# A Fibrational Theory of Geometric Morphisms

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# Introduction

Category theory can be viewed as an elementary, i.e. essentially first order, theory independent from set theory. In an elementary topos, i.e. a category satisfying a number of elementary axioms, one can perform all constructions that one performes with sets in everyday mathematics. Nevertheless, the language of category theory is not expressive enough to capture those categorical notions that make reference to set theory. Amongst those are: (co-)completeness, (local) smallness, existence of a small set of generators and well-poweredness. If we want to replace the category of sets by a category $\mathbb{B}$ whose objects are regarded as index sets we need an abstract theory of families. Such a theory is the theory of fibred categories.* We can choose $\mathbb{B}$ as a topos but for most purposes it suffices that $\mathbb{B}$ has pullbacks. A category fibred over $\mathbb{B}$ is a functor $P : \mathbb{E} \to \mathbb{B}$ satisfying a certain elementary requirement. An object $X$ in $\mathbb{E}$ should be thought of as a family indexed by the object $I = PX$ in $\mathbb{B}$. If $\mathbb{B}$ is the category of sets then every category $\mathbb{C}$ gives rise to a so called family fibration $P : \mathbf{Sets}(\mathbb{C}) \to \mathbf{Sets}$ where the objects in $\mathbf{Sets}(\mathbb{C})$ over some set $I$ are $I$-indexed families $(X_i)_{i \in I}$ of objects $X_i$ in $\mathbb{C}$. This class of examples is prototypical in the sense that the definition of categorical notions will be extended to fibred categories in such a way that the category $\mathbb{C}$ satisfies a property if and only if the fibration $P : \mathbf{Sets}(\mathbb{C}) \to \mathbf{Sets}$ satisfies the extension of this property to fibrations.

In this diploma thesis we present a detailed investigation and a slight generalization of a widely unknown result of J.L. Moens in his PhD thesis [Moe82a]. Let $F : \mathbb{B} \to \mathbb{C}$ be a functor to a category $\mathbb{C}$ with pullbacks. Then the functor $\mathsf{gl}(F) := \partial_1 : \mathbb{C}/F \to \mathbb{B}$ is a fibration. In Section 4 we will characterize in purely fibrational terms all fibrations $P$ of the form $\mathsf{gl}(F)$ where $F$ is a finite limit preserving functor between categories with finite limits. They will be given the name *lextensive fibrations*, as we have reformulated Moens' characterization as an extensivity condition for sums (in a fibrational sense) in $P$. The notion of extensivity and how it relates to other properties of sums has been described in [CLW93] for ordinary categories. These definitions and results can be extended to fibrations. In Section 5 we make precise the correspondence between finite limit preserving functors $F$ with source $\mathbb{B}$ and lextensive fibrations over $\mathbb{B}$ in a 2-categorical context. The requirement of preservation of finite limits by $F$ will be weakened to preservation of pullbacks in Section 6. In Section 8 we show that the functor $F$ has a right adjoint $U$ (i.e. $F$ is the inverse image part of a geometric morphism) if and only if the fibration $\mathsf{gl}(F)$ has Lawvere comprehension. Lextensive fibrations with Lawvere comprehension are called *geometric fibrations*.

After this we will concentrate on geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ between toposes. We prove that the fibration $\mathsf{gl}(F)$ has a small set of generators (in a fibrational sense) if and only if the geometric morphism $F \dashv U$ is bounded.

The correspondence between geometric morphisms and geometric fibrations originating from Moens' work provides a purely "logical" account of geometric morphisms. This alternative "logical" view has the advantage that it can be formulated as a precise mathematical 1-1-correspondence whereas the traditional "geometric" view is based on the formal analogy between inverse image parts of geometric morphism and frame morphisms generalizing continuous maps. Furthermore, via the "logical" account of geometric morphisms one may give a more conceptual reformulation of properties of geometric morphisms in fibrational terms.

*There is also an alternative approach, namely the theory of indexed categories, see [PS78]. Nevertheless we believe that fibred categories even though being more subtle and less straightforward have many advantages compared to indexed categories.

Typical such properties are *injective*, *surjective*, *connected*, *locally connected* or *local*. In this diploma thesis the cases of injectivity and surjectivity are treated.

The theory of fibred categories was introduced by A. Grothendieck and developed further by J. Giraud and J. Bénabou. In particular, Bénabou's work is of major importance for this diploma thesis. Most, if not all, of the results presented here are at least implicit in his work. Furthermore, much of this diploma thesis is covered in [Moe82a, Moe82b], we only provide more detailed and sometimes different proofs. Other people that have contributed to this particular topic in fibred category theory are M. Jibladze and J. Funk (see [Jib89] and [Funk90]).

An introduction to fibred category theory can be found in [Borc94], for an exposition of the underlying philosophy see [Bén85]. Unfortunately the lecture notes [Bén80] are not yet available in book form. For another introduction to fibred category theory, mainly investigating them from the point of view of categorical logic (and applied to semantics of type theory) see [Jacobs].

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Contents

# 1. Fibred Categories

## 1.1. Definitions.

**Definition 1.** Let $P : \mathbb{E} \to \mathbb{B}$ be a functor.

(i) For objects $X$ in $\mathbb{E}$ and $I$ in $\mathbb{B}$ we say that $X$ is *over* $I$ whenever $PX = I$. For morphisms $f : X \to Y$ in $\mathbb{E}$ and $u : I \to J$ in $\mathbb{B}$ we say that $f$ is *over* $u$ whenever $Pf = u$.

(ii) Let $u : I \to J$ be a morphism in $\mathbb{B}$, $X$ be over $I$ and $Y$ be over $J$. Then the collection of morphisms from $X$ to $Y$ over $u$ is denoted by $\mathbb{E}_u(X, Y)$.

(iii) For any object $I$ in $\mathbb{B}$ we define the *fibre over* $I$ to be the subcategory $\mathbb{E}_I$ of $\mathbb{E}$ consisting of objects over $I$ and morphisms over $\mathrm{id}_I$. A morphism in any such fibre is called a *vertical* morphism.

(iv) A morphism $f : X \to Y$ over $u : I \to J$ is called *cartesian* if it satisfies the following universal property: For any $g : Z \to Y$ over $v : K \to J$ and any $w : K \to I$ such that $u \circ w = v$ there exists a unique $h : Z \to X$ over $w$ such that $f \circ h = g$. In this situation we call $f$ a *cartesian lifting* or *terminal lifting* of $u$.

(v) The functor $P : \mathbb{E} \to \mathbb{B}$ is called a *fibration* if for every object $Y$ over $J$ and every morphism $u : I \to J$ there exists a cartesian lifting $f : X \to Y$ over $u$. The category $\mathbb{E}$ is then called the *entire* category and the category $\mathbb{B}$ is called the *base* category of the fibration. We shall also say that $\mathbb{E}$ is a category fibred over $\mathbb{B}$.

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "h"'] \arrow[drr, bend left, "g"] & & \\
 & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
K \arrow[dr, "w"'] \arrow[drr, bend left, "v"] & & \\
 & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

**Example 1 (Family Fibrations).** Let $\mathbb{C}$ be an arbitrary category. Then the category $\mathbf{Sets}(\mathbb{C})$ is defined in the following way.

- An object of $\mathbf{Sets}(\mathbb{C})$ is a set-indexed family $(X_i)_{i\in I}$ of objects in $\mathbb{C}$.
- A morphism from $(X_i)_{i\in I}$ to $(Y_j)_{j\in J}$ consists of a map $u : I \to J$ in $\mathbf{Sets}$ together with an $I$-indexed family $(f_i)_{i\in I}$ of morphisms $f_i : X_i \to Y_{u(i)}$ in $\mathbb{C}$.
- Let $(u, (f_i)_{i\in I}) : (X_i)_{i\in I} \to (Y_j)_{j\in J}$ and $(v, (g_j)_{j\in J}) : (Y_j)_{j\in J} \to (Z_k)_{k\in K}$ be morphisms in $\mathbf{Sets}(\mathbb{C})$. Then their composite is $(v \circ u, (g_{u(i)} \circ f_i)_{i\in I})$.

The functor $P : \mathbf{Sets}(\mathbb{C}) \to \mathbf{Sets}$ sending an object $(X_i)_{i\in I}$ to its index set $I$ and a morphism $(u, (f_i)_{i\in I})$ to $u$ is a fibration: Given a family $(Y_j)_{j\in J}$ and a map $u : I \to J$, there is a canonical cartesian lifting of $u$, namely $(\mathrm{id}_{Y_{u(i)}})_{i\in I} : (Y_{u(i)})_{i\in I} \to (Y_j)_{j\in J}$.

(i) The fibre over $I \in \mathbb{C}$ is the category $\mathbb{C}^I$.

(ii) A morphism $(u, (f_i)_{i\in I})$ in $\mathbf{Sets}(\mathbb{C})$ is cartesian exactly if each $f_i$ is an iso.

The family fibrations play the role of guiding examples. Whenever we define a property for fibrations, which is already defined for ordinary categories, we will do so in a way such that $\mathrm{Fam}(\mathbb{C})$ satisfies this property (in the fibrational sense) if and only if $\mathbb{C}$ satisfies this property (in the sense of ordinary category theory). Examples of such properties are (co-)completeness, (local) smallness, well-poweredness, existence of a small generating family; properties which make explicit reference to set theory and whose definitions we wish to extend reasonably to fibrations over other base categories than $\mathbf{Sets}$.

Another example in the same spirit is the following one. A map $p : X \to I$ between sets can be considered as a family of sets, indexed over $I$. The set with index $i \in I$ is simply $p^{-1}(i) \subseteq X$. If $u : J \to I$ is a map then the pullback of $p$ along $u$ gives rise to a family of sets such that the set with index $j \in J$ w.r.t. $u^*p$ can be identified with the set indexed by $u(j)$ w.r.t. $p$. This idea leads us to considering a morphism $p : X \to I$ in an arbitrary category as a family, indexed over $I$, even though the notion of element of $I$ does not make sense in the general setting. Just as well, we consider a pullback of such a family along some $u$ as a form of *reindexing* of the family along $u$.

**Example 2 (Codomain Fibrations).** Let $\mathbb{B}$ be an arbitrary category. Consider the codomain functor $\partial_1 : \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$. Then

(i) the fibre over $I \in \mathbb{B}$ is the slice $\mathbb{B}/I$,
(ii) the cartesian morphisms are exactly the pullback squares,
(iii) the codomain functor is a fibration if and only if $\mathbb{B}$ has pullbacks.

As monomorphisms are stable under pullbacks, the previous example gives rise to the following.

**Example 3 (Mono Fibrations, Subobject Fibrations).** Let $\mathbb{B}$ be an arbitrary category and consider the full subcategory $\mathbf{Mono}(\mathbb{B})$ of $\mathbb{B}^{\mathbf{2}}$. Then the restriction of the codomain functor $\partial_1 : \mathbf{Mono}(\mathbb{B}) \to \mathbb{B}$ is a fibration if and only if pullbacks of monomorphisms along arbitrary morphisms exist in $\mathbb{B}$.

Passing to equivalence classes of monomorphisms into the same object leads to the *subobject fibration* $\partial_1 : \mathbf{Sub}(\mathbb{B}) \to \mathbb{B}$. In this case, the fibre over some $I \in \mathbb{B}$ is the (posetal) category of subobjects of $I$.

**Example 4 (Small Fibrations).** Let $\mathbb{B}$ be a category with pullbacks and let $\mathbb{C} = (C_0, C_1, \partial_0, \partial_1, i, c)$ be an internal category in $\mathbb{B}$. Then one can construct a category $\mathbb{B}(\mathbb{C})$ and a functor $P : \mathbb{B}(\mathbb{C}) \to \mathbb{B}$ which is a fibration. The category $\mathbb{B}(\mathbb{C})$ is defined in the following way

- An object of $\mathbb{B}(\mathbb{C})$ is a pair $(I, m)$, where $I$ is an object of $\mathbb{B}$ and $m$ is a morphism $m : I \to C_0$ in $\mathbb{B}$.
- A morphism from $(I, m)$ to $(J, n)$ is a pair $(u, f)$, where $u : I \to J$ and $f : I \to C_1$ are morphisms in $\mathbb{B}$ such that $\partial_0 \circ f = m$ and $\partial_1 \circ f = n \circ u$.
- Let $(u, f)$ be a morphism from $(I, m)$ to $(J, n)$ and $(v, g)$ be a morphism from $(J, n)$ to $(K, l)$. Then $f : I \to C_1$ and $g \circ u : I \to C_1$ induce a map $h : I \to D$ into the domain of definition of the (internal) composition. Now the composite of $(u, f)$ and $(v, g)$ is defined to be $(v \circ u, c \circ h)$.

The functor $P : \mathbb{B}(\mathbb{C}) \to \mathbb{B}$ sending an object $(I, m)$ to $I$ and a morphism $(u, f)$ to $u$ is a fibration: Given an object $(J, n)$ and a morphism $u : I \to J$, there is a canonical cartesian lifting of $u$, namely $(u, i \circ n \circ u) : (I, n \circ u) \to (J, n)$.

There are two ways of looking at the prior example. The objects of $\mathbb{B}(\mathbb{C})$ can be seen as families of objects of $\mathbb{C}$, but also as generalized elements of the object $C_0$ of objects of $\mathbb{C}$.

Note also that when applied to a small category $\mathbb{C}$, i.e. an internal category in $\mathbf{Sets}$, the previous construction and the family construction coincide. So there is no ambiguity in writing $\mathbf{Sets}(\mathbb{C})$.

**Example 5.** Let $\mathbb{C}$ be a category. Then

(i) the functor $\partial_0 : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$ is a fibration.
(ii) the functor $\mathrm{Id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is a fibration.
(iii) the functor $!_{\mathbb{C}} : \mathbb{C} \to \mathbb{1}$ is a fibration.

Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration and $u : I \to J$ a morphism in $\mathbb{B}$. Then we will sometimes denote a cartesian lifting of $u$ with codomain $X$ by $\bar{u}X : u^*X \to X$. The star-notation is motivated by the codomain fibrations, where a cartesian lifting of $u$ is simply a pullback of $X$ along $u$. Now, any simultaneous choice of cartesian liftings $\bar{u}X : u^*X \to X$ of $u$ for all objects $X$ in $\mathbb{E}_J$ gives rise to a functor $u^* : \mathbb{E}_J \to \mathbb{E}_I$ taking each object $X$ of $\mathbb{E}_J$ to the domain $u^*X$ of the chosen cartesian lifting with codomain $X$ and each morphism $f : X \to Y$ in $\mathbb{E}_J$ to the unique vertical arrow $u^*f$, such that the diagram below commutes. Functoriality is easily verified using the uniqueness of the vertical fill-ins.

$$\begin{tikzcd} u^*X \arrow[r, "\bar{u}X"] \arrow[d, dashed, "u^*f"'] & X \arrow[d, "f"] \\ u^*Y \arrow[r, "\bar{u}Y"'] & Y \end{tikzcd}$$

Any two such choices induce naturally isomorphic functors. These functors are called *reindexing* functors or *inverse image* functors.

**Definition 2 (Cleavage).** A *cleavage* is an assignment of a cartesian morphism $\bar{u}Y : u^*Y \to Y$ over $u$ to each $u : I \to J$ in $\mathbb{B}$ and each $Y$ over $J$ A *cloven fibration* is a fibration, equipped with a cleavage.

Even when talking about a fibration which is not equipped with a cleavage we will sometimes write $\bar{u}Y : u^*Y \to Y$ in order to indicate that $\bar{u}Y$ is some cartesian lifting of $u$.

The concept of cartesianness can be dualized, requiring an initial lifting property.

**Definition 3 (Cocartesian Morphism).** Let $P : \mathbb{E} \to \mathbb{B}$ be a functor. A morphism $f : X \to Y$ over $u : I \to J$ is called *cocartesian* if for any $g : X \to Z$ over $v : I \to K$ and any $w : J \to K$ with $w \circ u = v$ there exists a unique $h : Y \to Z$ over $w$ such that $h \circ f = g$.

**Definition 4.** A functor $P : \mathbb{E} \to \mathbb{B}$ is a *cofibration* if for every object $X$ over $I$ and every morphism $u : I \to J$ there exists a cocartesian morphism $f : X \to Y$ over $u$. A functor $P : \mathbb{E} \to \mathbb{B}$ is a *bifibration* if it is both a fibration and a cofibration.

The following lemmas are basic for working with fibrations and their proofs are straightforward.

**Lemma 1.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration.*

(i) *A morphism $f : X \to Y$ over some $u : I \to J$ is cartesian if and only if for any $g : Z \to Y$ over $u$ there exists a unique vertical morphism $h : Z \to X$ such that $f \circ h = g$.*
(ii) *A morphism $f : X \to Y$ over some $u : I \to J$ is cocartesian if and only if for any $g : X \to Z$ over $u$ there exists a unique vertical morphism $h : Y \to Z$ such that $h \circ f = g$.*

Mind that for an arbitrary functor $P$ the above mentioned properties are weaker than cartesianness and cocartesianness. Nevertheless, whenever $P$ is a fibration, both equivalences hold.

**Remark 1.** For an arbitrary functor $P$ a morphism satisfying (i) is called *precartesian*. It is easy to prove, that $P$ is a fibration if and only if it admits precartesian liftings and in addition the class of precartesian morphisms is closed under composition.

**Lemma 2.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration and consider a commutative square in $\mathbb{E}$ and its image in $\mathbb{B}$ as shown in the diagram. If $\varphi$ and $\psi$ are cartesian then the upper square is a pullback if and only if the lower square is a pullback.*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & Y \arrow[dr, "g"] & \\ & Z \arrow[r, "\psi"'] & W \end{tikzcd}$$

$$\begin{tikzcd} PX \arrow[r, "P\varphi"] \arrow[dr, "Pf"'] & PY \arrow[dr, "Pg"] & \\ & PZ \arrow[r, "P\psi"'] & PW \end{tikzcd}$$

Given a fibration over a base category with pullbacks, the preceding lemma gives rise to a method of constructing pullbacks of cartesian morphisms along arbitrary morphisms in the total category of this fibration. If $\psi : Z \to W$ is cartesian and $g : Y \to W$ is arbitrary, then one projects them into the base category, builds a pullback there, lifts the arrow parallel to $P\psi$ in the pullback square to some cartesian morphism $\varphi : X \to Y$ and defines $f : X \to Z$ to be the unique fill-in over the arrow parallel to $Pg$.

**Corollary 1.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration and $\mathbb{B}$ have pullbacks. Then pullbacks in $\mathbb{E}$ of cartesian morphisms along arbitrary morphisms exist and are cartesian, again. Furthermore, $P$ preserves pullbacks along cartesian morphisms.*

**Proposition 1.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a functor and $\varphi$, $\psi$ composable morphisms in $\mathbb{E}$. Then*

(i) *if $\varphi$ and $\psi$ are (co-)cartesian, then $\varphi \circ \psi$ is (co-)cartesian.*
(ii) *if $\varphi$ and $\varphi \circ \psi$ are cartesian, then $\psi$ is cartesian.*
(iii) *if $\psi$ and $\varphi \circ \psi$ are cocartesian, then $\varphi$ is cocartesian.*

In the case of codomain fibrations, (i) and (ii) reduce to the well known pullback lemma.

**Lemma 3.** *If $P : \mathbb{E} \to \mathbb{B}$ is a fibration, then every morphism in $\mathbb{E}$ factors as a vertical morphism, followed by a cartesian morphism.*

*If $P : \mathbb{E} \to \mathbb{B}$ is a cofibration, then every morphism in $\mathbb{E}$ factors as a cocartesian morphism, followed by a vertical morphism.*

*Both these factorizations are unique up to vertical isomorphism.*

1.2. **Closure Properties.** Fibrations are closed under some elementary constructions.

**Proposition 2.** *Fibrations are closed under composition. If $Q : \mathbb{D} \to \mathbb{C}$ and $P : \mathbb{C} \to \mathbb{B}$ are fibrations then $PQ$ is a fibration. A morphism $f$ in $\mathbb{D}$ is $PQ$-cartesian if and only if $f$ is $Q$-cartesian and $Qf$ is $PQ$-cartesian.*

**Proposition 3.** *Fibrations are stable under pullbacks (in* **Cat***). If $P : \mathbb{E} \to \mathbb{C}$ is a fibration and $F : \mathbb{B} \to \mathbb{C}$ is a functor then $F^*P : F^*\mathbb{E} \to \mathbb{B}$ is a fibration.*

$$\begin{tikzcd} F^*\mathbb{E} \arrow[r] \arrow[d, "F^*P"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "P"] \\ \mathbb{B} \arrow[r, "F"'] & \mathbb{C} \end{tikzcd}$$

*A morphism in $F^*\mathbb{E}$ is cartesian if and only if its image in $\mathbb{E}$ is cartesian.*

**Proposition 4.** *Fibrations are stable under exponentiation by a category. Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration and let $\mathbb{D}$ be some category. Then the functor $P^{\mathbb{D}} : \mathbb{E}^{\mathbb{D}} \to \mathbb{B}^{\mathbb{D}}$ is a fibration.*

In general, this involves axiom of choice, but we are only going to make use of this construction in the harmless case when $\mathbb{D}$ is finite.

## 2. Fibred Functors

For some fixed base category $\mathbb{B}$ we are now going to define appropriate notions of functors and natural transformations making the collection of fibrations over $\mathbb{B}$ into a 2-category $\mathrm{Fib}(\mathbb{B})$.

