Under consideration for publication in Math. Struct. in Comp. Science

Impredicativity entails Untypedness

PETER LIETZ and THOMAS STREICHER

Fachbereich Mathematik, Technische Universität Darmstadt,
Schloßgartenstr. 7, 64289 Darmstadt, Germany.

Received 12 October 2000

In standard realizability one works with respect to an untyped universe of realizers called a partial combinatory algebra (pca). It is well known that a pca $\mathcal{A}$ gives rise to a categorical model of impredicative type theory via the category $\mathbf{Asm}(\mathcal{A})$ of assemblies over $\mathcal{A}$ or the realizability topos over $\mathcal{A}$.
Recently, in (Longley 1999b) John Longley has introduced a typed version of pca's. The above mentioned construction of categorical models extends to the typed case. However, in general these are not impredicative anymore. We show that for a typed pca $\mathcal{T}$ the ensuing models are impredicative if and only if $\mathcal{T}$ has a universal type U. Such a type U can be endowed with the structure of an untyped pca such that U and $\mathcal{T}$ induce equivalent realizability models, in other words a typed pca $\mathcal{T}$ with a universal type is essentially untyped. Thus, *a posteriori* it turns out that nothing is lost by restricting to (untyped) pca's as far as realizability models of impredicative type theories are concerned.
For instance, we show that for a typed pca $\mathcal{T}$ the fibred category of discrete families in $\mathbf{Asm}(\mathcal{T})$ is small if and only if $\mathcal{T}$ has a universal type, hence is essentially untyped. As the category of $\neg\neg$–separated objects of the modified realizability topos is equivalent to $\mathbf{Asm}(\mathcal{T})$ for an appropriate typed pca $\mathcal{T}$ without a universal type it follows that the discrete families in the subcategory of $\neg\neg$–separated objects of the modified realizability topos do not provide a model of polymorphic λ–calculus.

1. Typed Partial Combinatory Algebras

In this introductory section we review the notion of *typed partial combinatory algebra* and give some examples.

Definition 1.1. A *typed partial combinatory algebra (tpca)* is a non-empty set $\mathcal{T}$ of types together with

1. binary operations $\times$ and $\to$ on $\mathcal{T}$,
2. a set $|T|$ of realizers of type T for every $T \in \mathcal{T}$
3. a partial application function $\cdot_{S,T} : |S \to T| \times |S| \longrightarrow |T|$ for all $S, T \in \mathcal{T}$

such that for all $S, T, U \in \mathcal{T}$ there are elements

$$\mathsf{k}_{S,T} \in |S \to T \to S|, \qquad \mathsf{s}_{S,T,U} \in |(S \to T \to U) \to (S \to T) \to (S \to U)|$$
$$\mathsf{pair}_{S,T} \in |S \to T \to S \times T|, \qquad \mathsf{fst}_{S,T} \in |S \times T \to S|, \qquad \mathsf{snd}_{S,T} \in |S \times T \to T|$$

satisfying

$$\begin{array}{ll} \mathsf{k}\,a\,b = a, \qquad \mathsf{s}\,a\,b\downarrow, & \mathsf{s}\,a\,b\,c \succeq a\,c\,(b\,c) \\ \mathsf{fst}\,(\mathsf{pair}\,a\,b) = a, & \mathsf{snd}\,(\mathsf{pair}\,a\,b) = b \end{array}$$

for all appropriately typed a, b, c.

Notice that $e \succeq e'$ (as in (Freyd and Scedrov 1991)) means that whenever e is defined then e' is defined and equal to e. As usual, $\to$ associates to the right and $\times$ binds stronger than $\to$, type annotations are often omitted, application associates to the left and is often denoted by mere juxtaposition, as above. In the following, for any expression e the statements $e \in |T|$ and $e \Vdash x$ are supposed to imply that e is defined.

Remark 1.2. The notion of typed pca was introduced by John Longley in (Longley 1999b) under the name *partial combinatory type structure*. Although typed combinators as well as (untyped) partial combinatory algebra have been studied for a long time, surprisingly enough a combination of both had not been considered before. Our definition differs from that given in (Longley 1999b) only in the respect that Longley requires that every type is inhabited, whereas we decided not to do so in view of some of the following examples. However, as soon as there is some type T there also is an inhabited type, as $T \to T$ is inhabited by $\mathsf{i} = \mathsf{skk}$.

The notion of tpca should be viewed as open, as it is possible to extend it with additional features such as sum types, arithmetic, recursion etc. But these possible extensions are not needed for the purposes of the current paper.

Remark 1.3. A typed version of combinatory completeness holds for every tpca, i.e., for any polynomial $e : T$ (a well-typed term built up from typed variables and constants) and any variable $x{:}S$ there is a polynomial $\lambda^* x{:}S.\, e : S \to T$ whose variables are those of e excluding x such that $\lambda^* x{:}S.\, e\downarrow$ and $(\lambda^* x{:}S.\, e)\, a \succeq e[a/x]$ for all $a \in |T|$ and all valuations of the free variables.

Examples of tpca's are abundant. First of all, any pca can be made into a tpca with one single type. Examples arising from typed functional programming languages are studied in (Longley 1999a).

Example 1.4. Let $\mathcal{A}$ be a pca.

(i) We define the tpca $\mathcal{P}(\mathcal{A})$ as follows. The underlying set of types is the power set of $\mathcal{A}$ and for any type S let $|S| = S$. For types S, T we define

$$\begin{array}{rcl} |S \times T| & = & \{a \in \mathcal{A} \mid \mathsf{p}_0\, a \in |S| \text{ and } \mathsf{p}_1\, a \in |T|\} \\ |S \to T| & = & \{a \in \mathcal{A} \mid \forall b \in |S|.\, a\, b \in |T|\} \end{array}$$

where $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1 \in \mathcal{A}$ are appropriate combinators for pairing and projection. The typed application operations are defined as restrictions of the application operation in $\mathcal{A}$ to the respective subsets and, therefore, are total.

