}\omega$ making the following diagram commute:

$$\begin{tikzcd}
Z \times Y \arrow[r, "\overline{g}"] & \mathcal{P}\omega \\
Z \times X \arrow[u, "\mathrm{id}_Z \times i"] \arrow[ur, "g"'] &
\end{tikzcd}$$

By transposition, this yields a continuous $\overline{f} : Z \to \mathcal{P}\omega^Y$ such that $\mathcal{P}\omega^i \circ \overline{f} = f$ and so $\mathcal{P}\omega^i$ is ω-projecting topological quotient map. $\square$

Notice that in every cartesian-closed category, for any object Y, the contravariant functor $Y^{(-)}$ takes colimits to limits. Thus, if $q : X \twoheadrightarrow Q$ is ω-projecting in **Lim**, then, by lemma 2.2.3, it is a regular epi and so for each limit space Y, $Y^q : Y^Q \to Y^X$ is a regular mono (and thus a **Lim**-pre-embedding).

Lemma 2.2.13. *For all cb-spaces X, Y, the exponential Y^X, taken in* **Seq**, *is a PQ-space.*

Proof: Let $i_X : X \to \mathcal{P}\omega$ and $i_Y : Y \to \mathcal{P}\omega$ be topological pre-embeddings. Consider the following pullback in **Seq**:

$$\begin{tikzcd}
P \arrow[r, "h"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}\omega^{\mathcal{P}\omega} \arrow[d, "\mathcal{P}\omega^{i_X}"] \\
Y^X \arrow[r, "{i_Y}^X"'] & \mathcal{P}\omega^X
\end{tikzcd}$$

By lemma 2.2.9, ${i_Y}^X$ and h are **Lim**-pre-embeddings and, by lemma 2.2.12, $\mathcal{P}\omega^{i_X}$ is an ω-projecting topological quotient map. As $\mathcal{P}\omega^{\mathcal{P}\omega}$ is countably-based, by lemma 2.2.11, P is countably-based as well. Finally, by lemmas 2.2.4 and 2.2.5, q is ω-projecting topological quotient map and thus Y^X is in **PQ**. $\square$

Proposition 2.2.14. **PQ** *has exponentials and the inclusion functor I :* **PQ** $\hookrightarrow$ **Seq** *preserves them*

Proof: Suppose Q, R are PQ-spaces. Then there exist cb-spaces X, Y and ω-projecting quotient maps $q : X \twoheadrightarrow Q$ and $r : Y \twoheadrightarrow R$. Since r is w-projecting, so is $r^X : Y^X \twoheadrightarrow R^X$ by lemma 2.2.5. By lemma 2.2.13, Y^X is a PQ-space and so R^X also is a PQ-space. Since $q : X \twoheadrightarrow Q$ is ω-projecting, $R^q : R^Q \to R^X$ is a regular mono. But by proposition 2.2.7, **PQ** has all its regular **Lim**-subobjects and so R^Q

is a PQ-space. The situation is illustrated by the following diagram:

$$\begin{tikzcd}
P' \arrow[r, tail] \arrow[dd, tail] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, tail] \\
 & Y^X \arrow[d, tail, "r^X"] \\
R^Q \arrow[r, tail, "R^q"'] & R^X
\end{tikzcd}$$

where P is the cb-space constructed in lemma 2.2.13, and P' is countably-based.

□

Putting all together, we get:

Corollary 2.2.15. **PQ** *is bi-cartesian closed, has equalizers and the inclusion functor* $I : \mathbf{PQ} \hookrightarrow \mathbf{Seq}$ *preserves this structure.*

Proof: By propositions 2.2.1 and 2.2.7, **PQ** has finite limits and sums, which are preserved by I.
Finally, by proposition 2.2.14, **PQ** inherits exponentials from **Seq**. □

In order to obtain a good category of predomains in chapter 4, we have to distinguish those sequential spaces (and PQ-spaces) that satisfy T_0. We can transfer the notion of a T_0-space to **Lim** using the following property

(†) $x = y$ if for all sequences (x_n), $(x_n) \to x$ if and only if $(x_n) \to y$.

It is easy to see that a sequential space X satisfies (†) if and only if it satisfies T_0 as a topological space.

Definition 2.2.16. $\mathbf{Seq}_0$ is the full subcategory of **Seq** whose objects X satisfy (†).
$\mathbf{PQ}_0$ is the full subcategory of **PQ** whose objects satisfy (†).

It is a straightforward verification that $\mathbf{Seq}_0$ is an exponential ideal in **Seq**, and similarly $\mathbf{PQ}_0$ an exponential ideal of **PQ**. Thus we get the following corollary.

Corollary 2.2.17. $\mathbf{PQ}_0$ *is bi-cartesian closed, has equalizers and the inclusion functor* $I : \mathbf{PQ}_0 \hookrightarrow \mathbf{Seq}_0$ *preserves this structure.*

Proof: $\mathbf{0}_{\mathbf{Seq}}$ and $\mathbf{1}_{\mathbf{Seq}}$ and finite sums of T_0-spaces are T_0-spaces. Also subspaces (regular subobjects) and products of T_0-spaces satisfy T_0, hence their sequentializations satisfy T_0. Finally, $\mathbf{PQ}_0$ is an exponential ideal of **PQ** and thus has exponentials. □

2.3 Characterization of PQ as QCB

In this section, we will show that the objects of **PQ** are exactly the qcb-spaces. It is clear that every PQ-space is a qcb-space, so we have to show that each qcb-space appears as ω-projecting quotient of some cb-space.

Lemma 2.3.1. *If Q is a qcb-space and R is a PQ-space, then the exponential R^Q, taken in* **Seq***, is a PQ-space.*

Proof: There exists a cb-space X and a topological quotient map $q : X \twoheadrightarrow Q$, which, by lemma 2.1.5, is a morphism in **Seq**. This morphism is a regular epi in **Sp**, hence arises as co-equalizer of continuous maps $f, g : Y \to X$. Denote by Y^* the topological space, whose underlying set is the one of Y and whose topology has as base $f^{-1}(\mathcal{B}) \cup g^{-1}(\mathcal{B})$, where $\mathcal{B}$ is some countable base of X. Then Y^* is a cb-space, hence a sequential space and $f, g : Y^* \to X$ are continuous. Thus all these spaces and morphisms are in **Seq**, hence in **Lim**. Moreover, it is easily verified that q is a co-equalizer of f and g in **Lim**, making $R^q : R^Q \to R^X$ to a regular mono in **Lim**. But, by corollary 2.2.15, R^X is a PQ-space and, by proposition 2.2.7, **PQ** has all its regular **Lim**-subobjects. Thus R^Q is a PQ-space. $\square$

Let Σ denote the Sierpinski-space (the two-element space whose topology is given by $\{\emptyset, \{\top\}, \{\bot, \top\}\}$, i.e. any sequence converges to $\bot$ and a sequence converges to $\top$ if and only if it is eventually $\top$). Then for any sequential space X, we can construct Σ^X in **Seq**. This will be a sequential space consisting of all open subsets of X.

Now we get an evaluation map $\varepsilon : X \times \Sigma^X \to \Sigma$, that has an exponential transpose, which we will denote by $\eta_X : X \to \Sigma^{\Sigma^X}$. In topological terms, η_X assigns to each $x \in X$ its open neighbourhood filter $\mathcal{U}(x)$.

The following lemma and its consequences will not only be used in this section, but also play an important role in chapter 4.

Lemma 2.3.2. *For any object X of* **Seq***, $\eta_X : X \to \Sigma^{\Sigma^X}$ is a* **Lim***-pre-embedding.*

Proof: We have to show that $(\eta_X(x_n)) \to \eta_X(x)$ in Σ^{Σ^X} implies $(x_n) \to x$. So assume $(\eta_X(x_n)) \to \eta_X(x)$. Then for each (sequentially) open $U \subseteq X$, $(\eta_X(x_n))(U) \to \eta_X(x)(U)$. Let $x \in U$, then $\eta_X(x)(U) = \top$, hence there exists $i \in \mathbb{N}$ such that $\eta_X(x_n)(U) = \top$ for $n \geq i$. This gives $x_n \in U$ for $n \geq i$ and, since X was sequential, yields $(x_n) \to x$ in X. $\square$

Corollary 2.3.3. *For any object X of* $\mathbf{Seq}_0$*, $\eta_X : X \to \Sigma^{\Sigma^X}$ is a regular mono in* **Lim***.*

Proof: Clearly for a T_0-space X, η_X is injective, since (†) is equivalent to the property that distinct $x, y \in X$ can be separated by a morphism $f : X \to \Sigma$. Since regular monos in **Lim** are given by **Lim**-pre-embeddings which in addition are injective, η_X is a regular mono in **Lim**. $\square$

Note that the above results hold in particular for all objects of **PQ** and $\mathbf{PQ}_0$.

Theorem 2.3.4. *Q is a qcb-space if and only if it is a PQ-space.*

Proof: Let Q be a PQ-space. Then there exists a cb-space X and an ω-projecting topological quotient map $q : X \twoheadrightarrow Q$, hence Q is a qcb-space.
Conversely, suppose Q is a qcb-space. By lemma 2.3.1, Σ^{Σ^Q} is a PQ-space. Consider the following pullback,

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\eta_Q^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ Q \arrow[r, "\eta_Q"'] & \Sigma^{\Sigma^Q} \end{tikzcd}$$

where f is ω-projecting and B a cb-space. By the previous lemma, η_Q is a **Lim**-pre-embedding. Thus, by lemma 2.2.9, g also is a **Lim**-pre-embedding and so A is a cb-space. By lemma 2.2.5, $\eta_Q^* f$ is ω-projecting, hence Q is a PQ-space. □

The preceding theorem yields that **PQ** coincides with **QCB**, the category of qcb-spaces and continuous maps. Similarly, one has that $\mathbf{PQ}_0$ coincides with $\mathbf{QCB}_0$, the latter consisting of the qcb-spaces which satisfy T_0. From now on we will use **QCB** instead of **PQ**, and $\mathbf{QCB}_0$ instead of $\mathbf{PQ}_0$, since the notion of a qcb-space is more familiar than the notion of a PQ-space.

2.4 QCB as topological subcategory

In the previous sections, we have looked at **QCB** as a subcategory of **Seq**. It makes also sense to look at **QCB** as a subcategory of other cartesian-closed subcategories of **Sp**. In fact, we will see that there exist many cartesian-closed categories $\mathbf{Sp}_{\mathcal{C}}$ of topological spaces, containing **QCB**, for which the inclusion $I : \mathbf{QCB} \hookrightarrow \mathbf{Sp}_{\mathcal{C}}$ is a cartesian-closed functor. This result was shown in a recent paper by M. Escardó, J. Lawson and A. Simpson [ELS04], and we will give a sketch of their results without any proofs.

A main ingredient of the final result of this section is yet another characterization of **QCB** given by the following.

Definition 2.4.1. Let X be a topological space. A *sequential pseudobase* of X, is a family $\mathcal{B}$ of subsets of X, satisfying that for each converging sequence $(x_n) \to x$ and any open subset U containing x, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$ and (x_n) is eventually in B.

Theorem 2.4.2. *A topological space X is a qcb-space if and only if it is sequential and has a countable sequential pseudobase.*

The two parts of the proof of this theorem are due to A. Bauer [Bau01] and J. Lawson [ELS04] and both are based on results given by M. Schröder [Sch02].