### 2.1. Definitions.

**Definition 5 (Fibred Functor).** Let $P : \mathbb{C} \to \mathbb{B}$ and $Q : \mathbb{D} \to \mathbb{B}$ be fibrations over the same base category $\mathbb{B}$. Then a *fibred functor* from $P$ to $Q$ is a functor $F : \mathbb{C} \to \mathbb{D}$ such that $QF = P$ and $F$ preserves cartesianness of morphisms.

**Definition 6 (Fibred natural transformation).** Let $F$ and $G$ be two fibred functors from $P : \mathbb{C} \to \mathbb{B}$ to $Q : \mathbb{D} \to \mathbb{B}$. Then a *fibred natural transformation* from $F$ to $G$ is a natural transformation $\alpha$ from $F$ to $G$ such that $Q\alpha = \mathrm{id}_P$, i.e. such that all of its components are vertical w.r.t. $Q$.

We obtain a 2-category $\mathrm{Fib}(\mathbb{B})$ by defining the objects to be the fibrations over $\mathbb{B}$, the 1-cells to be the fibred functors and the 2-cells to be the fibred natural transformations. Now 2-categorical notions like equivalences, adjunctions and (co-)monads make sense in the setting of $\mathrm{Fib}(\mathbb{B})$.

Note that due to the commutativity requirement, each fibred functor $F$ induces a family of functors $F_I : \mathbb{C}_I \to \mathbb{D}_I$. Likewise, every fibred natural transformation $\alpha$ from $F$ to $G$ induces a family of natural transformations $\alpha_I : F_I \to G_I$.

**Example 6.** A fibration over some category $\mathbb{B}$ is called *small fibration* if it is isomorphic to $P : \mathbb{B}(\mathbb{C}) \to \mathbb{B}$ for some internal category $\mathbb{C}$ in $\mathbb{B}$.

### 2.2. Fibred Adjunctions.

Explicitly, a fibred adjunction between fibrations $P : \mathbb{C} \to \mathbb{B}$ and $Q : \mathbb{D} \to \mathbb{B}$ consists of fibred functors $F : \mathbb{C} \to \mathbb{D}$ and $U : \mathbb{D} \to \mathbb{C}$ such that $QF = P$ and $PU = Q$ and natural transformations $\eta : \mathrm{Id}_{\mathbb{C}} \to UF$ and $\epsilon : FU \to \mathrm{Id}_{\mathbb{D}}$ such that $P\eta = \mathrm{id}_P$, $Q\epsilon = \mathrm{id}_Q$ (i.e. the components of $\epsilon$ and $\eta$ are vertical) and the triangular equations $\epsilon F \circ F\eta = \mathrm{id}_F$ and $U\epsilon \circ \eta U = \mathrm{id}_U$ are satisfied.

$$\begin{tikzcd} \mathbb{C} \arrow[rr, bend left, "F"] \arrow[dr, "P"'] \arrow[rr, phantom, "\perp"] & & \mathbb{D} \arrow[ll, bend left, "U"] \arrow[dl, "Q"] \\ & \mathbb{B} & \end{tikzcd}$$

The fact that the unit and the counit of a fibred adjunction are vertical implies that if $f : FX \to Y$ is over some $u$ then so is its transpose $f^\vee : X \to UY$ (and if $g : X \to UY$ is over $u$ then so is $g^\wedge : FX \to Y$). Actually, if $\eta$ is vertical so is $\epsilon$ and vice versa. The following Proposition shows that this definition of a fibred adjunction contains even more redundancy.

**Proposition 5.** *Consider a commutative diagram of categories and functors as above, where $(F, U, \eta, \epsilon)$ is an adjunction and $\eta$ and $\epsilon$ are vertical. Then $U$ preserves cartesianness and $F$ preserves cocartesianness of morphisms.*

*Proof.* We will show, that $U$ preserves cartesianness. Let $f : X \to Y$ be a morphism over $u : I \to J$ in $\mathbb{D}$. We have to show, that $Uf : UX \to UY$ is cartesian. Consider a morphism $g : Z \to UY$ over $v : K \to J$ and a morphism $w : K \to I$ in $\mathbb{B}$ such that $uw = v$. The transpose $g^\wedge : FZ \to Y$ of $g$ is again over $u$ and induces a morphism $h : FZ \to X$ over $w$, as $f$ is cartesian. The transpose $h^\vee : Z \to UX$ of $h$ is the desired morphism. We have to check, that $Ff \circ h^\vee = g$. By naturality $(Ff \circ h^\vee)^\wedge = fh$ and by definition of $h$, $fh = g^\wedge$. Uniqueness follows as all transformations performed on $g$ were bijective. This completes the proof. $\square$

In particular, if $P$ and $Q$ are fibrations and $F$ is a fibred functor then any ordinary right adjoint with vertical counit which commutes with $P$ and $Q$, is in fact a fibred right adjoint. The following counterexample shows that the analogue for left adjoints does not hold.

**Example 7.** Consider the fibration $P$ over the category $\mathbf{2}$, symbolized by the following diagram. The only nontrivial cartesian morphism in it is the horizontal one.

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \\
\bullet \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

We also know, that the identity functor $\mathrm{Id}_\mathbf{2}$ on $\mathbf{2}$ is a fibration. Then the unique fibred functor from $P$ to $\mathrm{Id}_\mathbf{2}$ has a left adjoint, commuting with the projections, with vertical unit and counit. But it is not a fibred left adjoint, because it maps the nontrivial morphism of $\mathbf{2}$, which is cartesian, to the diagonal morphism in the above fibration, which is not.

The following lemma describes how a fibred adjoint can be constructed fibrewise.

**Lemma 4.** *Let $P : \mathbb{C} \to \mathbb{B}$ and $Q : \mathbb{D} \to \mathbb{B}$ be fibrations and $F : \mathbb{C} \to \mathbb{D}$ be a fibred functor. Then $F$ has a fibred right adjoint if and only if:*

(i) *For all $I \in \mathbb{B}$ and for each $A \in \mathbb{D}_I$ there is a universal arrow $(A', \epsilon_A : F_I A' \to A)$ from $F_I$ to $A$, i.e. a terminal object in the comma category $F_I/A$.*

(ii) *Whenever $(A', \epsilon_A : F_I A' \to A)$ is universal, $u : J \to I$ in $\mathbb{B}$ and $\varphi : u^*A \to A$, $\varphi' : u^*A' \to A'$ are cartesian over $u$ then $u^*A'$, together with the unique vertical arrow making the following diagram commute is universal, too.*

$$\begin{tikzcd}
A' & FA' \arrow[r, "\epsilon_A"] & A \\
u^*A' \arrow[u, "\varphi'"] & Fu^*A' \arrow[u, "F\varphi'"] \arrow[r, dashed] & u^*A \arrow[u, "\varphi"']
\end{tikzcd}$$

*An analogue of the lemma holds for fibred left adjoints.*

*Proof.* By requirement (i), for each $A$ over $I$ there is a universal morphism $(A', \epsilon_A : F_I A' \to A)$ from $F_I$ to $A$. Using requirement (ii), one can show that the morphism $(A', \epsilon_A : F_I A' \to A)$ is also universal from $F$ to $A$. This gives rise to a right adjoint with vertical counit, commuting with $P$ and $Q$. Also by requirement (ii) this right adjoint preserves cartesianness. [†] $\square$

In essence, Lemma 4 says that a fibred functor $F$ admits a fibred right adjoint if and only if (i) each functor $F_I$ admits a right adjoint $G_I$ and (ii) the counits of these local adjunctions are stable under reindexing.

Mind, that condition (ii) is absolutely necessary. Even if each of the functors $F_I$ admits a right adjoint, $F$ need not admit a fibred right adjoint, as shown by the following example.

**Example 8.** Consider the fibration $Q$, symbolized by the following diagram, and the unique fibred functor from $Q$ to $\mathrm{Id}_{\mathbf{2}}$.

$$\begin{tikzcd} \bullet & \\ \bullet \arrow[u] \arrow[r] & \bullet \\ & \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

The restrictions of this fibred functor to both fibres admit a right adjoint (in this case that means each fibre has a terminal object), but they do not extend to a fibred right adjoint.

Still, there is one case in which condition (ii) is satisfied automatically.

**Corollary 2.** *Let $F$ be a fibred functor such that for every $I \in \mathbb{B}$ the functor $F_I$ is an equivalence. Then $F$ admits a fibred right adjoint $G$.*

*Proof.* We have to prove, that condition (ii) of Lemma 4 is satisfied. If $F_I$ is an equivalence, then a morphism $(A', \epsilon_A : F_I A' \to A)$ is universal if and only if $\epsilon_A$ is an isomorphism. But the unique vertical fill-in is an isomorphism if $\epsilon_A$ is, because isomorphisms are stable under reindexing. $\square$

2.3. **The Family Monad.** According to Example 1, from every category $\mathbb{C}$ one can construct a fibration $P : \mathbf{Sets}(\mathbb{C}) \to \mathbf{Sets}$. Applying this construction to the category $\mathbf{Sets}(\mathbb{C})$ in the place of $\mathbb{C}$, we introduce a further level of indexing. An object over a set $J$ w.r.t. the fibration $P' : \mathbf{Sets}(\mathbf{Sets}(\mathbb{C})) \to \mathbf{Sets}$ is then a family $((X_{i,j})_{i \in I_j})_{j \in J}$ of families of objects of $\mathbb{C}$. This family can be identified with an object over $\sum_{j \in J} I_j$ w.r.t. $P$, where $\sum_{j \in J} I_j = \{(j,i) | j \in J, i \in I_j\}$. There is a canonical morphism $\pi_0 : \sum_{j \in J} I_j \to J$ and actually, every morphism in $\mathbf{Sets}$ is of that form. So up to isomorphism we can identify the objects over $J$ w.r.t. $P'$ with pairs $((X_k)_{k \in K}, p : K \to J)$, where $p$ is an arbitrary map. This definition makes sense also for base categories other than $\mathbf{Sets}$.

[†] If we were only interested in constructing fibred *right* adjoints we could also use Proposition 5 here.

**Definition 7.** Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration. Then $\mathrm{Fam}\, P : \mathrm{Fam}\, \mathbb{E} \to \mathbb{B}$ is given by the following diagram:

$$
\begin{tikzcd}
\mathrm{Fam}\,\mathbb{E} \arrow[r] \arrow[d] \arrow[dd, bend right, "\mathrm{Fam}\,P"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "P"] \\
\mathbb{B}^{\mathbf{2}} \arrow[r, "\partial_0"'] \arrow[d, "\partial_1"] & \mathbb{B} \\
\mathbb{B} &
\end{tikzcd}
$$

As fibrations are stable under pullbacks and composition $\mathrm{Fam}\, P$ is a fibration.

For a fibration $P : \mathbb{E} \to \mathbb{B}$ the total category $\mathrm{Fam}\, \mathbb{E}$ of $\mathrm{Fam}\, P$ consists of pairs $(X, v : I \to J)$, where $v : I \to J$ is a morphism in $\mathbb{B}$ and $X$ is an object over $I$ with respect to $P$. The functor $\mathrm{Fam}\, P$ maps $(X, v : I \to J)$ to $J$. By definition, a morphism in $\mathrm{Fam}\, \mathbb{E}$ from $(X, v : I \to J)$ to $(X', v' : I' \to J')$ consists of morphisms $r : I \to I'$ and $s : J \to J'$ such that $v'r = sv$, together with a morphism $f : X \to X'$ over $r$. Such a morphism is cartesian if and only if the square $v'r = sv$ is a pullback and $f$ is cartesian.

We can extend Fam to an endofunctor on $\mathrm{Fib}(\mathbb{B})$. Furthermore, there are natural transformations $\eta : \mathrm{Id}_{\mathrm{Fib}(\mathbb{B})} \to \mathrm{Fam}$ and $\mu : \mathrm{Fam}^2 \to \mathrm{Fam}$. For a fibration $P : \mathbb{E} \to \mathbb{B}$

- $\eta_P : P \to \mathrm{Fam}\, P$ sends an object $X$ over $I$ to the object $(X, \mathrm{id}_I)$.
- $\mu_P : \mathrm{Fam}^2\, P \to \mathrm{Fam}\, P$ sends an object $((X, v : I \to J), u : J \to K)$ to $(X, u \circ v : I \to K)$.

The morphism part is now obvious and fibredness of both functors is easily checked.

**Proposition 6.** *The category* $\mathrm{Fib}(\mathbb{B})$ *is equipped with a monad structure, defined by* $(\mathrm{Fam}, \eta, \mu)$.

2.4. **Change of base.** In Proposition 3 we have seen that fibrations are stable under pullback, so given a functor $F : \mathbb{B} \to \mathbb{C}$, any fibration $P : \mathbb{E} \to \mathbb{C}$ over $\mathbb{C}$ induces a fibration $F^*P : F^*\mathbb{E} \to \mathbb{B}$ over $\mathbb{B}$. In fact, $F^*$ extends to a 2-functor $F^* : \mathrm{Fib}(\mathbb{C}) \to \mathrm{Fib}(\mathbb{B})$. Later on we will examine, which requirements have to be imposed on the functor $F$ such that $F^*$ preserves certain 'good' properties of fibrations. Almost every good property of fibrations will be preserved by $F^*$ if $F$ preserves finite limits and has a right adjoint.

**Proposition 7.** *Let* $P : \mathbb{E} \to \mathbb{C}$ *and* $Q : \mathbb{D} \to \mathbb{B}$ *be fibrations and* $H : \mathbb{D} \to \mathbb{E}$ *and* $F : \mathbb{B} \to \mathbb{C}$ *be functors such that* $FQ = PH$ *and* $H$ *preserves cartesianness of morphisms. Then the canonical functor from from* $Q$ *to* $F^*P$ *preserves cartesian morphisms, i.e. is a fibred functor.*

$$
\begin{tikzcd}
\mathbb{D} \arrow[dr, dashed] \arrow[drr, bend left, "H"] \arrow[ddr, bend right, "Q"'] & & \\
& F^*\mathbb{E} \arrow[r] \arrow[d, "F^*P"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "P"] \\
& \mathbb{B} \arrow[r, "F"'] & \mathbb{C}
\end{tikzcd}
$$

## 3. Completeness Properties of Fibrations

In fibrations we deal with two different types of limits. The first type is defined with respect to some external (usually finite) diagram category. The second type is defined with respect to some internal category in the base category. We will not define internal limits in this generality but restrict our view to internal products.

3.1. **Fibred Limits.** In this subsection we will concentrate on (co-)limits with respect to some diagram category $\mathbb{D}$.

**Definition 8.** Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration and $\mathbb{D}$ be a category. Then the fibration $P^{(\mathbb{D})} : \mathbb{E}^{(\mathbb{D})} \to \mathbb{B}^{(\mathbb{D})}$ is constructed from $P^{\mathbb{D}} : \mathbb{E}^{\mathbb{D}} \to \mathbb{B}^{\mathbb{D}}$ by change of base along the canonical functor $\Delta_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}^{\mathbb{D}}$ (which sends an $I \in \mathbb{B}$ to the constant functor with value $I$), as indicated in the following diagram.

$$\begin{tikzcd} \mathbb{E}^{(\mathbb{D})} \arrow[r] \arrow[d, "P^{(\mathbb{D})}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E}^{\mathbb{D}} \arrow[d, "P"] \\ \mathbb{B} \arrow[r, "\Delta_{\mathbb{B}}"'] & \mathbb{B}^{\mathbb{D}} \end{tikzcd}$$

The fibration $P^{(\mathbb{D})}$ can be described as follows. An object over some $I$ in $\mathbb{B}$ w.r.t. $P^{(\mathbb{D})}$ is a diagram of shape $\mathbb{D}$ in $\mathbb{E}_I$. If $u$ is a morphism in $\mathbb{B}$ then a morphism between such diagrams in $\mathbb{E}^{(\mathbb{D})}$ over $u$ is a natural transformation whose components are over $u$ w.r.t. $P$.

According to Proposition 7, there is a canonical fibred functor $\Delta_P$ from $P$ to $P^{(\mathbb{D})}$.

**Definition 9 (Fibred Limits).** Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration and $\mathbb{D}$ be a category. We say that $P$ has fibred (co-)limits of shape $\mathbb{D}$, whenever the fibred functor $\Delta_P$ admits a fibred right (left) adjoint.

By Lemma 4 a fibration has fibred (co-)limits of shape $\mathbb{D}$ exactly if every fibre has (co-)limits of shape $\mathbb{D}$ and (co-)limiting cones are preserved under reindexing.

As in the definition of fibred limits the existence of a fibred right adjoint is required, we can make use of Proposition 5. E.g. let $\mathbb{D}$ be the empty category, then $P^{(\mathbb{D})}$ is the identical functor $\mathrm{Id}_{\mathbb{B}}$ and $P$ has fibred terminal objects if and only if the fibred functor $P$ from $P$ to $\mathrm{Id}_{\mathbb{B}}$ admits a fibred right adjoint. By Proposition 5 any ordinary right adjoint with vertical counit is a fibred functor. The counit being vertical w.r.t. $\mathrm{Id}_{\mathbb{B}}$ means, that the counit is the identity. *So a fibration has fibred terminal objects if and only if $P$ admits a right-adjoint-right-inverse.*

As far as fibred limits are concerned there is yet another characterization, given that the base category has limits of shape $\mathbb{D}$ already. Note that the analogue for colimits does not hold.

**Proposition 8.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration, $\mathbb{D}$ be a category and $\mathbb{B}$ have limits of shape $\mathbb{D}$. Then the following are equivalent:*

(i) *$\mathbb{E}$ has limits of shape $\mathbb{D}$ and $P$ preserves them.*
(ii) *The fibration $P$ has fibred limits of shape $\mathbb{D}$.*

The fibration $P$ defined in Example 7 shows that Proposition 8 does not hold for colimits. The total category of the fibration has an initial object, which is preserved by $P$. Still, $P$ does not have fibred initial objects.

Proposition 8 can be used to show, that in a fibration $P : \mathbb{E} \to \mathbb{B}$ with fibred pullbacks, a square of vertical morphisms over $I$ is a pullback in $\mathbb{E}_I$ if and only if it is a pullback in $\mathbb{E}$.

Of course, a fibred functor $F$ is said to preserve fibred limits of shape $\mathbb{D}$ if and only if for each object $I$ in the base, the functor $F_I$ preserves limits of shape $\mathbb{D}$. One can also prove, that $F$ preserves fibred limits of shape $\mathbb{D}$ if and only if it preserves limits of shape $\mathbb{D}$ as an ordinary functor.

In general, the keyword 'fibred' is supposed to indicate that each fibre has a certain structure and that reindexing preserves this structure. For example, a fibration is called a *fibred CCC* if each fibre is cartesian closed and reindexing preserves finite products, exponential objects and evaluation maps. There are plenty of interesting examples of fibred CCCs. The subobject fibrations of toposes are fibred CCCs. A codomain fibration over $\mathbb{C}$ is a fibred CCC if and only if $\mathbb{C}$ is locally cartesian closed.

3.2. **Internal Products.** Whereas in the preceding subsection we have concentrated on (co-)limits with respect to some external diagram category we will now use the base category as a means of indexing.

We shall try to reasonably define when a fibration has (co-)products, considering the objects of the base category as index sets and an object $X$ over some $I$ as a family indexed by $I$. We will call these (co-)products *internal*, as opposed to fibred limits, where the objects are indexed by some *external* diagram category $\mathbb{D}$. Taking as a guiding example $\mathbf{Sets}(\mathbb{C})$ one might be tempted to define a fibration to have internal products if for each object $I$ in the base category, the reindexing functor $(!_I)^*$ admits a right adjoint. There are reasons against this choice. It would require (and give undue prominence to) a terminal object in the base category. Furthermore, this definition turns out to be insufficient in general as we wish to perform products along an arbitrary morphism $u : I \to J$ in the base category. If the category $\mathbb{C}$ has small products, the product of an object $(X_i)_{i\in I}$ in $\mathbf{Sets}(\mathbb{C})$ along $u$ is defined to be the family $(\prod_{i\in u^{-1}(j)} X_i)_{j\in J}$. Taking the product along $u$ yields a right adjoint to the reindexing functor $u^*$. In addition, we shall require a so called *Beck-Chevalley condition.* This condition expresses some kind of commutation between (co-)products and substitution and is necessary for interpreting some type theory in a fibration. For a detailed account on this aspect see [Jacobs].

*When dealing with internal (co-)products, we will always assume the existence of pullbacks in the base category*, which is a very modest assumption anyway.