(ii) If we restrict the set of types to the nonempty subsets of $\mathcal{A}$ we obtain another tpca which we call $\mathcal{P}^+(\mathcal{A})$.

(iii) Let $a \in \mathcal{A}$ be such that $\mathsf{p}aa = a$ and $\forall x \in \mathcal{A}.\ ax = a$. Then we can further restrict the set of types to those subsets of $\mathcal{A}$ containing a and call the resulting tpca $\mathcal{P}^a(\mathcal{A})$.
(iv) Finally, we can restrict the set of types to the set of finite types over some chosen ground type. For appropriate choices of $\mathcal{A}$ this gives rise to HRO (hereditary recursive operations) and ICF (intensional continuous functionals) as discussed in (Troelstra 1973)).

Example 1.5. Interesting examples arise from cartesian closed categories. Let $\mathbb{C}$ be a CCC. We obtain a tpca as follows. Define the set of types to be the set of objects† of $\mathbb{C}$ and for any type T let $|T|$ be the set $\mathbb{C}(1, T)$ of global sections of T. For instance, the cartesian closed category of algebraic lattices gives rise to a notion of realizability investigated in (Bauer, Birkedal and Scott 1998). Other interesting examples arise from the cartesian closed category $\mathbf{PER}(\mathcal{A})$ of partial equivalence relations on a pca $\mathcal{A}$. In analogy to Example 1.4 (ii) and (iii) one can consider the subcategory $\mathbf{PER}^+(\mathcal{A})$ of nonempty per's and the subcategory $\mathbf{PER}^a(\mathcal{A})$ of those per's containing some suitable fixed element a in their carrier, as both are exponential ideals. Finally one can restrict to subcategories of finite types over some chosen ground type thus obtaining HEO and ECF (see (Troelstra 1973)).

Example 1.6.

(i) Fix a set $\mathcal{B}$ of base types and define the set of types as the set freely generated from $\mathcal{B}$ by $\times$ and $\to$. For any type T define $|T|$ to be the set of closed simply typed λ–terms of type T modulo β or $\beta\eta$ equality. Application is induced by the application of λ–calculus.
(ii) Let the set of types be those of Gödel's system T (with product types) and for each type T let $|T|$ be the set of closed terms of system T of type T modulo an appropriate conversion equality or observational equivalence.
(iii) Similarly one may take the closed types of Girard's system F (polymorphic λ–calculus) and for $|T|$ the closed terms of type T modulo β or $\beta\eta$ equality.

Remark 1.7. Notice that for example 1.4 (ii) and (iii) (but not (i)!) it suffices to require that $\mathcal{A}$ is a *conditional* pca (see (Hyland and Ong 1993)), in which the axiom $\mathsf{s}\,a\,b\downarrow$ need not hold. In a sense, the introduction of types makes up for the lack of definedness of $\mathsf{s}\,a\,b$ in $\mathcal{A}$. In Example 1.4(iii) instead of requiring that all types contain some fixed element a we can require that all types contain some fixed *right–absorptive set* Θ of $\mathcal{A}$ (see (Hyland and Ong 1993)).

Correspondingly, in Example 1.5 for the cases of $\mathbf{PER}^+(\mathcal{A})$ and $\mathbf{PER}^a(\mathcal{A})$ (but not $\mathbf{PER}(\mathcal{A})$!) it suffices that $\mathcal{A}$ is a conditional pca. Furthermore, instead of the category $\mathbf{PER}^a(\mathcal{A})$ we can consider the cartesian closed category $\mathbf{PER}^\Theta(\mathcal{A})$ of those per's R satisfying $\theta R \theta'$ for all $\theta, \theta' \in \Theta$ for some right–absorptive set Θ.

In Example 1.5 the category $\mathbb{C}$ need not be cartesian closed. Instead, one may take for $\mathbb{C}$ a weak version of a partial cartesian closed category as in (Birkedal 1999).

† ignoring size matters for the moment

2. Universal Types

Definition 2.1. Let $\mathcal{T}$ be a tpca. A type $U \in \mathcal{T}$ is called *universal* if for any type $T \in \mathcal{T}$ there are realizers $\mathsf{e}_T \in |T \to U|$ and $\mathsf{r}_T \in |U \to T|$ such that $\forall x \in |T|. \mathsf{r}_T(\mathsf{e}_T\, x) = x$ i.e. every type is a partial[‡] retract of the type U.

A universal type U in a tpca gives rise to a pca as in particular $U \to U$ is a retract of U. The underlying set of the pca is $|U|$ and for $x, y \in |U|$ application is defined as $\mathsf{r}_{U\to U} xy$. This structure is combinatory complete. For any untyped U-polynomial e and variables $x_1, \ldots, x_n$ define $\lambda^* x_1 \cdots x_n.\, e$ as $\mathsf{e}_{U\to U}\, \lambda^* x_1 : U. \cdots \mathsf{e}_{U\to U}\, \lambda^* x_n : U.\, e'$, where e' is e with applications MN replaced by $\mathsf{r}_{U\to U} MN$. In particular, combinators s and k can be defined for U in this way.

In (Longley 1999b) a 2-category of tpca's and applicative morphisms is introduced, containing the 2-category of pca's familiar from (Longley 1994) as a full subcategory. We will not repeat the definition of the 2-category here but rather state the induced notion of equivalence for typed pca's as a definition.

