A Universal Model for CPS_∞ in Locally Boolean Domains

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Outline

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- Locally Boolean Domains (à la Jim LAIRD)

- Universality Results

  - for SPCF_∞, i. e. Sequential PCF with First Order Oracles

  - for CPS_∞, i. e. an infinitary CPS target language

Introduction

Description of Main Result

Semantic reconstruction of Krivine's machine (Reus&Streicher,JFP'98) using continuation semantics*

$$D \cong [C \to \mathsf{O}] \qquad C \cong D \times C$$

or equivalently (as $C \cong D^\omega$)

$$D \cong [D^\omega \to \mathsf{O}]$$

which is isomorphic to Scott's O_∞.

Main Theorem The canonical solution of $D \cong [D^\omega \to \mathsf{O}]$ in Laird's Locally Boolean Domains gives rise to a **universal** model for CPS_∞, an infinitary untyped CPS target language with error element $\top$.

*where $\mathsf{O} = \{\bot, \top\}$ with $\bot \sqsubset \top$, the so-called *Sierpiński space* of **observations**

Curien-Lamarche Games

We could formulate our result in terms of

> *Observably Sequential Algorithms*
> with an *untrappable error* $\top$

as introduced by Cartwright, Curien and Felleisen (early 1990ies) and F. Lamarche (1992).

Curien-Lamarche Games are the simplest version of Game Semantics:
A game is a tuple (C, V, P) where C is a set of *cells*, V is a set of *values* and P is a prefix-closed set of alternating sequences of cells and values. The elements of P are called *positions*, odd-length positions are called *queries* and even-length positions are called *responses*.
A *generalized response* is either a response or a sequence of the form $q\bot$ or $q\top$ with q a query. (We assume that $\bot, \top \notin V$.)

C-L Games (ctd.)

A *configuration* of or *strategy* for the game (C, V, P) is a set σ of generalized responses s.t.

(1) σ is closed under even-length prefixes

(2) $qx, qy \in \sigma$ implies $x = y$

(3) if $r \in \sigma$ and $rc \in P$ for some $c \in C$ then there exists $x \in V \cup \{\bot, \top\}$ with $qcx \in \sigma$.

Notice that traditionally generalized queries of the form $q\bot$ are omitted from σ. But it's better to have them for exhibiting a most useful *duality*:
let $\neg\sigma$ be the strategy containing the same *proper* responses as σ and satisfying

- $q\bot \in \neg\sigma \quad$ iff $\quad q\top \in \sigma$
- $q\top \in \neg\sigma \quad$ iff $\quad q\bot \in \sigma$.

C-L Games (ctd.)

On the generalized responses of a C-L game we can define various orders:

- $r_1 \leq_s r_2$ iff $r_1 = r_2$ or there exists a query q s.t. $r_1 = q\bot$ and q is a prefix of r_2
- $r_1 \leq_c r_2$ iff $r_1 = r_2$ or there exists a query q s.t. $r_2 = q\top$ and q is a prefix of r_1
- $r_1 \sqsubseteq r_2$ iff $r_1 \leq_s r_2$ or $r_1 \leq_c r_2$
- $r_1 \leq_b r_2$ iff $r_1 \leq_s r_2$ and $r_1 \leq_c r_2$.

These orders can be extended to the set of strategies as follows

- $\sigma_1 \leq_s \sigma_2$ iff $\forall\, p_1 \in \sigma_1.\ \exists\, p_2 \in \sigma_2.\ p_1 \leq_s p_2$
- $\sigma_1 \leq_c \sigma_2$ iff $\forall\, p_1 \in \sigma_1.\ \exists\, p_2 \in \sigma_2.\ p_1 \leq_c p_2$
- $\sigma_1 \sqsubseteq \sigma_2$ iff $\forall\, p_1 \in \sigma_1.\ \exists\, p_2 \in \sigma_2.\ p_1 \sqsubseteq p_2$
- $\sigma_1 \leq_b \sigma_2$ iff $\forall\, p_1 \in \sigma_1.\ \exists\, p_2 \in \sigma_2.\ p_1 \leq_b p_2$.