**Definition 10.** Let $\mathbb{B}$ be a category with pullbacks and let $P : \mathbb{E} \to \mathbb{B}$ be a fibration. Then $P$ has internal products if for every $X$ over some $I$ and every $u : I \to J$ there is

- an object $\prod_u X$ over $J$ together with
- a cartesian morphism $\bar{u}(\prod_u X) : u^* \prod_u X \to \prod_u X$ over $u$ and
- a vertical morphism $\pi : u^* \prod_u X \to X$

such that

(i) for every $Y$ over $J$, every cartesian morphism $\bar{u}Y : u^*Y \to Y$ over $u$ and and every vertical morphism $\alpha : u^*Y \to Y$ there is a unique vertical morphism $\beta : Y \to \prod_u X$ such that $\pi \circ u^*\beta = \alpha$.

$$\begin{tikzcd}
X & u^*\prod_u X \arrow[l, "\pi"'] \arrow[r, "\bar{u}(\prod_u X)"] & \prod_u X \\
 & u^*Y \arrow[ul, "\alpha"] \arrow[u, dashed, "u^*\beta"'] \arrow[r, "\bar{u}Y"'] & Y \arrow[u, dashed, "\beta"']
\end{tikzcd}$$

(ii) for every pullback square

$$\begin{tikzcd} K \arrow[r, "v"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "t"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

in $\mathbb{B}$ the top row in the following diagram satisfies the universal property of (i), where $\bar{s}X$ and $\bar{s}(u^*\prod_u X)$ are cartesian over $s$ and $\bar{t}(\prod_u X)$ is cartesian over $t$ and where $s^*\pi$ and $\varphi$ are the unique fill-ins over $\mathrm{id}_K$ and $v$ respectively (note that by Lemma 2, $\varphi$ must be cartesian).

$$\begin{tikzcd} s^*X \arrow[d, "\bar{s}X"'] & s^*u^*\prod_u X \arrow[l, "s^*\pi"'] \arrow[r, "\varphi"] \arrow[d, "\bar{s}(u^*\prod_u X)"'] \arrow[dr, phantom, "\lrcorner", very near start] & t^*\prod_u X \arrow[d, "\bar{t}(\prod_u X)"] \\ X & u^*\prod_u X \arrow[l, "\pi"] \arrow[r, "\bar{u}(\prod_u X)"'] & \prod_u X \end{tikzcd}$$

**Remark 2.** Let $P : \mathbb{E} \to \mathbb{B}$ be a cloven fibration with internal products and let $u : I \to J$ be a morphism in the base, then any simultaneous choice of objects $\prod_u X$ and morphisms $\pi : u^*\prod_u X \to X$ satisfying the above requirements gives rise to a functor $\prod_u : \mathbb{E}_I \to \mathbb{E}_J$, that is right adjoint to $u^*$. Furthermore, for every pullback square $tv = us$ in the base, the canonical natural transformation from $t^*\prod_u$ to $\prod_v s^*$ is a natural isomorphism.

**Definition 11.** Let $\mathbb{B}$ be a category with pullbacks and let $P : \mathbb{E} \to \mathbb{B}$ be a fibration. Then $P$ has internal coproducts if for every $X$ over some $I$ and every $u : I \to J$ there is

- an object $\coprod_u X$ over $J$ together with
- a cartesian morphism $\bar{u}(\coprod_u X) : u^*\coprod_u X \to \coprod_u X$ over $u$ and
- a vertical morphism $\iota : X \to u^*\coprod_u X$

such that

(i) for every $Y$ over $J$, every cartesian morphism $\bar{u}Y : u^*Y \to Y$ over $u$ and and every vertical morphism $\alpha : Y \to u^*Y$ there is a unique vertical morphism $\beta : \coprod_u X \to Y$ such that $u^*\beta \circ \iota = \alpha$.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\alpha"'] & u^*\coprod_u X \arrow[r, "\bar{u}(\coprod_u X)"] \arrow[d, dashed, "u^*\beta"] & \coprod_u X \arrow[d, dashed, "\beta"] \\ & u^*Y \arrow[r, "\bar{u}Y"'] & Y \end{tikzcd}$$

(ii) for every pullback square

$$\begin{tikzcd} K \arrow[r, "v"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "t"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

in $\mathbb{B}$ the top row in the following diagram satisfies the universal property of (i), where $\bar{s}X$ and $\bar{s}(u^*\coprod_u X)$ are cartesian over $s$ and $\bar{t}(\coprod_u X)$ is cartesian over $t$, and where $s^*\iota$

and $\varphi$ are the unique fill-ins over $\mathrm{id}_K$ and $v$ respectively (note that by Lemma 2, $\varphi$ must be cartesian).

$$\begin{tikzcd}
s^*X \arrow[r, "s^*\iota"] \arrow[d, "\bar{s}X"'] & s^*u^*\coprod_u X \arrow[r, "\varphi"] \arrow[d, "\bar{s}(u^*\coprod_u X)"'] \arrow[dr, phantom, "\lrcorner", very near start] & t^*\coprod_u X \arrow[d, "\bar{t}(\coprod_u X)"] \\
X \arrow[r, "\iota"'] & u^*\coprod_u X \arrow[r, "\bar{u}(\coprod_u X)"'] & \coprod_u X
\end{tikzcd}$$

An analogue of Remark 2 holds for internal coproducts. In this case, the induced functor $\coprod_u : \mathbb{E}_I \to \mathbb{E}_J$ is left adjoint to $u^*$ and the canonical natural transformation from $\coprod_v s^*$ to $t^* \coprod_u$ is a natural isomorphism.

Fortunately, for the case of internal coproducts there are two further characterizations which are simpler than the original definition.

**Theorem 1.** *Let $\mathbb{B}$ be a category with pullbacks and let $P : \mathbb{E} \to \mathbb{B}$ be a fibration. Then the following are equivalent.*

(i) *$P$ has internal coproducts.*
(ii) *$P$ is a bifibration and cocartesian morphisms are stable under pullbacks along cartesian morphisms.*
(iii) *The fibred functor $\eta_P : P \to \mathrm{Fam}\, P$ admits a fibred left adjoint.*

*Proof.* (i) $\Longrightarrow$ (ii): Given a morphism $u : I \to J$ in $\mathbb{B}$ and an object $X$ over $I$, then composition of $\iota : X \to u^* \coprod_u X$ with $\bar{u}(\coprod_u X) : u^* \coprod_u X \to \coprod_u X$ yields a cocartesian morphism. By the Beck-Chevalley condition, cocartesian morphisms are stable under pullback along cartesian morphisms.

(ii) $\Longrightarrow$ (i): Given a morphism $u : I \to J$ in $\mathbb{B}$ and an object $X$ over $I$, let $\varphi : X \to Y$ be a cocartesian lifting of $u$. Then factorizing $\varphi$ as a vertical morphism, followed by a cartesian morphism, yields a coproduct. As cocartesian morphisms are stable under pullback along cartesian morphisms, the Beck-Chevalley condition is satisfied.

(ii) $\Longleftrightarrow$ (iii): Let $(X, u : I \to J)$ be an object in $\mathrm{Fam}\,\mathbb{E}$. A vertical morphism in $\mathrm{Fam}\,\mathbb{E}$ is universal from $(X, u : I \to J)$ to $\eta_P$ if and only if the component lying in $\mathbb{E}$ is cocartesian over $u$. The stability of cocartesian morphisms under pullback along cartesian morphisms is equivalent to the preservation of cartesian morphisms by the right adjoint of $\eta_P$. □

In comparison, characterization (i) has the advantage of being dual to the definition of internal products. In what follows we will mainly work with characterization (ii) as it is very handy. In this context the stability of cocartesian morphisms under pullbacks along cartesian morphisms will also be referred to as Beck-Chevalley condition. Also, we will use the words coproduct and sum interchangeably. Characterization (iii) has the advantage that it entails the Beck-Chevalley condition instead of requiring it. Note that the existence of internal products is *not* equivalent to $\eta_P$ having a fibred right adjoint, in contrast to what one might be inclined to suppose.

## 4. Extensivity

4.1. **Extensive Categories.** In this subsection, we will introduce some particular properties of sums and examine how these are related, as has been done in [CLW93] for sums in ordinary categories. Namely, these properties are 'universality' (also known as 'stability'), 'disjointness' and 'extensivity'. In the next subsection we turn to fibrations and give fibrational definitions of these properties with respect to internal sums and prove analogous results.

**Definition 12.** A category $\mathbb{C}$ with finite coproducts is said to have *universal sums* if it has pullbacks along injections and whenever $\iota_1 : X_1 \to X_1 + X_2 \leftarrow X_2 : \iota_2$ is a coproduct cone and $h : Y \to X_1 + X_2$ is a morphism, then the top row of the following diagram is a coproduct cone, too.

$$\begin{tikzcd} Y_1 \arrow[r, "\kappa_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "h"] & Y_2 \arrow[l, "\kappa_2"'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ X_1 \arrow[r, "\iota_1"'] & X_1 + X_2 & X_2 \arrow[l, "\iota_2"] \end{tikzcd}$$

I.e. coproducts are stable under pullbacks.

**Definition 13.** A category $\mathbb{C}$ with finite coproducts is said to have *disjoint sums* if pullbacks along injections exist, all injections are monic and the pullback of different injections (i.e. their intersection) is an initial object in $\mathbb{C}$.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_2 \arrow[d, "\iota_2"] \\ X_1 \arrow[r, "\iota_1"'] & X_1 + X_2 \end{tikzcd}$$

**Definition 14.** A category $\mathbb{C}$ is said to have *extensive sums* or to be *extensive* for short if it has finite coproducts and for every pair $X_1$, $X_2$ of objects the functor

$$\mathbb{C}/X_1 \times \mathbb{C}/X_2 \longrightarrow \mathbb{C} \;/\; X_1 + X_2 \qquad (f,g) \mapsto f + g$$

is an equivalence.

One might suppose that one would have to add a zero-ary version of extensivity, saying that the functor from $\mathbb{1}$ to $\mathbb{C}/0$ sending the unique object of $\mathbb{1}$ to $\mathrm{id}_0$ is an equivalence. This is tantamount to saying that initials in $\mathbb{C}$ are *strict* (i.e. every morphism into 0 is an isomorphism). Interestingly, the (binary) extensivity condition above already implies strictness of initial objects.

**Remark 3.** Given an arbitrary functor $F : \mathbb{A} \to \mathbb{B}$, every object $X$ in $\mathbb{A}$ induces a functor $F/X : \mathbb{A}/X \to \mathbb{B}/FX$ between slice categories. Note, that the functor, referred to in the definition of extensivity, is of that type. It is the functor $+/(X_1, X_2)$, where $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is the coproduct functor.

The following theorem clarifies the connection between all these definitions. Its proof can be found in [CLW93].

**Theorem 2.** *For a category $\mathbb{C}$ with finite coproducts, the following are equivalent:*

(i) $\mathbb{C}$ *is extensive.*

(ii) *In $\mathbb{C}$ pullbacks along injections exist and for every diagram*

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A \arrow[d, "f"'] & A_2 \arrow[l, "a_2"'] \arrow[d, "f_2"'] \\ X_1 \arrow[r, "\iota_1"'] & X_1 + X_2 & X_2 \arrow[l, "\iota_2"] \end{tikzcd}$$

*whose bottom row is a coproduct cone the top row is a coproduct cone if and only if both squares are pullbacks.*

(iii) *The sums in $\mathbb{C}$ are universal and disjoint.*

Note, that in an extensive category, the existence of certain finite limits is guaranteed. A category that has all finite limits is sometimes called a *lex* category (the word 'lex' is an abbreviation of *left exact*, a notion originating from homological algebra). This motivates the following definition.

**Definition 15.** A category is called *lextensive* if it is extensive and has all finite limits.

These definitions can of course be extended to the case of arbitrary set-indexed coproducts. For example, in a locally cartesian closed category $\mathbb{C}$, all existing sums are universal. This follows, as for every morphism $u : I \to J$ the functor $u^* : \mathbb{C}/J \to \mathbb{C}/I$ has a right adjoint $\prod_u$ and thus preserves arbitrary coproducts. If, moreover, $\mathbb{C}$ is a topos, then all sums existing are disjoint (a proof can be found in [John77]).

4.2. **Extensive Fibrations.** We are going to define extensivity, universality and disjointness for fibrations with internal coproducts. We do this in a way such that $\mathbf{FinSets}(\mathbb{C})$ has universal/disjoint coproducts or is extensive as a fibration if and only if $\mathbb{C}$ has *finite* universal/disjoint coproducts or is extensive as a category, respectively. Moreover, $\mathbf{Sets}(\mathbb{C})$ will have universal/disjoint coproducts or will be extensive as a fibration if and only if $\mathbb{C}$ has arbitrary coproducts and the canonical generalizations of universality, disjointness or extensivity, respectively, to infinite coproducts hold.

*Note, that we will define universality, disjointness and extensivity with respect to internal coproducts, rather than potentially existing fibred coproducts.*

First of all, what is a reasonable way to generalize the requirement that pullbacks along injections exist for a fibration $P : \mathbb{E} \to \mathbb{B}$ with internal coproducts? Examining the case of family fibrations leads to the idea, that the appropriate requirement is that pullbacks of cocartesian morphisms along vertical morphisms exist and that $P$ preserves these pullbacks. An equivalent formulation is the following: For every cocartesian morphism $\varphi : X \to Y$ and every vertical morphism $\beta : B \to Y$, there is a pullback square

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\beta"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

such that $\alpha$ is vertical.

Remember that every morphism in $\mathbb{E}$ factors as a vertical morphism, followed by a cartesian morphism and that pullbacks along cartesian morphisms exist and are preserved by $P$.

So the requirement above implies, that pullbacks of cocartesian morphisms along arbitrary morphisms exist and are preserved by $P$ and vice versa.

**Definition 16.** A fibration with internal coproducts is said to have *universal sums* if pullbacks of cocartesian morphisms along vertical morphisms exist and are cocartesian again. Furthermore, $P$ is required to preserve these pullbacks.

Together with the Beck-Chevalley condition for coproducts, this means that pullbacks of cocartesian morphisms along arbitrary morphisms exist *and are cocartesian* and that these pullbacks are preserved by $P$.

**Definition 17.** A fibration $P : \mathbb{E} \to \mathbb{B}$ with internal coproducts is said to have *disjoint sums* if pullbacks of cocartesian morphisms along arbitrary morphisms exist and are preserved by $P$ and if for every cocartesian morphism $\varphi : X \to Y$ in $\mathbb{E}$, the mediating arrow $\delta_\varphi$ from $X$ into the kernel pair of $\varphi$ induced by $(\mathrm{id}_X, \mathrm{id}_X)$ is again cocartesian.

$$\begin{tikzcd}
X \arrow[dr, "\delta_\varphi"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& A \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
& X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

Bearing in mind Remark 3, the following definition needs no further motivation.

**Definition 18.** A fibration $P : \mathbb{E} \to \mathbb{B}$ is said to have *extensive sums* or to be *extensive* for short if it has internal coproducts and for every $u : I \to J$ in $\mathbb{B}$ and every $X$ over $I$ the functor

$$\coprod_u /X : \mathbb{E}_I/X \longrightarrow \mathbb{E}_J/\coprod_u X \qquad \alpha \longmapsto \coprod_u \alpha$$

is an equivalence.

**Remark 4.** A functor $F : \mathbb{A} \to \mathbb{B}$ preserves pullbacks if and only if for each $X \in \mathbb{A}$ the functor $F/X : \mathbb{A}/X \to \mathbb{B}/FX$ preserves finite limits. As in extensive fibrations all functors $\coprod_u /X$ are equivalences, in particular all functors $\coprod_u$ preserve pullbacks.

Now there follow some technical lemmas which will be used in the proof of Theorem 3 below.

**Lemma 5.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration with internal coproducts. Consider a commutative square*

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

*with $\alpha$ and $\beta$ vertical and $\varphi$, $\psi$ over some $u : I \to J$ in $\mathbb{B}$. Then the square is a pullback if and only if for every $C$ over $I$, for every vertical $\gamma : C \to X$ and for every $g : C \to B$ such that $\varphi \circ \gamma = \beta \circ g$ there exists a unique $h : C \to A$ such that $\alpha \circ h = \gamma$ and $\psi \circ h = g$.*

*Proof.* Clearly, the condition is necessary. On the other hand, assume arbitrary morphisms $\gamma : C \to X$ and $g : C \to B$ such that $\varphi \circ \gamma = \beta \circ g$. Let $v := P\gamma$. There is a cocartesian morphism $\xi : C \to C'$ over $v$. Then $\gamma$ induces a unique vertical morphism $\gamma' : C' \to X$ such that $\gamma' \circ \xi = \gamma$ and $g$ induces a unique $g' : C' \to B$ over $u$ such that $g' \circ \xi = g$. Furthermore, the equation $\varphi \circ \gamma' = \beta \circ g'$ holds. The morphisms $h' : C' \to A$ such that $\alpha \circ h' = \gamma'$ and $\psi \circ h' = g'$ correspond uniquely to the morphisms $h : C \to A$ such that $\alpha \circ h = \gamma$ and $\psi \circ h = g$. □

**Lemma 6.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration with internal coproducts. Consider a cocartesian morphism $\varphi : X \to \coprod_u X$ over some $u : I \to J$. Then the following are equivalent:*

(i) *The functor $\coprod_u /X : \mathbb{E}_I/X \to \mathbb{E}_J/\coprod_u X$ admits a right adjoint.*
(ii) *Pullbacks of vertical morphisms along $\varphi$ exist and are preserved by $P$.*

*Proof.* For some vertical morphism $\beta : B \to \coprod_u X$, i.e. an object of $\mathbb{E}_J/\coprod_u X$, a universal morphism from $\coprod_u /X$ to $\beta$ consists of

- an object $\beta' : B' \to X$ in $\mathbb{E}_I/X$
- a cocartesian morphism $\psi : B' \to \coprod_u B'$ and
- a morphism $\epsilon : \coprod_u \beta' \to \beta$ in $\mathbb{E}_J/\coprod_u X$

such that, given

- an object $\gamma : C \to X$ in $\mathbb{E}_I/X$
- a cocartesian morphism $\xi : C \to \coprod_u C$ and
- a morphism $\alpha : \coprod_u \gamma \to \beta$

in $\mathbb{E}_J/\coprod_u X$ there exists a unique $\alpha' : \gamma \to \beta'$ such that $\epsilon \circ \alpha'' = \alpha$, where $\alpha'' = \coprod_u /X\ (\alpha')$. Using the previous lemma, it is easy to check that the morphisms $\beta'$ and $\epsilon \circ \psi$ constitute a pullback of $\varphi$ and $\beta$ if $\beta'$, $\psi$ and $\epsilon$ satisfy the universality condition.

On the other hand, given a pullback square whose left-hand morphism is vertical and factorizing the upper morphism as a cocartesian, followed by a vertical morphism yields a universal arrow.

$$
\begin{tikzcd}
C \arrow[rr, "\xi"] \arrow[dr, dashed, "\alpha'"] \arrow[ddr, bend right, "\gamma"'] & & \coprod_u C \arrow[dr, dashed, "\alpha''"] \arrow[drr, bend left, "\alpha"] \arrow[ddrr, "\coprod_u \gamma"] & & \\
& B' \arrow[rr, "\psi"] \arrow[d, "\beta'"] & & \coprod_u B' \arrow[r, "\epsilon"] \arrow[dr, "\coprod_u \beta'"] & B \arrow[d, "\beta"] \\
& X \arrow[rrr, "\varphi"'] & & & \coprod_u X
\end{tikzcd}
$$

□

**Lemma 7.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration with universal and disjoint internal coproducts. Consider a cocartesian morphism $\varphi : X \to Y$ and an arbitrary morphism $\psi : W \to X$ in $\mathbb{E}$.*

*Then the unique morphism $\gamma$, making the following diagram commute, is cocartesian.*

$$
\begin{tikzcd}
W \arrow[dr, "\gamma"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\mathrm{id}_W"'] & & \\
& Q \arrow[r, "\theta"'] \arrow[d, "\varphi'"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "\varphi"] \\
& W \arrow[r, "\varphi\circ\psi"'] & Y
\end{tikzcd}
$$

*Proof.* Let us paste together the pullback in the following way

$$
\begin{tikzcd}
W \arrow[r, "\psi"] \arrow[dr, "\gamma"] \arrow[ddr, bend right, "\mathrm{id}_W"'] & X \arrow[dr, "\delta_\varphi"] \arrow[drr, bend left, "\mathrm{id}_X"] & & \\
& Q \arrow[r, "\psi'"'] \arrow[d, "\varphi'"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "\varphi"] \\
& W \arrow[r, "\psi"'] & X \arrow[r, "\varphi"'] & Y
\end{tikzcd}
$$

By definition $\pi_1 \circ \delta_\varphi = \mathrm{id}_X$, so the big square consisting the two small squares on the left is a pullback. Thus the upper left square is a pullback, too. By disjointness, $\delta_\varphi$ is cocartesian and, thus, by universality $\gamma$ is cocartesian, too. $\square$

**Lemma 8.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration with universal and disjoint internal coproducts. Consider composable morphisms $\varphi : X \to Y$ and $\psi : W \to X$ in $\mathbb{E}$. If $\varphi$ and $\varphi \circ \psi$ are cocartesian then so is $\psi$.*

*Proof.* Again, consider the diagram

$$
\begin{tikzcd}
W \arrow[dr, "\gamma"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\mathrm{id}_W"'] & & \\
& Q \arrow[r, "\theta"'] \arrow[d, "\varphi'"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "\varphi"] \\
& W \arrow[r, "\varphi\circ\psi"'] & Y
\end{tikzcd}
$$

By Lemma 7, $\gamma$ is cocartesian and by universality, $\theta$ is cocartesian. Thus, being the composition of cocartesian morphisms $\psi$ is cocartesian, too. $\square$

**Remark 5.** Together with Proposition 1 we obtain the following result: Given composable morphisms $\varphi$ and $\psi$ in the total category of a fibration with universal and disjoint sums, if two of the morphisms $\varphi$, $\psi$, $\varphi \circ \psi$ are cocartesian, then so is the third.

The following theorem is a fibrational version of Theorem 2 and contains the latter as a special case when $\mathbb{B}$ is the category **FinSets** of finite sets.

**Theorem 3.** *For a fibration $P : \mathbb{E} \to \mathbb{B}$ with internal coproducts the following are equivalent:*

(i) *$P$ is extensive.*

(ii) *In $\mathbb{E}$ pullbacks of cocartesian morphisms along vertical morphisms exist and are preserved by $P$ and a commutative square*

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

*with $\alpha$, $\beta$ vertical and $\varphi$ cocartesian is a pullback if and only if $\psi$ is cocartesian.*

(iii) *$P$ has universal and disjoint sums.*

*Proof.* (i) $\iff$ (ii): By Lemma 6, all functors of the form $\coprod_u /X$ have right adjoints if and only if all pullbacks of cocartesian morphisms along vertical morphisms exist and are preserved by $P$.