Definition 2.2. Two tpca's $\mathcal{T}$ and $\mathcal{S}$ are *equivalent* whenever there are functions $f : \mathcal{T} \to \mathcal{S}$ and $g : \mathcal{S} \to \mathcal{T}$ together with a function $u_T : |T| \to |f(T)|$ for every $T \in \mathcal{T}$ and a function $v_S : |S| \to |g(S)|$ for every $S \in \mathcal{S}$ satisfying the following properties.

For every pair $T, T' \in \mathcal{T}$ there is some $q_{T,T'} \in |f(T \to T') \to f(T) \to f(T')|$ such that $q_{T,T'}\, u_{T\to T'}(a)\, u_T(b) \succeq u_{T'}(ab)$ for all $a \in |T \to T'|, b \in |T|$. Likewise, for every pair $S, S' \in \mathcal{S}$ there is some $r_{S,S'} \in |g(S \to S') \to g(S) \to g(S')|$ such that $r_{S,S'}\, v_{S\to S'}(a)\, v_S(b) \succeq v_{S'}(ab)$ for all $a \in |S \to S'|, b \in |S|$.

Furthermore, for every $T \in \mathcal{T}$ there are elements $i_T \in |T \to g(f(T))|$ and $j_T \in |g(f(T)) \to T|$ satisfying $i_T\, a = v_{f(T)}(u_T(a))$ and $a = j_T\, v_{f(T)}(u_T(a))$ for all $a \in |T|$ and for every $S \in \mathcal{S}$ there are elements $k_S \in |S \to f(g(S))|$ and $l_S \in |f(g(S)) \to S|$ satisfying $k_S\, a = u_{g(S)}(v_S(a))$ and $a = l_S\, u_{g(S)}(v_S(a))$ for all $a \in |S|$.

For instance, the tpca's $\mathcal{P}^0(K_1)$ and $\mathcal{P}^+(K_1)$ are equivalent. The following proposition says that a tpca is equivalent to a pca if and only if it has a universal type.

Proposition 2.3. If a tpca $\mathcal{T}$ has a universal type U, then it is equivalent to the induced pca on U. Conversely, if a tpca is equivalent to a pca, then it has a universal type.

Proof. straightforward exercise. □

For a tpca $\mathcal{T}$ satisfying $\mathsf{s}\, a\, b\, c \simeq a\, c\, (b\, c)$ instead of just $\mathsf{s}\, a\, b\, c \succeq a\, c\, (b\, c)$ one consequence of the existence of a universal type is the existence of fixpoint combinators for arbitrary types.

Proposition 2.4. Let $\mathcal{T}$ be a tpca satisfying $\mathsf{s}\, a\, b\, c \simeq a\, c\, (b\, c)$ with a universal type $U \in \mathcal{T}$. Then for every type $T \in \mathcal{T}$ there is an element $\mathsf{Y}_T \in |(T \to T) \to T|$ satisfying $\forall f \in |T \to T|. f(\mathsf{Y}_T f) \simeq (\mathsf{Y}_T f)$. [§]

[‡] meaning that r_T may be partial although e_T is still total

[§] At present, we do not know whether in the general case one can at least prove the existence of combinators $\mathsf{Y}_T \in |(T \to T) \to T|$ satisfying $\forall f \in |T \to T|. (\mathsf{Y}_T f) \succeq f(\mathsf{Y}_T f)$. However, as all examples

Proof. First define

$$\mathsf{Y}_U = \lambda^* f : U \to U.\, (\lambda^* x : U.\, f(\mathsf{r}_{U\to U} x x))(\mathsf{e}_{U\to U} \lambda^* x : U.\, f(\mathsf{r}_{U\to U} x x)) \quad .$$

Then, for each type $T \in \mathcal{T}$ define

$$\mathsf{Y}_T = \lambda^* f : T \to T.\, \mathsf{r}_T(\mathsf{Y}_U(\lambda x : U.\, \mathsf{e}_T(f(\mathsf{r}_T x)))) \quad .$$

□

The existence of a universal type is a rather strong requirement for a tpca. As shown in the following proposition, a lot of natural examples of tpca's do not have a universal type.

Proposition 2.5.

(i) The tpca $\mathcal{P}(\mathcal{A})$ has no universal type.
(ii) The tpca's $\mathcal{P}^+(\mathcal{A})$ and $\mathcal{P}^a(\mathcal{A})$ have no universal type unless $\mathcal{A}$ has only one element.
(iii) The tpca $\mathcal{P}^\Theta(\mathcal{A})$ has no universal type unless $\mathcal{A} = \Theta$.
(iv) The tpca's HRO and ICF have no universal type.
(v) The tpca induced by a well-pointed CCC has a universal type if and only if there is an object U such that any object A in the category is a retract of U.
(vi) The tpca induced by the cartesian closed category of algebraic lattices has no universal type.
(vii) The tpca $\mathbf{PER}(\mathcal{A})$ has no universal type
(viii) The tpca's $\mathbf{PER}^+(\mathcal{A})$ and $\mathbf{PER}^a(\mathcal{A})$ have no universal type unless $\mathcal{A}$ has only one element.
(ix) The tpca's HEO and ECF have no universal type
(x) None of the tpca's in example 1.6 has a universal type.