Definition 2.4.3. A topological space X is called *exponentiable*, if for each topological space Y, there exists a topological space Y^X, whose elements are the continuous maps between X and Y, and for every topological space A, there is a natural bijection $f \leftrightarrow \overline{f}$ between the set of continuous maps $A \times X \to Y$ and the set of continuous maps $A \to Y^X$.
A class $\mathcal{C}$ of topological spaces is called *productive*, if all spaces in $\mathcal{C}$ are exponentiable.

It is known that the largest productive class is formed by the core-compact spaces. Other productive classes are the compact spaces, the compact Hausdorff spaces or the quotients of locally compact spaces.

Definition 2.4.4. Let $\mathcal{C}$ be a class of topological spaces.
For a topological space X, a continuous map $f : A \to X$, where A is in $\mathcal{C}$, is called *$\mathcal{C}$-probe* over X.
The *$\mathcal{C}$-generated topology over X* is the finest topology for which all $\mathcal{C}$-probes over X are continuous.
$\mathbf{Sp}_{\mathcal{C}}$ is the full subcategory of $\mathbf{Sp}$, consisting of those topological spaces for which the topology is $\mathcal{C}$-generated. ($\mathbf{Sp}_{\mathcal{C}}$ is called the category of $\mathcal{C}$-generated spaces.)

Clearly, the $\mathcal{C}$-generated topology is determined uniquely and there is a well-defined functor $\mathcal{C} : \mathbf{Sp} \to \mathbf{Sp}_{\mathcal{C}}$, sending X to $\mathcal{C}X$, the space whose topology is the $\mathcal{C}$-generated topology over X. It is also easy to see that the smaller $\mathcal{C}$ is, the finer is the $\mathcal{C}$-generated topology. The main theorem about categories of $\mathcal{C}$-generated spaces is the following.

Theorem 2.4.5. *If $\mathcal{C}$ is productive, then $\mathbf{Sp}_{\mathcal{C}}$ is cartesian-closed.*

In fact, since $\mathbf{Sp}$ itself has all colimits, $\mathbf{Sp}_{\mathcal{C}}$ is bi-cartesian-closed whenever $\mathcal{C}$ is productive. The following theorem gives a nice description of the $\mathcal{C}$-generated spaces.

Theorem 2.4.6. *A topological space X is $\mathcal{C}$-generated if and only if it is the colimit (in $\mathbf{Sp}$) of generating spaces, i.e. if and only if it is the quotient of a sum of generating spaces.*

Recall the definition of the topological space $\mathcal{N}$, given in section 1.4. For $\mathcal{S} = \{\mathcal{N}\}$, **Seq** is the category of $\mathcal{S}$-generated spaces and the $\mathcal{S}$-generated topology over X is just the topology of sequentially open sets of X.

Definition 2.4.7. For a topological space X, define a relation $\ll$ on the subsets of X, by $S \ll U$, if any open cover of U contains a finite subcover of S.
A family of $\mathcal{B}$ of subsets of X is called a *$\ll$-pseudobase* of X, if whenever $S \ll U \subseteq X$ and U is open, then there exists $B \in \mathcal{B}$ with $S \subseteq B \subseteq U$.

The following lemma is a combination of several lemmas from [ELS04].

Lemma 2.4.8. *Let $\mathcal{C}$ denote the class of core-compact spaces and $\mathbf{Sp}_{\mathcal{C}}$ the category of core-compacly generated spaces..*

(i) If $\mathcal{B}$ is a $\ll$-pseudobase of X, then it is a sequential pseudobase of X.

(ii) If $\mathcal{B}$ is a sequential pseudobase of X, then its closure under finite unions is a $\ll$-pseudobase of X.

(iii) If X is a topological space with a countable $\ll$-pseudobase, then $\mathcal{C}X$ carries the sequential topology over X.

(iv) The sequential and core-compactly generated topology agree for any space with countable $\ll$-pseudobase.

Now combining the fact that **Seq** is cartesian-closed, theorem 2.4.2, lemma 2.4.8 and the observation that if $\mathcal{C}$ contains $\mathcal{N}$, then the $\mathcal{C}$-generated topology on X is coarser than the sequential topology and finer than the core-compactly generated topology, yields the following result.

Theorem 2.4.9. *If $\mathcal{C}$ is productive and contains $\mathcal{N}$, then* **QCB** *is a subcategory of* $\mathbf{Sp}_{\mathcal{C}}$ *and the inclusion $I : \mathbf{QCB} \hookrightarrow \mathbf{Sp}_{\mathcal{C}}$ preserves the bi-cartesian-closed structure.*

Note that neither the previous result can be extended to the whole of **Seq**, nor can lemma 2.4.8 (iii) and (iv) to spaces with an arbitrary $\ll$-pseudobase. Counterexamples are given in [ELS04]. The authors also provide an alternative proof of the cartesian-closedness of **QCB**, involving $\ll$-pseudobases and $\ll$-*pseudosubbases.*

2.5 QCB as subcategory of Asm($\mathcal{P}\omega$)

As mentioned in section 1.3, **QCB** not only is a subcategory of **Lim** but also of $\mathbf{Asm}(\mathcal{P}\omega)$. In this section we will introduce the inclusion $\mathbf{QCB} \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$ and show that it preserves the bi-cartesian-closure of **QCB**. We start by observing that **QCB** is a full reflective subcategory of $\omega\mathbf{Equ}$.

Let $(X, \sim)$ be an ω-equlogical space, then $X/\sim$ is a qcb-space. Moreover, any equivariant $f : (X, \sim) \to (Y, \approx)$ map between ω-equilogical spaces is a continuous map between the quotients $X/\sim$ and $Y/\approx$. Hence we get a functor $R : \omega\mathbf{Equ} \to \mathbf{QCB}$, by $R((X, \sim)) = X/\sim$ and $R(f) = f$.
Conversely, suppose Q is a qcb-space and $q : X \twoheadrightarrow Q$ an ω-projecting topological quotient map for a cb-space X. If $\sim_q$ denotes the kernel relation of q, $(X, \sim_q)$ becomes an ω-equilogical space. Moreover, for any other ω-projecting topological quotient map $q' : X' \twoheadrightarrow Q$ such that X' is countably-based, the properties of ω-projecting maps ensure that $(X, \sim_q) \cong (X', \sim_{q'})$ in $\omega\mathbf{Equ}$. Furthermore, if $f : Q \to R$ is a continuous map between qcb-spaces and $q : X \twoheadrightarrow Q$ and $r : Y \twoheadrightarrow R$ are ω-projecting topological quotient maps, then f is an equivariant map between $(X, \sim_q)$ and $(Y, \sim_r)$, again by properties of ω-projecting maps. Hence $Q \mapsto (X, \sim_q)$ gives rise to a well-defined inclusion functor $\mathbf{QCB} \to \omega\mathbf{Equ}$, which we denote by $\widehat{J}$.
It is straightforward to verify that these functors form an adjunction $R \dashv \widehat{J}$ and thus **QCB** becomes a full reflective subcategory of $\omega\mathbf{Equ}$.

Proposition 2.5.1. *$\widehat{J}$ preserves finite products and sums and equalizers. Furthermore* **QCB** *contains all its regular ω***Equ***-subobjects.*

Proof: It is easily checked that $\widehat{J}$ preserves initial and terminal objects. Since it is a right adjoint, $\widehat{J}$ preserves finite products and equalizers, and from the fact that ω-projecting maps in **Lim** are closed under sums by proposition 2.2.7, it follows that for qcb-spaces P and Q, $\widehat{J}(P+Q) \cong \widehat{J}(P) + \widehat{J}(Q)$.
For the last part, observe that ω**Equ** has a regular subobject classifier $\nabla\mathbf{2}$, given by the two-element space equipped with the discrete topology and the identity relation. Clearly $\nabla\mathbf{2}$ is in the image of $\widehat{J}$ and thus, by the above, **QCB** contains all its regular ω**Equ**-subobjects. $\square$

Using the equivalence of ω**Equ** and $\mathbf{Asm}(\mathcal{P}\omega)$, we get a functor $J : \mathbf{QCB} \to \mathbf{Asm}(\mathcal{P}\omega)$. For an explicit description of J, let $q : X \twoheadrightarrow Q$ be an ω-projecting topological quotient map for a cb-space X and $i_X : X \hookrightarrow \mathcal{P}\omega$ be a topological pre-embedding. Set $J(Q) = (Q, \|\cdot\|)$, where $\|x\| = \{i_X(x') |\, q(x') = x\}$. On morphisms J acts as identity. For if $f : Q \to R$ is a morphism in **QCB**, then the following diagram

$$\begin{tikzcd}
\mathcal{P}\omega \arrow[r, "\exists a"] & \mathcal{P}\omega \\
X \arrow[u, "i_X"] \arrow[r, "\exists\overline{f\circ q}"] \arrow[d, two heads, "q"'] & Y \arrow[u, "i_Y"'] \arrow[d, two heads, "r"] \\
Q \arrow[r, "f"'] & R
\end{tikzcd}$$

ensures that f is a realizable map. Moreover, it is clear that J is full and faithful and thus provides an inclusion into $\mathbf{Asm}(\mathcal{P}\omega)$. Together with the results above, we get the following.

Proposition 2.5.2. *J preserves exponentials.*

Proof: Recall the pullback diagram, given in proposition 2.2.14 to construct the exponential R^Q in **Seq**. There we constructed a cb-space P' and an ω-projecting topological quotient map $s : P' \twoheadrightarrow R^Q$. Calculating P' in the pullback yields $J(R^Q) \cong (R^Q, \|\cdot\|_{R^Q})$, where $a \in \|f\|_{R^Q}$ if and only if a realizes f, as in the diagram above. Thus $J(R^Q) \cong J(R)^{J(Q)}$. $\square$

These results show:

Corollary 2.5.3. *There exists an inclusion functor $J : \mathbf{QCB} \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$ preserving the bi-cartesian-closed structure and equalizers.*

Proof: Since J is equivalent to $\widehat{J}$, it preserves finite limits and sums by proposition 2.5.1. By the preceding lemma, it also preserves exponentials, hence it is a bi-cartesian-closed functor. $\square$

Corollary 2.5.4. *There exists an inclusion functor* $J : \mathbf{QCB}_0 \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$ *preserving the bi-cartesian-closed structure and equalizers.*

Proof: Obviously, the above results restrict to $\mathbf{QCB}_0$. $\square$

In fact, the following theorem is true, from which it follows that **QCB** and $\mathbf{QCB}_0$ are exponential ideals of $\mathbf{Asm}(\mathcal{P}\omega)$, by 1.857 of [FS90].

