Denotational Completeness Revisited

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Abstract

We define a notion of *Kripke logical predicate* for models of classical linear logic. A Kripke logical predicate on a type A will be a set of *generalised* elements of A satisfying certain closure properties. Denotations of proofs of A will be characterised as those *global* elements of A satisfying all Kripke logical predicates on A.

Key words: denotational completeness, linear logic, Kripke logical predicates.

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1 Introduction

The aim of this paper is to prove a *denotational completeness* result for linear logic as in [4], but by different, more traditional tools of denotational semantics, namely logical predicates of "varying arity" *aka* Kripke logical predicates. These were introduced originally by A. Jung and J. Tiuryn in [6] for the purpose of characterising those set theoretic functions which are definable in typed λ–calculus (for arbitrary interpretations of the base type).

It is more or less folklore how to adapt the methods of [6] in order to characterise the λ–definable morphisms for models in arbitrary cartesian closed category $\mathbb{C}$. The key idea is to consider logical predicates in $\widehat{\mathbb{V}}$, the category of presheaves over the category $\mathbb{V}$ of *variable substitutions* in $\mathbb{C}$ [2], and then to show that the λ–definable generalised elements form such a Kripke logical predicate.

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[2] The same technique has been used in [1] for the purpose of a categorical reconstruction of "normalisation by evaluation" à la Berger–Schwichtenberg making use of a particular Kripke logical predicate ensuring normalisation.

The purpose of this paper is to adapt the Jung–Tiuryn technique to Classical Linear Logic (CLL) by identifying an appropriate notion of Kripke logical predicate for it. For an arbitrary model $\mathbb{C}$ of classical linear logic we characterise for arbitrary type A the *proof objects* of $[\![A]\!]_{\mathbb{C}}$, i.e. the *global* elements of $[\![A]\!]_{\mathbb{C}}$ arising as denotations of proofs of A, as those global elements of $[\![A]\!]_{\mathbb{C}}$ contained in all Kripke logical predicates $P_{\mathcal{B}}(A)$ on the *generalised* elements of $[\![A]\!]_{\mathbb{C}}$ where $\mathcal{B}$ ranges over certain "well-behaved" sets of generalised elements of $[\![\perp]\!]_{\mathbb{C}}$. Notice that "generalised" elements of A are objects of A "relative to an arbitrary context" whereas "global" elements of A are those elements of A "relative to the empty context".

Our Kripke logical predicates $P_{\mathcal{B}}$ appear as particular instances of the more general notion introduced recently in [5] for ILL (Intuitionistic Linear Logic). However, making intrinsic use of linear negation typical for CLL our restricted class of Kripke logical predicates can be defined in a simpler way appearing as a *proof-relevant* version of Girard's phase semantics for linear logic as in [3] (where our "well-behaved" $\mathcal{B} \subseteq [\![\perp]\!]_{\mathbb{C}}$ correspond to the arbitrary subsets of the monoid in *loc.cit.*). As phase semantics is already "built-in" to our approach we can avoid adding it as an additional component as in [4].

In section 2 we define our notion of Kripke logical predicate for arbitrary models of CLL (without quantifiers). In section 3 we show soundness, i.e. that proof objects satisfy all logical predicates under consideration. In section 4 we show denotational completeness, i.e. that every element satisfying all Kripke logical predicates appears as the denotation of some derivation. In particular we show that for every type A we have $Pr(A) = P_{Pr(\perp)}(A)$ where for arbitrary formulas C the set $Pr(C)$ consists of those generalised elements of C appearing as denotations of derivations. In section 5 we sketch how our characterisation of proof objects can be used to build denotationally complete models where *every type contains only proof objects*. Finally, in section 5 we compare our proof method with the original one of [4] and discuss the relevance of our achievements.

We think that our proof of denotational completeness—besides its simplicity—provides a new, though not unexpected link between traditional techniques of denotational semantics and linear logic in the sense that methods of the former may be fruitfully applied to the study of the latter. Although the skeleton of our proof follows quite closely the pattern of [6], or rather its adaptation to general models as given by ccc's, the flesh around this skeleton is fresh as we have to define a new notion of Kripke logical predicate for Classical Linear Logic which amounts to a "proof–relevant" version of Girard's phase semantics. Moreover, our proofs are not just a straightforward adaptation of those of [6] as the basic ingredients of *classical* linear logic are fairly different from typed λ–calculus.

2 Kripke Logical Predicates

In this section we define a notion of Kripke logical predicate for models of quantifier-free classical linear logic referred to simply as "linear logic" for the rest of this paper. Readers not feeling at ease with general categorical models of linear logic may well think of their favourite concrete model, for example the coherence space model of [3], without missing anything essential. For a precise definition of categorical model for Linear Logic see [7] or rather the corrected version in [2] (though the difference is not relevant for our purposes as we do not consider equality of proof terms).

For ease of exposition we sometimes employ a 2–sided sequent calculus for linear logic because this does not make any difference w.r.t. the interpretation of derivations, the so–called "proof objects". Accordingly, we distinguish between *left* contexts $\Gamma \vdash$ and *right* contexts $\vdash \Gamma$ which are interpreted differently

$$[\![A_1, \ldots, A_n \vdash]\!] = [\![A_1]\!] \otimes \ldots \otimes [\![A_n]\!]$$

$$[\![\vdash A_1, \ldots, A_n]\!] = [\![A_1]\!] \wp \ldots \wp [\![A_n]\!] \quad .$$

Nevertheless, in the rest of the paper we will often omit semantic brackets, i.e. for formulas A of linear logic we often write A for $[\![A]\!]$, the interpretation of A (e.g. in the coherence space model). Moreover, if Γ is a sequence of formulas we often write simply Γ for $[\![\Gamma \vdash]\!]$ or $[\![\vdash \Gamma]\!]$ and it (hopefully !) will always be clear from the context what is the intended reading. Moreover, we freely identify A with $A^{\perp\perp}$ which is valid in most models as e.g. the coherence space model.