Proof. Statements (i), (ii) and (iii) hold for cardinality reasons. Let us suppose $\mathcal{T}$ is one of the tpca's $\mathcal{P}(\mathcal{A})$, $\mathcal{P}^+(\mathcal{A})$, $\mathcal{P}^a(\mathcal{A})$ or $\mathcal{P}^\Theta(\mathcal{A})$. Assume there exists a universal type $U \in \mathcal{T}$. Then for any $T \in \mathcal{T}, T = \{\mathsf{r}_T\, u \mid u \in U\}$. Thus, the number of types were bounded by the cardinality of $\mathcal{A}$ which is not the case.

Similarly for (vii) and (viii). Statements (iv), (ix) and (x) hold because of Proposition 2.4. All application functions are total but there are realizable endomaps without fixpoint. Statement (v) is only the expansion of the definition of universal type. Finally, (vi) follows from (v) as there clearly is no universal algebraic lattice for cardinality reasons. □

3. Realizability Models

Now we generalize various constructions of realizability models from pca's to typed pca's.

Definition 3.1. Let $\mathcal{T}$ be a tpca.

of typed or untyped pca's considered in this paper do satisfy the requirement $\mathsf{s}\,a\,b\,c \simeq a\,c\,(b\,c)$ this is not a severe restriction.

(i) The category $\mathbf{Asm}(\mathcal{T})$ of *assemblies over* $\mathcal{T}$ is defined as follows. An object X consists of an underlying set I together with an underlying type $T \in \mathcal{T}$ and a realizability relation $\Vdash_X \subseteq |T| \times I$ such that $\forall i \in I. \exists t \in |T|.\, t \Vdash_X i$. For any $i \in I$ we define the set of realizers of i, $\|i\|_X$ to be the set $\{t \in |T| \mid t \Vdash_X i\}$. A morphism from X to X' is a map $f : I \to I'$ which is tracked by some $a \in |T \to T'|$, i.e. $\forall i \in I. \forall b \in |T|.\, b \Vdash_X i \supset a\, b \Vdash_{X'} f(i)$.

(ii) The category $\mathbf{D}(\mathcal{T})$ of *discrete sets over* $\mathcal{T}$ is the full subcategory of $\mathbf{Asm}(\mathcal{T})$ on those objects $X = (I, T, \Vdash_X)$ which satisfy $\forall t \in |T|. \forall i, i' \in I.\, t \Vdash_X i \wedge t \Vdash_X i' \supset i = i'$.

(iii) The fibration $\mathbf{UFam}(\mathbf{D}(\mathcal{T}))$ over $\mathbf{Asm}(\mathcal{T})$ of *uniform families of discrete sets* induced by $\mathcal{T}$ is defined as follows. An object in the total category is an assembly $X = (I, T, \Vdash_X)$ together with an I-indexed familiy of discrete sets $(D_i)_{i \in I}$ over one common type S. A morphism from $(X, S, (D_i)_{i \in I})$ to $(X', S', (D'_i)_{i \in I'})$ consists of a morphism of assemblies $f : X \to X'$ together with a family of morphisms of discrete sets $(f_i : D_i \to D'_{f(i)})_{i \in I}$ which is uniformly tracked, i.e. there is a realizer $a \in |T \to S \to S'|$ such that $\forall i \in I. \forall b \in |T|.\, b \Vdash_X i \supset a\, b \Vdash f_i$. The fibration itself is given by the projection functor sending $(X, S, (D_i)_{i \in I})$ to X and $(f, (f_i)_{i \in I})$ to f.

(iv) The hyperdoctrine $\mathcal{H}(\mathcal{T})$ over $\mathbf{Set}$ is defined as follows. An element in the fibre over a set I is a type $T \in \mathcal{T}$ together with a function $\varphi : I \to \mathcal{P}(|T|)$. The entailment $\varphi \vdash_I \varphi'$ holds if there is some $a \in |T \to T'|$ such that $\forall i \in I. \forall b \in \varphi(i).\, a\, b \in \varphi'(i)$. Reindexing is given by precomposition.

(v) The *realizability category over* $\mathcal{T}$, $\mathbf{RC}(\mathcal{T})$, is defined as $\mathbf{Set}[\mathcal{H}(\mathcal{T})]$, i.e. the category whose objects are formal partial equivalence relations w.r.t. $\mathcal{H}(\mathcal{T})$ and whose morphisms are formal strict functional relations (see (Pitts 1999) for the exact definition).

One can prove that all constructions of Definition 3.1 applied to equivalent tpca's yield equivalent (fibred) categories. For pca's the constructions above coincide with the usual ones. Next we discuss which of the properties of realizability models lift to the typed case.

The categories $\mathbf{Asm}(\mathcal{T})$ and $\mathbf{D}(\mathcal{T})$ are regular and locally cartesian closed. The global sections functor $\Gamma : \mathbf{Asm}(\mathcal{T}) \to \mathbf{Set}$ has a right adjoint $\nabla : \mathbf{Set} \to \mathbf{Asm}(\mathcal{T})$ which can be described concretely as follows. Let $T \in \mathcal{T}$ be a type inhabited by $* \in |T|$. Then for any set I define ∇I as the assembly with underlying set I, underlying type T and $\|i\|_{\nabla_I} = \{*\}$ for all $i \in I$. An assembly $X = (I, T, \Vdash_X)$ is called *codiscrete* iff it is isomorphic to ∇I, i.e. iff there exists a $t \in |T|$ such that $t \Vdash_X i$ for all $i \in I$. Note that an object of $\mathbf{Asm}(\mathcal{T})$ is in $\mathbf{D}(\mathcal{T})$ if and only if it is internally orthogonal to $\nabla(2)$. Furthermore, the fibration $\mathbf{UFam}(\mathbf{D}(\mathcal{T}))$ over $\mathbf{Asm}(\mathcal{T})$ is a split fibration equivalent to the fibration of *discrete families* in $\mathbf{Asm}(\mathcal{T})$, i.e. those families that are internally orthogonal to $\nabla(2)$. Thus it follows immediately that the fibration is complete and a fibred CCC (see (Hyland, Robinson and Rosolini 1990)). However, there need not exist a generic family.¶

¶ Notice that we call an object X in the total category of a fibration *generic* whenever for each object Y in the total category there is a cartesian morphism from Y to X. We do not require uniqueness in any form.