CL-games (ctd.)

Obviously, negation is *involutive* ($\neg\neg\sigma = \sigma$) and *antitone* ($\sigma \sqsubseteq \tau \Rightarrow \neg\tau \sqsubseteq \neg\sigma$).
If generalized responses of the form $q\bot$ are deleted from strategies the stable order $\leq_s$ boils down to $\subseteq$. This, however, hides the important extensional order $\sqsubseteq$.
N.B. These orders coincide in absence of $\top$!

The stable order $\leq_s$ (and the costable order $\leq_c$) can be defined in terms of $\sqsubseteq$ and $\neg$ as follows: for strategies σ define

$$\sigma_\bot = \sigma \wedge \neg\sigma \quad \text{and} \quad \sigma^\top = \sigma \vee \neg\sigma$$

i. e. replace all $\top$ by $\bot$ and all $\bot$ by $\top$, resp.
$\sigma \uparrow \tau$ iff $\sigma \sqsubseteq \tau^\top$ and $\tau \sqsubseteq \sigma^\top$ (*stably coherent*) then we have

$$\sigma \leq_s \tau \quad \text{iff} \quad \sigma \sqsubseteq \tau \text{ and } \sigma \uparrow \tau$$

(and $\sigma \leq_c \tau$ iff $\neg\tau \leq_s \neg\sigma$).
We write $x \downarrow y$ for $\neg x \uparrow \neg y$ and $x \updownarrow y$ iff $x \uparrow y$ and $x \downarrow y$.

Domains vs. Games

Cartwright, Curien and Felleisen observed that the category OSA of Curien-Lamarche Games with $\top$ is **well-pointed**.
Thus **strategies** (intensional) can be identified as particular) **functions** (extensional).

Question *What is the* **structure functions have to preserve** *such that they arise from strategies?*

has been answered by Jim **Laird** as follows

Answer A function f between (domains of configurations of) C-L games comes from a strategy iff the following conditions hold

- f is (Scott) **continuous** w.r.t. $\sqsubseteq$
- f is **stable** w.r.t. $\leq_s$
- f is **bistable**, i. e. from $x \updownarrow y$ it follows that $f(x) \updownarrow f(y)$ and $f(x \vee y) = f(x) \vee f(y)$.

Domains vs. Games (ctd.)

Moreover, J. Laird has axiomatized those orders with negation that (up to isomorphism) arise as domains of configurations of a Curien-Lamarche game.
The ensuing notion is that of a

Locally Boolean Domain

However, we need some intermediary steps to arrive at its precise formulation.

Then **Main Theorem** about locally boolean domains says that

> Every locally Boolen Domain is isomorphic to one of the form
> $$\Pi_{i \in I} \Sigma_{j \in J_i} D_{ij}$$
> where the D_{ij} are again locally boolean domains.

Iterating this and taking inverse limits allows one to reconstruct all lbd's as C-L games.

Locally

Boolean Domains

Locally Boolean Orders

A *locally boolean order* is a triple $(A, \sqsubseteq, \neg)$ where $\sqsubseteq$ is a partial order on A (called *extensional order*) and $\neg : A \to A$ is an involutive ($\neg\neg x = x$) and antitone ($\neg y \sqsubseteq \neg x$ whenever $x \sqsubseteq y$) operation such that

(1) for every $x \in A$ the set $\{x, \neg x\}$ has a least upper bound $x^\top = x \vee \neg x$ (and, thus, also a greatest lower bound $x_\bot = \neg(x^\top) = x \wedge \neg x$)

(2) (i) if $x \uparrow y$ then $x \vee y$ and $x \wedge y$ exist (w.r.t. $\sqsubseteq$) and $x \uparrow x \wedge y$

(ii) whenever $\uparrow \{x, y, z\}$, i.e. $x \uparrow y$, $x \uparrow z$ and $y \uparrow z$, then
$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ and
$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

where $x \uparrow y$ stands for $x \sqsubseteq y^\top$ and $y \sqsubseteq x^\top$.

