A Relational Characterisation of Syntactic Definability in Models of System **F**

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We define a notion of Kripke logical relation for models of Girard's system **F** (polymorphic λ–calculus) which allows for a syntax-free characterisation of syntactic definability.

1 Models for System F

Let **CCC** be the category of cartesian closed categories and functors preserving the cartesian closed structure on the nose.

Definition 1.1 (Seely model)
A Seely model of system **F** *or* PL–hyperdoctrine *is a functor* $\mathcal{P} : \mathbb{T}^{\mathsf{op}} \to \mathbf{CCC}$ *satisfying the following conditions.*

(1) *The category* $\mathbb{T}$ *is an algebraic theory in the sense of Lawvere, i.e. a category with finite products where all objects are finite powers of a distinguished single object* Ty *of types.*

(2) *For all projections* $\pi_\Theta : \Theta\times\mathsf{Ty} \to \Theta$ *in* $\mathbb{T}$ *there is a right adjoint* $\forall_\Theta : \mathcal{P}(\Theta\times\mathsf{Ty}) \to \mathcal{P}(\Theta)$ *to* $\mathcal{P}(\pi_\Theta)$*, i.e. for every* $A \in \mathcal{P}(\Theta\times\mathsf{Ty})$ *there is a morphism* $\mathsf{Ev}_A : \mathcal{P}(\pi_\Theta)(\forall_\Theta(A)) \to A$ *in* $\mathcal{P}(\Theta\times\mathsf{Ty})$ *such that for all* $f : \mathcal{P}(\pi_\Theta)(B) \to A$ *in* $\mathcal{P}(\Theta\times\mathsf{Ty})$ *there is a unique morphism* $\Lambda(f) : B \to \forall_\Theta(A)$ *with* $\mathsf{Ev}_A \circ \mathcal{P}(\pi_\Theta)(f) = f$*, and, moreover, these data satisfy the Beck–Chevalley Condition (BCC) on the nose, i.e. for pullbacks of*

the form

$$\begin{tikzcd} \Theta' \times \mathsf{Ty} \arrow[r, "\sigma \times \mathsf{Ty}"] \arrow[d, "\pi_{\Theta'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \times \mathsf{Ty} \arrow[d, "\pi_\Theta"] \\ \Theta' \arrow[r, "\sigma"'] & \Theta \end{tikzcd}$$

it holds that $\mathcal{P}(\sigma)(\mathsf{Ev}_A) = \mathsf{Ev}_{\mathcal{P}(\sigma\times\mathsf{Ty})(A)}$ *for all* $A \in \mathcal{P}(\Theta\times\mathsf{Ty})$.

(3) *There is an* object classifier $\mathsf{El} \in \mathcal{P}(\mathsf{Ty})$ *meaning that for every* $A \in \mathcal{P}(\Theta)$ *there is a unique morphism* $\chi_A : \Theta \to \mathsf{Ty}$ *with* $A = \mathcal{P}(\chi_A)(\mathsf{El})$, *called the* characteristic map of A.

Because of condition (3) *we will not distinguish notationally between a type* $A \in \mathcal{P}(\Theta)$ *and its characteristic map* $\chi_A : \Theta \to \mathsf{Ty}$. *Moreover, in order to facilitate notation we often write* σ^* *instead of* $\mathcal{P}(\sigma)$ *when* σ *is a morphism in* $\mathbb{T}$.

We say that a Seely model $\mathcal{P} : \mathbb{T}^{\mathsf{op}} \to \mathbf{CCC}$ has definable types *iff every morphism* $\mathsf{Ty}^n \to \mathsf{Ty}$ *in* $\mathbb{T}$ *appears as interpretation of some type in context* $\mathbb{T}^n$. *As* $\mathbb{T}$ *is an algebraic theory this entails that every morphism in* $\mathbb{T}$ *appears as interpretation of some finite list of syntactically definable types.* ♢

The restriction to Seely models with definable types is a very mild one as for any given Seely model $\mathcal{P} : \mathsf{Ty}^{\mathsf{op}} \to \mathbf{CCC}$ one may construct a Seely model $\mathcal{P}_{\mathsf{typdef}}$ with definable types by (1) restricting $\mathbb{T}$ to the subcategory $\mathbb{T}_{\mathsf{typdef}}$ consisting of those morphisms which arise as interpretations of (tuples of) type expressions and (2) restricting each fibre $\mathcal{P}(\mathbb{T}^n)$ to the full subcategory $\mathcal{P}_{\mathsf{typdef}}(\mathbb{T}^n)$ on those objects which arise as interpretations of (tuples of) type expressions. Obviously, the interpretation of system $\mathbf{F}$ in $\mathcal{P}$ factors through $\mathcal{P}_{\mathsf{typdef}}$. As we are interested in a syntax–free characterisation of syntactic definability of morphisms between system $\mathbf{F}$ types the restriction to Seely models with definable types is not essential.