The hyperdoctrine $\mathcal{H}(\mathcal{T})$ is a model for the $(\top, \wedge, \supset, \forall, \exists, =)$–fragment of first order predicate logic. However, in general $\mathcal{H}(\mathcal{T})$ need not be a tripos (see (Hyland, Johnstone and Pitts 1980; Pitts 1981)) as there need not exist a generic predicate allowing for an interpretation of higher order logic. Moreover, the fibres need not have finite suprema allowing for an interpretation of $(\perp, \vee)$.

The category $\mathbf{RC}(\mathcal{T})$ is locally cartesian closed and exact. It contains $\mathbf{Asm}(\mathcal{T})$ as a full reflective subcategory closed under subobjects and the inclusion preserves the locally cartesian closed structure. The category $\mathbf{RC}(\mathcal{T})$ is the *exact completion of the regular category* $\mathbf{Asm}(\mathcal{T})$ and was in fact defined this way in (Longley 1999b). Every object of $\mathbf{RC}(\mathcal{T})$ is covered via a regular epi by an object of $\mathbf{Asm}(\mathcal{T})$ (see (Freyd and Scedrov 1991; Carboni 1995; Carboni and Vitale 1998)). In general $\mathbf{RC}(\mathcal{T})$ need not be a topos.

4. Impredicativity entails Untypedness

We will show now that a tpca $\mathcal{T}$ has a universal type, i.e. is essentially untyped, if and only if the ensuing realizability models of Definition 3.1 have the usual impredicative features.

For this purpose we need the following somewhat technical lemma.

Lemma 4.1. Suppose there exists a morphism $m : A \to X$ in $\mathbf{Asm}(\mathcal{T})$ such that for every subobject $n : B \to Y$ of a codiscrete Y there is a pullback diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "m"] \\ Y \arrow[r] & X \end{tikzcd}$$

Then $\mathcal{T}$ has a universal type.

Proof. Let U be the type of A. We claim that U is universal. For an arbitrary type $T \in \mathcal{T}$ let Δ_T be the assembly with underlying set $|T|$, underlying type T and $\|t\|_{\Delta_T} = \{t\}$ for all $t \in |T|$. Let $i : \Delta_T \to \nabla|T|$ be the morphism whose underlying map is the identity on $|T|$. As i is monic and $\nabla|T|$ is codiscrete there is a pullback diagram

$$\begin{tikzcd} \Delta_T \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "m"] \\ \nabla|T| \arrow[r, "u"'] & X \end{tikzcd}$$

due to our assumption on m. Let $\mathsf{e}_T \in |T \to U|$ be a realizer of f. Note that e_T is defined for all $t \in |T|$. Let D be the assembly with underlying set $|T|$, underlying type U and $\|t\|_D = \{\mathsf{e}_T t\}$ for all $t \in |T|$. Let $i' : D \to \nabla|T|$ and $f' : D \to A$ be the morphisms whose underlying maps are those of i and f, respectively. Therefore, $u \circ i' = m \circ f'$. Let $g : D \to \Delta_T$ be the mediating morphism, i.e. $i \circ g = i'$ and $f \circ g = f'$. Notice that $g(t) = t$ for all $t \in |T|$, as $i \circ g = i'$. Let r_T be a realizer of g. Then for all $t \in |T|$ we have $\mathsf{r}_T(\mathsf{e}_T t) \Vdash_{\Delta_T} g(t) = t$ and, therefore, $\mathsf{r}_T(\mathsf{e}_T t) = t$, as desired. $\square$

Theorem 4.2. Let $\mathcal{T}$ be a tpca. The following are equivalent.

(i) $\mathcal{T}$ has a universal type
(ii) $\mathbf{Asm}(\mathcal{T})$ has a generic mono
(iii) $\mathcal{H}(\mathcal{T})$ is a tripos
(iv) $\mathbf{RC}(\mathcal{T})$ is a topos
(v) $\mathbf{UFam}(\mathbf{D}(\mathcal{T}))$ over $\mathbf{Asm}(\mathcal{T})$ has a generic family.

Proof. If $\mathcal{T}$ has a universal type then it is equivalent to a pca. As all constructions considered in (ii) - (v) preserve equivalence, the induced structures are equivalent to ones induced by a pca. For these the properties (ii) - (v) are well known and, therefore, condition (i) implies conditions (ii) - (v).

Assume (ii) and let m be a generic mono. As any mono arises as a pullback of m, this holds in particular for subobjects of codiscrete objects. Therefore, condition (i) follows by Lemma 4.1.

Clearly (iii) implies (iv).

Assume (iv). Let $\top : 1 \to \Omega$ be a subobject classifier in $\mathbf{RC}(\mathcal{T})$. Then Ω is covered by an assembly X via an epi $e : X \to \Omega$. Let $m : A \to X$ be the pullback of $\top$ along e. As $\mathbf{Asm}(\mathcal{T})$ is closed under subobjects A is an assembly, too. We show that all subobjects of codiscrete objects arise as pullbacks of m. Let $n : B \to C$ be a subobject of a codiscrete assembly C. As C is projective in $\mathbf{RC}(\mathcal{T})$ the classifying morphism $\chi : C \to \Omega$ for n factors through e and, therefore, in the diagram

$$\begin{tikzcd}
B \arrow[r, dashed] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left] & A \arrow[r] \arrow[d, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
C \arrow[r, dashed] \arrow[rr, bend right, "\chi"'] & X \arrow[r, "e"] & \Omega
\end{tikzcd}$$

the left square is a pullback, too. Thus, by Lemma 4.1 the tpca $\mathcal{T}$ has a universal type. Hence (iv) entails (i).