Definition 1 *Let $\mathbb{C}$ be the underlying category of a model of linear logic, e.g. the category of coherence spaces and linear maps, and $\mathbb{V}$ be the subcategory of $\mathbb{C}$ whose objects are denotations of left contexts of linear logic and whose morphisms are denotations of proofs using only the structural rules of weakening, contraction*[3] *and permutation. The morphisms of $\mathbb{V}$ will be refered to as* variable substitutions*.*
For a type A a generalised element of A at stage Γ *is a linear map $\Gamma \multimap A$* [4]*.*
We write $GE(A)$ for the collection of all generalised elements of A (at arbitrary stage Γ) and $GE(A)(\Gamma)$ for the collection of generalised elements of A at stage Γ. If $P \subseteq GE(A)$ then we write $P(\Gamma)$ for the collection of all generalised elements of A at stage Γ that are contained in P.
A subset $P \subseteq GE(A)$ will be called stable *iff P is "closed under variable substitutions", i.e. $a \circ \sigma \in P(\Delta)$ whenever $a \in P(\Gamma)$ and $\sigma : \Delta \to \Gamma$ in $\mathbb{V}$.*

[3] which, of course, are only applicable to banged formulas
[4] i.e., more precisely, a morphism from $[\![\Gamma \vdash]\!]$ to $[\![A]\!]$ in $\mathbb{C}$

A typical example of a stable subset of $GE(A)$ is given by the collection of so–called "proof objects" of A.

Definition 2 *Let A be a type, i.e. formula, of linear logic. An element of $GE(A)(\Gamma)$ is called a* proof object *if it is of the form $[\![\pi]\!]$ where π is a proof of $\Gamma \vdash A$. We write $Pr(A)$ for the stable subset of $GE(A)$ consisting of proof objects of A.*

Kripke logical predicates on A will be certain stable subsets of $GE(A)$ which, however, will be required to be "denotational facts" w.r.t. some arbitrary stable subset of $GE(\bot)$.

Definition 3 *If $a \in GE(A)$, $b \in GE(A^\bot)$ we write $\langle\, a \,|\, b \,\rangle$ for $\mathsf{ev} \circ (a \otimes b)$ in $GE(\bot)$ where $\mathsf{ev} : A \otimes A^\bot \multimap \bot$ is the linear evaluation map* [5]*.*
Let $\mathcal{B}$ be a stable subset of $GE(\bot)$. For $a \in GE(A)$, $b \in GE(A^\bot)$ we say that $a \perp_\mathcal{B} b$ iff $\langle\, a \,|\, b \,\rangle \in \mathcal{B}$. If $P \subseteq GE(A)$ then $P^{\perp_\mathcal{B}}$ is the set of all $b \in GE(A^\bot)$ such that $a \perp_\mathcal{B} b$ for all $a \in A$.
A denotational $\mathcal{B}$–fact of A *or* Kripke logical predicate on A of kind $\mathcal{B}$ *is a subset $P \subseteq GE(A)$ with $P = P^{\perp_\mathcal{B}\perp_\mathcal{B}}$.*

Lemma 4 *If $\mathcal{B}$ is a stable subset of $GE(\bot)$ then all denotational $\mathcal{B}$–facts of A are stable.*

PROOF. Let $P = P^{\perp_\mathcal{B}\perp_\mathcal{B}} \subseteq GE(A)$. For showing stability of P suppose that $a \in P(\Gamma)$ and $\sigma \in \mathbb{V}(\Delta, \Gamma)$. But then $a \circ \sigma \in P = P^{\perp_\mathcal{B}\perp_\mathcal{B}}$, too, as for $b \in P^{\perp_\mathcal{B}}(\Delta)$ we have $\mathsf{ev} \circ (a \otimes b) = \langle\, a \,|\, b \,\rangle \in \mathcal{B}$ and, therefore, also

$$\langle\, a \circ \sigma \,|\, b \,\rangle = \mathsf{ev} \circ ((a \circ \sigma) \otimes b) = \mathsf{ev} \circ (a \otimes b) \circ (\sigma \otimes \Delta) \in \mathcal{B}$$

as $\mathcal{B}$ is stable by assumption and variable substitutions are closed under $\otimes$. $\square$

Now we will define for every *stable* subset $\mathcal{B}$ of $GE(\bot)$ a Kripke logical predicate of kind $\mathcal{B}$ on the model under consideration.

Definition 5 *Let $\mathcal{B}$ be a stable subset of $GE(\bot)$. Now we define a Kripke logical relation $P_\mathcal{B}$ endowing every formula A with a denotational $\mathcal{B}$–fact $P_\mathcal{B}(A)$ by recursion on the complexity of formulas as follows*

- $P_\mathcal{B}(\bot) = \mathcal{B}$
- $P_\mathcal{B}(1) = \mathcal{B}^{\perp_\mathcal{B}}$
- $P_\mathcal{B}(A \otimes B) = \{a \otimes b \mid a \in P_\mathcal{B}(A), b \in P_\mathcal{B}(B)\}^{\perp_\mathcal{B}\perp_\mathcal{B}}$
- $P_\mathcal{B}(A \wp B) = \{a \otimes b \mid a \in P_\mathcal{B}(A^\bot), b \in P_\mathcal{B}(B^\bot)\}^{\perp_\mathcal{B}}$
- $P_\mathcal{B}(\top) = GE(\top)$

[5] E.g. in the coherence space model we have $\mathsf{ev}(z) = \emptyset$ iff $(x, x) \notin z$ for all $x \in |A|$.

- $P_{\mathcal{B}}(0) = GE(\top)^{\perp_{\mathcal{B}}}$
- $P_{\mathcal{B}}(A \oplus B) = (\{\mathsf{inl}(a) | a \in P_{\mathcal{B}}(A)\} \cup \{\mathsf{inr}(b) | b \in P_{\mathcal{B}}(B)\})^{\perp_{\mathcal{B}}\perp_{\mathcal{B}}}$
- $P_{\mathcal{B}}(A \& B) = (\{\mathsf{inl}(a) | a \in P_{\mathcal{B}}(A^{\perp})\} \cup \{\mathsf{inr}(b) | b \in P_{\mathcal{B}}(B^{\perp})\})^{\perp_{\mathcal{B}}}$
- $P_{\mathcal{B}}(!A) = \{\, \mathsf{prom}(a) \mid a \in P_{\mathcal{B}}(A)\}^{\perp_{\mathcal{B}}\perp_{\mathcal{B}}}$
- $P_{\mathcal{B}}(?A) = \{\, \mathsf{prom}(a) \mid a \in P_{\mathcal{B}}(A^{\perp})\}^{\perp_{\mathcal{B}}}$.

where $\mathsf{prom}(a) = !(a) \circ \delta_{!\Gamma}$ *for* $a \in GE(A)(!\Gamma)$ *(where* $\delta : ! \circ ! \to !$ *is the comultiplication of the comonad* $!$*) and* $\mathsf{prom}(a)$ *is undefined if the stage of* a *is not a banged context, i.e. not of the form* $!\Gamma$*.*[6]

The following properties are easily established for arbitrary $P_{\mathcal{B}}$. Notice that from now on we often write $f(a)$ for $\mathsf{ev} \circ (f \otimes a)$ whenever $f \in GE(A \multimap B)$ and $a \in GE(A)$.

