A Topos for Computable Analysis

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Both in $PER(\mathcal{P}\omega)$ and in $PER(RE)$ all functions from 2^N to N are continuous w.r.t. the initial segment topology. But these two realisability models differ in the following essential aspect: in the first case every $F : 2^N \to N$ has a uniform modulus of continuity, i.e., there exists $n \in N$ such that $F(\alpha) = F(\beta)$ for all α, β which coincide for argument less than n, whereas in the second case such a modulus of continuity does not exist in general. This follows from the existence of the so-called *Kleene tree* T_{Kl}, an infinite recursive binary tree which is wellfounded w.r.t. recursive 01-sequences (but, of course, not w.r.t. arbitrary 01-sequences). Considering the program

$$F_0 \equiv \lambda\alpha : 2^N.\, \mu n.\, \overline{\alpha}(n) \in T_{Kl}$$

of type N^{2^N} its interpretation in $PER(RE)$ gives rise to a total map from 2^N to N which does not admit a uniform modulus of continuity as its image is infinite.

Thus in $PER(RE)$ functions from 2^N to N are always continuous w.r.t. the initial segment topology but in general they aren't uniformly continuous. Under the assumption of Church's Thesis one can even show that Cantor space 2^N and Baire space N^N are effectively isomorphic. [1]

This has severe consequences for computable analysis as it entails the existence of (necessarily) continuous functions from $[0, 1]$ to the reals which are not bounded and, therefore, not uniformly continuous. Fortunately, these counterexamples can be avoided if one considers the effective functionals above as defined on *all* total elements and not just the effective ones. The point is that Kleene's functional F_0 is not defined for all $f \in 2^N$ as if α is some non-recursive infinite path in T_{Kl} then $F_0(\alpha)$ is undefined. Similarly, by requiring definedness for *all* elements one can rule out those functions $f : [0, 1] \to \mathbb{R}$ which aren't uniformly continuous.

An approach to computable analysis where maps are required to be computable but have to be defined also on non-computable arguments has been advertised by Pour-El and Richards in their book.

As the counterexample of Kleene goes through in all realisability toposes over effective partial combinatory algebras there arises the question what is is an appropriate topos for computable analysis.

In this note we define a tripos whose associated topos $\mathcal{CA}$ covers the realisability topos $RT(RE)$ in the sense that there is a surjective geometric morphism from $\mathcal{CA}$ to $RT(RE)$.

Theorem 0.1 *Let D be $\mathcal{P}\omega$ and $D_c = RE$ be the set of computable elements of D.*

[1] Let $e : N \to \{0, 1\}^*$ be a repetition-free enumeration of the terminal nodes of the Kleene tree. Then an effective bijection between N^N and 2^N is given by sending $\alpha \in N^N$ to the 01-sequence $e(\alpha(0)) * e(\alpha(1)) * \ldots * e(\alpha(n)) * \ldots$.

The following data define a tripos $\mathsf{H}_{\mathcal{CA}}$ *over* Set.
For any set I *let* $\mathsf{H}_{\mathcal{CA}}(I) := (\mathcal{P}(D)^I, \vdash_I)$ *be the preorder where*

$$A \vdash_I B \quad :\Leftrightarrow \quad \exists e \in D_c. \forall i \in I. \forall a \in A_i.\, ea \in B_i$$

for $A, B \in \mathsf{H}_{\mathcal{CA}}(I)$. *For any function* $f : J \to I$ *let* $\mathsf{H}_{\mathcal{CA}}(f)(A) := A \circ f$.

Proof: We first show that $\mathsf{H}_{\mathcal{CA}}(I)$ is a Heyting preorder for every set I.
The terminal object $\top_I$ is given by $\top_I(i) = D$. If $A, B : I \to \mathcal{P}(\mathcal{D})$ then $A \wedge B$ is given by $(A \wedge B(i)(i) := A_i \times B_i$. Of course, $A \wedge B \vdash A$ and $A \wedge B \vdash B$ are realised by $\pi_1, \pi_2 : D \to D$, respectively.
The Heyting implication $A \to B$ is given by

$$(A \to B)(i) := \{e \in D \,|\, \forall a \in A_i.\, ea \in B_i\}$$

where $(A \to B) \wedge A \vdash_I B$ is realised by $a \mapsto \pi_1(a)(\pi_2(a))$.
If $f : J \to I$ is a map in Set and $A \in \mathsf{H}_{\mathcal{CA}}(J)$ then $\forall_f(A) \in \mathsf{H}_{\mathcal{CA}}(I)$ is given by

$$\forall_f(A)_i = \bigcap_{j \in J} \mathsf{Eq}_{i,f(j)} \to A_j$$

where $\mathsf{Eq}_{x,x} = D$ and $\mathsf{Eq}_{x,y} = \emptyset$ whenever $x \neq y$. □

Theorem 0.2 *Let* H_{RE} *be the tripos over* **Set** *with*

$$\mathsf{H}_{RE}(I) = (\mathcal{P}(D_c)^I, \vdash_I)$$

where $\vdash_I$ *and reindexing is defined as for* $\mathsf{H}_{\mathcal{CA}}$.
There is a surjective geometric morphism

$$(F, U) : \mathsf{H}_{\mathcal{CA}} \to \mathsf{H}_{RE}$$

where $F_I : \mathcal{P}(\mathcal{P}\omega)^I \hookrightarrow \mathcal{P}(RE)^I$ *with right adjoint* U_I *given by* $U_I(A)_i = RE \cap A_i$.

Proof: U_I is right adjoint to F_I as computable $e \in D$ map computable elements to computable elements.
The F_I reflect isos and are moreover full and faithful as $\vdash_I$ is defined the same way for $\mathsf{H}_{\mathcal{CA}}$ and H_{RE}. □

Corollary 0.1 *There is a surjective geometric morphism from the topos* $\mathcal{CA}$ *induced by* $\mathsf{H}_{\mathcal{CA}}$ *to the topos* $RT(RE)$ *induced by* H_{RE}.