Theorem 2.5.5. *The reflection functor* $R : \omega\mathbf{Equ} \to \mathbf{QCB}$ *preserves finite products.*

Proof: Let $\sim_X$ and $\sim_Y$ be equivalence relations on cb-spaces X and Y, and $q_X : X \twoheadrightarrow X/{\sim_X}$ and $q_Y : Y \twoheadrightarrow Y/{\sim_Y}$ be topological quotient maps. Then for the relation $\sim_{X\times Y}$ on $X \times Y$, given by $(x, y) \sim_{X\times Y} (x', y')$ if and only if $x \sim_X x'$ and $y \sim_Y y'$, we get, by a straightforward calculation of the corresponding bases, that the topological product $X/{\sim_X} \times Y/{\sim_Y}$ is isomorphic (in **Sp** and thus also in **Seq**) to the topological quotient $X \times Y/{\sim_{X\times Y}}$.
Now the result follows, since $R((X, \sim_X)) \times R((Y, \sim_Y)) = X/\sim_X \times Y/\sim_Y$, and $R((X, \sim_X) \times (Y, \sim_Y)) = X \times Y/{\sim_{X\times Y}}$ $\square$

Thus, by **QCB** being a bi-cartesian-closed full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$ and **Seq** (and also of the categories considered in section 2.4), we get a very rich structure and theory on a category containing most of the spaces considered in mathematics and theoretical computer science. These are remarkable results, since on first sight the categories **Seq** and $\mathbf{Asm}(\mathcal{P}\omega)$ do not have much in common, except that both include $\omega\mathbf{Sp}$ as a subcategory. In fact, the following result of [MS02] holds.

Let $Q : \omega\mathbf{Equ} \to \mathbf{Sp}$ be the functor that maps an ω-equilogical space $(X, \sim)$ to the topological quotient $X/{\sim}$ and an equivariant f to its continuous counterpart. Notice that Q is not a full functor.

Theorem 2.5.6. **QCB** *is the largest common full subcategory* **C** *of* $\omega\mathbf{Equ}$ *and* **Sp***, that contains* $\omega\mathbf{Sp}$ *as a subcategory and for which the inclusion* $I_{\mathbf{C}} : \mathbf{C} \hookrightarrow \mathbf{Sp}$ *factors through* Q*, as in the following diagram:*

$$\begin{tikzcd}
\omega\mathbf{Equ} \arrow[rr, "Q"] & & \mathbf{Sp} \\
& \mathbf{C} \arrow[ul, hook, "\widehat{J}_{\mathbf{C}}"] \arrow[ur, hook', "I_{\mathbf{C}}"'] &
\end{tikzcd}$$

Proof: Suppose **C** is a common full subcategory of $\omega\mathbf{Equ}$ and **Sp** and the inclusion functor $I_{\mathbf{C}} : \mathbf{C} \hookrightarrow \mathbf{Sp}$ factors through Q. Furthermore let $\widehat{J}_{\mathbf{C}} : \mathbf{C} \hookrightarrow \omega\mathbf{Equ}$

denote the inclusion into ω**Equ**, for which $Q \circ \widehat{J}_{\mathbf{C}} = I_C$ holds. Suppose A is an object of **C** and $(X, \sim) = \widehat{J}_{\mathbf{C}}(A)$. We show that the quotient map $q : X \to X/\sim$ is ω-projecting.
Let Z be a cb-space and $f : Z \to X/\sim$ be any continuous map, where $X/\sim$ is equipped with the quotient topology. By **C** being a full subcategory of ω**Equ** and **Sp**, it follows that Q restricted to the image of $\widehat{J}_{\mathbf{C}}$ is full. Thus if $\sim_Z$ denotes the identity relation on Z, there exists an equivariant $g : (Z, \sim_Z) \to (X, \sim)$ such that $Q(g) = f$. Thus there exists a continuous $\overline{g} : Z \to X$ such that $q \circ \overline{g} = f$. Hence q is ω-projecting. $\square$

It follows that **QCB** is the largest category of topological spaces, that lives "naturally" in **Seq** and ω**Equ** (and hence in $\mathbf{Asm}(\mathcal{P}\omega)$).

Chapter 3

Expers and Complete Assemblies

3.1 Facts about $\mathbf{Mod}(\mathcal{P}\omega)$ and $\mathbf{Asm}(\mathcal{P}\omega)$

In this chapter, we will study some subcategories of the category of modest sets over $\mathcal{P}\omega$. We will start by showing some categorial facts about $\mathbf{Mod}(\mathcal{P}\omega)$ and $\mathbf{Asm}(\mathcal{P}\omega)$. Again we allow ourselves to simply write X for an assembly (or modest set) instead of $(X, \|\cdot\|_X)$, when no confusion may occur.

Notice that $\mathbf{Mod}(\mathcal{P}\omega)$ is a full reflective subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$. To see this, let X be an assembly. Let $\sim_X$ be the least equivalence relation on X containing $\{(x,x')|\ \|x\|_X \cap \|x'\|_X \neq \emptyset\}$. Furthermore define $\|[x]_{\sim_X}\|_{X/\sim_X} = \bigcup_{x' \in [x]_{\sim_X}} \|x'\|_X$. Then $(X/\sim_X, \|\cdot\|_{X/\sim_X})$ becomes a modest set.
For a morphism $f : X \to Y$ in $\mathbf{Asm}(\mathcal{P}\omega)$, tracked by $a \in \mathcal{P}\omega$, denote by $\tilde{f}$ the map $(X/\sim_X) \to (Y/\sim_Y)$ given by $f([x]_{\sim_X}) = [f(x)]_{\sim_Y}$. It is straightforward to check that this map is well-defined and tracked by a, as well. Thus $\tilde{f} : (X/\sim_X) \to (Y/\sim_Y)$ is a morphism in $\mathbf{Mod}(\mathcal{P}\omega)$.
Setting $R(X) = X/\sim_X$ and $R(f) = \tilde{f}$ yields the reflection functor $R : \mathbf{Asm}(\mathcal{P}\omega) \to \mathbf{Mod}(\mathcal{P}\omega)$, which is left adjoint to the obvious inclusion functor $I : \mathbf{Mod}(\mathcal{P}\omega) \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$.

Lemma 3.1.1. *$\mathbf{Mod}(\mathcal{P}\omega)$ and $\mathbf{Asm}(\mathcal{P}\omega)$ both have coequalizers.*

Proof: Assume $f, g : X \to Y$ are morphisms in $\mathbf{Asm}(\mathcal{P}\omega)$. Define $\approx$ to be the least equivalence relation on Y containing $\{(f(x), g(x))|\ x \in X\}$. Furthermore define $\|[y]_\approx\|_{Y/\approx} = \bigcup_{y' \in [y]_\approx} \|y'\|_Y$, so $(Y/\approx, \|\cdot\|_{Y/\approx})$ becomes an assembly. Then $[\cdot]_\approx : Y \to (Y/\approx)$ is a coequalizer for f and g, tracked by i. For if there is an $h : Y \to Z$, tracked by $a \in \mathcal{P}\omega$, such that $h \circ f = h \circ g$, then there is a mediating arrow $k : (Y/\approx) \to Z$ in $\mathbf{SET}$, which is tracked by a, too.
It is also clear that $Y/\approx$ is a modest set, if Y is. $\square$

Theorem 3.1.2. *The inclusion functor $I : \mathbf{Mod}(\mathcal{P}\omega) \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$ preserves finite limits and colimits and exponentials.*

Proof: This is a direct consequence of the fact that all constructions coincide in those categories. □

Proposition 3.1.3. *If $m : A \rightarrowtail X$ is a regular mono in $\mathbf{Asm}(\mathcal{P}\omega)$, whose codomain is a modest set, then it is already a regular mono in $\mathbf{Mod}(\mathcal{P}\omega)$.*

Proof: Suppose m is an equalizer of $f, g : X \to Y$. By the construction of equalizers given in theorem 1.2.3, A is a modest set. Notice that $\mathbf{Mod}(\mathcal{P}\omega)$ has pushouts and the inclusion functor preserves them. Consider the following pushout diagram:

$$\begin{tikzcd}
A \arrow[r, tail, "m"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "p.o."] & X \arrow[d, "f'"] \arrow[ddr, "f"] & \\
X \arrow[r, "g'"'] \arrow[drr, "g"'] & Z \arrow[dr, dashed] & \\
& & Y
\end{tikzcd}$$

Then Z is a modest set, and one easily checks that m is an equalizer of f' and g', hence a regular mono in $\mathbf{Mod}(\mathcal{P}\omega)$. □

Finally, we want to show that regular monos in $\mathbf{Asm}(\mathcal{P}\omega)$ and $\mathbf{Mod}(\mathcal{P}\omega)$ are closed under composition. For this we need the following definitions.

Definition 3.1.4. $\nabla\mathbf{2}$ is the assembly $(\{\bot, \top\}, \|\cdot\|)$, where $\|\bot\| = \|\top\| = \mathcal{P}\omega$. For an assembly X, $\top_X : X \to \nabla\mathbf{2}$ is the morphism with constant value $\top$.

Definition 3.1.5. An assembly A is said to be a *canonical subobject* (or subobject in *canonical form*) of an assembly B, if there exists a mono $m : A \rightarrowtail B$ such that $\|x\|_A = \|mx\|_B$ for all $x \in A$.

Using the equivalence of $\mathbf{Mod}(\mathcal{P}\omega)$ and $\mathbf{PER}(\mathcal{P}\omega)$, R is a canonical subobject of S in $\mathbf{PER}(\mathcal{P}\omega)$, if $\mathsf{carrier}(\mathsf{R}) \subseteq \mathsf{carrier}(\mathsf{S})$ and $[u]_{\mathsf{R}} = [u]_{\mathsf{S}}$ for all $u \in \mathsf{carrier}(\mathsf{R})$.

Now $\nabla\mathbf{2}$ becomes a regular subobject classifier in $\mathbf{Asm}(\mathcal{P}\omega)$ in the following sense: If X' is a regular subobject of X in canonical form, and $m : X' \rightarrowtail X$ the corresponding regular mono, then the following square is a pullback,

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, tail, "\top"] \\
X \arrow[r, "\chi_{X'}"'] & \nabla\mathbf{2}
\end{tikzcd}$$

where $\chi_{X'}$ is the function that maps an element x to $\top$ if and only if x is in the image of m. All these maps are tracked by i, hence morphisms in $\mathbf{Asm}(\mathcal{P}\omega)$. Clearly $\top$ is a regular mono and so is m.
Moreover, by the description of equalizers, given in theorem 1.2.3, it follows that each regular subobject in $\mathbf{Asm}(\mathcal{P}\omega)$ is isomorphic to one in canonical form. Thus all regular monos appear as pullbacks of $\top$, and so $\nabla\mathbf{2}$ becomes a regular subobject classifier in $\mathbf{Asm}(\mathcal{P}\omega)$. It also follows that all canonical subobjects are regular subobjects.

Proposition 3.1.6. *Compositions of regular monos in* $\mathbf{Mod}(\mathcal{P}\omega)$ *and* $\mathbf{Asm}(\mathcal{P}\omega)$ *are regular monos.*

Proof: Regular monos in canonical form are obviously closed under composition. Since each regular mono is isomorphic to one in canonical form, we get the desired result. □

Definition 3.1.7. $\mathbf{N}$ is the assembly $(\mathbb{N}, \|\cdot\|)$, where $\|n\| = \{\{n\}\}$.

Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be given by

$$f(n) = n + 1, \qquad g(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{if } n \neq 0. \end{cases}$$

Then it is clear that $\varphi : \mathcal{P}\omega \to \mathcal{P}\omega$ and $\psi : \mathcal{P}\omega \to \mathcal{P}\omega$, given by

$$\varphi(\{n_0, n_1, \ldots\}) = \{f(n_0), f(n_1), \ldots\}, \quad \psi(\{n_0, n_1, \ldots\}) = \{g(n_0), g(n_1), \ldots\},$$

are continuous. Let $\mathsf{fun}(\varphi)$ be a realizer for $\mathsf{succ} : \mathbf{N} \to \mathbf{N}$ and $\mathsf{fun}(\psi)$ be a realizer for $\mathsf{pred} : \mathbf{N} \to \mathbf{N}$. Then succ is a successor function and pred a predecessor function on $\mathbf{N}$. It is not hard to check that $\mathbf{N}$ together with succ and $z : \mathbf{1} \to \mathbf{N}$, $z(*) = 0$, becomes a natural numbers object (NNO) in $\mathbf{Mod}(\mathcal{P}\omega)$ and $\mathbf{Asm}(\mathcal{P}\omega)$.