Lemma 6 *For every stable* $\mathcal{B}$ *subset of* $GE(\perp)$ *and formulas* A,B *we have that*

(1) $P_{\mathcal{B}}(A)^{\perp_{\mathcal{B}}} = P_{\mathcal{B}}(A^{\perp})$
(2) $P_{\mathcal{B}}(A \& B) = \{a \& b \mid \exists\, \Gamma.\ a \in P_{\mathcal{B}}(A)(\Gamma) \wedge b \in P_{\mathcal{B}}(B)(\Gamma)\}$
(3) $f \in P_{\mathcal{B}}(A \multimap B)$ *iff* $\forall a \in P_{\mathcal{B}}(A).\ f(a) \in P_{\mathcal{B}}(B)$
(4) $f \in P_{\mathcal{B}}(A \otimes B \multimap C)$ *iff* $\forall a \in P_{\mathcal{B}}(A), b \in P_{\mathcal{B}}(B).\ f(a \otimes b) \in P_{\mathcal{B}}(C)$
(5) $f \in P_{\mathcal{B}}(!A \multimap B)$ *iff* $\forall a \in P_{\mathcal{B}}(A).\ f(\mathsf{prom}(a)) \in P_{\mathcal{B}}(B)$
(6) $id_1 \in P_{\mathcal{B}}(1)$.

PROOF. Claim (1) is obvious from the definition of $P_{\mathcal{B}}$.

For (2) suppose that $c \in GE(A \& B)$. Then, we have $c \in P_{\mathcal{B}}(A \& B)$ iff $\langle\, c \,|\, \mathsf{inl}(a) \,\rangle = \langle\, \mathsf{pr}_1 \circ c \,|\, a \,\rangle \in \mathcal{B}$ for all $a \in P_{\mathcal{B}}(A^{\perp})$ and $\langle\, c \,|\, \mathsf{inr}(b) \,\rangle = \langle\, \mathsf{pr}_2 \circ c \,|\, b \,\rangle \in \mathcal{B}$ for all $b \in P_{\mathcal{B}}(B^{\perp})$, i.e. iff $\mathsf{pr}_1 \circ c \in P_{\mathcal{B}}(A)$ and $\mathsf{pr}_2 \circ c \in P_{\mathcal{B}}(B)$.

For (3) suppose that $f \in GE(A \multimap B) = GE(A^{\perp} \wp B)$. Then $f \in P_{\mathcal{B}}(A^{\perp} \wp B)$ iff $\langle\, f \,|\, a \otimes b \,\rangle = \mathsf{ev} \circ ((\mathsf{ev} \circ (f \otimes a)) \otimes b) \in \mathcal{B}$ for all $a \in P_{\mathcal{B}}(A)$ and $b \in P_{\mathcal{B}}(B^{\perp})$, i.e. iff $f(a) = \mathsf{ev} \circ (f \otimes a) \in P_{\mathcal{B}}(B)$ for all $a \in P_{\mathcal{B}}(A)$.

Claim (4) is immediate from (3) as $GE(A \otimes B \multimap C) = GE(A \multimap B \multimap C)$.

For (5) suppose that $f \in GE(!A \multimap B) = GE(!A \multimap B^{\perp} \multimap \perp)$. Applying the already established equivalence (3) twice we get that $f \in P_{\mathcal{B}}(!A \multimap B) = P_{\mathcal{B}}(!A \multimap B^{\perp} \multimap \perp)$ iff $f(c)(b) \in \mathcal{B}$ for all $c \in P_{\mathcal{B}}(!A)$ and $b \in P_{\mathcal{B}}(B^{\perp})$. Now for arbitrary, but fixed $b \in P_{\mathcal{B}}(B^{\perp})$ we have $\lambda c{:}\,!A.f(c)(b) \in P_{\mathcal{B}}(!A \multimap \perp) = P_{\mathcal{B}}(?(A^{\perp})) = \{\, \mathsf{prom}(a) \,|\, a \in P_{\mathcal{B}}(A)\,\}^{\perp_{\mathcal{B}}}$ if and only if $f(\mathsf{prom}(a))(b) = \langle\, \lambda c{:}!A.f(c)(b) \,|\, \mathsf{prom}(a) \,\rangle \in \mathcal{B}$ for all $a \in P_{\mathcal{B}}(A)$. Thus, $f \in P_{\mathcal{B}}(!A \multimap B)$ iff

[6] The acronym prom refers to *promotion rule*.

$f(\mathsf{prom}(a))(b) \in \mathcal{B}$ for all $b \in P_{\mathcal{B}}(B^\perp)$ and $a \in P_{\mathcal{B}}(A)$, i.e., iff $f(\mathsf{prom}(a)) \in P_{\mathcal{B}}(B)$ for all $a \in P_{\mathcal{B}}(A)$.

For (6) we have to show that $\langle\, b \,|\, id_1 \,\rangle \in \mathcal{B}$ all $b \in \mathcal{B}$. If $b : \Gamma \to \perp$ then $\langle\, b \,|\, id_1 \,\rangle = b \circ \iota$ where ι is the canonical isomorphism between Γ and $\Gamma \otimes 1$. As ι is a variable substitution and $\mathcal{B}$ is stable it follows that $\langle\, b \,|\, id_1 \,\rangle = b \circ \iota \in \mathcal{B}$ as $b \in \mathcal{B}$ by assumption. $\Box$

Notice that by (2) and (3) of the previous lemma 6 we get that our notion of Kripke logical predicate conservatively extends the usual one for simply typed λ-calculus (as intuitionistic application is a particular case of linear application.

Notice also that our Kripke logical predicates are instances of M. Hasegawa's notion of Kripke logical predicates for models of intuitionistic linear logic as described in [5] [7]. However, we define Kripke logical predicates for the full quantifier-free part of classical linear logic where the presence of an involutive linear negation allows for considerable simplifications.