Finally, assume (v). Remember that the fibration $\mathbf{UFam}(\mathbf{D}(\mathcal{T}))$ is equivalent to the fibration of discrete families in $\mathbf{Asm}(\mathcal{T})$. As any mono is a discrete family this implies that every mono appears as a pullback of a fixed morphism in $\mathbf{Asm}(\mathcal{T})$. Thus by Lemma 4.1 condition (i) follows. □

In the proof of (v) $\Rightarrow$ (i) we have actually shown more. Whenever there is a subfibration of the codomain fibration of $\mathbf{Asm}(\mathcal{T})$ that contains all monos and has a generic family, then $\mathcal{T}$ has a universal type.

Moreover, whenever there is an arbitrary fibration over $\mathbf{Asm}(\mathcal{T})$ that contains the subobject fibration of $\mathbf{Asm}(\mathcal{T})$ as a *definable* subfibration and has a generic family, then $\mathcal{T}$ has a universal type. This is because a definable subfibration as opposed to an arbitrary subfibration inherits the property of having a generic family. See (Borceux 1994) for a treatment of definability in fibrations.

But, equally important, from Theorem 4.2 and Proposition 2.5 we get quite a few examples of realizability models which are not impredicative. We just discuss those examples which have already been considered in other contexts.

As the category **ALat** of algebraic lattices does not have a universal object the exact completion of the regular category **Asm**(**ALat**) is not a topos. Observe that as **Asm**(**ALat**) is the regular completion of the category **Sp** of topological spaces (see (Rosolini 1999)) it follows immediately that the exact completion of **Sp** as a lex category is not a topos. Moreover, the fibration **UFam**(**D**(**ALat**)) has no generic family and, therefore, is not a model of polymorphic λ–calculus or even Calculus of Constructions. This confirms a conjecture raised in (Bauer, Birkedal and Scott 1998).

Next notice that the category of $\neg\neg$–separated objects of the modified realizability topos **Mod** (see (Grayson 1981; van Oosten 1997)) is equivalent to the category $\mathbf{Asm}(\mathcal{P}^0(K_1))$ where K_1 is the 1^{st} Kleene algebra and $\mathcal{P}^0(K_1)$ is the tpca defined in Example 1.4. In this case the underlying type of an assembly is called its *set of potential realizers*.

More generally, whenever $\mathcal{A}$ is a conditional pca and $\Theta \subseteq \mathcal{A}$ is a right–absorptive set then the category of $\neg\neg$–separated objects in the modified realizability topos over $(\mathcal{A}, \Theta)$ (see (Hyland and Ong 1993; Ong and Ritter 1994)) is equivalent to the category $\mathbf{Asm}(\mathcal{P}^\Theta(\mathcal{A}))$ (see Remark 1.7). As tpca's of the form $\mathcal{P}^\Theta(\mathcal{A})$ do not have a universal type it follows from Theorem 4.2 that there is no generic family for the fibration of discrete families in the category of $\neg\neg$–separated objects of a modified realizability topos. Another consequence of Theorem 4.2 is that a modified realizability topos does not arise as the exact completion of its regular subcategory of $\neg\neg$–separated objects as the exact completion is not a topos.

All these considerations apply as well to extensional versions of modified realizability as considered in (Grayson 1981) replacing $\mathcal{P}^0(K_1)$ by $\mathbf{PER}^0(K_1)$.

5. Fibrations of PER–extension pairs

In order to get a semantic strong normalization proof for a type theory to work as proposed in (Hyland and Ong 1993; Ong and Ritter 1994) it is necessary to build a model of the respective type theory based on a right–absorptive c-pca (SN_*, Θ) consisting of (equivalence classes of) strongly normalizing λ–terms with a distinguished constant $*$. As we have seen, there is no generic family for the fibration of discrete families in $\mathbf{Asm}(\mathcal{P}^\Theta(\mathcal{A}))$ and, therefore, this fibration cannot be used for a semantic strong normalization proof for an impredicative type theory. There are, however, fibrations built upon the notion of right–absorptive c-pca that do have a generic family. In (Hyland and Ong 1993) the fibration of *PER–extension pairs* over **Set** is introduced. This fibration is a complete fibred CCC with a generic family and gives rise to a semantic strong normalization proof for system F.

In (Ong and Ritter 1994) this fibration is extended to the fibration of PER–extension pairs over $\mathbf{Asm}(\mathcal{P}^\Theta(\mathcal{A}))$. Roughly, the difference between the fibration of PER–extension pairs over $\mathbf{Asm}(\mathcal{P}^\Theta(\mathcal{A}))$ on the one hand and the fibration of uniform families of discrete sets over $\mathbf{Asm}(\mathcal{P}^\Theta(\mathcal{A}))$ on the other hand is that in the former there need not be one single set of potential realizers common to all elements of a family but the set of potential realizers may depend on the index. Therefore, the fibration of PER–extension pairs over $\mathbf{Asm}(\mathcal{P}^\Theta(\mathcal{A}))$ does have a generic family. However, as we shall see this fibration does not have *Lawvere comprehension* any more (i.e. the terminal object functor of the fibration has no right–adjoint). Accordingly, it cannot arise as the fibration of propositions induced by a model of the Calculus of Constructions in $\mathbf{Asm}(\mathcal{P}^{\Theta}(\mathcal{A}))$ in contradiction to what is claimed in (Ong and Ritter 1994).

