Zawadowski's Cartesian Bifibrations

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If $\mathbb{C}$ is a category with finite limits then $P_{\mathbb{C}} = \partial_1 : \mathbb{C}{\downarrow}\mathbb{C} \to \mathbb{C}$ is a bifibration such that

(1) $P_{\mathbb{C}}$ is a fibration of finite limit categories

(2) for all $u : J \to I$ the functor $u_!$ (left adjoint to the reindexing functor u^*) preserves pullbacks

(3) for all $u : J \to I$ the unit and counit of the adjunction $u_! \dashv u^*$ are cartesian[1].

If $F : \mathbb{B} \to \mathbb{C}$ is a functor between categories with finite limits and F preserves terminal objects then $F^*P_{\mathbb{C}}$ is a bifibration over $\mathbb{B}$ still satisfying the conditions (1)-(3) above. We call such fibrations *cartesian bifibrations.*[2]

We now show that every cartesian bifibration $P : \mathbb{X} \to \mathbb{B}$ is equivalent to one of the form $F^*P_{\mathbb{C}}$ for some terminal object preserving functor $F : \mathbb{B} \to \mathbb{C}$ between finite limit categories. For this purpose the following Lemma will be crucial.

Lemma If $P : \mathbb{X} \to \mathbb{B}$ is a cartesian bifibration and

$$\begin{tikzcd}
Y \arrow[r, "\psi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(*)"] & Z \arrow[d, "\beta"] \\
X \arrow[r, "\text{cocart.}", "\varphi"'] & u_!X
\end{tikzcd}$$

[1]a natural transformation $\tau : F \Rightarrow G$ is cartesian iff for all $\alpha : X \to Y$ the square

$$\begin{tikzcd}
FX \arrow[r, "\tau_X"] \arrow[d, "F\alpha"'] & GX \arrow[d, "G\alpha"] \\
FY \arrow[r, "\tau_Y"] & GY
\end{tikzcd}$$

is cartesian, i.e. a pullback

[2]They were introduced by M. Zawadowski in his paper *Lax Monoidal Fibrations* (2009).

is a commuting square in $\mathbb{X}$ with α and β vertical and φ cocartesian then this square is a pullback if and only if ψ is cocartesian.

Proof. Suppose ψ is cocartesian. Consider the diagram

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & u^*u_!Y \arrow[r, "\text{cart.}"] \arrow[d, "u^*u_!\alpha"] \arrow[dr, phantom, "\lrcorner", very near start] & u_!Y \arrow[d, "\beta"] \\
X \arrow[r, "\eta_X"'] & u^*u_!X \arrow[r, "\text{cart.}"] & u_!X
\end{tikzcd}$$

where the bottom and top arrow are φ and ψ, respectively. Since both squares are pullbacks so is the rectangle.

Now suppose that (*) is a pullback. Consider the diagram

$$\begin{tikzcd}
u_!Y \arrow[r, "u_!\gamma"] & u_!u^*Z \arrow[drr, "\varepsilon_Z"] & & \\
Y \arrow[u, "\text{cocart.}"] \arrow[r, "\gamma"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & u^*Z \arrow[u, "\text{cocart.}"'] \arrow[rr, "\text{cart.}"] \arrow[d, "u^*\beta"] \arrow[drr, phantom, "\lrcorner", very near start] & & Z \arrow[d, "\beta"] \\
X \arrow[r, "\eta_X"'] & u^*u_!X \arrow[rr, "\text{cart.}"] & & u_!X
\end{tikzcd}$$

where the bottom and top arrow of the rectangle are φ and ψ, respectively. Thus, for showing that ψ is cocartesian it suffices to show that $\varepsilon_Z \circ u_!\gamma$ is an isomorphism. For showing the latter consider the diagram

$$\begin{tikzcd}
u_!Y \arrow[r, "u_!\gamma"] \arrow[d, "u_!\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & u_!u^*Z \arrow[r, "\varepsilon_Z"] \arrow[d, "u_!u^*\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "\beta"] \\
u_!X \arrow[r, "u_!\eta_X"'] & u_!u^*u_!X \arrow[r, "\varepsilon_{u_!X}"'] & u_!X
\end{tikzcd}$$

where the left square is a pullback because $u_!$ preserves pullbacks and the right square is a pullback since ε is cartesian. Thus, the rectangle is a pullback, too. Since the bottom arrow of the rectangle is an identity arrow it follows that the top arrow of the rectangle is an isomorphism as claimed. $\square$

From this Lemma it follows that P is equivalent to $\Delta^* P_{\mathbb{X}_1}$ where $\Delta : \mathbb{B} \to \mathbb{X}_1$

is given by

$$\begin{tikzcd} X \arrow[r, "\psi_X"] \arrow[d, "\alpha"'] & \Delta(X) \arrow[d, "\Delta(\alpha)"] \\ Y \arrow[r, "\psi_Y"] & \Delta(Y) \end{tikzcd}$$

where ψ_X and ψ_Y are cocartesian over terminal projections.

Notice that Δ preserves pullbacks iff cocartesian arrows are stable under pullbacks along cartesian arrows, i.e. P has internal sums.