Stable/Costable Order

In lbo's one can define a stable and costable order as follows

- $x \leq_s y$ iff $x \sqsubseteq y$ and $x \uparrow y$
- $x \leq_c y$ iff $\neg y \leq_s \neg x$

The bistable order is defined as the intersection of $\leq_s$ and $\leq_c$.

Further, one can show that (*Duality at Work!*)

- $x \leq_s y$ iff
  $x \sqsubseteq y$ and $x_\perp \sqsubseteq y_\perp$ iff
  $x \sqsubseteq y \sqsubseteq x^\top$

- $x \leq_c y$ iff
  $x \sqsubseteq y$ and $x^\top \sqsubseteq y^\top$ iff
  $y_\perp \sqsubseteq x \sqsubseteq y$

- $x \leq_b y$ iff
  $x \sqsubseteq y$ and $x^\top = y^\top$ iff
  $x \sqsubseteq y$ and $x_\perp = y_\perp$.

Stable/Costable (ctd.)

In terms of Curien-Lamarche games

> $x \leq_s y$ means that y is obtained from x by expanding some occurrences of $\bot$ in x

and dually

> $x \leq_c y$ means that x is obtained from y by expanding some occurrences of $\top$ in y.

Moreover, $\neg$ swaps $\bot$ and $\top$.

Later for locally boolean domains it will hold that the extensional order is the least partial order generated by the union of the stable and the costable order, i.e.

$$\sqsubseteq = (\leq_s \cup \leq_c)^*$$

Properties of LBOs

Let A be a lbo.

For $X \subseteq A$ we write $\uparrow X$ as abbrevation for $\forall\, x, y \in X.\ x \uparrow y$

Lemma Let X be a finite non-empty subset of A with $\uparrow X$.
Then the supremum $\bigvee X$ w.r.t. $\sqsubseteq$ exists and is a supremum also w.r.t. $\leq_s$.
The analogue for infima holds as well.
Moreover, we have $\uparrow X$ iff X has an upper bound w.r.t. $\leq_s$.

Thus, for $x \in A$ the set $\{y \in A \mid y \leq_s x\}$ is a *distributive* lattice (w.r.t. $\leq_s$).

For $x \in A$ the set $[x] = \{y \in A \mid y \updownarrow x\}$ is a *boolean* lattice whose negation is given by $\neg$ (restricted to $[x]$).

Constructions on LBOs

Let $(A_i)_{i \in I}$ be a family of lbo's. Then we can form their **product** $\Pi_{i \in I} A_i$ and their **coproduct** $\coprod_{i \in I} A_i$ as usual.

For a lbo A its **lifting** $A_\uparrow$ is obtained from A by adding (new) $\bot$ and $\top$ which are swapped by $\neg$. (On A negation behaves as before!)

The **separated sum** of a family $(A_i)_{i \in I}$ is defined as

$$\Sigma_{i \in I} A_i = \left(\coprod_{i \in I} A_i \right)_\uparrow$$

N.B. O is the empty separated sum.

It will turn out that all locally boolean domains(!) are generated (from nothing) by the following operations

- products
- separated sums
- inverse limits.

Primality and Completeness

Def. An element p of a lbo A is **prime** iff for all $x, y \in A$, whenever $x \uparrow y$ or $x \downarrow y$ and $p \sqsubseteq x \vee y$ then $p \sqsubseteq x$ or $p \sqsubseteq y$.

Def. A lbo A is **complete** iff every $\sqsubseteq$-directed $X \subseteq A$ has a supremum $\bigvee A$ w.r.t. $\sqsubseteq$.