Theorem 5.1. The fibration of PER–extension pairs over $\mathbf{Asm}(\mathcal{P}^{\Theta}(\mathcal{A}))$ does not have Lawvere comprehension unless $\mathcal{A} = \Theta$.

Proof. Assume that $\mathcal{C}$ is a right–adjoint of the terminal object functor. Let X be the family over $\nabla(\mathcal{P}^{\Theta}(\mathcal{A}))$ that associates with any $T \in \mathcal{P}^{\Theta}(\mathcal{A})$ the PER–extension pair $\langle =_T, T\rangle$. Let U be the set of potential realizers of $\mathcal{C}X$. Now, for any $T \in \mathcal{P}^{\Theta}(\mathcal{A})$ let $f = (f_t)_{t\in T} : \mathbf{1}_{\mathcal{C}X} \to X$ be the morphism over $\mathrm{const}_T : \Delta_T \to \nabla(\mathcal{P}^{\Theta}(\mathcal{A}))$ defined by $f_t(*) = \{t\}$. As $\mathcal{C}$ is right-adjoint to $\mathbf{1}$ the diagram can be completed as indicated. From the realizers of $f^{\vee}$ and ϵ we obtain elements $\mathsf{e}_T \in |T \to U|$ and $\mathsf{r}_T \in |U \to T|$ satisfying $\mathsf{r}_T(\mathsf{e}_T t) = t$ for all $t \in |T|$. By Proposition 2.5 this is a contradiction.

$$\begin{tikzcd}
\mathbf{1}_{\Delta_T} \arrow[dr, dashed, "\mathbf{1}_{f^{\vee}}"'] \arrow[drr, bend left, "f"] & & \\
& \mathbf{1}_{\mathcal{C}X} \arrow[r, "\epsilon"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{C}X \arrow[r, "P\epsilon"] & \nabla(\mathcal{P}^{\Theta}(\mathcal{A})) \\
\Delta_T \arrow[ur, dashed, "f^{\vee}"] \arrow[urr, bend right, "Pf=\mathrm{const}_T"'] & &
\end{tikzcd}$$

□

Instead of the modified realizability topos over a right–absorptive c-pca $(\mathcal{A}, \Theta)$ we propose to use a different topos. It can be defined by a tripos that has $\Sigma = \{(A, P) \mid A \subseteq P \subseteq \mathcal{A},\ A = \emptyset \Rightarrow P = \emptyset,\ A \neq \emptyset \Rightarrow \Theta \subseteq P\}$ as truth values. Entailment is defined as for the tripos representing the topos $\mathbf{Eff}^{\to}$. The category of $\neg\neg$–separated objects of this topos does provide a model of the Calculus of Constructions. The details will appear in the first author's PhD thesis.

6. Discussion of related work

Since the Realizability Workshop in Trento in July 1999 quite a few papers have been written dealing with similar questions as discussed in our paper.

L. Birkedal (Birkedal 1999) has proved a result analogous to ours but for his *wcpc*'s instead of typed pca's. Birkedal's *wcpc*'s are a bit more general than tpca's as the former cover also the relative realizability toposes of (Birkedal 1999). His proof, however, relies on

A. Pitts' (Pitts 1999) characterization of those hyperdoctrines which give rise to a topos when applying the "tripos–to–topos construction". This use of Pitts' characterization Theorem emerged as a result of discussions between L. Birkedal and the first author in the autumn of 1999. Accordingly, in a previous version of our paper we proved our result using Pitts' result of (Pitts 1999).

When reading a draft version of the paper "An abstract look at realizability" (De Marchi, Robinson and Rosolini 1999) from November 1999 and trying to understand some proof in there it appeared to us that we can reuse a result we had already before the Trento meeting, namely that a tpca $\mathcal{T}$ has a universal type if and only if $\mathbf{Asm}(\mathcal{T})$ has a generic mono, with almost no modification. The point was that we saw that (1) a subobject classifier in $\mathbf{RC}(\mathcal{T})$ gives rise to a mono m in $\mathbf{Asm}(\mathcal{T})$ from which one may obtain every subobject of a codiscrete object ∇I via pullback and (2) that from such a mono m we can construct a universal type in $\mathcal{T}$ in literally the same way as from a generic mono in $\mathbf{Asm}(\mathcal{T})$. Using this method one can prove the respective results of (Birkedal 1999) and (De Marchi, Robinson and Rosolini 1999) in a simpler way even though these are formulated for more general settings than just tpca's. In particular, the intensional notion of computation employed in (De Marchi, Robinson and Rosolini 1999) is given by just a category $\mathbb{C}$ together with a faithful functor U from $\mathbb{C}$ to the category of sets and partial functions both satisfying some elementary conditions but without any assumptions about the existence of weak partial function spaces. Using a result from (Carboni and Rosolini 1998) it follows that $\mathbb{C}$ has weak dependent products whenever the ensuing realizability category is locally cartesian closed.

One also should mention the work of M. Menni. In (Menni 1999) he gave a very nice characterization of those finite limit categories $\mathbb{C}$ whose exact completion is a topos. In the subsequent paper (Menni 2000) under the assumption that the global sections functor $\Gamma : \mathbb{C} \to \mathbf{Set}$ has a full and faithful right adjoint ∇ (a so–called "chaotic situation") he could simplify the characterization of (Menni 1999) as follows: the exact completion of $\mathbb{C}$ is a topos if and only if $\mathbb{C}$ has weak dependent products and a generic mono. As partitioned assemblies over a typed pca (or arising from the more general situations considered in (Birkedal 1999; De Marchi, Robinson and Rosolini 1999)) are instances of a "chaotic situation" the result of (Menni 2000) provides a conceptual explanation for the central role that generic monos play — explicitly or implicitly — in the above mentioned papers.