As usual we can apply the well known *Grothendieck construction* to $\mathcal{P}$ giving rise to the category $\mathbf{G} = \mathsf{Gr}(\mathcal{P})$ defined as follows. The objects of $\mathbf{G}$ are pairs $[\Theta, \Gamma]$ where Θ is an object of $\mathbb{T}$ and Γ is an object of $\mathcal{P}(\Theta)$. A morphism from $[\Theta', \Gamma']$ to $[\Theta, \Gamma]$ in $\mathbf{G}$ is a pair $[\vec{A}, f]$ where $\vec{A} : \Theta' \to \Theta$

is a morphism in $\mathbb{T}$ and $f : \Gamma' \to \mathcal{P}(\vec{A})(\Gamma)$ is a morphism in $\mathcal{P}(\Theta')$. For morphisms $[\vec{A}, f] : [\Theta', \Gamma'] \to [\Theta, \Gamma]$ and $[\vec{B}, g] : [\Theta'', \Gamma''] \to [\Theta', \Gamma']$ their composition $[\vec{A}, f] \circ [\vec{B}, g]$ is defined as $[\vec{A} \circ \vec{B}, \mathcal{P}(\vec{B})(f) \circ g]$. Obviously, the identity morphisms on $[\Theta, \Gamma]$ are given by $[\mathsf{id}_\Theta, \mathsf{id}_\Gamma]$.

For subsequent use we define for Seely models $\mathcal{P}$ with definable types the following subcategory $\mathbf{V}$ of $\mathsf{Gr}(\mathcal{P})$ which is full on objects.

Definition 1.2 (Variable Substitutions)
For a Seely model $\mathcal{P} : \mathbb{T}^{\mathsf{op}} \to \mathbf{CCC}$ with definable types the category $\mathbf{V}$ of variable substitutions *is defined as the subcategory of $\mathbf{G}{=}\mathsf{Gr}(\mathcal{P})$ consisting of those arrows $[\vec{A}, f] : [\Theta', \Gamma'] \to [\Theta, \Gamma]$ where $f : \Gamma' \to \mathcal{P}(\vec{A})(\Gamma)$ is a target tupling of projections in $\mathcal{P}(\Theta')$.* $\Diamond$

2 Kripke Logical Relations for System F

Next we are heading for a notion of Kripke logical relation for models of system $\mathbf{F}$ generalising those for models of simply typed λ–calculus as introduced for the special case of well–pointed ccc's by Jung and Tiuryn in [JT].