Let $u : I \to J$ be a morphism in $\mathbb{B}$ and $\varphi : X \to Y$ be some cocartesian morphism over $u$. Then the unit of the adjunction $\coprod_u /X \dashv \varphi^*$ is a natural isomorphism if and only if the cocartesianness of $\psi$ implies the pullback property for all commutative squares under consideration. Conversely, the counit is a natural isomorphism if and only if the pullback property implies the cocartesianness of $\psi$.

(ii) $\implies$ (iii): Universality is exactly one direction of the equivalence stated in (ii).

In order to prove disjointness, we have to show that if $\varphi : X \to Y$ is cocartesian, then so is $\delta_\varphi$. First, factor $\delta_\varphi$ as $\alpha \circ \psi$, where $\alpha : A' \to A$ is vertical and $\psi : X \to A'$ is cocartesian. By universality we know, that the $\pi_i$ are cocartesian, too. Pick one of them, say $\pi_1$, let $\xi : A' \to Q$ be a cocartesian morphism over the same map as $\pi_1$ and let $\beta : Q \to X$ be the unique vertical morphism such that $\beta \circ \xi = \pi_1 \circ \alpha$. Then $\beta$ is an isomorphism, because both $\xi \circ \psi$ and $\mathrm{id}_X = \pi_1 \circ \delta_\varphi$ are cocartesian morphisms over the same map. As $\xi$ is cocartesian by (ii) the right square in the diagram is a pullback and so $\alpha$ is an isomorphism as $\beta$ is an isomorphism. But then $\delta_\varphi = \alpha \circ \psi$ is a cocartesian morphism.

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[dr, "\delta_\varphi"'] & A' \arrow[r, "\xi"] \arrow[d, "\alpha"'] & Q \arrow[d, "\beta"] \\
 & A \arrow[r, "\pi_1"'] & X
\end{tikzcd}$$

(iii) $\implies$ (ii): By universality, pullbacks of cocartesian morphisms along vertical morphisms exist and are cocartesian again. So what remains to be shown is that the cocartesianness of $\psi$ implies that the square is a pullback.

Consider the following diagram where $\beta'$ is vertical.

$$\begin{tikzcd}
A \arrow[dr, "\gamma"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\alpha"'] & & \\
 & C \arrow[r, "\varphi'"'] \arrow[d, "\beta'"] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "\beta"] \\
 & X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

By universality, $\varphi'$ is cocartesian. As $\varphi' \circ \gamma = \psi$, it follows from Lemma 8 that $\gamma$ is cocartesian. As $\gamma$ is vertical and cocartesian it is an isomorphism. So the original square is already a pullback. $\square$

**Definition 19.** Let $\mathbb{B}$ be a category with pullbacks. Then a fibration $P : \mathbb{E} \to \mathbb{B}$ is called *lextensive* if it has fibred finite limits and is extensive.

One can easily show, that a lextensive category is distributive, i.e. for objects $X, Y, Z$ the canonical morphism from $(X \times Z) + (Y \times Z)$ to $(X + Y) \times Z$ is an isomorphism. The adequate fibred analogue of distributivity is the so called *Frobenius reciprocity*. It says that, given a morphism $u : I \to J$ in the base, an object $X$ over $I$ and $Z$ over $J$, then the canonical morphism from $\coprod_u (X \times u^* Z)$ to $(\coprod_u X) \times Z$ is an isomorphism (where the first $\times$ stands for the product in the fibre over $I$ and the second stands for the product in the fibre over $J$). As expected, the following proposition holds.

**Proposition 9.** *Frobenius reciprocity holds in lextensive fibrations.*

*Proof.* Let $u : I \to J$ be a morphism in the base, $X$ an object over $I$ and $Z$ an object over $J$. Let $\xi : u^* Z \to Z$ be a cartesian morphism over $u$ in the following diagram.

$$\begin{tikzcd} X \times u^* Z \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & u^* Z \arrow[r, "\xi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ X \arrow[r] & u^* 1_J \arrow[r] & 1_J \end{tikzcd}$$

Now, let $\varphi : X \to \coprod_u X$ be a cocartesian morphism over $u$ and let $\psi$ be the morphism induced by $\varphi \circ p$ and $\xi \circ q$.

$$\begin{tikzcd} X \times u^* Z \arrow[r, "\psi"] \arrow[d, "p"'] & (\coprod_u X) \times Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ X \arrow[r, "\varphi"'] & \coprod_u X \arrow[r] & 1_J \end{tikzcd}$$

Note, that by construction the outer rectangles of both diagrams coincide. The left hand square of the previous diagram is a pullback as the outer rectangle is a pullback by construction. Thus, by extensivity $\psi$ is cocartesian. $\square$

Now, that we know some interesting properties of fibrations, we take another look at codomain fibrations.

**Proposition 10.** *Let $\mathbb{C}$ be a category with pullbacks and let $P := \partial_1 : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$ be the corresponding codomain fibration. Then $P$ is a lextensive fibration, i.e.*

(i) *$P$ has fibred finite limits.*
(ii) *$P$ has extensive internal sums.*

*Proof.* As limits in $\mathbb{C}^{\mathbf{2}}$ are constructed componentwise it is clear that $P$ has fibred finite limits by Proposition 8.

The cocartesian morphisms in $\mathbb{C}^{\mathbf{2}}$ are exactly those squares, whose upper edge is an isomorphism. So for $x : X \to I$ and $u : I \to J$, there is a canonical cocartesian lift, namely:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "x"'] & X \arrow[d, "u \circ x"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

As isomorphisms are stable under pullback the Beck-Chevalley condition is satisfied. For extensivity, we have to show that the functor $\coprod_u /x : \mathbb{C}/I/x \to \mathbb{C}/J/(u \circ x)$ is an equivalence. Actually, it is an isomorphism because the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}/I/x \arrow[rr, "\coprod_u/x"] \arrow[dr, "\cong"'] & & \mathbb{C}/J/(u \circ x) \arrow[dl, "\cong"] \\ & \mathbb{C}/X & \end{tikzcd}$$

□

Now, let us take a look at the change of base functors $F^* : \mathrm{Fib}(\mathbb{C}) \to \mathrm{Fib}(\mathbb{B})$ induced by functors $F : \mathbb{B} \to \mathbb{C}$ and examine which requirements have to be imposed on $F$ such that $F^*$ preserves 'good' properties of fibrations.

**Proposition 11.** *Let $P : \mathbb{E} \to \mathbb{C}$ be a fibration, $F : \mathbb{B} \to \mathbb{C}$ be a functor and $\mathbb{D}$ be a category. Whenever $P$ has fibred limits of shape $\mathbb{D}$, then $F^*P$ has fibred limits of shape $\mathbb{D}$.*

**Definition 20.** Let $\mathbb{C}$ be a category with pullbacks and $F : \mathbb{B} \to \mathbb{C}$ be a functor. Then by $\mathsf{Gl}(F)$ we denote the comma category $\mathbb{C}/F$. It is the total category of the fibration $\mathsf{gl}(F)$, arising from the codomain fibration $\partial_1 : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$ by change of base along $F$.

$$\begin{tikzcd} \mathsf{Gl}(F) \arrow[r] \arrow[d, "\mathsf{gl}(F)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}^{\mathbf{2}} \arrow[d, "\partial_1"] \\ \mathbb{B} \arrow[r, "F"'] & \mathbb{C} \end{tikzcd}$$

An object of $\mathsf{Gl}(F)$ is an object $I$ of $\mathbb{B}$ together with a morphism $x : X \to FI$ in $\mathbb{C}$. (If $I$ is clear from the context, we will sometimes leave it implicit.) A morphism from $(I, x : X \to FI)$ to $(J, y : Y \to FJ)$ is a morphism $u : I \to J$ together with a morphism $f : X \to Y$ such that the following square commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\ FI \arrow[r, "Fu"'] & FJ \end{tikzcd}$$

Such a morphism in $\mathsf{Gl}(F)$ is cartesian if and only the square is a pullback and it is cocartesian if and only if $f$ is an isomorphism.

**Theorem 4.** *Let $\mathbb{B}$ and $\mathbb{C}$ be categories with pullbacks and $F : \mathbb{B} \to \mathbb{C}$ be a functor. Then the following are equivalent:*

(i) *The functor $F^* : \mathrm{Fib}(\mathbb{C}) \to \mathrm{Fib}(\mathbb{B})$ preserves the property of having internal coproducts.*
(ii) *The fibration $\mathbf{gl}(F) : \mathbf{Gl}(F) \to \mathbb{B}$ has internal coproducts.*
(iii) *The functor $F$ preserves pullbacks*

*Moreover, if $F$ satisfies the properties (i) to (iii) then $F^*$ preserves the property of having extensive sums.*

*Proof.* Clearly, as the codomain fibration has internal coproducts (i) implies (ii).

(iii) $\implies$ (i): Let $P : \mathbb{E} \to \mathbb{C}$ be a fibration with internal coproducts and consider the following change of base situation.

$$\begin{tikzcd}
F^*\mathbb{E} \arrow[r, "Q"] \arrow[d, "F^*P"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{E} \arrow[d, "P"] \\
\mathbb{B} \arrow[r, "F"'] & \mathbb{C}
\end{tikzcd}$$

The functor $Q$ preserves and reflects cocartesianness and cartesianness. Furthermore $Q$ reflects pullback squares over pullback squares, i.e. any commutative square in $F^*\mathbb{E}$, which is mapped to pullback squares by $F^*P$ and $Q$ is itself a pullback. From these facts we can conclude that $F^*P$ is a bifibration and that cocartesian morphisms in $F^*\mathbb{E}$ are stable under pullback along cartesian morphisms.

(ii) $\implies$ (iii): Consider a pullback square in $\mathbb{B}$. We have to show that its image under $F$ in $\mathbb{C}$ is a pullback, or equivalently, that the unique fill-in $\alpha$ from the image into the actual pullback is an isomorphism.

$$\begin{tikzcd}
I \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & J \arrow[d, "u"] \\
K \arrow[r, "v"'] & L
\end{tikzcd}
\qquad
\begin{tikzcd}
FI \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "Fq"] \arrow[ddr, bend right, "Fp"'] & & \\
& P \arrow[r, "q'"] \arrow[d, "p'"] \arrow[dr, phantom, very near start, "\lrcorner"] & FJ \arrow[d, "Fu"] \\
& FK \arrow[r, "Fv"'] & FL
\end{tikzcd}$$

Now the following cube is a pullback square in $\mathbf{Gl}(F)$ by Lemma 2. Its front face is cocartesian, its sides are both cartesian and so the rear face must be cocartesian by the Beck-Chevalley condition for $\mathbf{gl}(F)$. But that means that $\alpha$ is an isomorphism.

$$\begin{tikzcd}
FI \arrow[rr, dashed, "\alpha"] \arrow[dr, "Fq"] \arrow[dd, equal] & & P \arrow[dr, "q'"] \arrow[dd, "p'"'] & \\
& FJ \arrow[rr, equal] \arrow[dd, equal] & & FJ \arrow[dd, "Fu"] \\
FI \arrow[rr, "Fp"] \arrow[dr, "Fq"'] & & FK \arrow[dr, "Fv"] & \\
& FJ \arrow[rr, "Fu"'] & & FL
\end{tikzcd}$$

Finally, $F^*$ preserves the property of having extensive sums, as the functor $\coprod_u : (F^*\mathbb{E})_I \to (F^*\mathbb{E})_J$ is isomorphic to the functor $\coprod_{Fu} : \mathbb{E}_{FI} \to \mathbb{E}_{FJ}$. $\square$

For the next theorem only we assume that $\mathbb{B}$ has all finite limits.

**Theorem 5.** *Let $\mathbb{B}$ be a category with finite limits and let $P : \mathbb{E} \to \mathbb{B}$ be a fibration. Then the following are equivalent:*

(i) *The fibration $P$ is lextensive.*
(ii) *There is a category $\mathbb{C}$ with finite limits and a finite limit preserving functor $F : \mathbb{B} \to \mathbb{C}$ such that $P$ is equivalent to $\mathsf{gl}(F)$.*

*Furthermore, the category $\mathbb{C}$ and the functor $F$ are determined uniquely up to equivalence.*

*Proof.* The preceding propositions about codomain fibrations and preservation of properties under change of base make clear that (ii) implies (i).

Given a lextensive fibration $P : \mathbb{E} \to \mathbb{B}$, define $\mathbb{C} := \mathbb{E}_1$, i.e. the fibre of $P$ over a terminal object 1 of $\mathbb{B}$. For every object $X$ of $\mathbb{E}$ over some $I$ choose a cocartesian morphism $\chi_X : X \to \coprod_I X$ over the terminal projection of $I$ to 1.

As a first step, let us define a functor $S : \mathbb{E} \to \mathbb{C}$ as follows: for each object $X$ over $I$, let $SX$ be the codomain of $\chi_X$ and for each morphism $f : X \to Y$ over $u : I \to J$ let $Sf$ be the unique vertical morphism making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\chi_X"] \arrow[d, "f"'] & \coprod_I X = SX \arrow[d, "Sf"] \\ Y \arrow[r, "\chi_Y"'] & \coprod_J Y = SY \end{tikzcd}$$

The functoriality of $S$ follows from the universal property of the cocartesian morphisms $\chi_X$.

As $P$ has fibred terminal objects, there is a right adjoint right inverse of $P$, called $\mathbf{1} : \mathbb{B} \to \mathbb{E}$. Define $F = D_P$ as $S \circ \mathbf{1}$.

$$\begin{tikzcd} & \mathbb{E} \arrow[dr, "S"] & \\ \mathbb{B} \arrow[ur, "1"] \arrow[rr, "D_P"'] & & \mathbb{C} \end{tikzcd}$$

Let us now construct the fibred equivalence $\Phi_P$ from $P$ to $\mathsf{gl}(D_P)$.

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "\Phi_P"] \arrow[dr, "P"'] & & \mathsf{Gl}(D_P) \arrow[dl, "\mathsf{gl}(D_P)"] \\ & \mathbb{B} & \end{tikzcd}$$

For every object $X$ in $\mathbb{E}$ over some $I$, define $\mathrm{t}_X : X \to \mathbf{1}_I$ to be the terminal projection of $X$ in $\mathbb{E}_I$. The functor $\Phi_P(X)$ sends an object $X$ over $I$ to $S\,\mathrm{t}_X : SX \to D_P(X)$ and a morphism $f : X \to Y$ over $u : I \to J$ to the right hand square below, which emerges from the left one by application of $S$. This is clearly a morphism over $u$ in $\mathsf{Gl}(D_P)$.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{t}_X"'] & Y \arrow[d, "\mathrm{t}_Y"] \\
1_I \arrow[r, "1_u"'] & 1_J
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
SX \arrow[r, "Sf"] \arrow[d, "S\,\mathrm{t}_X"'] & SY \arrow[d, "S\,\mathrm{t}_Y"] \\
D_P(X) \arrow[r, "D_P(u)"'] & D_P(Y)
\end{tikzcd}
$$

The restriction of the functor $\Phi_P$ to any fibre $\mathbb{E}_I$ is an equivalence because $P$ has extensive sums, so by Corollary 2 we will know that $\Phi_P$ is a fibred equivalence, as soon as we have shown that $\Phi_P$ is a fibred functor. This means, we have to show, that if $f$ is cartesian, then the right hand square above is a pullback. In order to do so, assume $f$ cartesian and build the actual pullback of $D_P(u)$ and $S\,\mathrm{t}_Y$ in $\mathbb{C}$ (the right hand side of the cube below). This is in fact a pullback in the category $\mathbb{E}$. So there is a canonical morphism $\psi$ from $X$ to $D_P(u)^*SY$. It suffices to show that $\psi$ is cocartesian.

$$
\begin{tikzcd}
X \arrow[rr, dashed, "\psi"] \arrow[dr, "f"] \arrow[dd, "\mathrm{t}_X"'] & & D_P(u)^*SY \arrow[dr] \arrow[dd] & \\
& Y \arrow[rr, "\chi_Y"] \arrow[dd, "\mathrm{t}_Y"] & & SY \arrow[dd, "S\,\mathrm{t}_Y"] \\
1_I \arrow[rr, "\chi_{1_I}"] \arrow[dr, "1_u"'] & & D_P(I) \arrow[dr, "D_P(u)"] & \\
& 1_J \arrow[rr, "\chi_{1_J}"] & & D_P(J)
\end{tikzcd}
$$

The left face of the cube is a pullback by Lemma 2, the front face is a pullback by Theorem 3(ii) and the right face is a pullback by construction. So the rear face is a pullback, too. Again by Theorem 3(ii), $\psi$ is cocartesian.

As the fibration $\mathsf{gl}(D_P)$ is equivalent to $P$, it has internal coproducts and, therefore, by Theorem 4 the functor $D_P$ preserves pullbacks. Clearly, the way we have defined it, the functor $D_P$ preserves terminal objects. So $D_P$ preserves finite limits.

We finish the proof by showing that the functor $F$ is uniquely determined. This follows from the following two facts.

- If $P \simeq P'$ then $D_P \simeq D_{P'}$
- $D_{\mathsf{gl}(F)} \cong F$

$\square$

## 5. The 2-category LextFib($\mathbb{B}$)

Both the assignment of the functor $D_P$ to some lextensive fibration $P$ over $\mathbb{B}$ and the assignment of the fibration $\mathbf{gl}(F)$ to some finite limit preserving functor $F$ with source $\mathbb{B}$ extend to 2-functors between suitable 2-categories to be defined below.

**Definition 21.** Let $\mathbb{B}$ be a category with pullbacks. Then the 2-category LextFib($\mathbb{B}$) is defined as follows.

- An object is a lextensive fibration over $\mathbb{B}$.
- A 1-cell is a fibred functor which preserves all features of lextensive fibrations, i.e., a functor which preserves cartesianness and cocartesianness of morphisms and fibred finite limits.
- The 2-cells are fibred natural transformations.

**Definition 22.** Let $\mathbb{B}$ be a category with finite limits. Then the 2-category $\mathbb{B}/\mathbf{Lex}$ is defined as follows.

- An object is a finite limit preserving functor $F : \mathbb{B} \to \mathbb{C}$ to some category $\mathbb{C}$ with finite limits.
- A 1-cell from $F : \mathbb{B} \to \mathbb{C}$ to $F' : \mathbb{B} \to \mathbb{C}'$ is a functor $H : \mathbb{C} \to \mathbb{C}'$ together with a natural isomorphism $\alpha : HF \to F'$.
- Let $(H, \alpha)$ and $(K, \beta)$ be morphisms from $F$ to $F'$. Then a 2-cell from $(H, \alpha)$ to $(K, \beta)$ is a natural transformation $\xi : H \to K$ such that $\beta \circ \xi F = \alpha$.

**Definition 23.** Let $\mathbb{B}$ be a category with finite limits. Then the 2-functor $\mathbf{gl} : \mathbb{B}/\mathbf{Lex} \to$ LextFib($\mathbb{B}$) is defined as follows.

- A functor $F : \mathbb{B} \to \mathbb{C}$ is mapped to the fibration $\mathbf{gl}(F) : \mathbf{Gl}(F) \to \mathbb{B}$.
- A morphism $(H, \alpha)$ from $F : \mathbb{B} \to \mathbb{C}$ to $F' : \mathbb{B} \to \mathbb{C}'$ is mapped to the fibred functor from $\mathbf{gl}(F)$ to $\mathbf{gl}(F')$ sending the left square to the outside of the right rectangle in the diagram below.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
FI \arrow[r, "Fu"'] & FJ
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
HX \arrow[r, "Hf"] \arrow[d, "Hx"'] & HY \arrow[d, "Hy"] \\
HFI \arrow[r, "HFu"'] \arrow[d, "\alpha_I"', "\cong"] & HFJ \arrow[d, "\alpha_J", "\cong"'] \\
F'I \arrow[r, "F'u"'] & F'J
\end{tikzcd}
$$

- Let $(H, \alpha)$ and $(K, \beta)$ be morphisms from $F$ to $F'$ Then a 2-cell $\xi$ from $(H, \alpha)$ to $(K, \beta)$ is mapped to the fibred natural transformation, whose component at stage $x : X \to FI$ is the following rectangle.

$$
\begin{tikzcd}
HX \arrow[r, "\xi_X"] \arrow[d, "Hx"'] & KX \arrow[d, "Kx"] \\
HFX \arrow[r, "\xi_{FI}"'] \arrow[d, "\alpha_I"'] & KFI \arrow[d, "\beta_I"] \\
F'I \arrow[r, equal] & F'I
\end{tikzcd}
$$

The essence of Theorem 5 and its proof is that the functor **gl** is full and faithful and that for every object $P$ in LextFib($\mathbb{B}$) there exists an object $F$ in $\mathbb{B}/\mathbf{Lex}$ such that $P$ is equivalent to $\mathbf{gl}(F)$. Assuming a strong enough axiom of choice, we can use the method of constructing a functor $D_P$ from a lextensive fibration $P$ in a uniform manner, yielding a 2-functor as described below.

**Definition 24.** Let $\mathbb{B}$ be a category with finite limits. Then the 2-functor $D : \mathrm{LextFib}(\mathbb{B}) \to \mathbb{B}/\mathbf{Lex}$ is defined as follows.