3.2 The category $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$

Now we will define the category $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$, which will be shown to be equivalent to $\mathbf{QCB}_0$ in chapter 4. Then we will provide the required properties of $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$, needed to obtain this result. We start by defining a modest set Σ, which will correspond to Sierpinski space under the equivalence to be constructed later. Notice that most of the results below do not depend on that particular choice of Σ.

Definition 3.2.1. Σ is the modest set $(\{\perp, \top\}, \|\cdot\|)$ given by $\|\perp\| = \{\emptyset\}$ and $\|\top\| = \mathcal{P}\omega\backslash\{\emptyset\}$.
A Σ-*extensional object* in $\mathbf{Mod}(\mathcal{P}\omega)$ is a modest set X, which is a regular subobject of some power of Σ.

Since $\mathbf{Mod}(\mathcal{P}\omega)$ is equivalent to $\mathbf{PER}(\mathcal{P}\omega)$, the full subcategory of extensional objects with respect to Σ in $\mathbf{Mod}(\mathcal{P}\omega)$ is denoted by $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$, and the objects of $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ are called *expers*. The definition of expers goes back to [FMRS92], where a less categorial description of $\mathbf{ExPer}_\Sigma(K_1)$ is given. Note that it is possible to show that $\mathbf{ExPER}_{\boldsymbol{\Sigma}}(\mathcal{P}\omega)$ is a full reflective subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$, see e.g. [Str99], where expers are called $S-$ *seperable*.

In the following, we will work in the category $\mathbf{PER}(\mathcal{P}\omega)$. For a per R, we set $\mathsf{carrier}(\mathsf{R}) = \{u \in \mathcal{P}\omega |\ u\,\mathsf{R}\,u\}$. Then $\mathsf{carrier}(\mathsf{R}) \subseteq \mathcal{P}\omega$ and R restricts to a total equivalence relation on $\mathsf{carrier}(\mathsf{R})$. The per corresponding to Σ is given by $u\,\Sigma\,v$ if and only if $u = \emptyset = v$ or $u \neq \emptyset \neq v$.

In the following, we will will write $\Sigma(-)$ instead of $\Sigma^{(-)}$ and $\Sigma^{n+1}(-)$ instead of $\Sigma^{\Sigma^n(-)}$.

Definition 3.2.2. For a per R, the map $\eta_\mathsf{R} : \mathsf{R} \to \Sigma^2(\mathsf{R})$, given by $\lambda x.\lambda p.px$.

It is easy to see that η_R is a well-defined $\mathbf{PER}(\mathcal{P}\omega)$-morphism for each per R.

Lemma 3.2.3. *For all pers* R*,* $\Sigma(\eta_\mathsf{R}) \circ \eta_{\Sigma(\mathsf{R})} = id_{\Sigma(R)}$*.*

Proof: We have to show that $(\Sigma(\eta_\mathsf{R}) \circ \eta_{\Sigma(\mathsf{R})})(a)(u) = a(u)$ for $a \in \Sigma(\mathsf{R})$ and $u \in \mathsf{R}$. This follows from:

$$\begin{aligned}(\Sigma(\eta_\mathsf{R}) \circ \eta_{\Sigma(\mathsf{R})})(a)(u) &= (\eta_{\Sigma(\mathsf{R})}(a) \circ \eta_\mathsf{R})(u)\\ &= \eta_{\Sigma(\mathsf{R})}(a)(\eta_\mathsf{R}(u))\\ &= \eta_\mathsf{R}(u)(a)\\ &= a(u).\end{aligned}$$

□

Lemma 3.2.4. *η is a natural transformation* $\mathbf{Id} \to \Sigma^2(-)$*, i.e. the following diagram commutes for all pers* R *and* S*:*

$$\begin{tikzcd} \mathsf{R} \arrow[r, "\eta_\mathsf{R}"] \arrow[d, "a"'] & \Sigma^2(\mathsf{R}) \arrow[d, "\Sigma^2(a)"] \\ \mathsf{S} \arrow[r, "\eta_\mathsf{S}"'] & \Sigma^2(\mathsf{S}) \end{tikzcd}$$

Proof: Clear, as η is the unit of the adjunction $\Sigma(-)^{op} \dashv \Sigma(-) : \mathbf{PER}(\mathcal{P}\omega)^{op} \to \mathbf{PER}(\mathcal{P}\omega)$:

$$\frac{\dfrac{\Sigma(A) \to \Sigma(A)}{\Sigma(A) \times A \to \Sigma}}{A \to \Sigma^2(A)}$$

□

For each $I \subseteq \mathcal{P}\omega$, there is a per, given by $u\,I\,v$ if and only if $u = v$ and $u \in I$. The corresponding modest set is given by $(I, \|\cdot\|_I)$, $\|x\|_I = \{x\}$.

Lemma 3.2.5. *For each per* R, $\Sigma(\mathsf{R})$ *is a canonical subobject of* $\Sigma(\mathsf{carrier}(\mathsf{R}))$.

Proof: Suppose $a\,\Sigma(\mathsf{R})\,a$, then for all $u \in \mathsf{carrier}(\mathsf{R})$, we have $a(u)\,\Sigma\,a(u)$. Hence $\mathsf{carrier}(\Sigma(\mathsf{R})) \subseteq \mathsf{carrier}(\Sigma(\mathsf{carrier}(\mathsf{R})))$. Moreover, $a\,\Sigma(\mathsf{R})\,b$ implies $a\,\Sigma(\mathsf{carrier}(\mathsf{R}))\,b$ and conversely, if $a \in \mathsf{carrier}(\Sigma(\mathsf{R}))$ and $a\,\Sigma(\mathsf{carrier}(\mathsf{R}))\,b$, then for all $u\,\mathsf{R}\,v$, one has $b(u)\,\Sigma\,a(u)\,\Sigma\,a(v)\,\Sigma\,b(v)$, hence $a\,\Sigma(\mathsf{R})\,b$. Thus for all $a \in \mathsf{carrier}(\Sigma(\mathsf{R}))$, we have $[a]_{\Sigma(\mathsf{R})} = [a]_{\Sigma(\mathsf{carrier}(\mathsf{R}))}$, hence $\Sigma(\mathsf{R})$ is a canonical subobject of $\Sigma(\mathsf{carrier}(\mathsf{R}))$. $\square$

For the next theorem, recall that it is a category-theoretic fact that if $m : A \rightarrowtail B$ is a regular mono and $m = f \circ m'$, then $m' : A \to B'$ is a regular mono, as well. For if m is an equalizer of h, k, then m' is an equalizer of $h \circ f, k \circ f$.

Theorem 3.2.6. *In* $\mathbf{PER}(\mathcal{P}\omega)$*, the following are equivalent for any object* R*:*

(i) there is a per S *and a regular mono* $m : \mathsf{R} \rightarrowtail \Sigma(\mathsf{S})$*,*

(ii) $\eta_{\mathsf{R}} : \mathsf{R} \to \Sigma^2(\mathsf{R})$ *is a regular mono,*

(iii) there is a subset $I \subseteq \mathcal{P}\omega$ *and a regular mono* $m : \mathsf{R} \rightarrowtail \Sigma(I)$*.*

Proof:

- $(i) \Rightarrow (ii)$: Suppose (i) is true, then, by lemmas 3.2.3 and 3.2.4, we get

$$m = \Sigma(\eta_{\mathsf{S}}) \circ \eta_{\Sigma(\mathsf{S})} \circ m = \Sigma(\eta_{\mathsf{S}}) \circ \Sigma^2(m) \circ \eta_{\mathsf{R}}.$$

  Thus η must be a regular mono in this case.

- $(ii) \Rightarrow (iii)$: Set $I = \mathsf{carrier}(\Sigma(\mathsf{R}))$ and the claim follows from lemmas 3.1.6 and 3.2.5.

- $(iii) \Rightarrow (i)$: Clear. $\square$

Using the equivalence of $\mathbf{Mod}(\mathcal{P}\omega)$ and $\mathbf{PER}(\mathcal{P}\omega)$, we get the following corollary.

Corollary 3.2.7. *In* $\mathbf{Mod}(\mathcal{P}\omega)$*, the following are equivalent for any object* A*:*

(i) A *is an exper,*

(ii) $\eta_A : A \to \Sigma^2(A)$ *is a regular mono,*

(iii) there is a subset $I \subseteq \mathcal{P}\omega$ *and a regular mono* $m : A \rightarrowtail \Sigma(I)$*.*

3.3 Complete Objects in $\mathbf{Asm}(\mathcal{P}\omega)$

The equivalence of $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ and $\mathbf{QCB}_0$ will give a nice connection between a category obtained from a realizability model and a category of sequential (hence topological) spaces. However in order to obtain a good category of (pre)domains, some more requirements have to be imposed. In classical domain theory, one has suprema of ascending chains, which is used to model recursion. In this section, we will provide a similar concept within our realizability model $\mathbf{Asm}(\mathcal{P}\omega)$. This will yield the category $\mathbf{C}(\mathcal{P}\omega)$ of complete assemblies. In chapter 4, the intersection of $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ and $\mathbf{C}(\mathcal{P}\omega)$, denoted by $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$, will give us the required category of predomains, and we will see that these predomains in fact have a nice description in topological terms.

In classical domain theory, the poset ω, the set of natureal numbers equipped with the usual order, forms a universal ascending chain and the poset $\overline{\omega}$, ω extended by a greatest point ∞, a universal ascending chain with a supremum. So we will start by defining analogues ω and $\overline{\omega}$ to our realizability model and defining completeness. Then we will define a lifting functor in an appropriate way.

Definition 3.3.1. $\overline{\omega}$ is the assembly $(\mathbb{N}\cup\{\infty\}, \|\cdot\|)$, where $\|n\| = \{\{k \in \mathbb{N} | k \leq n\}\}$ and $\|\infty\| = \{\omega\}$. ω is the regular subobject of $\overline{\omega}$, given by $(\mathbb{N}, \|\cdot\|)$, where $\|\cdot\|$ is inherited from $\overline{\omega}$.

Notice that there is an obvious mono $\iota : \omega \to \overline{\omega}$ tracked by i. We can think of a morphism $f : \omega \to X$ as an ω-*chain* in X.

Definition 3.3.2. If a morphism $f : \omega \to X$ extends uniquely along ι, as in the following diagram,

$$\begin{tikzcd}
\overline{\omega} \arrow[r, dashed, "\overline{f}"] & X \\
\omega \arrow[u, "\iota"] \arrow[ur, "f"'] &
\end{tikzcd}$$

then $\overline{f}(\infty)$ is called the *limit* of the ω-chain, given by f.