The main problem to overcome is that the closed types of system $\mathbf{F}$ are not inductively defined and, therefore, one has to take into account open types as well, i.e. "types in context". Categorically speaking that means that for a Seely model $\mathcal{P} : \mathbb{T}^{\mathsf{op}} \to \mathbf{CCC}$ one must not just consider the objects of $\mathcal{P}(1)$ but also the objects of *arbitrary* fibres $\mathcal{P}(\mathsf{Ty}^n)$. But notice that the collection of "types in context" are the generalised elements of the object $[\mathsf{Ty}, 1_{\mathsf{Ty}}]$ in $\mathbf{G} := \mathsf{Gr}(\mathcal{P})$. As is well known these most naturally organise into the representable presheaf $\mathsf{Yon}_{\mathbf{G}}([\mathsf{Ty}, 1_{\mathsf{Ty}}]) = \mathbf{G}(-, ([\mathsf{Ty}, 1_{\mathsf{Ty}}])$ which is an object in $\widehat{\mathbf{G}}$, the topos of presheaves over $\mathbf{G}$. Recall from [Pi] that the PL–hyperdoctrine $\mathcal{P}$ gives rise to a category $\mathcal{C}$ internal to $\widehat{\mathbf{G}}$ such that $\mathsf{Ob}(\mathcal{C})$ is given by $\mathsf{Yon}_{\mathbf{G}}([\mathsf{Ty}, 1_{\mathsf{Ty}}])$ and $\mathsf{Mor}(\mathcal{C})$ is given by $\mathsf{Yon}_{\mathbf{G}}([\mathsf{Ty}{\times}\mathsf{Ty}, \pi_2^*\mathsf{El}{\to}\pi_2^*\mathsf{El}]$ where π_1 and π_2 are the first and second projection, respectively, from $\mathsf{Ty}{\times}\mathsf{Ty}$ to Ty. Similarly, the generalised elements of "types in context" organise into the representable presheaf $\mathsf{Yon}_{\mathbf{G}}([\mathsf{Ty}, \mathsf{El}])$. Notice that $[\mathsf{id}_{\mathsf{Ty}}, !_{\mathsf{El}}] : [\mathsf{Ty}, \mathsf{El}] \to [\mathsf{Ty}, 1_{\mathsf{Ty}}]$ induces via Yoneda a morphism from $\mathsf{Yon}_{\mathbf{G}}([\mathsf{Ty}, \mathsf{El}])$ to $\mathsf{Yon}_{\mathbf{G}}([\mathsf{Ty}, 1_{\mathsf{Ty}}])$ by which the former appears as fibred over the latter. One easily sees that this map represents the family $\mathcal{C}(1, C)_{C \in \mathsf{Ob}(\mathcal{C})}$.

Motivated by these considerations one might be inclined to define a *Kripke logical relation on* $\mathcal{P}$ simply as a subpresheaf R of the representable presheaf $\mathsf{Yon}_{\mathbf{G}}([\mathsf{Ty},\mathsf{El}])$. However, this requirement appears as too strong – even if $\mathcal{P}$ has definable types – when we want to consider as a particular Kripke logical relation the collection of generalised elements which are syntactically definable by terms of system $\mathbf{F}$. These do not organise into a presheaf over $\mathbf{G}$ as $\mathsf{id}_{[\mathsf{Ty},\mathsf{El}]}$ is syntactically definable wheras $\mathsf{id}_{[\mathsf{Ty},\mathsf{El}]}\circ[A,f]$ is syntactically definable only if $[A,f]$ is syntactically definable. However, the syntactically definable generalised elements of $\mathsf{Yon}_{\mathbf{G}}([\mathsf{Ty},\mathsf{El}])$ are stable under precomposition by arrows from $\mathbf{V}$ (cf. Definition 1.2). For this reason it turns out as necessary to consider presheaves over arbitrary subcategories of $\mathbf{G}$ which we call "arities".

Definition 2.1 (Arities, Kripke Relations)
Let $\mathcal{P}$ *be a* PL*–hyperdoctrine. An* arity *is a subcategory* $\mathbf{A}$ *of* $\mathbf{G}{=}\mathsf{Gr}(\mathcal{P})$*. Then a* Kripke relation on $\mathcal{P}$ of arity $\mathbf{A}$ *is a subpresheaf of* $\mathsf{Yon}_{\mathbf{A}}([\mathsf{Ty},\mathsf{El}]) \in \widehat{\mathbf{A}}$ *where* $\mathsf{Yon}_{\mathbf{A}}([\mathsf{Ty},\mathsf{El}])$ *is the restriction of* $\mathsf{Yon}([\mathsf{Ty},\mathsf{El}]) : \mathbf{G}^{\mathrm{op}} \to \mathbf{Set}$ *to* $\mathbf{A}^{\mathrm{op}}$*.* $\diamondsuit$

As a morphism $[A,f] : [\Theta,\Gamma] \to [\mathsf{Ty},\mathsf{El}]$ is uniquely determined by its second component f (namely as the codomain of f) from now on in most cases **we write simply f instead of the more clumsy $[A,f]$.** Accordingly, a Kripke relation on $\mathcal{P}$ of arity $\mathbf{A}$ can be understood equivalently as a collection R of vertical arrows $f : \Gamma{\to}A$ over some Θ with $[\Theta,A] \in \mathsf{Ob}(\mathbf{A})$ satisfying the following closure property: if $f : A{\to}B$ over Θ is in R then $f[\vec{A}] \circ g$ is in R for all $[\vec{A},g] : [\Theta',C]{\to}[\Theta,A]$ in $\mathbf{A}$.

