How Unique are Ground Models?

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Models $\mathcal{E}$ over Ground Models $\mathcal{S}$

Often in semantics one builds a new model $\mathcal{E}$ over a *ground model* $\mathcal{S}$ as e.g. in forcing, classical realizability, topos theory...

and there is a so-called *constant objects* functor

$$F : \mathcal{S} \to \mathcal{E}$$

describing how the ground model $\mathcal{S}$ sits within the new model $\mathcal{E}$.

Typically this F faithfully represents the construction of $\mathcal{E}$ from $\mathcal{S}$.

Iteration of constructions as composition of CO functors.

To which extent is F determined by $\mathcal{E}$ when $\mathcal{S} = \mathbf{Set}$?

Boolean and Heyting Valued Sets

Let A be a complete Heyting (or boolean) algebra in a base topos $\mathcal{S}$ then the topos $Sh_{\mathcal{S}}(A)$ of sheaves over A contains the base $\mathcal{S}$ via $F : \mathcal{S} \to \mathcal{E}$ sending I to the "constant sheaf" with value I. Thinking of "$\mathcal{E}$ as A-valued sets" we have $F(I) = (I, eq_I)$ where $eq_I(i,j) = \bigvee\{1_A \mid i = j\}$.

The CO functor F preserves finite limits, has a right adjoint U and every $X \in \mathcal{E}$ appears as subquotient of some FI.

Such adjunctions $F \dashv U : \mathcal{E} \to \mathbf{Set}$ are called "localic geometric morphisms" since the latter condition says that subobjects of $1_{\mathcal{E}}$ generate. Under these assumptions $\mathcal{E}$ is equivalent to $Sh_{\mathcal{S}}(U\Omega_{\mathcal{E}})$

Since maps maps $I \to U\Omega_{\mathcal{E}}$ correspond to maps $FI \to \Omega_{\mathcal{E}}$, i.e. subobjects of FI, the *externalization* of $U\Omega_{\mathcal{E}}$ is given by $F^*\mathrm{Sub}_{\mathcal{E}}$ (where $\mathrm{Sub}_{\mathcal{E}}$ is the subobject fibration of $\mathcal{E}$).

The Moens-Jibladze Correspondence (1)

If $F : \mathcal{S} \to \mathcal{E}$ is a finite limit preserving functor between toposes we may consider the (Grothendieck) fibration P_F as in

$$\begin{tikzcd} \mathcal{E}{\downarrow}F \arrow[r] \arrow[d, "P_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}{\downarrow}\mathcal{E} \arrow[d, "P_{\mathcal{E}}"] \\ \mathcal{S} \arrow[r, "F"'] & \mathcal{E} \end{tikzcd}$$

where $P_{\mathcal{E}}$ (and thus also P_F) is the codomain functor.
All fibers of P_F are toposes and all reindexing functors are logical (i.e. preserve finite limits, exponentials and subobject classifiers) and P_F has internal sums (i.e. P_F is a cofibration where cocartesian arrows are stable under pullbacks along cartesian arrows in $\mathcal{E}$).

The Moens-Jibladze Correspondence (2)

Such fibrations $P : \mathcal{X} \to \mathcal{S}$ are called *fibered toposes with internal sums*.
M. Jibladze has shown that internal sums are necessarily *stable and disjoint* from which it follows by Moens's Theorem that $P : \mathcal{X} \to \mathcal{S}$ is equivalent to P_F where $F : \mathcal{S} \to \mathcal{E} = P(1)$ sends $u : J \to I$ to the unique vertical arrow Fu rendering the diagram

$$\begin{tikzcd} 1_J \arrow[r, "\varphi_J", "\text{cocart.}"'] \arrow[d, "1_u"'] & FJ \arrow[d, "Fu"] \\ 1_I \arrow[r, "\text{cocart.}", "\varphi_I"'] & FI \end{tikzcd}$$

commutative. Up to equivalence this F is determined by P, informally speaking it sends $I \in \mathcal{S}$ to $\coprod_I 1_I$.

Properties of P_F in terms of properties of F

Further fibrational properties of P_F can be reformulated as elementary properties of F as follows

1. P_F is locally small iff F has a right adjoint U
2. P_F has a small generating family iff there is a *bound* $B \in \mathcal{E}$ such that every $X \in \mathcal{E}$ appears as subquotient of some $B \times FI$.

In particular, P_F is a localic topos fibered over $\mathcal{S}$ iff P_F is locally small and $F \dashv U$ is bounded by $1_{\mathcal{E}}$.

Triposes as Generalized Localic Toposes (1)

A *tripos* over a base topos $\mathcal{S}$ is a functor F from $\mathcal{S}$ to a topos $\mathcal{E}$ such that

1. (Tr1) F preserves finite limits
2. (Tr2) every $A \in \mathcal{E}$ appears as subquotient of FI for some $I \in \mathcal{S}$
3. (Tr3) there is a subobject $\tau : T \rightarrowtail \Sigma$ such that every mono $m : P \rightarrowtail FI$ fits into a pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\tau"] \\ FI \arrow[r, "Fp"'] & F\Sigma \end{tikzcd}$$

for some (typically not unique) $p : I \to \Sigma$.