- A fibration $P$ is mapped to the functor $D_P$, whose construction is described in the proof of Theorem 5. (Mind that this involves choice.)
- A fibred functor $H$ from $P$ to $P'$ is mapped to its restriction to the fibre over 1 together with the canonical natural transformation from $H_1 D_P$ to $D_{P'}$, which is a natural isomorphism because $H$ preserves cocartesianness and fibred terminal objects.
- Let $H$ and $K$ be fibred functors from $P$ to $P'$. Then a fibred natural transformation $\xi$ from $H$ to $K$ is mapped to its restriction to the fibre over 1, which is a natural transformation from $H_1$ to $K_1$ satisfying the required equality. (This can be checked by a straightforward diagram chase.)

**Proposition 12.** *For every object $F : \mathbb{B} \to \mathbb{C}$ of $\mathbb{B}/\mathbf{Lex}$ there is an isomorphism $\Psi_F : D_{\mathbf{gl}(F)} \to F$ in $\mathbb{B}/\mathbf{Lex}$ and for every fibration $P$ in* LextFib($\mathbb{B}$) *there is an equivalence $\Phi_P : P \to \mathbf{gl}(D_P)$. The first family constitutes a natural transformation while the latter is natural only up to a canonical fibred natural isomorphism.*

**Definition 25.** Let $\mathbb{C}$ be a 2-category. Then we say that an object 0 of $\mathbb{C}$ is *2-initial* if

- for every object $X$ in $\mathbb{C}$ there exists a 1-cell $i : 0 \to X$.
- given two 1-cells $i_1, i_2 : 0 \to X$ there exists a unique 2-cell $\xi : i_1 \to i_2$.

In particular this implies that up to isomorphism there is only one 1-cell from 0 to $X$ and that all 2-initial objects are equivalent.

Clearly, the identity functor on $\mathbb{B}$ is 2-initial in the 2-category $\mathbb{B}/\mathbf{Lex}$. As can be expected, its image under the functor **gl**, the codomain fibration, is 2-initial in LextFib($\mathbb{B}$). In what follows we need only assume that $\mathbb{B}$ have pullbacks.

**Proposition 13.** *The codomain fibration $\partial_1 : \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ is 2-initial in the 2-category* LextFib($\mathbb{B}$).

*Proof.* Given a fibration $P$ in LextFib($\mathbb{B}$), let us take a look at the canonical 1-cell $\Delta_P$ from $\partial_1 : \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ to $P$. Every object $x : X \to I$ in $\mathbb{B}^{\mathbf{2}}$ is the codomain of a cocartesian morphism starting from a fibred terminal object. As a 1-cell in LextFib($\mathbb{B}$) is defined to preserve cocartesian morphisms and fibred terminal objects, $\Delta_P\, x$ is the codomain of a cocartesian morphism $\varphi$ over $x$ with domain $1_X$. This fact determines the functor $\Delta_P$ up to isomorphism.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "x"] \\
X \arrow[r, "x"'] & I \\
X \arrow[r, "x"'] & I
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1_X \arrow[r, "\varphi"] & \coprod_x 1_X = \Delta_P\, x \\
\\
X \arrow[r, "x"'] & I
\end{tikzcd}
$$

Having chosen the object part by choosing fibred terminal objects and cocartesian lifts, the morphism part is defined as follows. Consider an arbitrary morphism $(f, u)$ in $\mathbb{B}^{\mathbf{2}}$ as below.

Then its image $\Delta_P(f,u)$ is the unique morphism over $u$, making the right hand square in the diagram below commute.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{1}_X \arrow[r, "\varphi"] \arrow[d, "\mathbf{1}_f"'] & \coprod_x \mathbf{1}_X = \Delta_P\, x \arrow[d, "\Delta_P(f,u)"] \\
\mathbf{1}_Y \arrow[r, "\psi"'] & \coprod_y \mathbf{1}_Y = \Delta_P\, y
\end{tikzcd}
$$

Functoriality follows by the universal property of cocartesian morphisms. Now we show that $\Delta_P$ preserves cartesianness. Assume that the square $u \circ x = y \circ f$ is a pullback. Then the outer rectangle of the following diagram is a pullback (with cartesian sides).

$$
\begin{tikzcd}
\mathbf{1}_X \arrow[r, dashed, "\xi"'] \arrow[rr, bend left, "\mathbf{1}_x"] \arrow[d, "\mathbf{1}_f"'] & W \arrow[r, "p'"'] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1}_I \arrow[d, "\mathbf{1}_u"] \\
\mathbf{1}_Y \arrow[r, "\psi"] \arrow[rr, bend right, "\mathbf{1}_y"'] & \coprod_y \mathbf{1}_Y \arrow[r, "\alpha"] & \mathbf{1}_J
\end{tikzcd}
$$

As cartesian morphisms are stable under pullbacks $p$ is cartesian. And as $\psi$ is cocartesian, by the Beck-Chevalley condition $\xi$ is cocartesian. So $W$ is isomorphic to $\coprod_x \mathbf{1}_X$ and as $p$ is cartesian $\Delta_P(f,u)$ is.

The functor $\Delta_P$ also preserves cocartesian morphisms. Assume that $f$ is an isomorphism. Then $\mathbf{1}_f$ is an isomorphism and $\Delta_P(f,u)$ is cocartesian by Proposition 1.

We still need to show, that $\Delta_P$ preserves fibred finite limits. The preservation of fibred terminal objects is clear by the definition of $\Delta_P$ so it suffices to show that fibred pullbacks are preserved. As the proof is a little technical, we have placed it in Lemma 10 below, which is preceded by a little auxiliary lemma. $\square$

**Lemma 9.** *Let $\mathbb{B}$ be a category with pullbacks and let $P : \mathbb{E} \to \mathbb{B}$ be a fibration with universal sums. Consider the following diagram in $\mathbb{E}$, where $\varphi, \psi$ are cocartesian and $\alpha, \beta$ are vertical.*

$$
\begin{tikzcd}
P \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & Y \arrow[d, "\psi"] \\
 & P' \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "\beta"] \\
X \arrow[r, "\varphi"'] & X' \arrow[r, "\alpha"'] & Z
\end{tikzcd}
$$

*If the outside square is a pullback, then the canonical morphism from $P$ to $P'$ is cocartesian.*

*Proof.* Describe the outside pullback square as pasted together by four small pullback squares, the lower right one of them being the pullback of $\alpha$ and $\beta$. Then the pullback of $\varphi$ along $\beta'$ and the pullback of $\psi$ along $\alpha'$ are cocartesian due to the universality of sums. Again by the universality of sums, the upper left square consists solely of cocartesian morphisms and so the canonical morphism from $P$ to $P'$, being the compose of two cocartesian morphisms, is cocartesian. $\square$

**Lemma 10.** *Let $\mathbb{B}$ be a category with pullbacks and let $P : \mathbb{E} \to \mathbb{B}$ be a lextensive fibration. Let $I$ be an object in $\mathbb{B}$ and consider a pullback square in $\mathbb{B}/I$.*

$$\begin{tikzcd}
I_1 \arrow[r, "w"] \arrow[dd, "v"'] \arrow[rrd, "u_1"] \arrow[ddr, phantom, "\lrcorner", very near start] & I_2 \arrow[dd, "r"] \arrow[rd, "u_2"] & \\
 & & I \\
I_3 \arrow[r, "s"'] \arrow[rru, "u_3"] & I_4 \arrow[ru, "u_4"'] &
\end{tikzcd}$$

*(Recall, that the functor $\partial_0 : \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ preserves and reflects pullbacks, and so the square in $\mathbb{B}$ with corners $I_1$ to $I_4$ is a pullback.)*

*Now, consider the following diagram in $\mathbb{E}$,*

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[ddd, "g"'] \arrow[rrd, "\varphi_1"] \arrow[dddr, phantom, "\lrcorner", very near start] & X_2 \arrow[ddd, "q"] \arrow[rrd, "\varphi_2"] & & \\
 & & Y_1 \arrow[r, "f'"'] \arrow[d, "g'"'] & Y_2 \arrow[d, "q'"] \\
 & & Y_3 \arrow[r, "p'"] & Y_4 \\
X_3 \arrow[r, "p"'] \arrow[rru, "\varphi_3"] & X_4 \arrow[rru, "\varphi_4"'] & &
\end{tikzcd}$$

*where the square on the left is a pullback square above the square in the previous diagram, all $\varphi_i$ are cocartesian over $u_i$ and the square on the right consists of vertical morphisms.*

*Then the square on the right is a pullback.*

*Proof.* Factor both $p$ and $q$ as cocartesian morphisms $\varphi, \psi$, followed by vertical morphisms $\alpha, \beta$. Construct a pullback of $\alpha$ and $\beta$ with $\alpha', \beta'$ vertical.

$$\begin{tikzcd}
X_1 \arrow[rr, "f"] \arrow[ddd, "g"'] \arrow[rrrd, "\varphi_1"] \arrow[rdd, dashed] & & X_2 \arrow[dd, "\psi"] \arrow[rrd, "\varphi_2"] & & \\
 & & & Y_1 \arrow[r, "f'"] \arrow[dd, "g'"] & Y_2 \arrow[dd, "q'"] \\
 & X_1' \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] \arrow[rrru, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & X_2' \arrow[d, "\beta"] \arrow[rrru, dashed] & & \\
 & & & Y_3 \arrow[r, "p'"] & Y_4 \\
X_3 \arrow[r, "\varphi"'] \arrow[rrru, "\varphi_3"] & X_3' \arrow[r, "\alpha"'] \arrow[rru, dashed] & X_4 \arrow[rru, "\varphi_4"'] & &
\end{tikzcd}$$

Observe that by Lemma 9 the canonical morphism from $X_1$ to $X_1'$ is cocartesian. Due to the initial lifting property of cocartesian morphisms this induces four parallel morphisms over $u_4$ between the respective corners of the vertical pullback square on the left and the vertical square on the right side of the diagram. By Lemma 8 these morphisms are cocartesian. So the right square is the image of the vertical pullback square on the left under the functor $\coprod_{u_4}$ which is known to preserve pullbacks as the sums in the fibration were assumed to be extensive. $\square$

This proves that $\Delta_P$ preserves fibred pullbacks. Just choose each $X_i$ to be a terminal object in the fibre over $I_i$.

By the way, observe that Lemma 10 proves as well that the functor $S$ defined in the proof of Theorem 5 preserves pullbacks, by choosing $I$ to be a terminal object in $\mathbb{B}$. So we could have used Lemma 10 in order to show, that the functor $D_P$ preserves finite limits. In fact, the functor $D_P$ is exactly the restriction of $\Delta_P$ to the fibre over 1.

As an application, Proposition 13 allows us to check whether a fibration $P$ is equivalent to the codomain fibration. It is sufficient to check whether $\Delta_P$ is an equivalence. This can be formulated in a more elementary way as follows.

**Corollary 3.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration over a category $\mathbb{B}$ with pullbacks. Then $P$ is equivalent to the codomain fibration over $\mathbb{B}$ if and only if*

(i) *$P$ is lextensive*
(ii) *For every $X$ over $I$ there is a morphism $u : J \to I$ in $\mathbb{B}$ and a cocartesian morphism $\varphi : 1_J \to X$ over $u$.*
(iii) *Given morphisms $u : J \to I$, $u' : J' \to I'$ in $\mathbb{B}$, an object $X$ over $I$ and an object $Y$ over $I'$ and cocartesian morphisms $\varphi : 1_J \to X$ and $\psi : 1_{J'} \to Y$ over $u$ and $u'$ respectively, then for every $f : X \to Y$ there exists a unique $g : J \to J'$ such that the following diagram commutes*

$$\begin{tikzcd} 1_J \arrow[r, "\varphi"] \arrow[d, "1_g"'] & X \arrow[d, "f"] \\ 1_{J'} \arrow[r, "\psi"'] & Y \end{tikzcd}$$

Requirement (ii) could be interpreted as: every family $X$ consists of copowers of 1, whereas condition (iii) can be thought of as: The morphisms between two such copowers of 1 correspond to the morphisms between the index sets in the base.

## 6. A Category of Fractions

In the proof of Theorem 5 we have assumed that $\mathbb{B}$ has all finite limits. One can in fact prove an analogue of the theorem under the weaker assumption that $\mathbb{B}$ has pullbacks (that means, we no longer assume that $\mathbb{B}$ has a terminal object). But we have to use a completely different technique, to be introduced in this section.

Theorem 5 says that a fibration over a base with finite limits is lextensive if and only if it is equivalent to a fibration of the form $\mathbf{gl}(F)$, where $F : \mathbb{B} \to \mathbb{C}$ preserves finite limits and $\mathbb{C}$ has finite limits.

Now, if we consider categories $\mathbb{B}$ and $\mathbb{C}$ with pullbacks and a pullback preserving functor $F$ then still the fibration $\mathbf{gl}(F)$ is lextensive. On the other hand, if $P$ is a lextensive fibration over a category $\mathbb{B}$ with pullbacks one can construct a pullback preserving functor $F$ to a category with pullbacks $\mathbb{C}$ such that $P$ is equivalent to $\mathbf{gl}(F)$. In the presence of a terminal object 1 in $\mathbb{B}$, $\mathbb{C}$ can be taken as the fibre of $P$ over 1. When no terminal object is present in $\mathbb{B}$, we can still recover the category $\mathbb{C}$ by *universally adding formal inverses* to all cocartesian morphisms in $\mathbb{E}$. This can be seen as a generalization of the method of constructing a field of fractions from a ring (more precisely, an integral domain) by adding formal inverses of all nonzero elements.

**Definition 26.** Let $\mathbb{E}$ be a category and $\Sigma$ a class of morphisms in $\mathbb{E}$. Then a category $\mathbb{E}[\Sigma^{-1}]$ together with a functor $P_\Sigma : \mathbb{E} \to \mathbb{E}[\Sigma^{-1}]$ is called a category of fractions of $\mathbb{E}$ w.r.t. $\Sigma$ if it maps all morphisms in $\Sigma$ to isomorphisms and if it is 2-initial in the full subcategory of $\mathbb{E}/\mathbf{Cat}$ consisting of all functors with this property, i.e.,

- if $F : \mathbb{E} \to \mathbb{C}$ is a functor that maps all morphisms in $\Sigma$ to isomorphisms then there exists a functor $\bar{F} : \mathbb{E} \to \mathbb{C}$ and a natural isomorphism $\alpha : \bar{F}P_\Sigma \to F$
- given functors $\bar{F}_1 : \mathbb{E} \to \mathbb{C}$ and $\bar{F}_2 : \mathbb{E} \to \mathbb{C}$ with natural isomorphisms $\alpha_1 : \bar{F}_1 P_\Sigma \to F$ and $\alpha_2 : \bar{F}_2 P_\Sigma \to F$ there exists a unique natural transformation $\xi : \bar{F}_1 \to \bar{F}_2$ such that $\alpha_2 \circ \xi P_\Sigma = \alpha_1$.

In particular, categories of fractions are determined uniquely up to equivalence.

**Definition 27.** Let $\mathbb{E}$ be a category. Then a class $\Sigma$ of morphisms in $\mathbb{E}$ is said to admit a *calculus of right fractions* if

(i) $\Sigma$ contains all identity morphisms and is closed under composition
(ii) given morphisms $f : X \to Z$ in $\mathbb{E}$ and $\sigma : Y \to Z$ in $\Sigma$, there exist morphisms $\tau : T \to X$ in $\Sigma$ and $g : T \to Y$ in $\mathbb{E}$ such that $f\sigma = \tau g$
(iii) given morphisms $f, g : X \to Y$ in $\mathbb{E}$ and $\sigma : Y \to Z$ in $\Sigma$ with $f\sigma = g\sigma$, there exists a morphism $\tau : T \to X$ such that $f\tau = g\tau$

If a class $\Sigma$ of morphisms in $\mathbb{E}$ admits a calculus of right fractions, there exists an easy description of the category of fractions $\mathbb{E}[\Sigma^{-1}]$.

**Definition 28.** Let $\mathbb{E}$ be a category and let $\Sigma$ be class of morphisms admitting a calculus of right fractions. Then a category of fractions of $\mathbb{E}$ w.r.t. $\Sigma$ can be defined as follows.

- The objects of $\mathbb{E}[\Sigma^{-1}]$ are those of $\mathbb{E}$
- A morphism from $X$ to $Y$ in $\mathbb{E}[\Sigma^{-1}]$ is an equivalence class of pairs $(\sigma, f)$ of morphisms in $\mathbb{E}$ where $\sigma : T \to X$ is in $\Sigma$ and $f : T \to Y$ is an arbitrary morphism in $\mathbb{E}$.

  Two morphisms $(\sigma, f)$ and $(\sigma', f')$ are equivalent if there exist morphisms $\alpha$ and $\alpha'$ in $\mathbb{E}$

$$\begin{tikzcd}
 & T \arrow[dl, "\sigma"'] \arrow[dr, "f"] & \\
X & Z \arrow[u, "\alpha"'] \arrow[d, "\alpha'"] & Y \\
 & T' \arrow[ul, "\sigma'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

  such that the above diagram commutes and such that $\sigma\alpha = \sigma'\alpha'$ belongs to $\Sigma$.
- Given morphisms $(\sigma, f) : X \to Y$ and $(\sigma', f') : Y \to Z$, there exist morphisms $\tau$ in $\Sigma$ and $g$ in $\mathbb{E}$ such that $f\tau = \sigma' g$. The composed morphism is defined to be the equivalence class of $(\sigma\tau, f'g)$.
- The functor $P_\Sigma$ is the identity on objects and sends $f : X \to Y$ to $(\mathrm{id}_X, f)$.
- If $F : \mathbb{E} \to \mathbb{C}$ is a functor that maps all members of $\Sigma$ to isomorphisms, then $\bar{F} : \mathbb{E}[\Sigma^{-1}] \to \mathbb{C}$ is defined as $\bar{F}X = FX$ on objects and $\bar{F}(\sigma, f) = Ff \circ (F\sigma)^{-1}$ on morphisms.

**Definition 29.** Let $\mathbb{E}$ be a category with pullbacks. Then a class $\Sigma$ of morphisms in $\mathbb{E}$ is called a *pullback congruence* if

- every isomorphism of $\mathbb{E}$ belongs to $\Sigma$
- whenever $f$ and $g$ are composable morphisms in $\mathbb{E}$ and whenever two of the morphisms $f$, $g$, $fg$ belong to $\Sigma$, then so does the third
- $\Sigma$ is stable under pullbacks along arbitrary morphisms of $\mathbb{E}$.

Let $P : \mathbb{E} \to \mathbb{B}$ be a lextensive fibration over a category $\mathbb{B}$ with pullbacks. As we have seen in Remark 5 and Theorem 3, the class of cocartesian morphisms in $\mathbb{E}$ is a pullback congruence.

**Lemma 11.** *If $\mathbb{E}$ is a category with pullbacks and $\Sigma$ is a pullback congruence, then $\Sigma$ admits a calculus of right fractions.*

*Moreover, the category $\mathbb{E}[\Sigma^{-1}]$ has pullbacks and $P_\Sigma$ preserves all finite limits existing in $\mathbb{E}$. The class of morphisms $f$ in $\mathbb{E}$ such that $P_\Sigma f$ is an isomorphism is exactly $\Sigma$.*

*Proof.* see [Bén89]. □

The properties of $P_\Sigma$ and $\mathbb{E}[\Sigma^{-1}]$ stated above are in general not satisfied for categories of fractions w.r.t. arbitrary classes of morphisms, admitting a calculus of right fractions. But pullback congruences are particularly well behaved.

Recall the definition of the functor $S : \mathbb{E} \to \mathbb{E}_1$ in the proof of Theorem 5. We claim, that $\mathbb{E}_1$ together with $S$ is a category of fractions of $\mathbb{E}$ w.r.t. $\Sigma = \mathrm{Cocart}(P)$. In order to prove our claim, we give an equivalence of $P_\Sigma$ and $S$ in the 2-category $\mathbb{E}/\mathbf{Cat}$. As $S$ maps all cocartesian morphisms to isomorphisms, there is a canonical 1-cell $(\bar{S}, \mathrm{id}_S) : P_\Sigma \to S$. As a 1-cell in the opposite direction, we choose $(P_\Sigma|_{\mathbb{E}_1}, \alpha) : S \to P_\Sigma$, where $P_\Sigma|_{\mathbb{E}_1} : \mathbb{E}_1 \to \mathbb{E}[\Sigma^{-1}]$ is simply the restriction of $P_\Sigma$ to $\mathbb{E}_1$ and where $\alpha$ is the natural isomorphism $\alpha : P_\Sigma|_{\mathbb{E}_1} S \to P_\Sigma$ with $\alpha_X = (\chi_X, \mathrm{id}_X)$, the latter being invertible by the definition of morphisms in $\mathbb{E}[\Sigma^{-1}]$. Composition of the two 1-cells in any order can be immediately seen to give 1-cells isomorphic to the respective identity 1-cells.

We are now going to state an analogue of Theorem 5 for fibrations over categories with pullbacks. The details of the proof are omitted. For a detailed proof see [Str97].