3 Soundness

In this section we prove that proof objects, i.e. global elements of the form $[\![\pi]\!]$, satisfy all Kripke logical predicates. In the next section we show the reverse inclusion.

Before we embark on the proof of the soundness theorem we introduce a notational convention that will prove useful in the sequel. If $\Gamma \equiv A_1, \ldots, A_n$ then we write $\Gamma^\perp$ and $!\Gamma$ for $A_1^\perp, \ldots, A_N^\perp$ and $!A_1, \ldots, !A_n$, respectively.

Theorem 7 (Soundness)
If π is a proof of $\vdash A_1, \ldots, A_n$ then $[\![\pi]\!] \in P_{\mathcal{B}}(A_1 \wp \ldots \wp A_n)$ for all stable $\mathcal{B} \subseteq GE(\perp)$.

PROOF. Let $\mathcal{B}$ be a stable subset of $GE(\perp)$. We proceed by induction on structure of derivation π.

(*Axiom*) If π is obtained by one application of the rule (axiom) then $[\![\pi]\!] = id_A$. From Lemma 6(3) it follows that $id_A \in P_{\mathcal{B}}(A \multimap A)$ as for all $a \in P_{\mathcal{B}}(A)$ we have that $id_A(a) = a \in P_{\mathcal{B}}(A)$.

[7] where we instantiate Hasegawa's $\mathbb{I}$ by the inclusion of the symmetric monoidal category $\mathbb{V}$ into the symmetric monoidal closed category $\mathbb{C}$

($Permutation$) Suppose that π' is a derivation of $A_1, \ldots, A_n$ satisfying the induction hypothesis $[\![\pi']\!] \in P_{\mathcal{B}}(A_1, \ldots, A_n)$. Let π be the derivation of the formula $A_{i_1} \wp \ldots \wp A_{i_n}$ obtained from π' by applying the permutation rule instantiated by permutation i. Let $\iota : A_1 \wp \ldots \wp A_n \to A_{i_1} \wp \ldots \wp A_{i_n}$ be the isomorphic variable substitution induced by the permutation i then $[\![\pi]\!] = \iota \circ [\![\pi']\!]$. For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(A_{i_1} \wp \ldots \wp A_{i_n})$ assume that $a_i \in P_{\mathcal{B}}(A_i^{\perp})$ for $i = 1, \ldots, n$. But then

$$\langle\, \iota \circ [\![\pi']\!] \mid a_{i_1} \otimes \ldots \otimes a_{i_n} \,\rangle = \langle\, [\![\pi']\!] \mid a_1 \otimes \ldots \otimes a_n \,\rangle \in \mathcal{B}$$

as $[\![\pi']\!] \in P_{\mathcal{B}}(A_1 \wp \ldots \wp A_n)$ by induction hypothesis.

(Cut) Suppose that π_1 and π_2 are proofs of Γ, A and $A^{\perp}, \Delta$, respectively, satisfying the induction hypotheses $[\![\pi_1]\!] \in P_{\mathcal{B}}(\Gamma, A) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap A)$ and $[\![\pi_2]\!] \in P_{\mathcal{B}}(A^{\perp}, \Delta) = P_{\mathcal{B}}(A \multimap \Delta)$.
Let π be the proof of Γ, Δ obtained from π_1 and π_2 by application of the rule (cut). Then $[\![\pi]\!] = [\![\pi_2]\!] \circ [\![\pi_1]\!] \in \Gamma, \Delta = \Gamma^{\perp} \multimap \Delta$. By Lemma 6(3) for $[\![\pi]\!] \in P_{\mathcal{B}}(\Gamma^{\perp} \multimap \Delta)$ it suffices to show that $[\![\pi]\!](g) = [\![\pi_2]\!]([\![\pi_1]\!](g)) \in P_{\mathcal{B}}(\Delta)$ for every $g \in P_{\mathcal{B}}(\Gamma^{\perp})$. But this latter condition holds as by the induction hypotheses and Lemma 6(3) both $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ preserve $P_{\mathcal{B}}$ under linear application.

($\otimes$) Suppose that π_1 and π_2 are proofs of Γ, A and Δ, B, respectively, satisfying the induction hypotheses $\pi_1 \in P_{\mathcal{B}}(\Gamma, A) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap A)$ and $\pi_2 \in P_{\mathcal{B}}(\Delta, B) = P_{\mathcal{B}}(\Delta^{\perp} \multimap B)$. Then for the proof π of $\Gamma, \Delta, A \otimes B$ obtained from π_1 and π_2 by application of the rule ($\otimes$) we have that $[\![\pi]\!] = [\![\pi_1]\!] \otimes [\![\pi_2]\!]$.
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(\Gamma, \Delta, A \otimes B) = P_{\mathcal{B}}(\Gamma^{\perp} \otimes \Delta^{\perp} \multimap A \otimes B)$ it suffices by Lemma 6(4) to show that $[\![\pi]\!](g \otimes d) = [\![\pi_1]\!](g) \otimes [\![\pi_2]\!](d) \in P_{\mathcal{B}}(A \otimes B)$ for all $g \in P_{\mathcal{B}}(\Gamma^{\perp})$ and $d \in P_{\mathcal{B}}(\Delta^{\perp})$. This, however, is the case as by induction hypothesis and Lemma 6(3) we have that $[\![\pi_1]\!](g) \in P_{\mathcal{B}}(A)$ and $[\![\pi_2]\!](d) \in P_{\mathcal{B}}(B)$.

(1) If π is the derivation consisting of one application of the rule (1) then $[\![\pi]\!] = id_1$ which is in $P_{\mathcal{B}}(1)$ by Lemma 6(6).

($\wp$) For the rule ($\wp$) nothing has to be shown as if π' is a proof of Γ, A, B and π is obtained from π' by one application of the rule ($\wp$) then by definition we have $[\![\pi]\!] = [\![\pi']\!] \in [\![\Gamma, A, B]\!] = [\![\Gamma, A \wp B]\!]$.

