Observations on Fibered Toposes

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1 Toposes Fibered over a Base Topos

Let $\mathcal{S}$ be a (base) topos. A fibration $P : \mathcal{X} \to \mathcal{S}$ is a fibration of toposes iff all fibers are toposes and all reindexing functors are logical.

Theorem 1.1 *A fibration $P : \mathcal{X} \to \mathcal{S}$ of toposes has internal sums iff $\mathcal{X}$ is a topos and P is a logical functor.*

Proof: By a theorem of Jibladze from 1988 a fibered topos $P : \mathcal{X} \to \mathcal{S}$ has internal sums iff P is equivalent to the fibration $P_F = \partial_1 : \mathcal{E}{\downarrow}F \to \mathcal{S}$ for some finite limit preserving functor $F : \mathcal{S} \to \mathcal{E}$ between toposes. Notice that F is unique up to isomorphism since $FI = \coprod_I 1_I$. From the theory of Artin glueing one knows that $\mathcal{E}{\downarrow}F$ is a topos and P_F is logical.

Suppose that $P : \mathcal{X} \to \mathcal{S}$ is a fibration of toposes with $\mathcal{X}$ a topos and P logical (or just preserving the cartesian closed structure). Since P is in particular a fibration of finite limit categories it has a full and faithful right adjoint 1 picking terminal objects in each fiber. Thus $P \dashv 1$ is an injective geometric morphism. Proposition A.4.5.1 of Johnstone's *Elephant* tells us that P is logical iff $P \dashv 1$ is equivalent to $U^* \dashv \Pi_U : \mathcal{X}/U \to \mathcal{X}$ for some subterminal object U. Notice that U is isomorphic to $\Sigma_U 1_{\mathcal{X}/U}$ where $\Sigma_U \dashv U^*$. From the theory of Artin glueing one knows that the open subtopos $U^* \dashv \Pi_U$ has a closed complement $j : \mathcal{E} \hookrightarrow \mathcal{X}$ induced by the closure operator $U \vee (-)$ on $\mathcal{X}$ and that the fibration $U^* : \mathcal{X} \to \mathcal{X}/U$ is equivalent to the fibration $P_F = \partial_1 : \mathcal{E}/F \to \mathcal{S}$ where $F = j^*\Pi_U$. But since F preserves finite limits the fibration P_F equivalent to P has internal sums. □

Since in the proof of the backwards direction we have just used that P preserves finite limits and exponentials we get the

Theorem 1.2 *A fibration $P : \mathcal{X} \to \mathcal{S}$ of toposes has internal sums iff $\mathcal{X}$ is a topos and P preserves finite limits and exponentials.*

2 A topos $\mathcal{E}$ has small sums iff $\mathrm{Fam}(\mathcal{E})$ is a topos

For every topos $\mathcal{E}$ the fibration $\mathit{Fam}(\mathcal{E}) : \mathrm{Fam}(\mathcal{E}) \to \mathbf{Set}$ is a fibration of toposes.

Theorem 2.1 *For an elementary topos $\mathcal{E}$ the category* $\mathrm{Fam}(\mathcal{E})$ *is cartesian closed iff $\mathcal{E}$ has small sums.*

Proof: If $\mathcal{E}$ has small sums then $\mathrm{Fam}(\mathcal{E})$ is equivalent to $\mathcal{E}\downarrow\Delta$ where $\Delta(I) = \coprod_I 1_{\mathcal{E}}$. Since $\Delta : \mathbf{Set} \to \mathcal{E}$ preserves finite limits $\mathcal{E}\downarrow\Delta$ is a topos and thus in particular cartesian closed.

Suppose $\mathrm{Fam}(\mathcal{E})$ is cartesian closed. Since it has a subobject classifier anyway it is also a topos. As observed by Peter Johnstone the fibration $\mathit{Fam}(\mathcal{E})$ is equivalent to $U^* : \mathrm{Fam}(\mathcal{E}) \to \mathrm{Fam}(\mathcal{E})/U$ for $U = (1,(0))$ (the initial object in the fiber over 1) since in $\mathcal{E}$ every morphism to 0 is an iso. Thus $\mathit{Fam}(\mathcal{E})$ is logical since U^* is logical from which it follows by Theorem 1.1 that $\mathit{Fam}(\mathcal{E})$ has internal sums, i.e. $\mathcal{E}$ has small sums. □

Thus $\mathrm{Fam}(\mathcal{E})$ is not cartesian closed if $\mathcal{E}$ is the free topos (with nno) or a nontrivial realizability topos though $\mathit{Fam}(\mathcal{E})$ certainly is a fibered topos.