**Theorem 6.** *Let $\mathbb{B}$ be a category with pullbacks and $P : \mathbb{E} \to \mathbb{B}$ be a fibration. Then the following are equivalent:*

(i) *The fibration $P$ is lextensive.*
(ii) *There exists a category $\mathbb{C}$ with pullbacks and a pullback preserving functor $F : \mathbb{B} \to \mathbb{C}$ such that $P$ is equivalent to $\mathbf{gl}(F)$.*

*Furthermore, the class of all functors $F : \mathbb{B} \to \mathbb{C}$ such that $P$ is equivalent to $\mathbf{gl}(F)$ has a 2-initial object.*

*outline of the proof.* Clearly, (ii) implies (i). For the other direction, let $\Sigma$ be the class of cocartesian morphisms in $\mathbb{E}$ and let $\mathbf{1} : \mathbb{B} \to \mathbb{E}$ be a terminal object functor. Then define $\mathbb{C}$ to be $\mathbb{E}[\Sigma^{-1}]$ and $F$ to be $P_\Sigma \mathbf{1}$. By 11, $\mathbb{E}[\Sigma^{-1}]$ has pullbacks and $P_\Sigma$ preserves them. Being a right adjoint to $P$, $\mathbf{1}$ preserves pullbacks, too. It can be shown that $P$ is equivalent to $\mathbf{gl}(F)$. □

Mind, that $F$ is not uniquely determined up to equivalence any more. For a counterexample, consider a category $\mathbb{B}$ with all finite limits and a functor $F : \mathbb{B} \to \mathbb{C}$, which preserves only pullbacks. Then the functors $F$ and $F/1 : \mathbb{B} \to \mathbb{C}/F\,1$ induce the same lextensive fibration but are not equivalent. In this case $F/1$ is 2-initial amongst all functors whose glueing is equivalent to $\mathbf{gl}(F)$.

## 7. Local Smallness

In ordinary category theory a category is called *locally small* if for all objects $X$ and $Y$ the class of morphisms from $X$ to $Y$ is a set. We will investigate in which way an object in the base category of a fibration can be said to represent the class of morphisms between two objects in the total category. As objects $X, Y$ in the total category over some $I$ in the base category should be thought of as $I$-indexed families, an object $\mathrm{Hom}_I(X,Y)$ in the base category should be equipped with a morphism $\delta$ to $I$, giving the index that the (hypothetical) element of $\mathrm{Hom}_I(X,Y)$ is defined at. If $\mathbb{C}$ is a locally small category then for objects $(X_i)_{i\in I}, (Y_i)_{i\in I}$ over $I$ in the family fibration $\mathbf{Sets}(\mathbb{C})$, a set $\mathrm{Hom}_I(X,Y)$ satisfying the requirements of the definition below will be the disjoint union of Hom sets $\{(i,f) | i \in I, f : X_i \to Y_i\}$ together with the obvious projection into $I$. Furthermore, there is a canonical vertical morphism over $\mathrm{Hom}_I(X,Y)$ from $(X_i)_{(i,f)\in \mathrm{Hom}_I(X,Y)}$ to $(Y_i)_{(i,f)\in \mathrm{Hom}_I(X,Y)}$, namely, the family of morphisms $(f)_{(i,f)\in \mathrm{Hom}_I(X,Y)}$. These objects and morphisms satisfy a universal property which can be formulated for arbitrary fibrations.

A fibration $P : \mathbb{E} \to \mathbb{B}$ will be called *locally small* if it satisfies the following requirements: For every object $I$ in $\mathbb{B}$ and every pair $X$, $Y$ of objects in $\mathbb{E}_I$ there are

- an object $\mathrm{Hom}_I(X,Y)$ in $\mathbb{B}$
- a morphism $\delta : \mathrm{Hom}_I(X,Y) \to I$ in $\mathbb{B}$
- cartesian morphisms $\bar{\delta}(X) : \delta^*X \to X$ and $\bar{\delta}(Y) : \delta^*Y \to Y$ over $\delta$ and
- a vertical morphism $\gamma : \delta^*X \to \delta^*Y$

such that, given

- an object $J$ in $\mathbb{B}$
- a morphism $u : J \to I$ in $\mathbb{B}$
- cartesian morphisms $\bar{u}(X) : u^*X \to X$ and $\bar{u}(Y) : u^*Y \to Y$ over $u$ and
- a vertical morphism $\gamma' : u^*X \to u^*Y$

there exists a unique morphism $v : J \to \mathrm{Hom}_I(X,Y)$ such that $\delta \circ v = u$ and $\gamma \circ \varphi_0 = \varphi_1 \circ \gamma'$, where $\varphi_0$ is the unique morphism $\varphi_0 : u^*X \to \delta^*X$ over $v$ such that $\bar{\delta}(X) \circ \varphi_0 = \bar{u}(X)$ and $\varphi_1$ is the unique morphism $\varphi_1 : u^*Y \to \delta^*Y$ over $v$ such that $\bar{\delta}(Y) \circ \varphi_1 = \bar{u}(Y)$.

$$\begin{tikzcd}
u^*X \arrow[r, dashed, "\varphi_0"'] \arrow[rr, bend left, "\bar{u}(X)"] \arrow[d, "\gamma'"'] & \delta^*X \arrow[r, "\bar{\delta}(X)"'] \arrow[d, "\gamma"] & X \\
u^*Y \arrow[r, dashed, "\varphi_1"] \arrow[rr, bend right, "\bar{u}(Y)"'] & \delta^*Y \arrow[r, "\bar{\delta}(Y)"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
J \arrow[r, dashed, "v"'] \arrow[rr, bend left, "u"] & \mathrm{Hom}_I(X,Y) \arrow[r, "\delta"'] & I
\end{tikzcd}$$

The form in which we have chosen to present local smallness is best suited for an understanding of the concept, but unfortunately it involves a complex arrangement of objects and morphisms. So, before giving the official definition we will make some simplifications. Without losing information, we can compose $\gamma$ with $\bar{\delta}(Y)$ and treat it as a single entity, as $\gamma$ and $\bar{\delta}(Y)$ can be recovered up to isomorphism by the usual factorization of a morphism as a vertical morphism followed by a cartesian one. The same can be done with $\gamma'$ and $\bar{u}(Y)$.

Furthermore, we will not name the morphisms in the base as they are determined by the morphisms in the entire category.

**Definition 30.** A fibration $P : \mathbb{E} \to \mathbb{B}$ is called *locally small* if for each pair $X, Y$ of objects in the same fibre there is a pair $(\varphi : W \to X, f : W \to Y)$ of morphisms where $P\varphi = Pf$ and $\varphi$ is cartesian satisfying the following universal property: For each pair $(\varphi' : W' \to X, f' : W' \to Y)$, where $P\varphi' = Pf'$ and $\varphi'$ is cartesian, there is a unique morphism $h : W' \to W$ such that $\varphi \circ h = \varphi'$ and $f \circ h = f'$.

That is, we require that a category of certain spans has a terminal object. If $(\varphi, f)$ is a terminal span and $W$ is the domain of $\varphi$ and $f$, we can take $PW$ for $\mathrm{Hom}_I(X, Y)$ and $P\varphi = Pf$ for $\delta$.

In the previous definition, we have considered $X$ and $Y$ in the same fibre. Instead, one might also want to represent the class of morphisms between two families over different index sets $I, J$ by an object $\mathrm{Hom}(X, Y)$ together with morphisms $\delta_0 : \mathrm{Hom}(X, Y) \to I$ and $\delta_1 : \mathrm{Hom}(X, Y) \to J$ giving the index of the domain and the codomain of a (hypothetical) element of $\mathrm{Hom}(X, Y)$. This gives rise to another possible formulation of local smallness, which is equivalent under the assumption that the base category has binary products and equalizers.

**Proposition 14.** *Let $\mathbb{B}$ be a category with binary products and equalizers. Then a fibration $P : \mathbb{E} \to \mathbb{B}$ is locally small if and only if for every arbitrary pair $X, Y$ of objects of $\mathbb{E}$ there is a pair $(\varphi : W \to X, f : W \to Y)$ of morphisms, where $\varphi$ is cartesian and $f$ is arbitrary, satisfying the following universal property: For each pair $(\varphi' : W' \to X, f' : W' \to Y)$, where $\varphi'$ is cartesian and $f$ is arbitrary, there is a unique morphism $h : W' \to W$ such that $\varphi \circ h = \varphi'$ and $f \circ h = f'$.*

*Proof.* Let $P$ be locally small. For some $X$ over $I$ and some $Y$ over $J$, construct the product $I \times J$ in the base with projections $\pi_0 : I \times J \to I$ and $\pi_1 : I \times J \to J$. Choose cartesian morphisms $\bar{\pi_0}(X) : \pi_0^* X \to X$ over $\pi_0$ and $\bar{\pi_1}(Y) : \pi_1^* Y \to Y$ over $\pi_1$. Consider a pair $(\varphi, f)$, terminal amongst all spans with $\varphi : W \to \pi_0^* X$ cartesian, $f : W \to \pi_1^* Y$ arbitrary and $P\varphi = Pf$. Then the pair $(\bar{\pi_0}(X) \circ \varphi, \bar{\pi_1}(Y) \circ f)$ satisfies the universal property stated in the proposition.

Conversely, let $X, Y$ be objects in the same fibre. Then by hypothesis there exists a pair $(\varphi, f)$, terminal amongst all spans with $\varphi : W \to X$ cartesian, $f : W \to Y$ and arbitrary. Let $e$ be an equalizer of $P\varphi$ and $Pf$ and let $\bar{e}(W)$ be a cartesian lifting of $e$. Then $(\varphi \circ \bar{e}(W), f \circ \bar{e}(W))$ is a terminal span in the sense of the definition of local smallness. □

**Proposition 15.** *Let $\mathbb{C}$ be a category with pullbacks. The codomain fibration $\partial_1 : \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$ is locally small if and only if $\mathbb{C}$ is locally cartesian closed.*

*Proof.* Let $x : X \to I$ and $y : Y \to I$ be objects over $I$. Then a span $(\varphi, f)$ can be defined as follows. Build the local exponent $y^x$ and define $\varphi$ as the pullback square of $x$ along $y^x$. Define $f$ as the local evaluation map from $y^x \times_I x$ to $y$. This span is in fact terminal and conversely, every terminal span gives rise to a local exponent. □

## 8. Lawvere Comprehension

The comprehension scheme in set theory allows one to construct from a set $X$ and a predicate $P$ on $X$ the set $\{x \in X | P(x)\}$ of all elements of $X$ satisfying $P$.

Recall the fibration $\partial_1 : \mathbf{Sub}(\mathbb{C}) \to \mathbb{C}$ from Example 3. An object in $\mathbf{Sub}(\mathbb{C})$ consists of an object $X$ in $\mathbb{C}$ together with a specified subobject $\iota : S \to X$, and the functor $\partial_1$ maps some object $(X, \iota : S \to X)$ of $\mathbf{Sub}(\mathbb{C})$ to the underlying object $X$. Clearly, the fibration has fibred terminal objects $\mathbf{1}_X = (X, \mathrm{id}_X)$. There is also a canonical functor $\partial_0 : \mathbf{Sub}(\mathbb{C}) \to \mathbb{C}$, mapping some $(X, \iota : S \to X)$ to $S$ which can be understood as comprehension. Furthermore, there is a canonical morphism $\iota : \mathbf{1}_S \to (X, \iota : S \to X)$ over $\iota$.

For another example which is essentially the same consider a topos $\mathcal{E}$ with truth value object $\Omega$. Then we can define the following fibration over $\mathcal{E}$: An object of the total category $\mathbf{Pred}(\mathcal{E})$ consists of an object $X$ of $\mathcal{E}$ together with a morphism $\chi : X \to \Omega$ (i.e. a predicate on $X$). A morphism from $(X, \chi)$ to $(X', \chi')$ is a morphism $f : X \to X'$ in $\mathcal{E}$ such that $\Rightarrow \circ \langle \chi, \chi' f \rangle = \mathrm{true}_X$. The functor $P : \mathbf{Pred}(\mathcal{E}) \to \mathcal{E}$ maps $f : (X, \chi) \to (X', \chi')$ to $f : X \to X'$. There is also a canonical functor, mapping an object $(X, \chi)$ to the (domain of) the subobject of $X$ which is classified by $\chi$. Again, there are fibred terminal objects $\mathbf{1}_X = (X, \mathrm{true}_X)$ and if $\chi$ classifies the subobject $\iota : S \to X$ then there is a canonical morphism $\iota : \mathbf{1}_S \to (X, \chi)$.

One may ask whether the behaviour of these functors can be described more abstractly in fibrational terms.

**Definition 31.** Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration with fibred terminal objects. We say that $P$ has *Lawvere comprehension* if for every object $X$ of $\mathbb{E}$ over some $I$ there is a morphism $\epsilon_X : \mathbf{1}_{\mathcal{C}X} \to X$ which is universal in the following sense: for every morphism $g : \mathbf{1}_J \to X$ over some $u : J \to I$ there exists a unique $v : J \to \mathcal{C}X$ such that $\epsilon_X \circ \mathbf{1}_v = g$ (where $\mathbf{1}_v$ is the unique morphism over $v$ from $\mathbf{1}_J$ to $\mathbf{1}_{\mathcal{C}X}$).

$$
\begin{tikzcd}
\mathbf{1}_J \arrow[dr, dashed, "\mathbf{1}_v"'] \arrow[drr, bend left, "g"] & & \\
& \mathbf{1}_{\mathcal{C}X} \arrow[r, "\epsilon_X"'] & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
J \arrow[dr, dashed, "v"'] \arrow[drr, bend left, "u"] & & \\
& \mathcal{C}X \arrow[r, "P\epsilon_X"'] & I
\end{tikzcd}
$$

The two fibrations mentioned above have Lawvere comprehension in the sense of Definition 31. Taking them as a guiding example for Lawvere comprehension, the object $\mathcal{C}X$ can be thought of as an analogue of the set $\{i \in I | X(i)\}$. For a different example consider the following.

**Example 9.** Let $\mathbb{B}$ be a category with pullbacks. Then the codomain fibration $\partial_1 : \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ has Lawvere comprehension. For an object $x : X \to I$ we may take for $\epsilon_x$ the square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "x"] \\ X \arrow[r, "x"'] & I \end{tikzcd}$$

**Theorem 7.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration with fibred terminal objects given by a right adjoint right inverse $\mathbf{1} : \mathbb{B} \to \mathbb{E}$ of $P$. Then the following are equivalent.*

(i) *$P$ has Lawvere comprehension.*
(ii) *$\mathbf{1} : \mathbb{B} \to \mathbb{E}$ has a right adjoint.*

*Moreover, if $P$ is lextensive then $P$ has Lawvere Comprehension if and only if*

(iii) *The fibred functor $\Delta_P$ from the codomain fibration over $\mathbb{B}$ to $P$ has a fibred right adjoint $\Gamma_P$.*

*Proof.* (i) and (ii) are obviously equivalent.

Assume now that $P$ is lextensive and consider the functor $\Delta_P$. Let $\mathbf{1}^{(\mathbb{B})} : \mathbb{B} \to \mathbb{B}^{\mathbf{2}}$ and $\mathbf{1}^{(P)} : \mathbb{B} \to \mathbb{E}$ be terminal object functors. As $\Delta_P$ preserves terminal objects, the composed functor $\Delta_P \circ \mathbf{1}^{(\mathbb{B})}$ is isomorphic to $\mathbf{1}^{(P)}$. We know, that $\mathbf{1}^{(\mathbb{B})}$ has a right adjoint $\partial_0$, so, as adjunctions compose, if $\Delta_P$ has a fibred right adjoint then $\mathbf{1}^{(P)}$ has a right adjoint.

On the other hand, assume that $\mathbf{1}^{(P)}$ has a right adjoint $\mathcal{C}$ with counit $\epsilon$. Then $\Delta_P$ has a fibred right adjoint $\Gamma_P$, mapping a morphism $f : X \to Y$ over $u : I \to J$ to the morphism

$$\begin{tikzcd} \mathcal{C}X \arrow[r, "\mathcal{C}f"] \arrow[d, "P\epsilon_X"'] & \mathcal{C}Y \arrow[d, "P\epsilon_Y"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

in $\mathbb{B}^{\mathbf{2}}$.

In order to prove adjointness, we define a vertical natural transformation $\tilde{\epsilon} : \Delta_P \Gamma_P \to P$, which will turn out to be the counit of the adjunction $\Delta_P \dashv \Gamma_P$. Recall from the definition of $\Delta_P$ that for an object $X$ over $I$, $\Delta_P \Gamma_P X = \Delta_P P\epsilon_X$ is the codomain of a chosen cocartesian morphism, say $\varphi$, over $P\epsilon_X$ with domain $\mathbf{1}^{(P)}_{\mathcal{C}X}$. Define $\tilde{\epsilon_X}$ to be the unique vertical morphism from $\Delta_P \Gamma_P X$ to $X$ such that $\tilde{\epsilon}_X \circ \varphi = \epsilon_X$. In order to check universality of $\tilde{\epsilon}_X$ consider an object $v : K \to L$ in $\mathbb{B}^{\mathbf{2}}$, a cocartesian morphism $\psi : \mathbf{1}^{(P)}_K \to \Delta_P\, v$ over $v$ and an arbitrary morphism $f : \Delta_P\, v \to X$ in $\mathbb{E}$. Then by the universal property of comprehension, there is a unique morphism $w : K \to \mathcal{C}X$ such that $\epsilon_X \circ \mathbf{1}^{(P)}_w = f \circ \psi$. By definition, $\Delta_P(Pf, w)$ is the unique morphism over $Pf$ from $\Delta_P\, v$ to $\Delta_P P\epsilon_X$ such that $(\Delta_P(Pf, w)) \circ \psi = \varphi \circ \mathbf{1}^{(P)}_w$. Finally, $\tilde{\epsilon}_X \circ \Delta_P(Pf, w) = f$ because both morphisms are over $Pf$ and $\tilde{\epsilon}_X \circ \Delta_P(Pf, w) \circ \psi = f \circ \psi$

and $\psi$ is cocartesian.

$$
\begin{tikzcd}
\mathbf{1}_K^{(P)} \arrow[r, "\psi"] \arrow[ddr, dashed, "\mathbf{1}_w^{(P)}"'] & \Delta_P\, v \arrow[dr, "f"] \arrow[ddr, dashed, "\Delta_P(Pf,w)"'] & \\
 & & X \\
 & \mathbf{1}_{\mathcal{C}X}^{(P)} \arrow[ur, "\epsilon_X"'] \arrow[r, "\varphi"'] & \Delta_P \Gamma_P X \arrow[u, "\tilde{\epsilon}_X"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
K \arrow[r, "v"] \arrow[ddr, dashed, "w"'] & L \arrow[ddr, "Pf"] & \\
 & & \\
 & \mathcal{C}X \arrow[r, "P\epsilon_X"'] & I
\end{tikzcd}
$$

From Lemma 5 we know, that any ordinary right adjoint of a fibred functor with vertical counit gives rise to a fibred right adjoint. $\square$

**Remark 6.** There is yet another characterization of Lawvere comprehension. Consider some object $X$ over $I$. Then the requirement imposed on the morphism $\epsilon_X : 1_{\mathcal{C}X} \to X$ in the definition of Lawvere comprehension is exactly the requirement imposed on a morphism $f$ of a terminal span $(\varphi, f)$ from $1_I$ to $X$ in the definition of local smallness, leaving the cartesian morphism $\varphi$ implicit by requiring the domain of $f$ to be a fibred terminal object. So, Lawvere comprehension is just a special case of local smallness and instead of saying that some fibration has Lawvere comprehension we could say it has *small global sections.* For an ordinary category, to have small global sections means, that for every object $P$ the class of morphisms from the terminal object to $P$ is a set.

**Lemma 12.** *Let $\mathbb{C}$ and $\mathbb{D}$ be categories with pullbacks and let $F : \mathbb{C} \to \mathbb{D}$ be a pullback preserving functor. Then for every object $I$ in $\mathbb{C}$ the functor $F_I := F/I : \mathbb{C}/I \to \mathbb{D}/FI$ preserves finite limits.*

*If $F$ has a right adjoint $U$ with unit $\eta$ and counit $\epsilon$ then for each object $I$ in $\mathbb{C}$ the functor $F_I$ has a right adjoint $U_I := \eta_I^* \circ U/FI$.*

*Moreover, if $\epsilon$ is a natural isomorphism then for each object $I$ in $\mathbb{C}$ the counit $\epsilon^{(I)}$ of the adjunction $F_I \dashv U_I$ is a natural isomorphism, too.*

*Proof.* For some $x : X \to FI$ in $\mathbb{D}/FI$, let $U_Ix$ be the arrow $\pi : P \to I$ in the left diagram below. Then the counit $\epsilon_x^{(I)} : F_IU_Ix \to x$ of the adjunction $F_I \dashv G_I$ is given by the outer rectangle of the right diagram below. Mind that the lower edge equals $\mathrm{id}_{FI}$ because of the triangular equalities for $F \dashv G$.

$$\begin{tikzcd} P \arrow[r, "\pi'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & UX \arrow[d, "Ux"] \\ I \arrow[r, "\eta_I"'] & UFI \end{tikzcd} \qquad \begin{tikzcd} FP \arrow[r, "F\pi'"] \arrow[d, "F\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & FUX \arrow[r, "\epsilon_X"] \arrow[d, "FUx"] & X \arrow[d, "x"] \\ FI \arrow[r, "F\eta_I"'] & FUFI \arrow[r, "\epsilon_{FI}"'] & FI \end{tikzcd}$$

The universal property of $\epsilon_x^{(I)}$ can be verified by simple diagram chasing.

Now assume that $\epsilon$ is a natural isomorphism. Then $\epsilon_{FI}$ is an iso. Therefore, $F\eta_I$ is an iso, too, and as isos are stable under pullbacks, $F\pi'$ is an iso. So the upper edge of the triangle $\epsilon_x^{(I)}$ is the composition of two isos hence $\epsilon_x^{(I)}$ is an isomorphism. Thus, $\epsilon^{(I)}$ is a natural isomorphism. □

**Remark 7.** If $\mathbb{C}$ has a terminal object 1, which is preserved by $F$, then the implications of Lemma 12 turn into equivalences as in this case $F_1$ is isomorphic to $F$.

We now characterize when for a pullback preserving functor $F$ its glueing $\mathsf{gl}(F)$ has Lawvere comprehension.