($\perp$) Suppose π' is a proof of Γ and satisfies the induction hypothesis $[\![\pi']\!] \in P_{\mathcal{B}}(\Gamma) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap \perp)$. Then for the proof π of $\Gamma, \perp$ obtained from π' by application of the rule ($\perp$) we have that $[\![\pi]\!]$ is the global element of $\Gamma^{\perp} \multimap \perp$ such that $[\![\pi]\!](g) = \langle\, [\![\pi']\!] \mid g \,\rangle$ for all $g \in GE(\Gamma^{\perp})$.
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(\Gamma, \perp) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap \perp)$ it suffices to show that $[\![\pi]\!](g) = \langle\, [\![\pi']\!] \mid g \,\rangle \in P_{\mathcal{B}}(\perp) = \mathcal{B}$ for all $g \in P_{\mathcal{B}}(\Gamma^{\perp})$ which, however, holds by induction hypothesis on π'.

($\&$) Suppose that π_1 and π_2 are derivations of Γ, A_1 and Γ, A_2, respectively, satisfying the induction hypotheses $[\![\pi_i]\!] \in P_{\mathcal{B}}(\Gamma, A_i) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap A_i)$ $(i = 1, 2)$. Then for the derivation π of $\Gamma, A_1 \& A_2$ obtained from π_1 and π_2 by application of the rule ($\&$) we have that $[\![\pi]\!] = [\![\pi_1]\!] \& [\![\pi_2]\!]$.
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(\Gamma, A_1 \& A_2) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap A_1 \& A_2)$ suppose that $g \in P_{\mathcal{B}}(\Gamma^{\perp})$. Then $[\![\pi]\!](g) = ([\![\pi_1]\!] \& [\![\pi_2]\!]) \circ g = ([\![\pi_1]\!] \circ g) \& ([\![\pi_1]\!] \circ g) \in P_{\mathcal{B}}(A_1 \& A_2)$ by Lemma 6(2) as by induction hypothesis we have that $\mathsf{pr}_i \circ [\![\pi]\!](g) = [\![\pi_i]\!] \circ g \in P_{\mathcal{B}}(\Gamma, A_i) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap A_i)$.

($\top$) Let π be the derivation of $\Gamma, \top$ obtained by application of the rule ($\top$). Then $[\![\pi]\!]$ is the unique morphism from $\Gamma^{\perp}$ to $\top$.
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(\Gamma, \top) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap \top)$ suppose that $g \in P_{\mathcal{B}}(\Gamma^{\perp})$. But $[\![\pi]\!](g) \in GE(\top) = P_{\mathcal{B}}(\top)$.

($\oplus$) Suppose that π' is a derivation of Γ, A satisfying the induction hypothesis $[\![\pi']\!] \in P_{\mathcal{B}}(\Gamma, A) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap A)$. Then for the derivation π of $\Gamma, A \oplus B$ obtained from π' by application of the left introduction rule for $\oplus$ we have that $[\![\pi]\!] = \mathsf{inl} \circ [\![\pi']\!]$.
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(\Gamma, A \oplus B) = P_{\mathcal{B}}(\Gamma^{\perp} \multimap A \oplus B)$ suppose that $g \in P_{\mathcal{B}}(\Gamma^{\perp})$. Then $[\![\pi]\!](g) = \mathsf{inl} \circ [\![\pi']\!] \circ g = \mathsf{inl}([\![\pi']\!] \circ g) \in P_{\mathcal{B}}(A \oplus B)$ as by induction hypothesis on π' we have that $[\![\pi']\!] \circ g \in P_{\mathcal{B}}(A)$ and, therefore, we have $[\![\pi]\!](g) = \mathsf{inl}([\![\pi']\!] \circ g) \in (\{\mathsf{inl}(a) | a \in P_{\mathcal{B}}(A)\} \cup \{\mathsf{inr}(b) | b \in P_{\mathcal{B}}(B)\})^{\perp_{\mathcal{B}} \perp_{\mathcal{B}}} = P_{\mathcal{B}}(A \oplus B)$.
The proof for the right introduction rule is analogous.

(*Weakening*) Suppose that π' is a derivation of Γ satisfying induction hypothesis $[\![\pi']\!] \in P_{\mathcal{B}}(\Gamma)$. Let π be the derivation of $?A, \Gamma$ obtained from π' by application of the weakening rule. Then $[\![\pi]\!] = [\![\pi']\!] \circ \mathsf{w}_{A^{\perp}}$ where $\mathsf{w}_{A^{\perp}} : !(A^{\perp}) \multimap 1$ is the counit of the commutative comonoid $!(A^{\perp})$.
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(?A, \Gamma) = P_{\mathcal{B}}(!(A^{\perp}) \multimap \Gamma)$ by Lemma 6(5) suppose that $c \in P_{\mathcal{B}}(A^{\perp})(!\Delta)$. Then $[\![\pi]\!](\mathsf{prom}(c)) = [\![\pi']\!] \circ \mathsf{w}_{A^{\perp}} \circ \mathsf{prom}(c) = [\![\pi']\!] \circ \mathsf{w}_{\Delta} \in P_{\mathcal{B}}(\Gamma)$ as by induction hypothesis $[\![\pi']\!] \in P_{\mathcal{B}}(\Gamma)$ and $P_{\mathcal{B}}(\Gamma)$ is stable.

(*Contraction*) Suppose that π' is a derivation of $?A, ?A, \Gamma$ satisfying the induction hypothesis $[\![\pi']\!] \in P_{\mathcal{B}}(?A, ?A, \Gamma) = P_{\mathcal{B}}(!(A^{\perp}) \otimes !(A^{\perp}) \multimap \Gamma)$. Then the interpretation of the proof π obtained from π' by application of the contraction rule is $[\![\pi]\!] = [\![\pi']\!] \circ \mathsf{c}_{A^{\perp}}$ where $\mathsf{c}_{A^{\perp}}$ is the comultiplication of the commutative comonoid $!(A^{\perp})$.
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(?A, \Gamma) = P_{\mathcal{B}}(!(A^{\perp}) \multimap \Gamma)$ by Lemma 6(5) suppose that $c \in P_{\mathcal{B}}(A^{\perp})(!\Delta)$. Then $[\![\pi]\!](\mathsf{prom}(c)) = [\![\pi']\!] \circ \mathsf{c}_{A^{\perp}} \circ \mathsf{prom}(c) = [\![\pi']\!] \circ (\mathsf{prom}(c) \otimes \mathsf{prom}(c)) \circ \mathsf{c}_{\Delta} \in P_{\mathcal{B}}(\Gamma)$ as $P_{\mathcal{B}}(\Gamma)$ is stable and $[\![\pi']\!](\mathsf{prom}(c) \otimes \mathsf{prom}(c)) \in P_{\mathcal{B}}(\Gamma)$ by induction hypothesis on π' and $\mathsf{prom}(c) \otimes \mathsf{prom}(c) \in P_{\mathcal{B}}(!(A^{\perp}) \otimes !(A^{\perp}))$.