A Kripke *logical* relation on $\mathcal{P}$ of arity $\mathbf{A}$ will be defined as a Kripke relation on $\mathcal{P}$ of arity $\mathbf{A}$ which satisfies some natural closure properties.

For formulating these closure conditions we introduce the following notational conventions. We write $\mathsf{ev}(f,a)$ for $\mathsf{ev} \circ \langle f,a\rangle$ whenever $f : C \to B^A$ and $a : C \to A$ and $\mathsf{Ev}(f,B)$ for

$$\langle \mathsf{id}_\Theta, B\rangle^*(\mathsf{Ev}_A \circ \pi_\Theta^* f) = \langle \mathsf{id}_\Theta, B\rangle^*(\mathsf{Ev}_A) \circ f$$

whenever $f : C \to \forall_\Theta(A)$ and $B : \Theta \to \mathsf{Ty}$.

Definition 2.2 (Kripke Logical Relations)
Let $\mathbf{A}$ *be an arity. A* Kripke logical relation on $\mathcal{P}$ of arity $\mathbf{A}$ *is a Kripke relation* R *on* $\mathcal{P}$ *such that*

(1) *the generalised element* $!_\Gamma : \Gamma \to 1_\Theta$ *(over* Θ*) is in* R

(2) *a generalised element* $t : \Gamma \to A \times B$ *over* Θ *is in* R *iff the generalised elements* $\pi_1 \circ t : \Gamma \to A$ *and* $\pi_2 \circ t : \Gamma \to B$ *are in* R*, too*

(3) *a generalised element* $t : \Gamma \to (A \to B)$ *over* Θ *is in* R *iff for all morphisms* $[\vec{A}, \vec{t}] : [\Theta', \Gamma'] \to [\Theta, \Gamma]$ *in* $\mathbf{A}$ *and* $s : \Gamma' \to A[\vec{A}]$ *in* R *the generalised element* $\mathsf{ev}(t[\vec{A}, \vec{t}], s)] : \Gamma' \to B[\vec{A}]$ *is in* R*, too*

(4) *the generalised element* $t : \Gamma \to \forall_\Theta(A)$ *over* Θ *is in* R *iff for all morphisms* $[\vec{A}, \vec{t}] : [\Theta', \Gamma'] \to [\Theta, \Gamma]$ *in* $\mathbf{A}$ *and* $B : \Theta' \to \mathsf{Ty}$ *in* $\mathbb{T}$ *the generalised element* $\mathsf{Ev}(t[\vec{A}, \vec{t}], B) : \Gamma[\vec{A}] \to A[\vec{A}, B]$ *is in* R*, too.* $\Diamond$

In the above definition we have used $(\_)[\vec{A}]$ as a shorthand for $\mathcal{P}(\vec{A})(\_)$ and $t[\vec{A}, \vec{t}]$ as a shorthand for $t[\vec{A}] \circ \vec{t}$. This notation emphasizes the common view that composition is a "semantic version of substitution".

A most typical example of a Kripke logical relation on a Seely model with definable types is the Kripke logical relation R_{def} of arity $\mathbf{V}$ consisting of all syntactically definable generalised elements.

Next we define what it means for a morphism $f : \Gamma \to A$ in $\mathcal{P}(\Theta)$ to be invariant under a Kripke logical relation R.

Definition 2.3 (R–invariance)
Let R *be a Kripke logical relation of arity* $\mathbf{A}$ *on a Seely model* $\mathcal{P}$ *of system* $\mathbf{F}$*. We say that a vertical arrow* $f : \Gamma \to A$ *in* $\mathcal{P}(\Theta)$ *is* invariant under R *or simply* R–invariant *iff for all* $\vec{A} : \Theta' \to \Theta$ *in* $\mathbb{T}$ *and* $g : \Gamma' \to \Gamma[\vec{A}]$ *over* Θ' *in* R *the generalised element* $f[\vec{A}] \circ g : \Gamma' \to A[\vec{A}]$ *is in* R*, too.* $\Diamond$

As usual we have a "Main Lemma" for Kripke logical relations.