This gives rise to a notion of compact element as usual: $e \in A$ is **compact** iff for every $\sqsubseteq$-directed $X \subseteq A$ with $e \sqsubseteq \bigvee X$ there exists an $x \in X$ with $e \sqsubseteq x$.

We write CP(A) for the set of *compact prime* elements of A. For $x \in A$ we write CP(x) for $\{p \in \mathrm{CP}(A) \mid p \leq_s x\}$.

N.B. A prime element need not be compact as e. g. ∞ in $\bar{\omega} \cong \bar{\omega}_{\uparrow}$ (the C-L game with a single infinite path).

Locally Boolean Domains

Def.A lbo D is a **locally boolean predomain** (lbpd) iff

(1) $x = \bigvee CP(x)$ for all $x \in A$

(2) CP(p) is finite for all $p \in$ CP(A)

(3) for $x, y \in A$ it holds that

$$x \sqsubseteq y \;\text{ iff }\; \forall\, p \in \mathrm{CP}(x).\, \exists\, q \in \mathrm{CP}(y).\, p \sqsubseteq q\ .$$

A **locally boolean domain** (lbd) is a lbpd D with a least element $\bot$.

N.B. Locally Boolean Domains are in particular dI-domains w.r.t $\leq_s$ (as $\bigvee CP(x)$ in (1) is a supremum also w.r.t. $\leq_s$!).
The elements of CP(D) are the **complete primes** of the dI-domain $(D, \leq_s)$.

Maps of Loc. Bool. Dom.'s

For locally boolean (pre)domains A and B a function $f : A \to B$ is called a **map of locally boolean (pre)domains)** iff

- f is **(Scott) continuous** w.r.t. $\sqsubseteq$
- f is **stable** w.r.t. $\leq_s$
- f is **bistable**, i. e. from $x \updownarrow y$ it follows that $f(x) \updownarrow f(y)$ and $f(x \vee y) = f(x) \vee f(y)$.

The ensuing category is denoted as **LBD**.

The category **LBD** is cartesian closed: the exponential $[A \to B] = B^A$ has underlying set **LBD**(A,B) ordered **pointwise** and negation is defined as

$$(\neg f)(x) = \bigvee \{\neg f(\neg e) \mid e \in \mathsf{C}(x)\}$$

where $\mathsf{C}(x) = \{e \in A \mid e \leq_s x \text{ and } e \text{ compact}\}$.

We can't define $(\neg f)(x) = \neg f(\neg x)$ since

Cocontinuity Fails

Consider the map $F : [\mathbb{N}{\to}O^O] \to$ O defined recursively as

$$F(f) = f(0)(F(\lambda n.f(n{+}1))$$

Let $f = \lambda n.\mathrm{id}_S$ and $f_n(k) = \mathrm{id}_S$ for $k < n$ and $f(k) = \lambda u.\top$ for $k \geq n$.

The set $X := \{f_n \mid n \in \mathbb{N}\}$ is costably coherent, co-directed and $f = \bigwedge X$. As f is a minimal solution of its defining equation we have $F(f) = \bot$ and $F(f_n) = \top$ for all n.

Accordingly, we have

$$F(\bigwedge X) = \bot \neq \top = \bigwedge F[X]$$

i.e. **cocontinuity fails**, which

breaks the symmetry between $\bot$ and $\top$!