7. Conclusion

We have seen that one can build realizability models over arbitrary typed pca's, i.e. typed models of computation, such as for example models of Gödel's system T. Traditional untyped pca's can be considered as particular instances of typed pca's, namely those with a universal type. It is well known that realizability over untyped pca's gives rise to models of impredicative type theory. In this paper we have shown that this implication can be reversed in the sense that a realizability model over a typed pca $\mathcal{T}$ is impredicative if and only if $\mathcal{T}$ admits a universal type, i.e. is essentially untyped.

There are, however, examples of typed pca's without a universal type whose categories of assemblies are nevertheless equivalent to categories of $\neg\neg$–separated objects of appropriate toposes. For example, the category of assemblies over $\mathcal{P}^0(K_1))$ turns out to be equivalent to the category of $\neg\neg$–separated objects of the modified realizability topos introduced independently by Hyland and Grayson in the early eighties and studied in greater detail by van Oosten (van Oosten 1997; van Oosten 1991). Another example is the category of assemblies over $\mathbf{PER}^0(K_1)$, which is equivalent to the category of $\neg\neg$–separated objects of the extensional modified realizability topos of (Grayson 1981). But it is not clear in general when a category of assemblies over a tpca without a universal type can be reconstructed as the category of $\neg\neg$–separated objects for some topos. For instance, this seems unlikely for the tpca arising from some term model of Gödel's system T. Nevertheless it may be worthwhile to investigate the logical properties of such genuinely predicative realizability models.

Acknowledgment

We gratefully acknowledge discussions with John Longley, Matias Menni, Pino Rosolini and Alex Simpson.

References

Bauer, A., Birkedal, L. and Scott, D. S. (1998) Equilogical spaces. submitted to TCS.

Birkedal, L. (1999) *Developing theories of types and computability via realizability.* PhD thesis, Carnegie Mellon University.

Borceux, F. (1994) *Handbook of Categorical Algebra 2, Categories and Structures*, volume 51 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press.

Carboni, A. (1995) Some free constructions in realizability and proof theory. *Journal of Pure and Applied Algebra*, 103:117–148.

Carboni, A. and Rosolini, G. (1998) Locally cartesian closed exact completions. *Journal of Pure and Applied Algebra.* to appear.

Carboni, A. and Vitale, E.M. (1998) Regular and exact completions. *Journal of Pure and Applied Algebra*, 125:79–116.

Freyd, P. J. and Scedrov, A. (1991) *Categories, Allegories.* North-Holland.

Grayson, R. J. (1981) Modified realizability toposes. handwritten notes from Münster University.

Hyland, J. M. E., Johnstone, P. T. and Pitts, A. M. (1980) Tripos theory. *Math. Proc. Camb. Phil. Soc.*, 88:205–232.

Hyland, J. M. E. and Ong, C.-H. L. (1993) Modified realizability toposes and strong normalization proofs. In J.F. Groote and M. Bezem, editors, *Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*, pages 179–194. Springer-Verlag.

Hyland, J. M. E., Robinson, E. P. and Rosolini, G. (1990) The discrete objects in the effective topos. *Proceedings of the London Mathematical Society*, 60.

Longley, J. (1994) *Realizability Toposes and Language Semantics.* PhD thesis, University of Edinburgh.

Longley, J. (1999a) Matching typed and untyped realizability. In L. Birkedal, J. van Oosten, G. Rosolini, and D.S. Scott, editors, *Workshop on Realizability Semantics and Applications*, volume 23 of *Electronic Notes in Computer Science.* Elsevier.

Longley, J. (1999b) Unifying typed and untyped realizability. Manuscript (available from `http://www.dcs.ed.ac.uk/home/jrl/unifying.txt`).

De Marchi, F., Robinson, E. P. and Rosolini, G. (1999) An abstract look at realizability. submitted for publication.

Menni, M. (1999) A characterization of the left exact categories whose exact completions are toposes. submitted for publication.

Menni, M. (2000) Generic proofs and chaotic situations (extended abstract). unpublished manuscript.

Ong, C.-H. L. and Ritter, E. (1994) A generic strong normalization proof: application to the calculus of constructions (extended abstract). In *Computer Science Logic: 7th Workshop, CSL'93, Swansea, UK, Sep. 1993, Selected Papers*, volume 832 of *Lecture Notes in Computer Science*, pages 261–279. Springer Verlag.

van Oosten, J. (1991) *Exercises in realizability.* PhD thesis, University of Amsterdam.

van Oosten, J. (1997) The modified realizability topos. *Journal of Pure and Applied Algebra*, 116:273–289.

Pitts, A. M. (1981) *The Theory of Triposes.* PhD thesis, Cambridge University.

Pitts, A. M. (1999) Tripos theory in retrospect. In L. Birkedal, J. van Oosten, G. Rosolini, and D.S. Scott, editors, *Workshop on Realizability Semantics and Applications*, volume 23 of *Electronic Notes in Computer Science*. Elsevier.

Rosolini, G. (1999) Equilogical spaces and filter spaces. to appear.

Troelstra, A. S., (ed.) (1973) *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis.* Springer Verlag. With contributions by A.S. Troelstra, C.A. Smoryński, J.I. Zucker and W.A. Howard.