Notice that this notion of a limit is stronger than that of a supremum in the usual sense. However, in the next chapter we will show that under the inclusion $J : \mathbf{PQ} \to \mathbf{Asm}(\mathcal{P}\omega)$ our definition of a limit of an ω-chain, in $\mathbf{Asm}(\mathcal{P}\omega)$, is in one-to-one correspondence to a supremum of a chain with respect to the specialization order, in $\mathbf{PQ}$. These considerations motivate the following definition.

Definition 3.3.3. An assembly X is said to be *complete* if the morphism $X^{\iota} : X^{\overline{\omega}} \to X^{\omega}$ is an iso.
The full subcategory of complete objects of $\mathbf{Asm}(\mathcal{P}\omega)$ is denoted by $\mathbf{C}(\mathcal{P}\omega)$.

Notice that with this definition for an assembly X to be complete is equivalent the requirement that for every assembly Z and every morphism $f : Z \times \omega \to X$, there is a unique morphism $\overline{f} : Z \times \overline{\omega} \to X$ making the following diagram commute:

$$\begin{tikzcd}
Z \times \overline{\omega} \arrow[r, dashed, "\overline{f}"] & X \\
Z \times \omega \arrow[u, "Z \times \iota"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Theorem 3.3.4. $\mathbf{C}(\mathcal{P}\omega)$ *is bi-cartesian-closed, has equalizers and the inclusion* $\mathbf{C}(\mathcal{P}\omega) \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$ *preserves the structure.*

Proof: As usual we will show that $\mathbf{C}(\mathcal{P}\omega)$ has initial and terminal objects, binary products and sums, exponentials and equalizers.

- Initial and terminal objects: Clear, as $\mathbf{0}^{\overline{\omega}} \cong \mathbf{0} \cong \mathbf{0}^{\omega}$ and $\mathbf{1}^{\overline{\omega}} \cong \mathbf{1} \cong \mathbf{1}^{\omega}$.

- Binary products: Assume X and Y are complete assemblies. Then $(X \times Y)^{\iota} \cong X^{\iota} \times Y^{\iota}$ is an iso, hence $X \times Y$ is complete.

- Binary Sums: Suppose again X and Y are complete assemblies. Furthermore suppose $f : Z \times \omega \to X + Y$ is a morphism. Straightforward considerations (on the realizer level) show that there exist assemblies Z_1 and Z_2 such that $Z \times \omega \cong (Z_1 + Z_2) \times \omega \cong (Z_1 \times \omega) + (Z_2 \times \omega)$ and $f \cong f_1 + f_2$, where $f_1 : Z_1 \times \omega \to X$ and $f_2 : Z_2 \times \omega \to Y$. The following diagram yields that $X + Y$ is complete as well:

$$\begin{tikzcd}
(Z_1 \times \overline{\omega}) + (Z_2 \times \overline{\omega}) \arrow[r, dashed, "\overline{f_1} + \overline{f_2}"] & X + Y \\
(Z_1 \times \omega) + (Z_2 \times \omega) \arrow[u, "(Z_1 \times \iota) + (Z_2 \times \iota)"] \arrow[ur, "f_1 + f_2"'] &
\end{tikzcd}$$

- Exponentials: We will show that $\mathbf{C}(\mathcal{P}\omega)$ is an exponential ideal of $\mathbf{Asm}(\mathcal{P}\omega)$. Let Y be a complete assembly and X be any assembly. Then $(Y^X)^{\iota} \cong (Y^{\iota})^X$ and $(Y^{\iota})^X : (Y^{\overline{\omega}})^X \to (Y^{\omega})^X$ is an iso, as Y^{ι} is an iso by assumption.

- Equalizers: Suppose X and Y are complete assemblies, and $e : E \to X$ is an equalizer of $f, g : X \to Y$. Then for each assembly Z and any $h : Z \times \omega \to A$, we have $e \circ h : Z \times \omega \to X$, and $f \circ e \circ h = g \circ e \circ h$, since $f \circ e = g \circ e$. Consider the following diagram:

$$\begin{tikzcd}
Z \times \overline{\omega} \arrow[dr, dashed, "\overline{h}"'] \arrow[drr, dashed, "\overline{e \circ h}"] & & & \\
 & E \arrow[r] & X \arrow[r, "f, g"] & Y \\
Z \times \omega \arrow[uu, "Z \times \iota"] \arrow[ur, "h"] \arrow[urr, "e \circ h"'] & & &
\end{tikzcd}$$

  By Y being complete, it follows that $f \circ \overline{e \circ h} = g \circ \overline{e \circ h}$ and thus the mediating arrow $\overline{h}$ is the required unique extension of h along $Z \times \iota$. □

We will now give the definition of a lifting functor $L : \mathbf{C}(\mathcal{P}\omega) \to \mathbf{C}(\mathcal{P}\omega)$.

Definition 3.3.5. The lifting functor $L : \mathbf{C}(\mathcal{P}\omega) \to \mathbf{C}(\mathcal{P}\omega)$ is given as follows:

- for an object $X = (X, \|\cdot\|_X)$, $LX = (X \cup \{\bot\}, \|\cdot\|_{LX})$ such that $\|x\|_{LX} = \{\langle \mathsf{i}, u\rangle \mid u \in \|x\|_X\}$ and $\|\bot\|_{LX} = \{\langle \emptyset, u\rangle \mid u \in \mathcal{P}\omega\}$.

- for a morphism $f : X \to Y$, $Lf : LX \to LY$ is given by $(Lf)(x) = f(x)$ and $(Lf)(\bot) = \bot$. If f is tracked by a, then Lf is tracked by $\lambda x.\langle \mathsf{fst} x, a(\mathsf{snd} x)\rangle$.

Note that the canonical morphism $X \to LX$ is tracked by $\lambda x.\langle \mathsf{i}, x\rangle$.

Lemma 3.3.6. *$L\omega \cong \omega$ and $L\overline{\omega} \cong \overline{\omega}$*

Proof: For this, check that the following functions $\Phi, \Psi : \mathcal{P}\omega \to \mathcal{P}\omega$ are continuous:

$$\Phi : u \mapsto \begin{cases} \{0\} \cup \mathsf{succ}(u) & \text{if } \mathsf{fst}(u) \neq \emptyset \\ \{0\} & \text{otherwise,} \end{cases}$$

$$\Psi : u \mapsto \begin{cases} \langle \mathsf{i}, \mathsf{pred}(u)\rangle & \text{if } 1 \in u \\ \langle \emptyset, \emptyset\rangle & \text{otherwise} \end{cases}$$

and set $a = \mathsf{fun}(\Phi)$ and $b = \mathsf{fun}(\Psi)$.
Then $j : L\overline{\omega} \to \overline{\omega}$, given by $j(\bot) = 0$ and $j(n) = n + 1$, is tracked by a, its inverse $h : \overline{\omega} \to L\overline{\omega}$ is tracked by b. Thus $L\overline{\omega} \cong \overline{\omega}$. Moreover, the restrictions of j and h to $L\omega$ and ω are tracked by the same elements, hence also $L\omega \cong \omega$. □

Notice that Σ together with $\top : \mathbf{1} \to \Sigma$, defined as in the previous section, is a dominance in the sense of [Lon94], and this dominance gives rise to the lifting functor L. Then it can be shown that the isos $j : L\omega \to \omega$ and $h : \overline{\omega} \to L\overline{\omega}$ form the initial, respectively terminal, algebra for the lifting monad L. It is also a straightforward observation that Σ satisfies the so-called strong completeness axiom, i.e. Σ and $L\Sigma$ are complete. The interested reader is referred to chapters 4 and 5 of [Lon94].

3.4 Replete Objects in $\mathbf{Asm}(\mathcal{P}\omega)$

There is another well-known subcategory, used to model (pre)domains in realizability models. It is the category $\mathbf{Rep}(\mathcal{P}\omega)$ of replete objects, which was introduced by Martin Hyland in [Hyl91]. We will introduce it here and show some basic properties.

Definition 3.4.1. A morphism $e : X \to Y$ in $\mathbf{Asm}(\mathcal{P}\omega)$ is called $\Sigma-$*iso* (or $(\Sigma-)$*equable*) if $\Sigma(e) : \Sigma(Y) \to \Sigma(X)$ is an iso.
An assembly A is called $(\Sigma-)$*replete* if A^e is an iso for all $\Sigma-$isos e.
$\mathbf{Rep}(\mathcal{P}\omega)$ is the full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$ consisting of the replete objects.

With this definition, we get that an assembly A is replete if and only if for all $\Sigma-$isos e and assemblies Z, each morphism $f : Z \times X \to A$ can be uniquely extended along $Z \times e$:

$$\begin{tikzcd} Z \times Y \arrow[r, dashed, "\overline{f}"] & A \\ Z \times X \arrow[u, "Z \times e"] \arrow[ur, "f"'] & \end{tikzcd}$$

In particular, each morphism $f : X \to A$ can be uniquely extended along e.

Theorem 3.4.2. *$\mathbf{Rep}(\mathcal{P}\omega)$ is a cartesian-closed subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$, has equalizers and the inclusion functor preserves this structure.*

Proof: We will show that $\mathbf{Rep}(\mathcal{P}\omega)$ has terminal object, binary products, exponentials and equalizers.

- Terminal object: Clear.
- Binary Products: Suppose A, B are replete objects, $e : X \to Y$ is a $\Sigma-$iso and $f : Z \times X \to A \times B$ any morphism. Let $f_i = \pi_i \circ f$ for $i = 1, 2$, then we get the following diagram:

$$\begin{tikzcd} Z \times Y \arrow[r, dashed, "\langle \overline{f_1}, \overline{f_2} \rangle"] & A \times B \\ Z \times X \arrow[u, "Z \times e"] \arrow[ur, "\langle f_1, f_2 \rangle"'] & \end{tikzcd}$$

  Set $\overline{f} = \langle \overline{f_1}, \overline{f_1} \rangle$ to get the required morphism.
  For the uniqueness part, assume $\widehat{f} : Z \times Y \to A \times B$ fulfills the required property, as well. Then for $\widehat{f_i} = \pi_i \circ \widehat{f}$, we get $\widehat{f_i} = \overline{f_i}$ for $i = 1, 2$. Thus $\widehat{f} = \overline{f}$.
- Exponentials: We show that $\mathbf{Rep}(\mathcal{P}\omega)$ forms an exponential ideal in $\mathbf{Asm}(\mathcal{P}\omega)$. Let B be replete and $e : X \to Y$ be a $\Sigma-$iso. We have to show that $\left(B^A\right)^e$ is an iso. But B^e is an iso by assumption, and thus $\left(B^A\right)^e \cong (B^e)^A : \left(B^X\right)^A \to \left(B^Y\right)^A$ is an iso, as well.
- Equalizers: Suppose $m : P \to A$ is an equalizer of $h, k : A \to B$ in $\mathbf{Asm}(\mathcal{P}\omega)$, and A, B are replete. Let $e : X \to Y$ be a $\Sigma-$iso and $f : Z \times X \to P$ be any morphism. Clearly, $h \circ m \circ f = k \circ m \circ f$, and thus, by uniqueness of the extension along $Z \times e$, we get $\overline{h \circ m \circ f} = \overline{k \circ m \circ f}$ and thus $h \circ \overline{m \circ f} = k \circ \overline{m \circ f}$. Now the mediating arrow, obtained as m is the equalizer, is the required unique extension of f along $Z \times e$, as in the following diagram:

$$\begin{tikzcd} Z \times Y \arrow[dr, dashed] \arrow[drr, dashed] \arrow[drrr, dashed] & & & \\ & P \arrow[r, "m"'] & A \arrow[r, "h, k"'] & B \\ Z \times X \arrow[uu, "Z \times e"] \arrow[ur, "f"'] & & & \end{tikzcd}$$

$\square$

Proposition 3.4.3. *Each replete object is complete.*

Proof: Clear, as $\iota : \omega \to \overline{\omega}$ is a $\Sigma-$iso. $\square$

It is known that $\mathbf{Rep}(\mathcal{P}\omega)$ is a full reflective subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$ and thus there exists a functor $R : \mathbf{Asm}(\mathcal{P}\omega) \to \mathbf{Rep}(\mathcal{P}\omega)$, called the *repletion*. Therefore, $\mathbf{Rep}(\mathcal{P}\omega)$ is one of the categories considered in Synthetic Domain Theory. Its objects can be viewed as predomains, as they have limits of ω-chains, by the previous proposition. Moreover, when only those replete objects with a least point are considered, a fixed point theorem is available (see [Tay91]).
However, there does not exist an easy description of the reflection functor R. In [Str99] the description of a map $r_X : X \to R(X)$ is given. It is also shown that $\mathbf{Rep}(\mathcal{P}\omega)$ is in fact the smallest full reflective subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$, containing Σ. Thus $\mathbf{Rep}(\mathcal{P}\omega)$ is contained in $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$. Furthermore, the following theorem is shown.