**Theorem 8.** *Let $\mathbb{B}$, $\mathbb{C}$ be categories with pullbacks and let $F : \mathbb{B} \to \mathbb{C}$ be a pullback preserving functor. Then the following are equivalent.*

(i) *The fibration $\mathsf{gl}(F)$ has Lawvere comprehension.*
(ii) *For each $I$ in $\mathbb{B}$ the functor $F_I$ admits a right adjoint $U_I : \mathbb{C}/FI \to \mathbb{B}/I$.*

*Proof.* (ii) $\Longrightarrow$ (i): Let $x : X \to FI$ be an object in $\mathsf{Gl}(F)$. Its comprehension can be taken as the morphism

$$\begin{tikzcd} F\mathcal{C}x \arrow[r, "\epsilon_x^{(I)}"] \arrow[d, "\mathrm{id}_{F\mathcal{C}x}"'] & X \arrow[d, "x"] \\ F\mathcal{C}x \arrow[r, "F_I U_I x"'] & FI \end{tikzcd}$$

$$\mathcal{C}x \xrightarrow[U_I x]{} I$$

in $\mathsf{Gl}(F)$, where $\mathcal{C}x$ is the domain of the morphism $U_I x$ and $\epsilon^{(I)}$ is the counit of the adjunction $F_I \dashv U_I$. The adjointness property of $F_I$ and $U_I$ *is literally the same as* the the universal property of comprehension for the pair pair $(\epsilon_x^{(I)}, U_I x)$.

(i) $\Longrightarrow$ (ii): By Theorem 7 the fibration $\mathsf{gl}(F)$ has Lawvere comprehension if and only if the functor $\Delta_{\mathsf{gl}(F)}$ has a fibred right adjoint $\Gamma_{\mathsf{gl}(F)}$. The restriction of $\Delta_{\mathsf{gl}(F)}$ to the fibre over $I$ is exactly the functor $F_I$, which has the restriction of $\Gamma_{\mathsf{gl}(F)}$ to the fibre over $I$ as a right adjoint. □

Theorem 8 can also be read as follows. Given a pullback preserving functor $F : \mathbb{B} \to \mathbb{C}$ the fibred functor $\Delta_{\mathsf{gl}(F)}$ admits a fibred right adjoint if and only if for each $I \in \mathbb{B}$ the restriction of $\Delta_{\mathsf{gl}(F)}$ to the fibre over $I$ admits an ordinary right adjoint.

**Definition 32.** Let $\mathbb{B}$ and $\mathbb{C}$ be categories with finite limits. A *geometric morphism* from $\mathbb{C}$ to $\mathbb{B}$ is a pair of functors $U : \mathbb{C} \to \mathbb{B}$ and $F : \mathbb{B} \to \mathbb{C}$ such that $F \dashv U$ and $F$ preserves finite limits.

$U$ is called the *direct image part* and $F$ the *inverse image part* of the geometric morphism. Note, that the direction of a geometric morphism that of its direct image part.

By Remark 7 the following corollary is immediate.

**Corollary 4.** *Let $\mathbb{B}$, $\mathbb{C}$ be categories with finite limits and let $F : \mathbb{B} \to \mathbb{C}$ be a finite limit preserving functor. Then the following are equivalent.*

(i) *The fibration $\mathbf{gl}(F)$ has Lawvere comprehension.*
(ii) *The functor $F$ is the inverse image part of a geometric morphism.*

**Proposition 16.** *Let $F : \mathbb{B} \to \mathbb{C}$ be a functor with the property that each of the functors $F_I$ admits a right adjoint $U_I : \mathbb{C}/FI \to \mathbb{B}/I$. Then $F^*$ preserves local smallness of fibrations.*

*Proof.* Assume that $P$ is a locally small fibration over $\mathbb{C}$ and let $X, Y$ be objects over some $I$ in $\mathbb{B}$ w.r.t. $F^*P$, that is, $X, Y$ are objects over $FI$ w.r.t. $P$. As $P$ is locally small, we can find a terminal span $(\varphi, f)$ over $\delta : \mathrm{Hom}_{FI}(X, Y) \to FI$. Define $\delta' = U_I\delta$ and let $\psi$ be a cartesian morphism over the top morphism in the triangle $\epsilon^{(I)}_\delta : F_I U_I \delta \to \delta$. Then $(\varphi \circ \psi, f \circ \psi)$ is a terminal span from $X$ to $Y$ in $F^*P$. □

**Corollary 5.** *Let $\mathbb{B}$ be a category with finite limits, $\mathbb{C}$ a locally cartesian closed category and let $F : \mathbb{B} \to \mathbb{C}$ be the inverse image part of a geometric morphism. Then the fibration $\mathbf{gl}(F)$ is locally small.*

*Proof.* By Proposition 15 the codomain fibration over $\mathbb{C}$ is locally small and by the previous proposition, $F^*$ preserves local smallness. □

For every category $\mathbb{B}$ with finite limits we are now going to define a whole 2-category, which has the geometric morphisms into $\mathbb{B}$ as its objects.

**Definition 33.** Let $\mathbb{B}$ be a category with finite limits. Then $\mathbf{GM}/\mathbb{B}$ is defined to be the following subcategory of $\mathbb{B}/\mathbf{Lex}$

- An object is a geometric morphism $F \dashv U : \mathbb{C} \to \mathbb{B}$
- A 1-cell from $F \dashv U : \mathbb{C} \to \mathbb{B}$ to $F' \dashv U' : \mathbb{C}' \to \mathbb{B}$ is a geometric morphism $H \dashv K : \mathbb{C} \to \mathbb{C}'$, together with a natural isomorphism $\alpha : HF' \to F$
- Let $(H \dashv K, \alpha)$ and $(H' \dashv K', \alpha')$ be 1-cells from $F \dashv U$ to $F' \dashv U'$. Then a 2-cell from $(H \dashv K, \alpha)$ to $(H' \dashv K', \alpha')$ is a natural transformation $\xi : H \to H'$ such that $\alpha' \circ \xi F' = \alpha$

In the case that $\mathcal{S}$ is a topos, the category $\mathbf{GM}/\mathcal{S}$ has as a full subcategory the very important category $\mathcal{T}op/\mathcal{S}$ of toposes defined over $\mathcal{S}$, or $\mathcal{S}$-toposes for short. Its objects are those geometric morphisms into $\mathcal{S}$ whose domain is a topos. Mind that the inclusion functor from $\mathbf{GM}/\mathbb{B}$ into $\mathbb{B}/\mathbf{Lex}$ is contravariant (on the level of 1-cells). The reason is, that from our point of view, the inverse image part of a geometric morphism is the more important one. The direct image part has to be seen rather as a property than as an extra structure. Nevertheless, we follow the tradition of letting the direct image part of a geometric fibration determine its direction. The origin of this notation lies in the fact that, if $X$ and $Y$ are topological spaces, then every continuous map $f : X \to Y$ induces a geometric morphism $f^* \dashv f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ between the respective sheaf toposes. But already when generalizing from topological spaces to complete Heyting algebras, the inverse image direction turns out to be the more important one.

Now we turn to the fibrational side and define a corresponding subcategory of $\mathrm{LextFib}(\mathbb{B})$.

**Definition 34.** A fibration is called *geometric* if it is lextensive and has Lawvere comprehension.

**Definition 35.** Let $P$ and $Q$ be fibrations with fibred finite limits. A *fibred geometric morphism* from $P$ to $Q$ is a pair of fibred functors $\Gamma : P \to Q$ and $\Delta : Q \to P$ such that $\Delta \dashv \Gamma$ is a fibred adjunction and $\Delta$ preserves fibred finite limits.

By Proposition 5 the functor $\Delta$ preserves cocartesian morphisms so, in particular, if $P$ and $Q$ are lextensive fibrations then the inverse image part of a fibred geometric morphism from $P$ to $Q$ is a 1-cell (from $Q$ to $P$) in $\mathrm{LextFib}(\mathbb{B})$.

**Definition 36.** Let $\mathbb{B}$ be a category with pullbacks. Then $\mathrm{GeomFib}(\mathbb{B})$ is defined to be the following subcategory of $\mathrm{LextFib}(\mathbb{B})$.

- The objects are geometric fibrations over $\mathbb{B}$
- The 1-cells are fibred geometric morphisms
- Given two fibred geometric morphisms $\Delta \dashv \Gamma, \Delta' \dashv \Gamma' : P \to Q$, a 2-cell from $\Delta \dashv \Gamma$ to $\Delta' \dashv \Gamma'$ is a fibred natural transformation from $\Delta$ to $\Delta'$ (which of course induces a fibred natural transformation from $\Gamma'$ to $\Gamma$.

Again, the inclusion functor from $\mathrm{GeomFib}(\mathbb{B})$ into $\mathrm{LextFib}(\mathbb{B})$ is contravariant on the level of 1-cells.

By Theorem 7 the 2-category $\mathrm{GeomFib}(\mathbb{B})$ has the codomain fibration over $\mathbb{B}$ as a 2-terminal object. In particular, for every geometric fibration $P$ over $\mathbb{B}$ there is a canonical fibred geometric morphism $\Delta_P \dashv \Gamma_P$ from $P$ to the codomain fibration over $\mathbb{B}$. Next, we take a look at special properties of the fibred adjunction $\Delta_P \dashv \Gamma_P$ and how these are related to properties of the fibration $P$.

**Definition 37.** Let $F \dashv U$ be a geometric morphism between categories with finite limits.

(i) The geometric morphism $F \dashv U$ is called *injective* if the following equivalent requirements are satisfied.
  (a) $U$ is full and faithful.
  (b) The counit of the adjunction $F \dashv U$ is a natural isomorphism

(ii) The geometric morphism $F \dashv U$ is called *surjective* if the following equivalent requirements are satisfied.
  (a) $F$ reflects isomorphisms.
  (b) $F$ is faithful.
  (c) The unit of the adjunction $F \dashv U$ is monic.

A fibred geometric morphism $\Delta \dashv \Gamma$ between fibrations with fibred finite limits is called *injective* resp. *surjective* if its restriction to each fibre is *injective* resp. *surjective* as an ordinary geometric morphism.

For a geometric fibration $P$ injectivity and surjectivity of the adjunction $\Delta_P \dashv \Gamma_P$ can be characterized by elementary properties of the fibration $P$.

**Proposition 17.** *Let $P : \mathbb{E} \to \mathbb{B}$ be a geometric fibration. Then*

(i) *The fibred adjunction $\Delta_P \dashv \Gamma_P$ is injective if and only if for every object $X$ in $\mathbb{E}$ the morphism $\epsilon_X : \mathbf{1}_{\mathcal{C}X} \to X$ is cocartesian.*

(ii) *The fibred adjunction $\Delta_P \dashv \Gamma_P$ is surjective if and only if for every $u : I \to J$ cocartesianness of $\mathbf{1}_u$ implies that $u$ is an iso.*

*Proof.* By inspection of the proof of Theorem 7 the counit $\tilde{\epsilon}$ of the adjunction $\Delta_P \dashv \Gamma_P$ is an iso if and only if for every $X \in \mathbb{E}$ the morphism $\epsilon_X : \mathbf{1}_{\mathcal{C}X} \to X$ is cocartesian.

For the second part of the proposition consider a commutative square

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "\varphi"'] & D \end{tikzcd}$$

in the total category of a lextensive fibration, where $\varphi$ and $\psi$ are cocartesian and $k$ is vertical. Then $k$ is an iso if and only if $h$ is cocartesian (for the 'only if' part use Lemma 8). Consider a morphism

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\ I \arrow[r, equal] & I \end{tikzcd}$$

over $I$ in the codomain fibration. Applying our little observation to the defining square of $\Delta_P(f, \mathrm{id}_I)$ we obtain that $\Delta_P(f, \mathrm{id}_I)$ is an iso if and only if $\mathbf{1}_f$ is cocartesian. So each fibre of the functor $\Delta_P$ reflects isos if and only if cocartesianness of $\mathbf{1}_f$ implies that $f$ is an iso for all morphisms $f$ in $\mathbb{B}$. $\square$

**Remark 8.** Requirement (i) can be read as 'every family is the sum of its global elements' whereas requirement (ii) can be read as 'families of 1s cannot be decomposed' in an appropriate fibred sense.

As a fibred adjunction is an equivalence if and only if it is both injective and surjective, a fibration is equivalent to the codomain fibration if and only if it is geometric and satisfies the requirements (i) and (ii) of Proposition 17.

In ordinary category theory, if $\mathcal{S}$ is the topos of sets and $\mathcal{E}$ is an arbitrary topos, then up to natural isomorphism there exists at most one geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$. It exists if and only if $\mathcal{E}$ has small copowers of 1 and small global sections (i.e. is locally small, by the cartesian closedness of $\mathcal{E}$). As $F$ has to preserve coproducts and terminal objects we have $FI = \coprod_I 1$. It is just as easy to see, that necessarily $UX = \mathcal{E}(1, X)$. Now, if $\mathcal{S}$ is an arbitrary topos and $\mathcal{E}$ is an $\mathcal{S}$-topos via the geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$, then there exists up to natural isomorphism exactly one geometric morphism *over* $\mathcal{S}$ from $\mathcal{S}$ (viewed as an $\mathcal{S}$-topos) to $\mathcal{E}$. It is notable that in the fibred adjunction $\Delta_{\mathbf{gl}(F)} \dashv \Gamma_{\mathbf{gl}F}$ from $\mathbf{gl}(F)$ to the codomain fibration over $\mathcal{S}$, the inverse image part $\Delta_{\mathbf{gl}(F)}$ is also of the form 'copowers of 1' (see the proof of Proposition 13) and the direct image part $\Gamma_{\mathbf{gl}(F)}$ is also of the form 'global sections' (see the proof of Theorem 7), but this time in a fibrational sense.

Next we examine the effect that the functor $\mathbf{gl}$ has on geometric morphisms over $\mathbb{B}$.

**Proposition 18.** *Let $\mathbb{B}$, $\mathbb{C}$, $\mathbb{C}'$ be categories with pullbacks, let $F : \mathbb{B} \to \mathbb{C}$, $F' : \mathbb{B} \to \mathbb{C}'$ and $H : \mathbb{C} \to \mathbb{C}'$ be pullback preserving functors and let $\alpha : HF \to F'$ be a natural isomorphism. Then $(H, \alpha)$ induces a fibred functor $\mathsf{gl}(H, \alpha)$ from $\mathsf{gl}(F)$ to $\mathsf{gl}(F')$.*

*If each of the functors $H_Y$ has a right adjoint $K_Y$, then $\mathsf{gl}(H, \alpha)$ has a fibred right adjoint.*

*Proof.* The functor $\mathsf{gl}(H, \alpha)$ sends an object $x : X \to FI$ in $\mathsf{gl}(F)$ to the object $\alpha_I \circ Hx : HX \to F'I$ in $\mathsf{gl}(F')$. For an object $x : X \to F'I$ in $\mathsf{gl}(F')$, define $U_I x$ as $K_{FI}(\alpha_I^{-1} \circ x)$. Let $\epsilon^{(Y)}$ be the counit of the adjunction $H_Y \dashv K_Y$. Then $\epsilon^{(FI)}_{\alpha_I^{-1} x}$ is a morphism from $\mathsf{gl}(H, \alpha)_I U_I x$ to $x$, in fact, it is a universal morphism from $\mathsf{gl}(H, \alpha)_I$ to $x$, which makes it the counit of the adjunction $\mathsf{gl}(H, \alpha)_I \dashv U_I$. Moreover, by a straightforward diagram chase, one shows that these counits are stable under reindexing. So by Lemma 4 the family of functors $U_I$ extends to a fibred right adjoint of $\mathsf{gl}(H, \alpha)$. $\square$

**Corollary 6.** *Let $\mathbb{B}$, $\mathbb{C}$, $\mathbb{C}'$ be categories with finite limits, let $F \dashv U : \mathbb{C} \to \mathbb{B}$, $F' \dashv U' : \mathbb{C}' \to \mathbb{B}'$ and $H \dashv K : \mathbb{C}' \to \mathbb{C}$ be geometric morphisms and let $\alpha : HF \to F'$ be a natural isomorphism.*

*Then the fibred functor $\mathsf{gl}(H, \alpha)$ has a fibred right adjoint.*

Altogether, by the previous observations we have established the following correspondence: The functors $\mathsf{gl}$ and $D$ restrict to the subcategory $\mathbf{GM}/\mathbb{B}$ of $\mathbb{B}/\mathbf{Lex}$ on the one hand and the subcategory $\mathrm{GeomFib}(\mathbb{B})$ of $\mathrm{LextFib}(\mathbb{B})$ on the other hand.

$$\begin{tikzcd}
\mathbb{B}/\mathbf{Lex} \arrow[r, shift left, "\mathsf{gl}"] & \mathrm{LextFib}(\mathbb{B}) \arrow[l, shift left, "D"] \\
(\mathbf{GM}/\mathbb{B})^{\mathrm{op}} \arrow[u, hook] \arrow[r, dashed, shift left, "\mathsf{gl}"] & \mathrm{GeomFib}(\mathbb{B})^{\mathrm{op}} \arrow[u, hook] \arrow[l, dashed, shift left, "D"]
\end{tikzcd}$$

## 9. Generating Families

**Definition 38.** Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration. An object $G$ over some $I$ is called a *generating family* if for vertical morphisms $\alpha, \beta : X \to Y$ over some $J$ it holds that $\alpha = \beta$ if and only if $\alpha \circ f = \beta \circ f$ for every cartesian morphism $\varphi : W \to G$ together with an arbitrary morphism $f : W \to X$.

$$\begin{tikzcd} G & W \arrow[l, "\varphi"'] \arrow[r, "f"] & X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & Y \\ I & K \arrow[l, "u"] \arrow[r, "v"'] & J & \end{tikzcd}$$

By contraposition, if $\alpha \neq \beta$, then there exists a reindexing of $G$ such that there is a morphism $f$ from the latter to $X$ with $\alpha \circ f \neq \beta \circ f$.

Let $(G_i)_{i \in I}$ be a generating family for some category $\mathbb{C}$. Then $(G_i)_{i \in I}$ is a generating family for the fibration $\mathbf{Sets}(\mathbb{C})$: if $\alpha \neq \beta$ over $J$, then there is an index $j \in J$ such that $\alpha_j \neq \beta_j$. As $(G_i)_{i \in I}$ is a generating family, there exists an index $i \in I$ and a morphism $f : G_i \to X_j$ such that $\alpha_j \circ f \neq \beta_j \circ f$. (Take $K$ to be singleton, $u$ to pick $i \in I$ and $v$ to pick $j \in J$ and let $\varphi$ be the canonical lifting of $U$.)

Conversely, let $\alpha \neq \beta$ be morphisms from $X$ to $Y$ in $\mathbb{C}$, viewed as vertical maps over $J$, where $J$ is chosen to be a singleton. Assume, that $(G_i)_{i \in I}$ is a generating family for the fibration $\mathbf{Sets}(\mathbb{C})$. Then there exist $W, \varphi$ and $f$ such that $\alpha \circ f \neq \beta \circ f$. That means, there exists an index $k \in K$ such that $\alpha \circ f_k \neq \beta \circ f_k$. Then $f_k \circ \varphi_k^{-1} : G_{u(k)} \to X$ is the desired morphism that distinguishes $\alpha$ and $\beta$.

If a category $\mathbb{C}$ is locally small, then a family $(G_i)_{i \in I}$ is generating if and only if for each object $X$ in $\mathbb{C}$, there exists a family $(f_k)_{k \in K}$ of morphisms with codomain $X$ and the domain being some object of $(G_i)_{i \in I}$, which is *collectively epic*, i.e. two morphisms $\alpha, \beta : X \to Y$ are equal if and only if $\alpha \circ f_k = \beta \circ f_k$ for all $k \in K$. We can take $K$ as the set $\{(i, h) \mid i \in I, h : G_i \to X\}$ and $f_{(i,h)}$ as $h$. This observation has a fibrational equivalent.

**Definition 39.** A morphism $f : W \to X$ in the total category of a fibration is called *collectively epic* if for every pair $\alpha, \beta : X \to Y$ of vertical morphisms, $\alpha = \beta$ if and only if $\alpha \circ f = \beta \circ f$.

**Remark 9.** In the above definitions instead of requiring $\alpha, \beta$ to be vertical we could have required them to be over the same morphism $u$ in the base, as $\alpha$ equals $\beta$ if and only if the unique vertical factorizations through some chosen cartesian lifting of $u$ with codomain $Y$ are equal.

**Proposition 19.** *Let $\mathbb{B}$ be a category with finite limits and $P : \mathbb{E} \to \mathbb{B}$ be a locally small fibration. Then an object $G$ in $\mathbb{E}$ is a generating family if and only if for every $X$ in $\mathbb{E}$ there is a cartesian morphism $\varphi : W \to G$ together with a collectively epic morphism $f : W \to X$.*

*Proof.* If $\varphi$ and $f$ exist with the above properties, then clearly $G$ is a generating object.