Lemma 2.1 (Main Lemma for Kripke Logical Relations)
For every term $\Theta | \Gamma \vdash t : A$ *of system* $\mathbf{F}$ *its interpretation* $[\![\Theta \mid \Gamma \vdash t : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$ *in a Seely model* $\mathcal{P}$ *is invariant under all Kripke logical relations on* $\mathcal{P}$*.*

Proof. We prove our claim by induction on the structure of (derivations of) terms in system **F**.

In case t is a variable declared in context Γ then the interpretation of $\Theta|\Gamma \vdash t : A$ of system **F** is a projection which, obviously, is invariant under all Kripke logical relations due to (2) of Definition 2.2.

Assume as induction hypothesis that $f = [\![\Theta|\Gamma, x{:}A \vdash t : B]\!]$ is invariant under all Kripke logical relations. We have to show that $\mathsf{curry}(f) = [\![\Theta|\Gamma \vdash \lambda x{:}A.t : A{\to}B]\!]$ is invariant under all Kripke logical relations, too. For this purpose suppose that R is a Kripke logical relation of arity $\mathbf{A}$, $\vec{A} : \Theta'{\to}\Theta$ and $g : \Gamma'{\to}\Gamma[\vec{A}]$ over Θ' is in R. We have to show that $\mathsf{curry}(f)[\vec{A}] \circ g : \Gamma'{\to}(A{\to}B)[\vec{A}]$ is in R. By (3) of Definition 2.2 this means to show that for all $[\vec{B}, h] : [\Theta'', \Gamma'']{\to}[\Theta', \Gamma']$ in $\mathbf{A}$ and $s : \Gamma''{\to}A[\vec{A}][\vec{B}]$ in R the generalised element $\mathsf{ev}((\mathsf{curry}(f[\vec{A}]){\circ}g)[\vec{B}, h], s) = \mathsf{ev}(\mathsf{curry}(f[\vec{A}{\circ}\vec{B}]){\circ}g[\vec{B}]{\circ}h, s) = f[\vec{A}{\circ}\vec{B}]{\circ}\langle g[\vec{B}]{\circ}h, s\rangle$ is in R. This, however, is the case as f is invariant under R by induction hypothesis and $\langle g[\vec{B}]{\circ}h, s\rangle$ is in R by (2) of Definition 2.2 as its components $g[\vec{B}]{\circ}h$ and s are in R.

Suppose as induction hypotheses that $f = [\![\Theta|\Gamma \vdash t : A{\to}B]\!]$ and $g = [\![\Theta|\Gamma \vdash t : A]\!]$ are invariant under all Kripke logical relations. We have to show that $\mathsf{ev}(f, g) = [\![\Theta|\Gamma \vdash t(s) : B]\!]$ is also invariant under all Kripke logical relations. For this purpose suppose that R is a Kripke logical relation of arity $\mathbf{A}$, $\vec{A} : \Theta'{\to}\Theta$ and $h : \Gamma'{\to}\Gamma[\vec{A}]$ over Θ' is in R. We have to show that $\mathsf{ev}(f, g)[\vec{A}]{\circ}h = \mathsf{ev}(f[\vec{A}]{\circ}h, g[\vec{A}]{\circ}h)$ is in R, too. Notice that both $f[\vec{A}]{\circ}h$ and $g[\vec{A}]{\circ}h$ are in R as both f and g are invariant under R and h is in R. Therefore, by (3) of Definition 2.2 the generalised element $\mathsf{ev}(f[\vec{A}]{\circ}h, g[\vec{A}]{\circ}h)$ is also in R as the identity on $[\Theta', \Gamma']$ is in $\mathbf{A}$.