Prenex Normal Form

Let D be a lbd with $\top \in \mathsf{CP}(D)$ and $\bot \neq \top$. Then $D^{\downarrow}$ is the lb**p**d obtained by *chopping off* $\bot$ and $\top$. We write $\min(D^{\downarrow})$ for the set of ($\leq_s$-)minimal prime elements of $D^{\downarrow}$.
For every $p \in \min(D^{\downarrow})$ the set

$$D_p = \{x \in D^{\downarrow} \mid p \leq_s x\}$$

is a lbd with $\sqsubseteq$ and $\neg$ inherited from D. Moreover, we have $D \cong \Sigma_{p \in \min(D^{\downarrow})} D_p$

Let D be a lbd where $\top$ is not prime. Let $\max(D)$ be the set of maximal elements of $[\top]$. For every $q \in \mathsf{CP}(D) \backslash \{\bot\}$ there exists a unique $p \in \max(D)$ with $q \sqsubseteq p$.
For $p \in \max(D)$ the set

$$D_p = \{x \in D \mid x \sqsubseteq p\}$$

is a lbd with $\sqsubseteq$ inherited from D and $\neg_{D_p} x = p \wedge \neg x$. Moreover, we have

$$D \xrightarrow{\cong} \Pi_{p \in \max(D)} D_p : x \mapsto (x_p)_{p \in \max(D)}$$

where $\quad x_p = \bigvee \{q \in \mathsf{CP}(x) \mid q \sqsubseteq p\}$.

Prenex Normal Form (ctd.)

Combining these two statements we get for every lbd D that

$$D \cong \mathsf{\Pi}_{p \in \mathsf{max}(D)} \mathsf{\Sigma}_{q \in \mathsf{min}((D{\downarrow})_p)} D_{p,q}$$

Applying this construction n-times and replacing the remaining summands by the empty lbpd we get a lbd $D^{(n)}$ which correctly reflects the behaviour of D "up to level $2n$".

Notice that $D^{(n)}$ is a retract of D via

$$\iota_n : D^{(n)} \trianglelefteq D : \pi_n$$

where π_n "chopps off" at level $\geq 2n$.

The π_n form a projective cone giving rise to

$$D \cong \mathsf{Lim}_{n \in \omega} D^{(n)}$$

exhibiting D as **inverse limit** of the $D^{(n)}$.

UNIVERSALITY RESULTS

Universality of $\Lambda_{\perp}^{\top}$

For $\Lambda_{\perp}^{\top}$, i.e. simply typed λ-calculus over base type O we have

Thm(J. Laird) $\Lambda_{\perp}^{\top}$ is universal for the finite type hierarchy over O in **LBD**.

The proof is based on the fact that for a (countable) set I, $\mathbf{LBD}(\Pi_{i\in I}\mathsf{O}, \mathsf{O})$ contains precisely the constant maps and the projections π_i.

Accordingly, we have an isomorphism

$$\mathsf{catch}_I : \mathsf{O}^{\Pi_{i\in I}\mathsf{O}} \xrightarrow{\cong} I_{\uparrow}$$

with $\mathsf{catch}_I(\pi_i) = i$ for all $i \in I$.

SPCF$^+$ and SPCF$_\infty$

In his CSL'03 paper *Bistability: an extensional characterization of sequentiality* J. Laird has introduced a language SPCF$^+$ whose types are given by

$$\sigma ::= \Pi_{i\in I}\sigma_i \mid \Sigma_{i\in I}\sigma_i \mid \sigma_1 \to \sigma_2$$

where $I \in \omega + 1$.
N.B. $I_\uparrow = \Sigma_{i\in I}1$ where 1 is the empty product. In particular, we have $\mathsf{O} = 0_\uparrow$.

Besides application and functional abstraction SPCF$^+$ terms are built using

- constructors and eliminators
  for I-ary products and (separated) sums
- the isos $\mathsf{catch}_I : \Big((\Pi_{i\in I}\mathsf{O}) \to \mathsf{O}\Big) \to \mathsf{O}$.

SPCF$_\infty$ is SPCF$^+$ extended by **recursive types** and **oracles**, i. e. constants for all elements of type $N{\to}N$ where $N = \Sigma_{i\in\omega}1$.

Universality of SPCF_∞

Laird (CSL'03) has shown that

Thm
The **LBD** model for SPCF^+ is fully abstract, i. e. all compact elements of SPCF^+ types are SPCF^+ definable.