Conversely, assume that $G$ is a generating object. Let $PG = I$ and let $X$ be an object over some $J$. By local smallness due to Proposition 14 there is a cartesian morphism $\varphi : W \to X$ and an arbitrary morphism $f : W \to Y$, terminal amongst all such spans. By the definition of a generating object, for every pair of vertical morphisms $\alpha \neq \beta$ from $X$ to $Y$ there exists a span, whose left leg is cartesian and whose right leg distinguishes $\alpha$ and $\beta$. As each such span factors through the terminal one, $f$ distinguishes $\alpha$ and $\beta$, that is, $f$ is collectively epic. $\square$

**Proposition 20.** *Let $\mathbb{B}, \mathbb{C}$ be categories with finite limits and $F : \mathbb{B} \to \mathbb{C}$ be a finite limit preserving functor. A morphism $(f, u)$ in $\mathbf{gl}(F)$ is collectively epic if and only if $f$ is epic in $\mathbb{C}$.*

*Proof.* If $f$ is epic in $\mathbb{C}$ then clearly $(u, f)$ is collectively epic. Conversely, assume that $(f, u)$ is collectively epic and let $\alpha, \beta : X \to Y$ be morphisms, where $X$ is the codomain of $f$, such that $\alpha f = \beta f$. We can make $\alpha$ and $\beta$ into the upper edges of morphisms in $\mathbf{Gl}(F)$ over the terminal projection of $J$, where $J$ is the codomain of $u$. Then $\alpha = \beta$ follows by Remark 9 as $(f, u)$ was assumed collectively epic. □

## 10. Well-Poweredness

For a fibration $P : \mathbb{E} \to \mathbb{B}$ a vertical morphism $\alpha$ over $I$ is monic in $\mathbb{E}_I$ if and only if it is monic in $\mathbb{E}$. In this section we are interested in fibrations where vertical monos are stable under reindexing. A sufficient condition is that $P$ has fibred pullbacks.

In ordinary category theory, a category $\mathbb{C}$ is called well-powered if for every object $X$ of $\mathbb{C}$ the subobjects of $X$ form a set. We want to extend this notion to fibrations. In a fibration of the form $\mathbf{Sets}(\mathbb{C})$ a vertical mono into some family $(X_i)_{i \in I}$ is a family of subobjects $Y_i \rightarrowtail X_i$ for each $i \in I$.

Let $\mathbb{C}$ be a well-powered category. Then for an object $(X_i)_{i \in I}$ over $I$ in $\mathbf{Sets}(\mathbb{C})$ we can define a set $\mathrm{Sub}_I(X)$ of pairs $(i, \iota)$ where $i \in I$ and $\iota : M \to X_i$ is a mono, such that for each $i \in I$ and each subobject of $X_i$ there exists a unique $(i, \iota)$ in $\mathrm{Sub}_I(X)$ with $\iota$ representing this subobject. Let $\sigma : \mathrm{Sub}_I(X) \to I$ be the projection map. Then there is a family over $\mathrm{Sub}_I(X)$ whose element at index $(i, \iota)$ is the domain of $\iota$, together with a vertical mono from that family into $\sigma^*(X_i)_{i \in I} = (X_i)_{(i,\iota) \in \mathrm{Sub}_I(X)}$, whose morphism at index $(i, \iota)$ is $\iota$. This vertical mono satisfies the universal property stated in the definition below.

**Definition 40.** Let $P : \mathbb{E} \to \mathbb{B}$ be a fibration in which monos are stable under reindexing. The fibration $P$ is *well-powered* if and only if for every object $X$ over $I$ there is a cartesian morphism $\varphi : Y \to X$ together with a vertical mono $m : Z \to Y$ such that, given a cartesian morphism $\varphi' : Y' \to X$ and a vertical mono $m' : Z' \to Y'$ there exists a unique morphism $h : Y' \to Y$ such that $h \circ m'$ factors through $m$ and $\varphi \circ h = \varphi'$.

$$\begin{tikzcd}
 & Z \arrow[d, tail, "m"] & Z' \arrow[l, dashed, tail] \arrow[d, tail, "m'"] \\
X & Y \arrow[l, "\varphi"'] & Y' \arrow[l, dashed, "h"'] \arrow[ll, bend left, "\varphi'"]
\end{tikzcd}$$

More informally, this means that there is a generic family of subobjects of $X$, i.e. every family of subobjects appears in a unique way as a reindexing of the generic family of subobjects. The object $PY$ can be thought of as representing the set of subobjects of $X$ and will also be denoted as $\mathrm{Sub}_I(X)$.

**Proposition 21.** *Let $\mathcal{S}$ be a category with finite limits. Then the fibration $\partial_1 : \mathcal{S}^{\mathbf{2}} \to \mathcal{S}$ is well-powered if and only if $\mathcal{S}$ is a topos.*

*Proof.* Similar to the proof of Proposition 15, one sees that a generic family of subobjects of $x : X \to I$ corresponds to a powerobject of $x$ in $\mathcal{S}/I$. □

**Proposition 22.** *If $\mathbb{B}, \mathbb{C}$ are categories with finite limits and $F : \mathbb{B} \to \mathbb{C}$ is the inverse image part of a geometric morphism, then $F^*$ preserves well-poweredness of fibrations.*

*Proof.* Assume that $P$ is a well-powered fibration over $\mathbb{C}$ and let $X$ be an object over some $I$ in $\mathbb{B}$ w.r.t. $F^*P$, that is, $X$ is an object over $FI$ w.r.t. $P$. As $P$ is well-powered, we can find a a generic family of subobjects for $X$, consisting of a cartesian morphism $\varphi : Y \to X$ over $\sigma : \mathrm{Sub}_{FI}(X) \to FI$ together with a vertical mono $m : Z \to Y$. We make use of the fact, that each functor $F_I$ admits a right adjoint $U_I$ with counit $\epsilon^{(I)}$. Define $\sigma' := U_I\sigma$, let $\psi : Y' \to Y$ and $\xi : Z' \to Z$ be cartesian over the upper side of the triangle $\epsilon^{(I)}_\delta : F_I U_I \sigma \to \delta$ and let $m' : Z' \to Y'$ be the unique vertical morphism such that $\psi m' = m\xi$. As monos are assumed stable under reindexing $m'$ is monic. Now the cartesian morphism $\varphi\psi$ together with the vertical mono $m'$ forms a generic family of subobjects of $X$ with respect to the fibration $F^*P$. $\square$

## 11. Boundedness

Let $\mathcal{E}$ be a topos over **Sets** via the geometric morphism $F \dashv U : \mathcal{E} \to$ **Sets**. This gives rise to two fibrations, namely the family fibration $\mathbf{Sets}(\mathcal{E})$, where the objects over some set $I$ are $I$-indexed families of objects of $\mathcal{E}$ and the fibration $\mathbf{gl}(F)$ obtained by glueing along $F$, where the objects over some set $I$ are the morphisms $x : X \to FI$. There is a canonical fibred functor from $\mathbf{gl}(F)$ to $\mathbf{Sets}(\mathcal{E})$ sending $x : X \to FI$ to the family $(X_i)_{i \in I}$, where $X_i$ arises from the pullback

$$\begin{tikzcd} X_i \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "x"] \\ F\{i\} \arrow[r, tail, "Fi"'] & FI \end{tikzcd}$$

and similarly for morphisms.

The two fibrations need not be equivalent as $\mathbf{gl}(F)$ is always a geometric fibration but in general $\mathbf{Sets}(\mathcal{E})$ is not. It is well known that a topos has all finite colimits but in general it need not have arbitrary external coproducts (i.e. internal coproducts with respect to the fibration $\mathbf{Sets}(\mathcal{E})$). Still, for certain infinite families we can show that their coproducts do exist. Let $\mathcal{E}$ be a topos over **Sets** via the geometric morphism $F \dashv U$ and let $X$ be an object in $\mathcal{E}$. Then arbitrary copowers of $X$ exist because every set $I$ is an $I$-indexed copower of 1, the functor $F$ preserves terminal objects and both $F$ and the functor $\_ \times X$ have a right adjoint and thus preserve colimits. So $FI \times X$ is an $I$-copower of $X$. From this one can show that in a topos over **Sets** all coproducts of bounded families exist.

**Definition 41.** A family $(X_i)_{i \in I}$ of objects in a category $\mathbb{C}$ is called *bounded* if there is an object $A$ such that each $X_i$ is a subobject of $A$.

**Lemma 13.** *Let $\mathcal{E}$ be a topos over* **Sets** *and $(X_i)_{i \in I}$ be a bounded family of objects in $\mathcal{E}$. Then the coproduct $\coprod_{i \in I} X_i$ exists.*

*Proof.* Let $A$ be an object of $\mathcal{E}$ with an $I$-indexed family of monomorphisms $m_i : X_i \to A$. We will define $\coprod_{i \in I} X_i$ as a subobject of the $I$-indexed copower of $A$. Let $\chi_i$ be the characteristic map of $m_i$. There is a unique morphism $\chi$ from $\coprod_I A$ to $\Omega$ with $\chi \circ \iota_i = \chi_i$, where $\iota_i$ denotes the injection of $A$ into $\coprod_I A$ at index $i$. Define $\coprod_{i \in I} X_i$ to be the subobject of $\coprod_I A$ classified by $\chi$ and define the injections $\kappa_i : X_i \to \coprod_{i \in I} X_i$ to be the pullbacks of $\iota_i$ along the inclusion of $\coprod_{i\in I} X_i$ into $\coprod_I A$. The domain of $\kappa_i$ can be chosen as $X_i$ by the pullback lemma, because $\iota_i \circ \chi = \chi_i$ classifies $m_i$.

$$\begin{tikzcd} X_i \arrow[r, tail, "\kappa_i"] \arrow[d, tail, "m_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_{i\in I} X_i \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\mathsf{true}"] \\ A \arrow[r, tail, "\iota_i"'] & \coprod_I A \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

Now, any family of morphisms $f_i : X_i \to B$ induces a family of morphisms $\tilde{f}_i : A \to \tilde{B}$ into the partial map classifier of $B$. The morphism $[\tilde{f}_i]_{i\in I} : \coprod_I A \to \tilde{B}$ restricts to the desired morphism from $\coprod_{i\in I} X_i$ to $B$ by pulling back along $\eta_B : B \to \tilde{B}$. □

**Remark 10.** In a topos existing coproducts are universal and disjoint. From this it follows that given a coproduct cone $(\iota_i : A_i \to A)_{i\in I}$, an arbitrary cone $(\kappa_i : X_i \to X)_{i\in I}$, and morphisms $(f_i : X_i \to A_i)_{i\in I}$ and $f : X \to A$ with $\iota_i f_i = f\kappa_i$ the family $(\kappa_i : X_i \to X)_{i\in I}$ is a coproduct cone if and only if all squares

$$\begin{tikzcd} X_i \arrow[r, "\kappa_i"] \arrow[d, "f_i"'] & X \arrow[d, "f"] \\ A_i \arrow[r, "\iota_i"'] & A \end{tikzcd}$$

are pullbacks.

The image of a family $x : X \to FI$ under the fibred functor from $\mathbf{gl}(F)$ to $\mathbf{Sets}(\mathcal{E})$ is always a bounded family, $X$ being a bound for $(X_i)_{i\in I}$. Restricting our view to the full subfibration $\mathbf{Sets}(\mathcal{E})_b$ of $\mathbf{Sets}(\mathcal{E})$ whose objects are bounded families we obtain the following result.

**Proposition 23.** *The fibred functor from* $\mathbf{gl}(F)$ *to* $\mathbf{Sets}(\mathcal{E})_b$ *is an equivalence.*

*Proof.* For every set $I$ the restriction of the fibred functor to the fibre over $I$ is an equivalence. There is a functor in the opposite direction sending a bounded family $(X_i)_{i\in I}$ to the morphism $\coprod_{i\in I} !_{X_i} : \coprod_{i\in I} X_i \to \coprod_I 1 \cong FI$, i.e. the unique morphism making all squares

$$\begin{tikzcd} X_i \arrow[r, "\kappa_i"] \arrow[d, "!_{X_i}"'] & \coprod_{i\in I} X_i \arrow[d, "\coprod_{i\in I} !_{X_i}"] \\ F\{i\} \arrow[r, "Fi"'] & FI \end{tikzcd}$$

commute. By Remark 10 this is an equivalence. □

Now that we have a more concrete understanding of the fibration $\mathbf{gl}(F)$ let us examine when it has a generating family in the sense of Definition 38. A bounded family of objects of $\mathcal{E}$ is generating in $\mathbf{Sets}(\mathcal{E})_b$ if and only if it is a generating family for $\mathcal{E}$ in the ordinary sense. So the fibration $\mathbf{gl}(F)$ admits a generating family if and only if there is a *bounded* generating family $(G_i)_{i\in I}$ for $\mathcal{E}$. By well-poweredness and local smallness of $\mathcal{E}$ this amounts to saying that there is an object $S$ in $\mathcal{E}$ such that for every object $X$ there is a family $(C_i)_{i\in I}$ of subobjects of $S$ together with a collectively epic family of morphisms $f_i : C_i \to X$. As $I$-indexed families $(C_i)_{i\in I}$ of subobjects of $S$ can be identified with subobjects $C = \coprod_{i\in I} C_i$ of $\coprod_I S = FI \times S$ and as collectively epic families $f_i : C_i \to X$ correspond to epic morphisms $f : C \to X$ the fibration $\mathbf{gl}(F)$ admits a generating family if and only if the geometric morphism $F \dashv U$ is *bounded*, which is defined as follows.

**Definition 42.** (i) Let $F \dashv U : \mathcal{E} \to \mathcal{S}$ be a geometric morphism between toposes. An object $S$ in $\mathcal{E}$ is called a *bound* or *separator* for $F \dashv U$ if for every object $X$ in $\mathcal{E}$, there exists an object $I$ in $\mathcal{S}$, a subobject $C$ of $FI \times S$ and an epimorphism from $C$ to $X$, i.e. every $X$ in $\mathcal{E}$ appears as a *subquotient* of $FI \times S$ for some $I \in \mathcal{S}$.

$$\begin{tikzcd} C \arrow[r, two heads] \arrow[d, tail] & X \\ FI \times S & \end{tikzcd}$$

(ii) A geometric morphism is a *bounded geometric morphism* if it admits a bound.

Note that these definitions make sense for an arbitrary base topos $\mathcal{S}$. In Theorem 10 we will also show that for a geometric morphism $F \dashv U$ between arbitrary toposes the fibration $\mathbf{gl}(F)$ admits a generating family if and only if $F \dashv U$ is bounded.

But first, let us take a look at examples of bounded and non-bounded geometric morphisms over **Sets**.

**Example 10.** Let $(\mathbb{C}, J)$ be a site in **Sets** and let $\mathrm{Sh}(\mathbb{C}, J)$ be the induced Grothendieck topos. We wish to show that the canonical geometric morphism from $\mathrm{Sh}(\mathbb{C}, J)$ to **Sets** is bounded. By the Yoneda lemma, every presheaf is a colimit of representable presheaves, so in particular the presheaf category $\mathbf{Sets}^{\mathbb{C}^{op}}$ has the representable presheaves as a small generating family. The inclusion functor from $\mathrm{Sh}(\mathbb{C}, J)$ into $\mathbf{Sets}^{\mathbb{C}^{op}}$ has a left adjoint, called the sheafification functor. Now the category $\mathrm{Sh}(\mathbb{C}, J)$ can be seen to have a small generating family, consisting of the associated sheaves of the representable presheaves in $\mathcal{S}^{\mathbb{C}^{op}}$. Moreover, as $\mathrm{Sh}(\mathbb{C}, J)$ has small coproducts, we can build the coproduct of the generating family, yielding a bound.

The previous example covers almost all toposes over **Sets** that one normally comes across. It is much harder to exhibit unbounded toposes over **Sets**. The following theorem, also referred to as *relative Giraud theorem* expresses the importance of the notion of boundedness of a geometric morphisms. Applied to the case that $\mathcal{S}$ is **Sets** it says, that the above considered class of examples is the most general possible.

**Theorem 9 (Diaconescu).** *A geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ between toposes is bounded if and only if there exists an internal site $(\mathbb{C}, J)$ in $\mathcal{S}$ such that $F \dashv U$ is equivalent to the canonical geometric morphism from $\mathrm{Sh}(\mathbb{C}, J)$ to $\mathcal{S}$ in $\mathcal{T}op/\mathcal{S}$.*

*Proof.* see [John77] □

Thus, a topos over **Sets** is bounded if and only if it is cocomplete and has a small generating family. (For the 'if' part take the coproduct of a generating family as a bound, for the 'only if' part use Theorem 9) Nevertheless, neither cocompleteness nor the existence of a small generating family alone imply boundedness, as the following examples show.

**Example 11.** The objects of the topos $\mathbf{Sets}^{\mathbb{Z}}$ can be described as sets $M$ together with a specified automorphism $r : M \to M$. Let $\mathcal{F}$ be the full subcategory of $\mathbf{Sets}^{\mathbb{Z}}$, consisting of those objects $(M, r)$ for which there exists a number $n \in \mathbb{N}$ such that $r^n = \mathrm{id}_M$ (which is equivalent to the existence of an upper bound for the length of the orbits in $(M, r)$). Then $\mathcal{F}$

is a topos and its inclusion functor into $\mathbf{Sets}^{\mathbb{Z}}$ is logical. Clearly, $\mathcal{F}$ has small copowers and is locally small, i.e. it is defined over **Sets**. Let $C_n$ be the cycle of length $n$. The family $(C_n)_{n\in\mathbb{N}}$ is a generating family, but it is not bounded and no bounded family can generate $\mathcal{F}$ because for each bounded family $(M_i, r_i)_{i\in I}$ there exists a number $n \in \mathbb{N}$ such that $r_i^n = \mathrm{id}_{M_i}$ for all $i \in I$. So there is no equivariant map from any $M_i$ to, say, $C_{n+1}$. Thus $\mathcal{F}$ is not bounded over **Sets**.

**Example 12.** An object of the category $\mathcal{A}$ is a triple $(A, S, g)$, where $A$ and $S$ are sets and $g$ is a function from $S$ to $\mathrm{Aut}(A)$. For $a \in A$ and $r \in \mathrm{Aut}(A)$ we write $a^r$ for the result of application of $r$ to $a$. For $s \notin S$ we put $g(s) = \mathrm{id}_A$. Given two objects $(A, S, g)$ and $(A', S', g')$, a morphism is a function $f : A \to A'$ such that $f(a^{g(s)}) = f(a)^{g'(s)}$ for all $s$. The category $\mathcal{A}$ is a topos over **Sets**. It can be thought of as a topos of presheaves over a large group. This topos is cocomplete, but there is no small generating family, whence the topos $\mathcal{A}$ is not bounded over **Sets**.

In order to finish this section we are going to prove the theorem announced in the discussion.

**Theorem 10.** *A geometric morphism $F \dashv U$ between toposes is bounded if and only if the associated fibration $\mathsf{gl}(F)$ admits a generating family.*

*Proof.* Assume that there is a generating family $g : G \to FI$ for the fibration $\mathsf{gl}(F)$. We claim that $G$ is a bound. Let $X$ be an object in $\mathcal{E}$. By Corollary 5 we know that the fibration $\mathsf{gl}(F)$ is locally small, so Proposition 19 applies and there is an object $c : C \to FJ$ in $\mathsf{Gl}(F)$ with a cartesian morphism to $g : G \to FI$ and a collectively epic morphism to $!_X : X \to F1$ as in the following diagram.

$$
\begin{tikzcd}
G \arrow[d, "g"'] & C \arrow[l] \arrow[r, "e"] \arrow[d, "c"] \arrow[dl, phantom, "\llcorner" very near start] & X \arrow[d, "!_X"] \\
FI & FJ \arrow[l, "Fu"] \arrow[r] & F1
\end{tikzcd}
$$

By Proposition 20 the morphism $e$ is epic. As $C$ is a subobject of $FJ \times G$ the object $X$ appears as a subquotient of $FJ \times G$.

Conversely, assume that $S$ is bound. Let $x : X \to FJ$ be an object in $\mathsf{Gl}(F)$. As $S$ is a bound, there is an $I$ in $\mathcal{S}$, a mono $m : C \to S \times FI$ and an epi $e : C \to X$.

$$
\begin{tikzcd}
C \arrow[r, "e", two heads] \arrow[d, "m"', tail] & X \\
S \times FI &
\end{tikzcd}
$$

This gives rise to the following diagram in $\mathsf{Gl}(F)$, where $\tilde{m}$ is the morphism induced by $m : C \to S \times FI$ and $xe : C \to FJ$, using that $F(I \times J) \cong FI \times FJ$.

$$
\begin{tikzcd}
& C \arrow[r, "e", two heads] \arrow[d, "\tilde{m}", tail] & X \arrow[dd, "x"] \\
S \arrow[d, "!_S"'] & S \times F(I \times J) \arrow[l, "p"'] \arrow[d, "p'"] \arrow[dl, phantom, "\llcorner" very near start] & \\
F1 & F(I \times J) \arrow[l] \arrow[r, "F\pi'"'] & FJ
\end{tikzcd}
$$

That means every family $x : X \to FJ$ can be covered by a family of subobjects of $!_S : S \to F\,1$. By the Propositions 21 and 22 we know, that $\mathbf{gl}(F)$ is well-powered. Consider a generic family of subobjects of $!_S : S \to F\,1$.

$$\begin{tikzcd}
& G \arrow[d, tail, "n"] \\
S \arrow[d, "!_S"'] & S \times F\,\mathrm{Sub}_1(S) \arrow[l, "q"'] \arrow[d, "q'"] \arrow[dl, phantom, "\llcorner"] \\
F\,1 & F\,\mathrm{Sub}_1(S) \arrow[l]
\end{tikzcd}$$

Then the morphism $g := q'n : G \to F\,\mathrm{Sub}_1(S)$ is a generating family for the fibration $\mathbf{gl}(F)$ as $p'\tilde{m} : C \to F(I \times J)$ is a reindexing of $q'n : G \to F\,\mathrm{Sub}_1(S)$ and the generating family $g : G \to F\,\mathrm{Sub}_1(S)$ was chosen independently from $x : X \to FJ$. $\square$

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