Suppose as induction hypothesis that $f = [\![\Theta, X|\Gamma \vdash t : A]\!] : \pi_\Theta^*\Gamma{\to}A$ is invariant under all Kripke logical relations and $\Theta \vdash \Gamma$, i.e. Γ does not depend on X. We have to show that $\mathsf{Curry}(f) = [\![\Theta|\Gamma \vdash \Lambda X.t : A]\!]$ is invariant under all Kripke logical relations. For this purpose suppose that R is a Kripke logical relation of arity $\mathbf{A}$, $\vec{A} : \Theta'{\to}\Theta$ and $g : \Gamma'{\to}\Gamma[\vec{A}]$ over Θ' is in R. We have to show that $\mathsf{Curry}(f)[\vec{A}]{\circ}g$ is in R. By condition (4) of Definition 2.2 this amounts to showing that for $[\vec{B}, \vec{t}] : [\Theta'', \Gamma'']{\to}[\Theta', \Gamma']$ in $\mathbf{A}$ and $B : \Theta''{\to}\mathsf{Ty}$ the generalised element $\mathsf{Ev}((\mathsf{Curry}(f)[\vec{A}]{\circ}g)[\vec{B}, \vec{t}], B) = \mathsf{Ev}(\mathsf{Curry}(f)[\vec{A}{\circ}\vec{B}]{\circ}g[\vec{B}]{\circ}\vec{t}, B)$ is in R. Notice that

$$\mathsf{Ev}(\mathsf{Curry}(f)[\vec{A}{\circ}\vec{B}]{\circ}g[\vec{B}]{\circ}\vec{t}, B) =$$

$$= \mathsf{Ev}(\mathsf{Curry}(f[(\vec{A}\circ\vec{B})\times\mathsf{id}_{\mathsf{Ty}}])\circ g[\vec{B}]\circ\vec{t}, B) =$$
$$= \mathsf{Ev}(\mathsf{Curry}(f[(\vec{A}\circ\vec{B})\times\mathsf{id}_{\mathsf{Ty}}]\circ\pi^*_{\Theta''}(g[\vec{B}]\circ\vec{t})), B) =$$
$$= \langle\mathsf{id}_{\Theta''}, B\rangle^*(f[(\vec{A}\circ\vec{B})\times\mathsf{id}_{\mathsf{Ty}}]\circ\pi^*_{\Theta''}(g[\vec{B}]\circ\vec{t})) =$$
$$= f[\langle\vec{A}\circ\vec{B}, B\rangle] \circ g[\vec{B}] \circ \vec{t}$$

and, therefore, it suffices to show that $f[\langle\vec{A}\circ\vec{B}, B\rangle] \circ g[\vec{B}] \circ \vec{t}$ is in R. Notice further that $(\pi^*_\Theta\Gamma)[\langle\vec{A}\circ\vec{B}, B\rangle] = \Gamma[\vec{A}\circ\vec{B}] = \Gamma[\vec{A}][\vec{B}]$ and $g[\vec{B}]\circ\vec{t} : \Gamma''\to\Gamma[\vec{A}][\vec{B}]$ is in R as g is in R and $[\vec{B}, \vec{t}]$ is a morphism in $\mathbf{A}$. Thus, as by induction hypothesis f is invariant under R it follows that $\mathsf{Ev}((\mathsf{Curry}(f)[\vec{A}]\circ g)[\vec{B}, \vec{t}], B) = f[\langle\vec{A}\circ\vec{B}, B\rangle] \circ g[\vec{B}] \circ \vec{t}$ is in R as desired.

Suppose as induction hypothesis that $f = [\![\Theta|\Gamma \vdash t : \forall X.A]\!]$ is invariant under all Kripke logical relations where $\Theta, X \vdash A$ and $\Theta \vdash B$. We have to show that $\mathsf{Ev}(f, [\![B]\!]) = [\![\Theta|\Gamma \vdash t(B) : A[B/X]]\!]$ is also invariant under all Kripke logical relations. For this purpose suppose that R is a Kripke logical relation of arity $\mathbf{A}$, $\vec{A} : \Theta'\to\Theta$ and $g : \Gamma'\to\Gamma[\vec{A}]$ over Θ' is in R. We have to show that $\mathsf{Ev}(f, [\![B]\!])[\vec{A}] \circ g$ is in R. Notice that $\mathsf{Ev}(f, [\![B]\!])[\vec{A}] \circ g = \mathsf{Ev}(f[\vec{A}], [\![B]\!][\vec{A}]) \circ g = \mathsf{Ev}(f[\vec{A}]\circ g, [\![B]\!][\vec{A}])$. As f is invariant under R and g is in R we conclude that $f[\vec{A}]\circ g$ is in R. From this it follows by condition (4) of Definition 2.2 that $\mathsf{Ev}(f, [\![B]\!])[\vec{A}] \circ g = \mathsf{Ev}(f[\vec{A}]\circ g, [\![B]\!][\vec{A}])$ is in R as the identity on $[\Theta', \Gamma']$ is in $\mathbf{A}$. □

3 Relational Characterisation of Syntactic Definability in System F

Now we are ready to give the following syntax–free relational characterisation of syntactic definability in Seely models with definable types.