A *weak tripos* over a base topos $\mathcal{S}$ is a functor F from $\mathcal{S}$ to a topos $\mathcal{E}$ validating just (Tr1) and (Tr2).

Triposes as Generalized Localic Toposes (2)

With every (strong) tripos $F : \mathcal{S} \to \mathcal{E}$ one can associate the fibered poset $\mathscr{P}_F = F^*\mathrm{Sub}_{\mathcal{E}}$ validating the conditions

1. $\mathscr{P}_F$ is a fibration of pre-Heyting-algebras
2. for every u in the base the reindexing map $u^* = \mathscr{P}_F(u)$ we have $\exists_u \dashv u^* \dashv \forall_u$ (as adjoints of maps of preorders) validating the (Beck-)Chevalley condition[1]
3. there is a generic $\tau \in \mathscr{P}_F(\Sigma)$ such that every $\varphi \in \mathscr{P}_F(I)$ is isomorphic to $f^*\tau$ for some $f : I \to \Sigma$

[1] we have $v^*\exists_u \dashv\vdash \exists_p q^*$ for every pullback

$$\begin{tikzcd} L \arrow[r, "q"] \arrow[d, "p"'] & J \arrow[d, "u"] \\ K \arrow[r, "v"'] & I \end{tikzcd}$$

in $\mathcal{S}$

Triposes as Generalized Localic Toposes (3)

If F is just a weak tripos then the third condition for triposes has to be weakened as follows:

for very $I \in \mathcal{S}$ there is a $P(I)$ in $\mathcal{S}$ and $\in_I$ in $\mathscr{P}_F(I \times P(I))$ such that for every ρ in $\mathscr{P}_F(I \times J)$

$$\text{(Comp)} \quad \forall j \in J. \exists p \in P(I). \forall i \in I.\, \rho(i,j) \leftrightarrow i \in_I p$$

holds in the logic of $\mathscr{P}_F$

This looks like the usual *comprehension principle* for HOL.
Its Skolemized (and thus stronger) version is equivalent to the existence of a generic subterminal $\tau : T \rightarrowtail F\Sigma$ (where Σ is $P(1)$).

Triposes as Generalized Localic Toposes (4)

For (weak) triposes $F : \mathcal{S} \to \mathcal{E}$ the CO functor $\mathcal{S} \to \mathcal{S}[\mathscr{P}_F]$ is equivalent to F and a (weak) tripos $\mathscr{P}$ is equivalent to $\mathscr{P}_F$ where F is the CO functor $\mathcal{S} \to \mathcal{S}[\mathscr{P}]$ as shown in Pitts's Thesis.

Here $\mathcal{S}[\mathscr{P}]$ is obtained from $\mathscr{P}$ by "adding quotients" defining morphisms as functional relations. The CO functor $\mathcal{S} \to \mathcal{S}[\mathscr{P}]$ sends I to (I, eq_I) where $eq_I = \exists_{\delta_I} \top_I$.

Uniqueness of Constant Objects Functors?

If $F_1, F_2 : \mathcal{S} \to \mathcal{E}$ are (weak) triposes is then $F_1 \simeq F_2$?
The answer is in general NO if $\mathcal{S}$ is not equal to **Set** since for sober (e.g. Hausdorff spaces) X and Y there are as many localic geometric morphism $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ as there are continuous maps from Y to X.

Conjecture CO functors from **Set** to $\mathcal{E}$ are in general not equivalent.

This holds for weak triposes since for natural numbers $n > 0$

$$F_n : \mathbf{Set} \to \mathbf{Set} : I \mapsto I^n$$

is a weak tripos and F_n and F_m are equivalent iff $n = m$.

Alas, the question is open for strong triposes!

Question even open for localic and realizability toposes!

Already in [HJP80] where triposes were introduced it was asked whether localic toposes $\mathsf{Sh}(A)$ over **Set** may be induced by triposes whose constant objects functor is not equivalent to $\Delta : \mathbf{Set} \to \mathsf{Sh}(A)$.

Maybe we get such examples via classical realizability? Krivine's criterion (absence of "parallel or") for a realizability algebra only guarantees that the associated tripos is not localic but not that the induced topos is not localic...e.g. possibly **Set**.

Also realizability toposes $\mathsf{RT}(\mathcal{A})$ over **Set** may be induced triposes whose constant objects functor is not equivalent to $\nabla : \mathbf{Set} \to \mathsf{RT}(\mathcal{A})$.

Non-Localic Grothendieck Toposes from Triposes over **Set**

If $\mathcal{E}$ is the topos of *reflexive graphs* $\mathbf{Set}^{\Delta_2^{\mathrm{op}}}$ or the topos $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ of *simplicial sets* then $\nabla : \mathbf{Set} \to \mathcal{E}$ (right adjoint to $\Gamma = \mathcal{E}(1, -)$) is a weak (but not) a strong tripos.

Every reflexive graph my be covered by a subobject of some $\nabla(S)$!

Summary

Ground models are typically not unique!

Set is induced by infinitely many non-equivalent weak triposes over **Set**.

Question open for triposes over **Set** even for localic and realizability toposes though there are canonical candidates Δ and ∇, respectively. But are these the only possibilities?

Maybe classical realizability gives rise to **Set** via a non-localic tripos?