We have shown that

Thm
SPCF_∞ is universal w.r.t. its **LBD** model, i. e. all elements of SPCF_∞ types arise as denotations of SPCF_∞ terms.

Proof (Sketch) : The type $U = [N \to N]$ is universal for countable based locally boolean domains. Moreover, for every SPCF_∞ type σ there are SPCF_∞ terms $\mathrm{IN}_\sigma : \sigma \to U$ and $\mathrm{OUT}_\sigma : U \to \sigma$ whose interpretations constitute a retraction pair in **LBD**.
As all elements of U are SPCF_∞ definable (by an oracle) universality at all types follows.

Its Operational Meaning

The elements of U code strategies (as in J. Longley's *Sequentially Realizable Functionals* APAL (2002)).

The terms OUT_σ can be understood as **interpreters** turning strategies into programs of type σ.

The terms IN_σ read off a strategy from a program of type σ in an extensional(!) way. This is possible due to *observable sequentiality* as ensured by $\top$.

Syn. & Sem. of CPS$_\infty$

With the domain equation

$$D = [D^\omega \to \mathsf{O}]$$

there is associated an infinitary language CPS$_\infty$ whose syntax is given by

$$\begin{aligned} M &::= x \mid \top \mid \lambda\vec{x}.t & & (\vec{x} = x_0, x_1, x_2, \ldots) \\ t &::= M\langle\vec{M}\rangle & & (\vec{M} = M_0, M_1, M_2, \ldots) \end{aligned}$$

or, more economically, by

$$M ::= x \mid \top \mid \lambda\vec{x}.M\langle\vec{M}\rangle \ .$$

The interpretation of CPS$_\infty$ in D is given by the following clauses where ϱ ranges over $\mathsf{Env} = D^{\mathsf{Var}}$

$$[\![\top]\!]\varrho = \top_D \qquad [\![x]\!]\varrho = \varrho(x)$$

$$[\![\lambda\vec{x}.t]\!]\varrho(\vec{d}) = [\![t]\!]\varrho[\vec{d}/\vec{x}]$$

$$[\![M\langle\vec{M}\rangle]\!]\varrho = [\![M]\!]\varrho\Big(\langle[\![M_i]\!]\varrho\rangle_{i\in\mathbb{N}}\Big)$$

λ-Calculus within CPS$_\infty$

The untyped λ-calculus can be interpreted within CPS$_\infty$ as follows

- variables are just variables
- $\lambda y.M$ gets translated as $\lambda y, \vec{x}.M\langle\vec{x}\rangle$
- MN gets translated as $\lambda\vec{x}.M\langle N, \vec{x}\rangle$ (where neither M nor N contains variables from $\vec{x}$)

As the fixpoint operator **Y** is available in untyped λ-calculus we obtain $\perp$ in CPS$_\infty$ as fixpoint of identity.

N.B. CPS$_\infty$ is more expressive than pure untyped λ-calculus as it doesn't contain a term semantically equivalent to

$$\lambda x, \vec{x}.x\langle\top, \top, \dots\rangle$$

which sends $\top_D$ to $\top_D$ and all other elements of D to $\perp_D$.

Realizability (à la Krivine)

Orthogonality between D and $S = D^\omega$:

$$d \perp k \quad \text{iff} \quad d(k) = \top$$

Who is responsible for raising $\top$?

- it's d if $d(k_\perp) = \top$
- it's k if $d(k_\perp) = \perp$

Define $X^\perp = \{d \in D \mid \forall\, k{\in}X.\ d \perp k\}$ for $X{\subseteq}S$. Subsets of D this form are closed under $\rightarrow$ and $\forall$ and, thus, **give rise to a model of classical AF$_2$ à la Krivine**.

Question

Is this model universal in the sense that every $d \in [\![A]\!]$ arises as denotation of an (infinite) term (possibly with $\top$) of type A (and as denotation of a $\top$-free term if $d(k_\perp) = \perp$ for all $k \in [\![A]\!]^\perp$)?