Theorem 3.1 (Relational Characterisation)
Let $\mathcal{P}$ be a Seely model for system $\mathbf{F}$ *with definable types. Then a vertical arrow $f : \Gamma\to A$ is syntactically definable in system* $\mathbf{F}$ *if and only if f is invariant under all Kripke logical relations on $\mathcal{P}$.*

Proof. From our Main Lemma 2.1 we know that syntactically definable arrows are invariant under all Kripke logical relations on $\mathcal{P}$.

Suppose that a vertical arrow $f : \Gamma \to A$ is invariant under all Kripke logical relations. Then, in particular, the arrow f is invariant under the Kripke logical relation R_{def} consisting of all syntactically definable generalised elements of $\mathcal{P}$. Thus, when instantiating $\vec{A}$ and g of Definition 2.3 by the syntactically definable arrows id_Θ and id_Γ, respectively, it follows that $[A, f]$ is in R_{def}, i.e. that f is syntactically definable. $\square$

Of course, in a Seely model with definable types a vertical arrow $f : \Gamma \to A$ is syntactically definable already if f is invariant under R_{def}. Therefore, requiring invariance under *all* Kripke logical relations may appear as sort of an "overkill". However, this "overkill" is motivated by the desire to avoid mentioning any syntactic notions in the characterisation of syntactic definability and, actually, "invariance under all Kripke logical relations" achieves this goal as in the definition of Kripke logical relation there is made no reference to syntactic definability in system **F**. Well, one restricts attention to Seely models with definable types but this is harmless in the light of the remark after Definition 1.1.

However, it might be interesting to restrict attention to Kripke logical relations of arity **V**. Of course, every definable vertical arrow is invariant under all Kripke logical relations of arity **V** and this is also a sufficient condition. Moreover, this restriction allows for a simpler definition of Kripke logical relations for the following reason. Already in [Pi] it has been shown that a Seely model $\mathcal{P}$ can be equivalently rephrased as a Moggi model $\mathcal{C}$ inside $\widehat{\mathbf{G}}$, i.e. $\mathcal{C}$ is a cartesian closed category internal to $\widehat{\mathbf{G}}$ that, moreover, has Ty–indexed products. It is easy to show that Pitts' construction from [Pi] can be adapted in a way that $\mathcal{P}$ gives rise to a Moggi model inside $\widehat{\mathbf{V}}$ (see e.g. [AHS]).[1] Obviously, a Kripke relation on $\mathcal{P}$ of arity **V** corresponds to a global element R of $\Pi A\colon \mathsf{Ob}(\mathcal{C}).\mathcal{P}(\mathcal{C}(1, A))$ in $\widehat{\mathbf{V}}$. One can also see almost immediately that the requirements (1)–(4) of Definition 2.2 are simply the externalisations of the following requirements for R

(1) $R_1(1) = \top$, i.e. R_1 is the constantly true predicate on $\mathcal{C}(1, 1)$

(2) $R_{A\times B}(\langle a, b\rangle)$ iff $R_A(a) \wedge R_B(b)$

[1]Notice also that in [AM] it has been shown that a Seely model $\mathcal{P} : \mathbb{T}^{\mathsf{op}} \to \mathbf{CCC}$ gives rise to a Moggi model inside $\widehat{\mathbb{T}}$ but that despite its inherent interest cannot be used for our current purposes.

(3) $R_{A\to B}(f)$ iff $\forall a{:}\,\mathcal{C}(1,A).\,R_A(a)\to R_B(f\,a)$

(4) $R_{\forall(A)}(f)$ iff $\forall B{:}\,\mathsf{Ob}(\mathcal{C}).R_{A(B)}(\mathsf{Ev}_A(f,B))$ for all $A\in\mathsf{Ob}(\mathcal{C})^{\mathsf{Ob}(\mathcal{C})}$.

which—formulated in the internal language of $\widehat{\mathbf{V}}$—express that R is preserved by all operations available for $\mathcal{C}$.

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