(*Deriliction*) Suppose that π' is a proof of A, Γ satisfying induction hypothesis $[\![\pi']\!] \in P_{\mathcal{B}}(A, \Gamma) = P_{\mathcal{B}}(A^{\perp} \multimap \Gamma)$. Let π be the proof of $?A, \Gamma$ obtained from π'

by the dereliction rule. Then $[\![\pi]\!] = [\![\pi']\!] \circ \mathsf{read}_{A^\perp} : !(A^\perp) \multimap \Gamma$ (where $\mathsf{read}_{A^\perp} : !(A^\perp) \multimap A^\perp$ is the counit of the comonad ! at $A^\perp$).
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(?A, \Gamma) = P_{\mathcal{B}}(!(A^\perp) \multimap \Gamma)$ by Lemma 6(5) it suffices to show that $[\![\pi]\!](\mathsf{prom}(c)) \in P_{\mathcal{B}}(\Gamma)$ for all $c \in P_{\mathcal{B}}(A^\perp)$. Suppose that $c \in P_{\mathcal{B}}(A^\perp)$. Then $[\![\pi]\!](\mathsf{prom}(c)) = [\![\pi']\!] \circ \mathsf{read}_{A^\perp} \circ \mathsf{prom}(c) = [\![\pi']\!] \circ c \in P_{\mathcal{B}}(\Gamma)$ by induction hypothesis on π'.

(*Promotion*) Suppose that π' is a derivation of $?\Gamma, A$ satisfying the induction hypothesis $[\![\pi']\!] \in P_{\mathcal{B}}(?\Gamma, A) = P_{\mathcal{B}}(!(\Gamma^\perp) \multimap A)$. Then for the proof π of $?\Gamma, !A$ obtained from π' by application of the promotion rule (!) we have $[\![\pi]\!] = \mathsf{prom}([\![\pi']\!])$.
For showing that $[\![\pi]\!] \in P_{\mathcal{B}}(?\Gamma, !A) = P_{\mathcal{B}}(!(\Gamma^\perp) \multimap !A)$ by Lemma 6(5) it suffices to show that $[\![\pi]\!](\mathsf{prom}(g)) \in P_{\mathcal{B}}(B)$ for all $g \in P_{\mathcal{B}}(\Gamma^\perp)$. Now suppose that $g \in P_{\mathcal{B}}(\Gamma^\perp)$. Then $[\![\pi]\!](\mathsf{prom}(g)) = [\![\pi]\!] \circ \mathsf{prom}(g) = \mathsf{prom}([\![\pi']\!]) \circ \mathsf{prom}(g) = \mathsf{prom}([\![\pi']\!] \circ \mathsf{prom}(g)) \in P_{\mathcal{B}}(!A)$ as $[\![\pi']\!] \circ \mathsf{prom}(g) \in P_{\mathcal{B}}(A)$ by Lemma 6(5), the induction hypothesis on π' and $\mathsf{prom}(g) \in P_{\mathcal{B}}(!(\Gamma^\perp))$ (as $g \in P_{\mathcal{B}}(\Gamma^\perp)$). □

4 Denotational Completeness

In this section we are going to prove the reverse implication to the Soundness Theorem 7, i.e. completness, saying that a global element a of A is a proof object iff a (considered as a generalised element at stage 1) is contained in $P_{\mathcal{B}}(A)$ for all stable $\mathcal{B} \subseteq GE(\perp)$. Actually, we will prove something stronger namely that $Pr(A) = P_{Pr(\perp)}(A)$ for all types A.

But before, we have to establish some properties of $Pr(A)$.

Lemma 8 *For every type A we have that*

(1) $Pr(A) = \{id_{A^\perp}\}^{\perp_{Pr(\perp)}}$
(2) $Pr(A^\perp) = Pr(A)^{\perp_{Pr(\perp)}}$.

PROOF. The first claim holds as for $a \in GE(A)(\Gamma)$ we have that $\langle\, a \,|\, id_{A^\perp} \,\rangle = \mathsf{ev} \circ (a \otimes id_{A^\perp}) \in Pr(\perp)$ iff $a \in Pr(A)$ as $a = \mathsf{curry}(\mathsf{ev} \circ (a \otimes id_{A^\perp}))$ and generalised elements coming from proofs are closed under currying.

Now to the second claim. Due to (1) we have $Pr(A^\perp) = \{id_A\}^{\perp_{Pr(\perp)}}$. As $\langle\, id_A \,|\, id_{A^\perp} \,\rangle \in Pr(\perp)$ we get $\{id_{A^\perp}\} \subseteq \{id_A\}^{\perp_{Pr(\perp)}} = Pr(A^\perp)$ from which it follows that $Pr(A)^{\perp_{Pr(\perp)}} = \{id_{A^\perp}\}^{\perp_{Pr(\perp)}\perp_{Pr(\perp)}} \subseteq Pr(A^\perp)$. The reverse inclusion $Pr(A^\perp) \subseteq Pr(A)^{\perp_{Pr(\perp)}}$ holds as $\langle\, f \,|\, a \,\rangle \in Pr(\perp)$ whenever $f \in Pr(A^\perp)$ and $a \in Pr(A)$. □

Theorem 9 *For every formula A we have that $Pr(A) = P_{Pr(\perp)}(A)$.*

PROOF. The proof is by induction on the structure of A.
First notice that $Pr(A) = P_{Pr(\perp)}(A)$ entails $Pr(A^\perp) = P_{Pr(\perp)}(A^\perp)$ as we have $Pr(A^\perp) = Pr(A)^{\perp_{Pr(\perp)}}$ by Lemma 8(2) and $P_{Pr(\perp)}(A)^{\perp_{Pr(\perp)}} = P_{Pr(\perp)}(A^\perp)$ by Lemma 6(1). Notice that this observation reduces the number of cases to be considered to the half.
As for the purposes of this proof we consider no other $\mathcal{B}$ than $Pr(\perp)$ there is no danger of confusion when writing $(\_)^\perp$ instead of $(\_)^{\perp_{Pr(\perp)}}$ to improve readability.