Lemma 3.4.4. *An assembly A is replete if and only if it is an exper and each $\Sigma-$iso $e : A \to B$ in $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ is already an iso.*

Using this lemma, T. Streicher describes the repletion map $e_A : A \to R(A)$ as the greatest extension of A in $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ such that e_A is a $\Sigma-$iso. In chapter 4, the notion of $\top\top-$closure will be introduced, which is one extension of this kind. However, we did not succeed in showing that it is in fact the greatest one, and thus the repletion.

Chapter 4

Topological Predomains

4.1 The Equivalence of QCB_0 and $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$

In chapter 2, we have seen that $\mathbf{QCB}$ and $\mathbf{QCB}_0$ appear as subcategories of $\mathbf{Asm}(\mathcal{P}\omega)$ via an inclusion functor $J : \mathbf{QCB} \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$. Furthermore the bi-cartesian-closed structure is preserved by J. In this section we will show that $\mathbf{QCB}_0$ is equivalent to $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$. For this, we need two more observations.

- Let $\Sigma_{\mathbf{QCB}}$ denote the Sierpinski space and Σ be the assembly, introduced in section 3.2. Then it is straightforward to verify that $J(\Sigma_{\mathbf{QCB}}) \cong \Sigma$.

- If X is a countably-based T_0-space, then there is a topologically isomorphic $I \subseteq \mathcal{P}\omega$. Thus $J(X) \cong J(I)$, and it is a straightforward observation that $J(I) \cong (I, \|\cdot\|_I)$, where $\|x\|_I = \{x\}$.

Theorem 4.1.1. $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ *and* $\mathbf{QCB}_0$ *are equivalent.*

Proof: Since both categories are full subcategories of $\mathbf{Asm}(\mathcal{P}\omega)$, it suffices to show that their objects coincide up to isomorphism.
Let Q be a qcb-space, satisfying T_0. Then, by corollary 2.3.3, Q is a regular subobject of $\Sigma^2_{\mathbf{QCB}}(Q)$. Since J preserves the bi-cartesian-closed structure and equalizers, by corollary 2.5.4, $J(Q)$ is a regular subobject of $\Sigma^2(J(Q))$, hence an exper.
Conversely, suppose X is an exper. Then, by corollary 3.2.7, there exists an $I \subseteq \mathcal{P}\omega$ and a regular mono $m : X \to \Sigma(I)$. But Σ and I are in the image $J(\mathbf{QCB})$, by the above observations, and, therefore, $\Sigma(I)$ is in the image of J, as J preserves bi-cartesian-closure and equalizers of $\mathbf{QCB}$. By proposition 2.5.1, $J(\mathbf{QCB})$ contains all its regular $\mathbf{Asm}(\mathcal{P}\omega)$-subobjects (up to isomorphism) and thus there exists a qcb-space Q, such that $J(Q) \cong X$. Now Q satisfies T_0, as $J^{-1}(m) : Q \to \Sigma_{\mathbf{QCB}}(I)$ is a regular mono and $\Sigma_{\mathbf{QCB}}(I)$ is a T_0-space. $\square$

4.2 Complete Expers and Topological Predomains

We have seen that $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ is a subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$, whose objects have a nice topological description via the equivalence with $\mathbf{QCB}_0$. On the other hand, we have $\mathbf{C}(\mathcal{P}\omega)$, a subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$, in which we defined a notion of chain-completeness and a lifting functor. In this section, we will obtain the category $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ by intersecting $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ with $\mathbf{C}(\mathcal{P}\omega)$, and observe that its objects have a nice topological description, as well. In particular, we will transfer the construction of $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ as a subcategory of $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ to $\mathbf{QCB}_0$.

Definition 4.2.1. The category $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ is the full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$, consisting of those assemblies which are objects of both, $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ and $\mathbf{C}(\mathcal{P}\omega)$.

The objects of $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ are called complete expers.

Theorem 4.2.2. *$\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ is bi-cartesian-closed and the inclusions $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega) \hookrightarrow \mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ and $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega) \hookrightarrow \mathbf{Asm}(\mathcal{P}\omega)$ preserve this structure. Moreover $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ is an exponential ideal in $\mathbf{Asm}(\mathcal{P}\omega)$.*

Proof: By theorem 3.3.4, the above holds for $\mathbf{C}(\mathcal{P}\omega)$, and in section 2.5, we have shown it for $\mathbf{QCB}_0$, and thus, by theorem 4.1.1, for $\mathbf{ExPER}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$. Now the result follows directly from the definition of $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$. $\square$

We are interested in giving a topological characterization of those qcb-spaces, whose image under J is a complete exper. Those spaces will be called *topological predomains* and form the following category.

Definition 4.2.3. The category $\mathbf{TP}$ is the full subcategory of $\mathbf{QCB}_0$, consisting of those objects Q for which $J(Q)$ is a complete exper.

By the above results, $\mathbf{TP}$ is bi-cartesian-closed, the inclusion $\mathbf{TP} \hookrightarrow \mathbf{QCB}_0$ preserves this structure and, moreover, it is an exponential ideal in $\mathbf{QCB}_0$.

Let $\omega_{\mathbf{QCB}}$ denote the qcb-space given by the poset ω and $\overline{\omega}_{\mathbf{QCB}}$ be the qcb-space, given by the poset $\overline{\omega}$. Recall the definition of the assemblies ω and $\overline{\omega}$, given in section 3.3. The observations of the previous section show that $J(\omega_{\mathbf{QCB}}) \cong \omega$ and $J(\overline{\omega}_{\mathbf{QCB}}) \cong \overline{\omega}$. Furthermore, let $i : \omega_{\mathbf{QCB}} \hookrightarrow \overline{\omega}_{\mathbf{QCB}}$ be the obvious embedding, then $J(i)$ is isomorphic to ι. Now we get the following characterization of topological predomains.

Lemma 4.2.4. *A qcb-space Q, satisfying T_0, is a topological predomain if and only if for every qcb-space Z, any morphism $f : Z \times \omega_{\mathbf{QCB}} \to Q$ extends uniquely along $Z \times i$, as visualized in the following diagram:*

$$\begin{tikzcd}
Z \times \overline{\omega}_{\mathbf{QCB}} \arrow[r, dashed, "\overline{f}"] & Q \\
Z \times \omega_{\mathbf{QCB}} \arrow[u, hook, "Z \times i"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proof: This follows directly from the definitions of **TP** and $\mathbf{C}(\mathcal{P}\omega)$. $\square$

In fact a nicer characterization for topological predomains can be given. For a topological space X, let $\sqsubseteq$ denote the specialization preorder. It follows that if X satisfies T_0, then $\sqsubseteq$ is a partial order. Recall that a partial order on X is said to be ω-complete, if each ascending sequence (x_n) in X has a least upper bound (or supremum) $\bigvee_n x_n$, and a subset $U \subseteq X$ is Scott-open, if it is an upper set with respect to $\sqsubseteq$ and for each ascending sequence (x_n) with $\bigvee_n x_n \in U$ there exists a $k \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq k$.

Theorem 4.2.5. *A topological space X is a topological predomain if and only if it satisfies the following conditions:*

- *X is a qcb-space,*
- *the specialization order $\sqsubseteq$ on X is an ω-complete partial order, i.e. $(X, \sqsubseteq)$ is a cpo,*
- *each open subset of X is Scott-open.*

Proof: Suppose X is a topological predomain. Then it is an object of $\mathbf{QCB}_0$, hence a qcb-space satisfying T_0. Moreover, if (x_n) is an ascending sequence in X, with respect to the specialization order $\sqsubseteq$, then $f : \omega_{\mathbf{QCB}} \to X$, given by $f(n) = x_n$ is a morphism in **QCB**. Thus f extends uniquely along i to a morphism $\overline{f} : \overline{\omega}_{\mathbf{QCB}} \to X$. We claim that $x := \overline{f}(\infty)$ is the supremum of (x_n) with respect to $\sqsubseteq$. It is clear that $x_n \sqsubseteq x$ for all $n \in \omega$. Conversely, if for some $y \in X$, $x_n \sqsubseteq y$ for all $n \in \omega$, and $U \subseteq X$ is open and contains x, then $\overline{f}^{-1}(U) \subseteq \overline{\omega}_{\mathbf{QCB}}$ is open and nonempty. Hence it is of the form $\{k \in \omega | \; k \geq n\}$ for some $n \in \omega$. Thus (x_n) is eventually in U and so $y \in U$, which yields $x \sqsubseteq y$, and thus x is the supremum of (x_n). So $\sqsubseteq$ in fact is an ω-complete partial order on X. Finally, suppose $U \subseteq X$ is open, and $\overline{f} : \overline{\omega}_{\mathbf{QCB}} \to X$ a morphism with $\overline{f}(\infty) \in U$. Then continuity of $\overline{f}$ ensures that $\overline{f}^{-1}(U)$ is of the form $\{n, n+1, \ldots, \infty\}$, so U is Scott-open.
Conversely, suppose X satisfies the three conditions above, Z a qcb-space and $f : Z \times \omega_{\mathbf{QCB}} \to X$ continuous. Then $f(z, n) \sqsubseteq f(z, n+1)$ for any $z \in Z$. Thus $\bigvee_n f(z, n)$ exists in X. We get $\overline{f} : Z \times \overline{\omega}_{\mathbf{QCB}} \to X$, by $\overline{f}(z, n) = f(z, n)$ and $\overline{f}(z, \infty) = \bigvee_n f(z, n)$. $\overline{f}$ is continuous, because each open $U \subseteq X$ is Scott-open. Since X is a $\mathbf{QCB}_0$ space, it is a topological predomain, by lemma 4.2.4. $\square$

In classical Domain Theory, one distinguishes pointed cpos, i.e. cpos having a least element. This motivates the following definition.