Universality of CPS_∞

(1) By induction on the structure of SPCF_∞ terms one can show that for every SPCF_∞ term $\Gamma \vdash t : \tau$ there exists a CPS_∞ term $\vec{x} \vdash M$ such that the diagram

$$\begin{tikzcd} D^n \arrow[r, "\llbracket \vec{x} \vdash M \rrbracket"] \arrow[d, "\mathrm{OUT}_\Gamma"'] & D \arrow[d, "\mathrm{OUT}_\tau"] \\ \llbracket \Gamma \rrbracket \arrow[r, "\llbracket \Gamma \vdash t \rrbracket"'] & \llbracket \tau \rrbracket \end{tikzcd}$$

commutes in **LBD**.

(2) For the recursive type $\delta := \mu X.[X^\omega \to \mathrm{O}]$ the map $\mathrm{OUT}_\delta : D \to \llbracket \delta \rrbracket$ is an isomorphism.

Suppose $d \in D$. Then $\mathrm{OUT}_\delta(d) \in \llbracket \delta \rrbracket$. By universality of SPCF_∞ there is term t with $\llbracket t \rrbracket = \mathrm{OUT}_\delta(d)$. By (1) there is a CPS_∞ term M with $\mathrm{OUT}_\delta(\llbracket M \rrbracket) = \llbracket t \rrbracket = \mathrm{OUT}_\delta(d)$. As by (2) the map OUT_δ is an isomorphism it follows that $\llbracket M \rrbracket = d$ as desired.

Full Completeness Fails

Question Does $[\![M_1]\!] = [\![M_2]\!]$ imply that M_1 and M_2 have the same infinite normal form?
NO!

For a closed CPS$_\infty$ term M consider

$$M^* \equiv \lambda\vec{x}.x_0\langle\bot,\lambda\vec{y}.x_0\langle M,\bot,\bot,\ldots\rangle,\bot,\bot,\ldots\rangle$$

For all $\vec{d} \in D^\omega$ we have

$$[\![M^*]\!](\vec{d}) = \top \quad \text{iff} \quad d_0 = \top$$

Suppose $d_0 \neq \top$. Then $d_0 = \bot$ or there is an n such that d_0 evaluates the n-th argument first. If $n = 1$ then $d_0\langle M,\bot,\bot,\ldots\rangle = \bot$, thus

$$d_0\langle\bot,\lambda\vec{y}.d_0\langle M,\bot,\bot,\ldots\rangle,\bot,\bot,\ldots\rangle = \bot$$

which is also the case if $n \neq 1$.
Suppose M_1 and M_2 have different infinite normal forms. Then M_1^* and M_2^* have also different infinite normal forms although we have $[\![M_1^*]\!] = [\![M_2^*]\!]$ by the above consideration.
Thus, the **Separation Theorem** fails for CPS$_\infty$ although it holds for untyped λ-calculus.

Open Questions

- Is there an "optimal" decompilation of elements of D?
  Many different CPS_∞ terms give rise to the same strategy, but is there a canonical one which is "optimal" in some intuitive operational sense?

- *Is there an extensional account of other paradigms of games?*
  One would have to be able to observe not only sequentiality (via $\top$) but also how often an argument is needed. (Ehrhard's Quantitative Semantics?)

- According to P.-L. Curien Girard's *Ludics* is an affine(!) variant of CPS_∞. In this case the Separation Theorem holds as opposed to CPS_∞ (see previous page!).
  Problem: There is an affine variant of $\mathbf{LBD}$ but in this model $\mathrm{O}\&\mathrm{O} \cong \mathrm{O}\otimes\mathrm{O}$ although $\lambda x{:}\mathrm{O}.\, x \otimes x$ is not affine in x !

- Universality w.r.t. AF_2 ?