($\perp$) $Pr(\perp) = P_{Pr(\perp)}(\perp)$ by definition of $P_{Pr(\perp)}$.

($\otimes$) Suppose as induction hypothesis that $\mathrm{Pr}(A_i) = P_{Pr(\perp)}(A_i)$ for $i = 1, 2$.
If $c \in Pr(A_1^\perp \wp A_2^\perp)$ and $a_i \in Pr(A_i)$ then $\langle\, c \,|\, a_1 \otimes a_2 \,\rangle \in Pr(\perp)$. Thus, $Pr(A_1^\perp \wp A_2^\perp) \subseteq \{\, a_1 \otimes a_2 \,|\, a_i \in P_{Pr(\perp)}(A_i) \,\}^\perp$ from which it follows that $\{a_1 \otimes a_2 | a_i \in P_{Pr(\perp)}(A_i)\}^{\perp\perp} \subseteq Pr(A_1^\perp \wp A_2^\perp)^\perp$. As by Lemma 8(2) we have $Pr(A_1 \otimes A_2) = Pr(A_1^\perp \wp A_2^\perp)^\perp$ it follows by definition of $P_{Pr(\perp)}$ that $P_{Pr(\perp)}(A_1 \otimes A_2) \subseteq Pr(A_1 \otimes A_2)$.
As $id_{A_1 \otimes A_2} = id_{A_1} \otimes id_{A_2} \in \{a_1 \otimes a_2 | a_i \in Pr(A_i)\}$ we have that $\{a_1 \otimes a_2 \,|\, a_i \in Pr(A_i)\}^\perp \subseteq \{id_{A_1 \otimes A_2}\}^\perp = Pr(A_1^\perp \wp A_2^\perp)$. Thus, we get that $Pr(A_1^\perp \wp A_2^\perp)^\perp \subseteq \{a_1 \otimes a_2 \,|\, a_i \in P_{Pr(\perp)}(A_i)\}^{\perp\perp} = P_{Pr(\perp)}(A_1 \otimes A_2)$. As by Lemma 8(2) we have $Pr(A_1 \otimes A_2) = Pr(A_1^\perp \wp A_2^\perp)^\perp$ it follows that $Pr(A_1 \otimes A_2) \subseteq P_{Pr(\perp)}(A_1 \otimes A_2)$.

($\top$) We have $Pr(\top) = GE(\top) = P_{Pr(\perp)}(\top)$ by definition $P_{Pr(\perp)}$.

($\&$) Suppose as induction hypothesis that $\mathrm{Pr}(A_i) = P_{Pr(\perp)}(A_i)$ for $i = 1, 2$.
If $c \in Pr(A_1^\perp \oplus A_2^\perp)$ and $a_i \in Pr(A_i)$ then $\langle\, c \,|\, a_1 \& a_2 \,\rangle \in Pr(\perp)$. Thus, $Pr(A_1^\perp \oplus A_2^\perp) \subseteq \{\, a_1 \& a_2 \,|\, a_i \in P_{Pr(\perp)}(A_i) \,\}^\perp$ from which it follows that $\{a_1 \& a_2 | a_i \in P_{Pr(\perp)}(A_i)\}^{\perp\perp} \subseteq Pr(A_1^\perp \oplus A_2^\perp)^\perp$. As by Lemma 8(2) we have $Pr(A_1 \& A_2) = Pr(A_1^\perp \oplus A_2^\perp)^\perp$ and, therefore, $P_{Pr(\perp)}(A_1 \& A_2) \subseteq Pr(A_1 \& A_2)$ by definition of $P_{Pr(\perp)}$.
As $id_{A_1 \& A_2} = id_{A_1} \& id_{A_2} \in \{a_1 \& a_2 \,|\, a_i \in Pr(A_i)\}$ we have $\{a_1 \& a_2 \,|\, a_i \in Pr(A_i)\}^\perp \subseteq \{id_{A_1 \& A_2}\}^\perp = Pr(A_1^\perp \oplus A_2^\perp)$. Thus, we get $Pr(A_1^\perp \oplus A_2^\perp)^\perp \subseteq \{a_1 \& a_2 | a_i \in P_{Pr(\perp)}(A_i)\}^{\perp\perp}$ and, therefore, $Pr(A_1 \& A_2) \subseteq P_{Pr(\perp)}(A_1 \& A_2)$ as $Pr(A_1 \& A_2) = Pr(A_1^\perp \& A_2^\perp)^\perp$ by Lemma 8(2).

($!$) Suppose that $\mathrm{Pr}(A) = P_{Pr(\perp)}(A)$.
If $c \in Pr(?(A^\perp))$ and $a \in Pr(A)$ then $\langle\, c \,|\, \mathsf{prom}(a) \,\rangle \in Pr(\perp)$. Thus, we have $Pr(?(A^\perp)) \subseteq \{\, \mathsf{prom}(a) \,|\, a \in P_{Pr(\perp)}(A) \,\}^\perp$ from which it follows that $\{\, \mathsf{prom}(a) \,| a \in P_{Pr(\perp)}(A)\}^{\perp\perp} \subseteq Pr(?(A^\perp))^\perp$. As by Lemma 8(2) we have $Pr(!A) = Pr(?(A^\perp))^\perp$ it follows by definition of $P_{Pr(\perp)}$ that $P_{Pr(\perp)}(!A) \subseteq Pr(!A)$.
As $id_{!A} = \mathsf{prom}(\mathsf{read}_A) \in \{\, \mathsf{prom}(a) \,|\, a \in Pr(A) \,\}$ we have $\{\, \mathsf{prom}(a) \,|\, a \in Pr(A)\}^\perp \subseteq \{id_{!A}\}^\perp = Pr(?(A^\perp))$ from which it follows that $Pr(?(A^\perp))^\perp \subseteq \{\, \mathsf{prom}(a) \,|\, a \in P_{Pr(\perp)}(A)\}^{\perp\perp}$. Thus, we have $Pr(!A) \subseteq P_{Pr(\perp)}(!A)$ as $Pr(!A) = Pr(?(A^\perp))^\perp$ by Lemma 8(2). $\square$

Theorem 10 (Completeness)
A global element a of A, i.e. $a \in GE(A)(1)$, is a proof object of A if and only if *$a \in P_{\mathcal{B}}(A)$ for all stable $\mathcal{B} \subseteq GE(\perp)$.*

PROOF. Immediate from the Soundness Theorem 7 and the Theorem 9. $\square$

5 A Denotationally Complete Model

Up to now we have used "biorthogonal closed Kripke logical predicates" for characterising "proof objects", i.e. those global elements of types that arise as interpretations of formal derivations. As usual one may adapt such a characterisation into a construction of a denotationally complete model (as was done also by J.-Y. Girard in [4]). Although for this purpose we might start with an arbitrary model $\mathbb{C}$ of linear logic, however, for reasons of concreteness we further assume that $\mathbb{C}$ is the widely known coherence space model described e.g. in [3]. Again we write $\mathbb{V}$ for the (non-full) sub-category of $\mathbb{C}$ of variable substitutions between (denotations) of contexts.