Definition 4.2.6. The category **TD** is the full subcategory of **TP**, consisting of those objects X, which have a least element with respect to the specialization order.
The objects of **TD** are called *topological domains.*

For a qcb-space X, let $X_\perp$ denote the topological space, obtained by adding an element $\perp$ to X and $U \subseteq X_\perp$ is open, whenever $\perp \notin U$ and U is already open in X or $U = X_\perp$. X is a topological domain if and only if $X \cong Y_\perp$ (in **Seq**) for some topological predomain Y. The next theorem will show that the lifting functor $L : \mathbf{C}(\mathcal{P}\omega) \to \mathbf{C}(\mathcal{P}\omega)$, introduced in definition 3.3.4, in fact restricts to the functor $(-)_\perp : \mathbf{TP} \to \mathbf{TD}$.

Theorem 4.2.7. *A topological space X is a topological domain if and only if $J(X) \cong LA$ for some complete exper A.*

Proof: Suppose X is a topological domain, then $X \cong Y_\perp$ for some topological predomain Y. Hence there exists a countably based T_0 space $I \subseteq \mathcal{P}\omega$ and an ω-projecting $q : I \twoheadrightarrow Y$. It is clear that $q' : I_\perp \to Y_\perp$ is ω-projecting, as well, where $q'(u) = q(u)$ for all $u \in I$ and $q'(\perp_I) = \perp_Y$. Moreover $i : I_\perp \to \mathcal{P}\omega$, given by $i(u) = \langle \mathsf{i}, u\rangle$ and $i(\perp_I) = \langle \emptyset, \emptyset\rangle$, is a topological embedding. Thus we get $J(X) \cong (Y_\perp, \|\cdot\|_{J(Y_\perp)})$, where

$$\|x\|_{J(Y_\perp)} = \{\langle \mathsf{i}, u\rangle \in \mathcal{P}\omega |\, u \in I \text{ and } q'(u) = x\}$$

for all $x \in Y$, and $\|\perp_Y\|_{J(Y_\perp)} = \{\langle \emptyset, \emptyset\rangle\}$.
On the other hand $LJ(Y) \cong (Y_\perp, \|\cdot\|_{LJ(Y)})$, where

$$\|x\|_{LJ(Y)} = \{\langle \mathsf{i}, u\rangle \in \mathcal{P}\omega |\, u \in I \text{ and } q(u) = x\}$$

for all $x \in Y$, and $\|\perp_Y\|_{LJ(Y)} = \{\langle \emptyset, u\rangle \in \mathcal{P}\omega |\, u \in \mathcal{P}\omega\}$. Thus $J(X) \cong LJ(Y)$ and by the previous results, $J(Y)$ is a complete exper.
Conversely, suppose $J(X) \cong LA$ for some complete exper A. Then there exists a topological predomain Y such that $J(Y) \cong A$. By the above result, $J(Y_\perp) \cong LA \cong J(X)$ and so, by J being an equivalence, $Y_\perp \cong X$. Hence X is a topological domain. □

It is easy to see that **TD** forms an exponential ideal in **TP** and it is straightforward to verify that **TD** is cartesian-closed. Also notice that **TD** has a least fixed point operator, that can be obtained as in classical Domain Theory.

However, sums of topological domains, taken in the usual way, will not be topological domains, since they have no least element. One can overcome this drawback by setting $Q \oplus R = S$, where $J(S) \cong L(X + Y)$ for $J(Q) \cong LX$ and $J(R) \cong LY$. Alternatively, one can equip $Q + R$ with a new least element, which yields $Q \boxplus R$ such that $J(Q \boxplus R) \cong L(J(Q + R))$.

It is also possible to define $\mathbf{TD}_\perp$, the subcategory of **TD** in which all morphisms are strict, i.e. $f(\perp_Q) = \perp_R$ for all $f : Q \to R$ in $\mathbf{TD}_\perp$.

4.3 Canonical expers and Sub-Cpos

In the following sections, we want to prove some results about complete expers and replete objects. For this we will have to restrict ourselves to certain canonical forms of objects of $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ and $\mathbf{Rep}(\mathcal{P}\omega)$, and the choice we make will not be closed under isomorphism. However, for each complete exper and replete object there will exist an isomorphic canonical one.

Recall the notion of a canonical subobject in $\mathbf{Asm}(\mathcal{P}\omega)$, given in definition 3.1.5.

Definition 4.3.1. A modest set A is said to be a *canonical exper* if it is a canonical subobject of $\Sigma(I)$ for some $I \subseteq \mathcal{P}\omega$.

Recall that it is straightforward to verify that each exper is isomorphic to a canonical one. For if A is an exper, then $\eta_A : A \rightarrowtail \Sigma^2(A)$ is a regular mono, hence equalizer of morphisms $f, g : \Sigma^2(A) \to \nabla\mathbf{2}$. Constructing the equalizer of f and g, as in theorem 1.2.3, will yield a canonical subobject A' of $\Sigma^2(A)$, which is isomorphic to A. By lemma 3.2.5, it follows that $\Sigma^2(A)$ is a canonical subobject of $\Sigma(I)$ for some $I \subseteq \mathcal{P}\omega$, hence A' is a canonical subobject of $\Sigma(I)$, i.e. a canonical exper.

Definition 4.3.2. Let $\mathsf{U} : \mathcal{P}\omega \to \Sigma(\mathcal{P}\omega)$ be the map given by $a \mapsto \mathsf{U}_a := \{u \in \mathcal{P}\omega |\, a \cdot u \neq \emptyset\}$.

Then U is continuous, as $\mathsf{U}_{\bigcup a_n} = \bigcup \mathsf{U}_{a_n}$ for all chains (a_n).

Now we can interpret canonicity as follows. For $I \subseteq \mathcal{P}\omega$, the underlying set of $\Sigma(I)$ is isomorphic to the set of open subsets of I (with respect to the subspace topology), where $\|U\|_{\Sigma(I)} = \{a \in \mathcal{P}\omega |\, U = \mathsf{U}_a \cap I\}$. Equipped with the inclusion order $\subseteq$, the set of open subsets of I becomes a complete lattice. The next theorem yields a one-to-one correspondence between ascending chains with respect to $\subseteq$ and morphisms $f : \omega \to \Sigma(I)$. Notice that by $\mathbf{CE}_{\mathbf{\Sigma}}(\mathcal{P}\omega)$ being an exponential ideal in $\mathbf{Asm}(\mathcal{P}\omega)$, it follows that $\Sigma(I)$ is a complete exper.

Proposition 4.3.3. *Let $I \subseteq \mathcal{P}\omega$. If (U_n) is an ω-chain of open subsets of I with respect to the inclusion order, then there exists a morphism $f : \omega \to \Sigma(I)$ in $\mathbf{Asm}(\mathcal{P}\omega)$ such that $f(n) = U_n$. Moreover, for the unique extension $\overline{f} : \overline{\omega} \to \Sigma(I)$, it holds that $\overline{f}(\infty) = \bigcup U_n$.*

Proof: For open $V \subseteq \mathcal{P}\omega$ let $\chi_V : \mathcal{P}\omega \to \mathcal{P}\omega$ denote the continuous function

$$\chi_V(u) = \begin{cases} \{0\} & \text{if } u \in V \\ \emptyset & \text{otherwise,} \end{cases}$$

Suppose (U_n) is an ascending chain of open subsets of I with respect to the inclusion order. Then there exist open sets $\widetilde{V_n} \subseteq \mathcal{P}\omega$ such that $U_n = \widetilde{V_n} \cap I$. Setting $V_n = \bigcup_{k \le n} \widetilde{V_n}$ yields an ω-chain (V_n) of open subsets of $\mathcal{P}\omega$ such that $U_n = V_n \cap I$. Set $a_n = \mathsf{fun}(\chi_{V_n})$, then $a_n \subseteq a_{n+1}$, by continuity of fun, and it is clear that $a_n \in \|U_n\|_{\Sigma(I)}$. Now define a map $g : \omega \to \mathcal{P}\omega$, by $g(n) = a_n$. By lemma 1.3.3, g can be extended to a continuous map $\overline{g} : \mathcal{P}\omega \to \mathcal{P}\omega$. Now $\mathsf{fun}(\overline{g})$ tracks $f : \omega \to \Sigma(I)$, $f(n) = U_n$, verifying the first claim.
Suppose now b tracks the unique extension $\overline{f} : \overline{\omega} \to \Sigma(I)$ of f, then b tracks f, as well. Let $a_n = b \cdot \{k \in \omega |\, k \le n\}$ and $a = b \cdot \omega$. Then $a_n \in \|\overline{f}(n)\|_{\Sigma(I)} = \|U_n\|_{\Sigma(I)}$, $a \in \|\overline{f}(\infty)\|_{\Sigma(I)}$ and $a = \bigcup_n a_n$. Thus for all $u \in I$, $a \cdot u \neq \emptyset$ if and only if

$\bigcup_n (a_n \cdot u) \neq \emptyset$, which holds if and only if $u \in \bigcup U_n$. Hence $\mathsf{U}_a \cap I = \bigcup U_n$, and thus $a \in \| \bigcup U_n \|_{\Sigma(I)}$, and so $\overline{f}(\infty) = \bigcup U_n$. $\square$

If $(A, \| \cdot \|_A)$ is a canonical subobject of $\Sigma(I)$, then A is isomorphic to a set of open subsets of $I \subseteq \mathcal{P}\omega$, and $\|U\|_A = \{a \in \mathcal{P}\omega | \; U = \mathsf{U}_a \cap I\}$, by canonicity. In fact, the next theorem shows that there is an equivalence between $(A, \| \cdot \|_A)$ being a complete exper and A, equipped with the inclusion order, forming a sub-cpo in the lattice of open subsets of I.

Theorem 4.3.4. *Let $I \subseteq \mathcal{P}\omega$. A canonical subobject $(A, \| \cdot \|_A)$ of $\Sigma(I)$ is a complete exper if and only if for all ω-chains (U_n) in A, with respect to the inclusion order, $\bigcup U_n \in A$.*

Proof: If $(A, \| \cdot \|_A)$ is a complete exper and (U_n) an ω-chain in A with respect to the inclusion order, then, as in the previous proposition, we can construct a morphism $f : \omega \to (A, \| \cdot \|_A)$ such that $f(n) = U_n$, and for the unique extension $\overline{f} : \omega \to (A, \| \cdot \|_A)$, it holds that $\overline{f}(\infty) = \bigcup U_n \in A$.
Conversely, suppose $\subseteq$ is an ω-complete partial order on A. Let $f : Z \times \omega \to A$ be a morphism in $\mathbf{Asm}(\mathcal{P}\omega)$, realized by $b \in \mathcal{P}\omega$, and $U_n = f(z, n)$. Then U_n is an ω-chain with respect to $\subseteq$, because the realizers of $\Sigma(I)$ restrict to A, and so if $a \in \|U\|_A$, $a' \in \|U'\|_A$ and $a \subseteq a'$ in $\mathcal{P}\omega$, then $U \subseteq U'$. Thus $\overline{f} : Z \times \overline{\omega} \to A$, given by $\overline{f}(z, n) = f(z, n)$ and $\overline{f}(z, \infty) = \bigcup f(z, n)$, is tracked by b, as well. Uniqueness of the extension follows from the fact that morphisms $\overline{\omega} \to (A, \| \cdot \|_A)$ are just morphisms $\overline{\omega} \to \Sigma(I)$ whose image is a subset of A. $\square$

Thus for canonical expers A, we established a one-to-one correspondence between A being complete in our realizability setting and the underlying set of A, equipped with an appropriate order, forming a sub-cpo of a complete lattice, given by the open sets of a countably-based T_0-space. However, in general not every Scott-continuous map between such cpos will be realizable, whereas each realizable map is continuous. This comes from the fact that the Scott-topology on A (i.e. the subspace topology inherited from $\Sigma(I)$) is in general finer then the topology given by $\Sigma(A)$.

4.4 Admissible and $\top\top$-closed relations

In this section, we will have a closer look at complete canonical expers. In particular, we will introduce the notions of admissible and $\top\top$-closed pers, and show that each complete canonical exper is admissible. We will also show that each replete canonical per is $\top\top$-closed. This section is based on ideas of Thomas Streicher and suggestions of Alex Simpson.

Definition 4.4.1. A per R on $\mathcal{P}\omega$ is said to be *admissible* if it is closed under suprema of ω-chains, i.e. if (a_n), (b_n) are ω-chains and $a_n \mathrel{\mathsf{R}} b_n$ for all $a \in \mathbb{N}$, then $\bigcup a_n \mathrel{\mathsf{R}} \bigcup b_n$.

However admissibility is not closed under isomorphism in $\mathbf{PER}(\mathcal{P}\omega)$. To see this consider the per Σ, given by $a \, \Sigma \, b$ whenever $a = \emptyset = b$ or $a \neq \emptyset \neq b$, and the per Σ'

given by $a\,\Sigma'\,b$ whenever $a\,\Sigma\,b$ and $a \neq \omega \neq b$. It is a straightforward verification that $\Sigma \cong \Sigma'$, but Σ clearly is admissible, whereas Σ' is not. Thus we will restrict ourselves to canonical expers.

Proposition 4.4.2. *Let A be a canonical exper. A is complete if and only if the corresponding per R is admissible.*

Proof: Suppose A is a complete canonical exper, say a canonical subobject of $\Sigma(I)$ for $I \subseteq \mathcal{P}\omega$. Let (a_n) and (b_n) be ω-chains in $\mathcal{P}\omega$ such that $a_n \,\mathsf{R}\, b_n$ for all $n \in \omega$. We have to show that $\bigcup a_n \,\mathsf{R}\, \bigcup b_n$. There exist $x_n \in A$ such that $a_n, b_n \in \|x_n\|_A$. By lemma 1.3.3, there exist $a, b \in \mathcal{P}\omega$ such that $a \cdot \{k \in \mathbb{N} |\, k \leq n\} = a_n$ and $b \cdot \{k \in \mathbb{N} |\, k \leq n\} = b_n$. Thus we can construct a morphism $f : \omega \to A$, such that $f(n) = x_n$ and f is tracked by a and b, and f has a unique extension $\overline{f} : \overline{\omega} \to A$. Now any realizer c of $\overline{f}$ realizes f, as well. Set $c_n := c \cdot \{k \in \mathbb{N} |\, k \leq n\}$. Then $\mathsf{U}_{a_n} \cap I = \mathsf{U}_{b_n} \cap I = \mathsf{U}_{c_n} \cap I$ for all $n \in \omega$. By continuity of U, then $\mathsf{U}_{\bigcup a_n} \cap I = \mathsf{U}_{\bigcup b_n} \cap I = \mathsf{U}_{\bigcup c_n} \cap I$. Hence $\bigcup a_n, \bigcup b_n, \bigcup c_n \in \|\overline{f}(\infty)\|_A$, by canonicity of A, and so $\bigcup a_n \,\mathsf{R}\, \bigcup b_n$.
Conversely, assume R is an admissible per. Let $f : Z \times \omega \to A$ be a morphism tracked by $a \in \mathcal{P}\omega$, $z \in Z$ be given and $u \in \|z\|_Z$. Then $a \cdot \langle u, \{k \in \mathbb{N} |\, k \leq n\}\rangle \in \|f(z, n)\|_A$. Hence $a \cdot \langle u, \{k \in \mathbb{N} |\, k \leq n\}\rangle \,\mathsf{R}\, a \cdot \langle u, \{k \in \mathbb{N} |\, k \leq n\}\rangle$. By R being admissible, this yields $a \cdot \langle u, \mathbb{N}\rangle \,\mathsf{R}\, a \cdot \langle u, \mathbb{N}\rangle$, so there exists $x \in A$ such that $a \cdot \langle u, \mathbb{N}\rangle \in \|x\|_A$. Setting $\overline{f}(z, \infty) = x$ yields a morphism $\overline{f} : Z \times \overline{\omega} \to A$, which is tracked by a, as well, and fulfills $f = \overline{f} \circ (Z \times \iota)$. Uniqueness of the extension follows again from the canonicity of A. Thus A is complete. $\square$

The following definition is due to Andrew Pitts and taken from [Aba00].

Definition 4.4.3. For a per R on $\mathcal{P}\omega$, define a relation $\mathsf{R}^{\top\top}$ on $\mathcal{P}\omega$ by

$$u \,\mathsf{R}^{\top\top} v \Leftrightarrow \forall f \,\Sigma(\mathsf{R})\, g \;\; f(u) \,\Sigma\, g(v).$$

R is said to be $\top\top$-*closed* if $\mathsf{R} = \mathsf{R}^{\top\top}$.

It is easy to see $\mathsf{R} \subseteq \mathsf{R}^{\top\top}$. In [Aba00], it is shown that in general being $\top\top$-closed is strictly stronger than admissibility, in general.

This is also true for $\mathbf{PER}(\mathcal{P}\omega)$, because being a canonical exper does not suffice for the corresponding relation to be $\top\top$-closed. To see this, let A be the canonical subobject of $\Sigma(\mathcal{P}\omega)$, whose underlying set is given by those open subsets of $\mathcal{P}\omega$ which are of the form

$$U_k = \mathcal{P}\omega \setminus \{\{k\}, \emptyset\}.$$

Then the per R corresponding to A is given by

$$a \,\mathsf{R}\, b \Leftrightarrow \exists k \in \omega.\, \mathsf{U}_a = \mathsf{U}_b = U_k$$

We claim $\mathsf{i} \,\mathsf{R}^{\top\top}\, \mathsf{i}$. Suppose this was not true. Then there exist $f \,\Sigma(\mathsf{R})\, g$ such that $f \cdot \mathsf{i} = \bot$ and $g \cdot \mathsf{i} = \top$. It is easy to verify that in this case, we could separate $\mathsf{U}_{\mathsf{i}} = \mathcal{P}\omega \setminus \{\emptyset\}$ simultaneously from all the U_k by some continuous $\Phi : \Sigma(\mathcal{P}\omega) \to \Sigma$.

But $\Phi(\mathsf{U_i}) = \top$ if and only if there exists $k \in \omega$ such that $\Phi(U_k) = \top$. Thus for all $f \, \Sigma(\mathsf{R}) \, g$ we have $f \cdot \mathsf{i} \, \Sigma \, g \cdot \mathsf{i}$, hence $\mathsf{i} \, \mathsf{R}^{\top\top} \, \mathsf{i}$. However, we do not have $\mathsf{i} \, \mathsf{R} \, \mathsf{i}$, and so R is not $\top\top$-closed.

Lemma 4.4.4. *For a per* R*, let* $e_{\mathsf{R}} : \mathsf{R} \to \mathsf{R}^{\top\top}$ *be given by* $[u]_{\mathsf{R}} \mapsto [u]_{\mathsf{R}^{\top\top}}$*. Then* $\Sigma^{e_{\mathsf{R}}} : \Sigma(\mathsf{R}^{\top\top}) \to \Sigma(\mathsf{R})$ *is an isomorphism.*

Proof: First we show that $\Sigma(\mathsf{R}^{\top\top}) = \Sigma(\mathsf{R})$.
Suppose $a \, \Sigma(\mathsf{R}^{\top\top}) \, b$, then, as $\mathsf{R} \subseteq \mathsf{R}^{\top\top}$, we have for all $u \, \mathsf{R} \, v$, $a \cdot u \, \Sigma \, b \cdot v$ and thus $a \, \Sigma(\mathsf{R}) \, b$. Conversely, let $a \, \Sigma(\mathsf{R}) \, b$, then, by definition of $\mathsf{R}^{\top\top}$, $u \, \mathsf{R}^{\top\top} \, v$ implies $a \cdot u \, \Sigma \, b \cdot v$ and hence $a \, \Sigma(\mathsf{R}^{\top\top}) \, b$.
It remains to show that $\Sigma^{e_{\mathsf{R}}} : \Sigma(\mathsf{R}^{\top\top}) \to \Sigma(\mathsf{R})$ is indeed the required iso. But since e_{R} is realized by i, we have $\Sigma^{e_{\mathsf{R}}}(f) = f \circ e_{\mathsf{R}} = f$. □

Proposition 4.4.5. *If A is a replete canonical exper, then* R*, the per corresponding to A, is* $\top\top$*-closed.*

Proof: Let B be the modest set corresponding to $\mathsf{R}^{\top\top}$. Then $e_{\mathsf{R}} : \mathsf{R} \to \mathsf{R}^{\top\top}$ is a mono, which is a Σ-iso, by the preceding lemma. Since A is replete, it follows that e_{R} is an iso. Canonicity of A ensures that $[u]_{\mathsf{R}} = [u]_{\mathsf{R}^{\top\top}}$ for all $u \in \mathsf{carrier}(\mathsf{R})$, hence R is $\top\top$-closed. □

It was our hope to prove the reverse direction of the proposition, i.e. to give a positive answer to the following question.

Question: Let A be a canonical exper and the corresponding per R be $\top\top$-closed. Is A replete?

We had two ideas to show this, but have encountered problems in both.

The first idea was to look at the following pullback,

$$\begin{tikzcd}
\mathsf{R} \arrow[dr, dashed] \arrow[drr, tail, "\eta_{\mathsf{R}}"] \arrow[ddr, tail, "m"'] & & \\
& \mathsf{S} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma^2(A) \arrow[d, "\Sigma^2(m)"] \\
& \Sigma(I) \arrow[r, tail, "\eta_{\Sigma(I)}"'] & \Sigma^3(I)
\end{tikzcd}$$

where $m : \mathsf{R} \rightarrowtail \Sigma(I)$ is a canonical mono and $I \subseteq \mathcal{P}\omega$. We hoped that S were isomorphic to $\mathsf{R}^{\top\top}$, and so the mediating arrow would be an iso, which would give the desired result, since $\mathbf{Rep}(\mathcal{P}\omega)$ is closed under pullbacks. However, we did not succeed in showing $\mathsf{S} \cong \mathsf{R}^{\top\top}$

The second idea was to directly find a morphism $p : \overline{\mathsf{R}} \to \mathsf{R}^{\top\top}$, where $\overline{\mathsf{R}}$ denotes the repletion of R. If $m : \mathsf{R}^{\top\top} \to \overline{\mathsf{R}}$ is the arrow, obtained in extending the reflection map $r_{\mathsf{R}} : \mathsf{R} \to \overline{\mathsf{R}}$ along e_{R}, it should be possible to show that $p \circ m \cong id_{\mathsf{R}^{\top\top}}$ and $m \circ p \cong id_{\overline{\mathsf{R}}}$. However, we did not succeed in constructing p.

So this question remains unanswered, but we suspect the answer to be negative. Hence the problem of finding a better construction of the repletion than the one given in [Str99] remains open.

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