Definition 11 *Let $\mathbb{M}$ be the category which is defined as follows.*
An object of $\mathbb{M}$ is a pair $A = (|A|, P(A))$ where $|A|$ is a coherence space and $P(A)$ is a family indexed by stable subsets $\mathcal{B}$ of $GE(\perp)$ such that $P(A)_{\mathcal{B}}$ is a Kripke logical predicate on $|A|$ of kind $\mathcal{B}$. A morphism in $\mathbb{M}$ from A to B is a linear map $f : |A| \to |B|$ such that $f \circ a \in P(B)_{\mathcal{B}}$ for all stable $\mathcal{B} \subseteq GE(\perp)$ and $a \in P(A)_{\mathcal{B}}$.
This category $\mathbb{M}$ can be endowed with the structure of a model of linear logic using that $\mathbb{C}$ itself is a model of linear logic and constructing the Kripke logical predicates as described in Definition 5. The required morphisms are constructed as in $\mathbb{C}$ and turn out to preserve all the required Kripke logical relations due to the Soundness Theorem 7.

Now the following is immediate.

Theorem 12 (Denotational Completeness)
If A is a type and a is a global section of $[\![A]\!]_{\mathbb{M}}$ in $\mathbb{M}$ then there exists a derivation π of A such that $[\![\pi]\!]_{\mathbb{M}} = a$.

PROOF. If a is a global element of $[\![A]\!]_{\mathbb{M}}$ in $\mathbb{M}$ then – by construction of $\mathbb{M}$ – a is a global element of $[\![A]\!]_{\mathbb{C}}$ in $\mathbb{C}$ satisfying all Kripke logical predicates. Therefore, by Theorem 10 it follows that $a = [\![\pi]\!]_{\mathbb{C}} = [\![\pi]\!]_{\mathbb{M}}$ for some derivation π of A. $\square$

Notice that this model $\mathbb{M}$ is not extensional although it is denotationally complete. This situation is comparable to the situation in game semantics where one constructs "intensionally full abstract models" that are not extensional though all morphism in the model arise as interpretations of terms. One might try to remedy this situation by restricting the coherence spaces to those elements which are invariant under all Kripke logical relations but, unfortunately, these restrictions are not coherence spaces anymore. [8]

6 Discussion

We consider our characterisation of proof objects via Kripke logical predicates for arbitrary models of linear logic as an alternative to J.-Y. Girard's characterisation of proof objects in the coherence space model using $\wp$–monoids combined with ordinary phase semantics, see [4]. His characterisation of proof objects is absolutley syntax–free whereas in our account we make essential use of the category $\mathbb{V}$ of variable substitutions which might be regarded as a "very faint shadow of syntax". But $\mathbb{V}$ is so trivial compared to the collection of all proof objects that we think it is not mere cheating what we achieved. However, as a reward for "cheating a bit" when using $\mathbb{V}$ we gain the following advantages.

Firstly, our proof of completeness is fairly simple compared to Girard's rather complicated construction of an appropriate $\wp$–monoid in his paper "On Denotational Completeness" [4] . Moreover, our methode works for arbitrary [9] models of linear logic and not only for the coherence space model.

Secondly, we perfectly can avoid considering an additional phase semantics component as this appears to be "built in" already. Namely, if A is a coherence space with an empty web [10] then in Girard's approach for every $\wp$–monoid $\mathbb{P}$ there is only the empty clique in $\mathbb{P}\wp A$ which forces him to add an ordinary monoid $\mathbb{M}$ providing the phase semantics component needed to distinguish those empty cliques which are proof objects from the empty cliques that don't arise as interpretations of proofs. In our approach there is no need for this as denotational facts of an A with empty web are simply (in obvious 1-1-correspondence with) certain *sets of contexts* as $GE(A) \cong |\mathbb{V}|$ if $|A| = \emptyset$. In particular, under this identification a denotational $\mathcal{B}$–fact of A is nothing but a fact w.r.t. the monoid of contexts (under concatenation) with $\perp := \mathcal{B}$.

[8] When constructing fully abstract models of PCF this methods works because the closure of a subset of cpo is a cpo again provided it contains the least element. Unfortunately, coherence spaces are not so robust under taking substructures.
[9] as e.g. Chu spaces or dePaiva's Dialectica categories
[10] as e.g. those of the form $A \otimes \top$, $A \otimes 0$, $A\wp\top$ and $A\wp 0$

Finally, notice that in the sense of "logical complexity" our characterisation of proof objects in the coherence space model by some sort of invariants does not provide "simpler" characterisation than the obvious one as syntactic definablity which, obviously, is an r.e. condition on finite cliques whereas ours is of much higher logical complexity. Nevertheless, a characterisation of syntactic definablity by invariants may be considered as more "elegant" in the sense that one "avoids any reference to syntax". But this is a merely aesthetic criterion which probably cannot be given a precise mathematical meaning. In particular, the achievements of denotational completeness do not seem to throw any light on the following question
Is it decidable whether a clique in the coherence space model comes from a proof ?

Full completness results of another flavour have been obtained e.g. in A. M. Tan's PhD Thesis [8] where it is proved that every dinatural transformation on an arbitrary model of multiplicative linear logic (MLL), as e.g. in particular the coherence space model, appears as denotation of a proof in MLL. As all denotations of derivations in MLL give rise to dinatural transformations these are precisely the proof objects of MLL. Alas, this doesn't give any handle on the question above as dinatural transformations are inherently infinite objects and, furthermore, it is not clear how to extend her results to all linear connectives, in particular the exponentials. However, a comparison and integration of both approaches is a topic worthwhile for future investigation.

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