Computing Optimal Discrete Morse Functions

Extended Abstract

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March 22, 2004

Abstract

The essential structural information of discrete Morse functions is captured by so-called Morse matchings. We show that computing optimal Morse matchings is $\mathcal{NP}$-hard and give an integer programming formulation for the problem. Then we present first polyhedral results for the corresponding polytope and report on some preliminary computational results.

1 Introduction

Discrete Morse functions where introduced by Forman [3] as a combinatorial analogy of classical smooth Morse theory and have many applications in combinatorial topology, e.g., they can be used to compute a compact representation of a simplicial complex as an CW-complex; for details and other applications see [3], Chari [1], and Joswig [6]. It turns out that the essential information of discrete Morse functions can be stored in a *Morse matching*. To be concise, we will therefore not give the definition of discrete Morse functions but state everything in terms of Morse matchings. In the applications one is interested in optimal Morse matchings, a problem which leads to a combinatorial optimization problem that we will describe in the following.

We first need some notation. Let Δ be a *(finite abstract) simplicial complex*, i.e., a set of subsets of a finite set V with the following property: if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$; hence Δ is an independence system with ground set V. In the following we will ignore $\varnothing$ as a member of Δ. The elements in V are called *vertices* and the elements of Δ are called *faces*. The *dimension* of a face F is $\dim F := |F| - 1$. In the following let $d = \max\{\dim F : F \in \mathcal{F}\}$ be the dimension of Δ. Let $\mathcal{F}$ be the set of faces of Δ and let $f_i = f_i(\Delta)$ be the number of faces of dimension $i \geq 0$. The maximal faces with respect to inclusion are called *facets* and Δ is *pure*, if all facets have the same dimension.

Consider the *Hasse diagram* $H = (\mathcal{F}, A)$ of Δ, that is, a directed graph on the faces of Δ with an arc $(F, G) \in A$ if $G \subset F$ and $\dim G = \dim F - 1$. It will be convenient not to distinguish between H and its underlying undirected graph, i.e., when we speak of matchings and (undirected) cycles we mean the corresponding structures in the underlying undirected graph.

Let $M \subset A$ be a matching in H and let $H(M)$ be the directed graph obtained from H by reversing the direction of the arcs in M. Then M is a *Morse matching* of Δ if $H(M)$ does not contain directed cycles, i.e., is acyclic (in the directed sense). Chari [1] showed that the essential structure of discrete Morse functions are contained in Morse matchings. As stated above, one is interested in maximum Morse matchings, i.e., the size of M is maximized. The complementary measure to $|M|$ is the number of *critical faces* of M, i.e., faces not matched by M. Hence, by maximizing $|M|$, we minimize the number of critical faces.

It seems helpful to briefly describe the case of Morse matchings for a one-dimensional simplicial complex Δ. Then Δ represents the incidences of a graph G. A Morse matching M of Δ matches edges with nodes of G. Let $\tilde{G}$ be the following oriented subgraph of G: take all edges which are matched in M and orient them towards its matched node. Since M is a matching this construction is well defined and the in-degree of each node is one. The acyclicity property shows that $\tilde{G}$ contains no directed cycles and hence is a branching. Therefore, the Morse matchings on a graph G are in one-to-one correspondence with orientations of subgraphs of G which are branchings. Generalizing this idea, Lewiner, Lopes, and Tavares [7] developed a heuristic for computing optimal Morse matchings, which works well for the data set which we also use in Section 4.

In the following we will show that computing optimal Morse matchings is $\mathcal{NP}$-hard. Then we will give an integer programming formulation for the problem and sketch polyhedral results for the corresponding polytope. We end with some preliminary computational results.

2 Hardness of Computing an Optimal Morse Matching

Eğecioğlu and Gonzalez [2] proved a hardness result which in terms of Morse matchings reads as follows: Given a pure 2-dimensional simplicial complex Δ and an integer K, it is $\mathcal{NP}$-complete to decide whether there exist a Morse matching with at most K critical 2-faces, i.e., faces of dimension 2. In fact we can remove the “restriction” to 2-faces and prove:

Theorem 2.1. *Given a simplicial complex Δ and an integer K, it is $\mathcal{NP}$-complete to decide whether there exists a Morse matching with at most K critical simplices.*

This result holds even when Δ is connected (i.e., H is connected), pure, 2-dimensional, and it can be embedded in $\mathbb{R}^3$. A crucial part in the proof of this theorem is the following lemma:

Lemma 2.2. *Given any Morse matching M on Δ, we can compute a Morse matching M' which has exactly one critical vertex and at most as many critical 2-faces as M.*

This lemma and the Euler equation make it possible to reduce the general case to the problem discussed by Eğecioğlu and Gonzalez. In fact, they proved strong inapproximability results for their problem. Lewiner, Lopes, and Tavares [7] claimed the same inapproximability results for computing Morse matchings with a minimum number of critical faces, but did not supply a reasoning as in the lemma. The reduction in our proof is not approximation preserving. Therefore, the approximability status seems to be open; the same holds for computing maximum Morse matchings.

3 An IP-formulation

In this section we will discuss an integer programming formulation of the problem to compute a maximum Morse matching. We introduce a binary variable x_a for every arc in H, where $x_a = 1$ if and only if a should be reversed in a Morse matching. The matching conditions are modeled by:

$$\boldsymbol{x}(\delta(F)) := \sum_{a \in \delta(F)} x_a \leq 1 \qquad \forall\, F \in \mathcal{F}. \tag{1}$$

To handle the acyclicity requirement, let M be a Morse matching and assume C to be a directed cycle in $H(M)$. Because of the matching property, the nodes in C can only belong to two levels in the Hasse diagram, i.e., $\{\dim F \,:\, F \in C\} = \{i, i+1\}$ for some $i \in \{0, \ldots, d-1\}$. Therefore define H_i to be the subgraph of H induced by the faces of dimension i and $i+1$, for $i \in \{0, \ldots, d-1\}$. Again by the matching property, the values x_a for the arcs in C alternate. A little thought reveals that the following constraints suffice to eliminate directed cycles:

$$\boldsymbol{x}(C) := \sum_{a \in C} x_a \leq \frac{|C|}{2} - 1 \qquad \forall\, C \text{ cycle in } H_i,\ i = 0, \ldots, d-1. \tag{2}$$

Hence, the convex hull of all incidence vectors of Morse matchings is the following polytope:

$$P_M := \big\{\boldsymbol{x} \in \{0,1\}^A \,:\, \boldsymbol{x} \text{ satisfies (1) and (2)}\big\}.$$

A Morse matching with incidence vector $\boldsymbol{x} \in P_M$ has $|\mathcal{F}| - 2\,\boldsymbol{x}(A)$ critical faces. The problem to compute an optimal Morse matching is then to solve $\max\{\boldsymbol{x}(A) \,:\, \boldsymbol{x} \in P_A\}$.

It is easy to see that P_M is a monotone, full dimensional polytope and that $x_a \geq 0$ defines a facet for every $a \in A$. Let us remark that the incidence vectors of Morse matchings do not have to be monotone if H is an arbitrary acyclic digraph. We can prove the following results:

Proposition 3.1. *The matching constraints $\boldsymbol{x}(\delta(F)) \leq 1$ define facets of P_M for $F \in \mathcal{F}$, except if $|\delta(F)| \leq 1$.*

It follows that the inequalities $x_a \leq 1$, $a \in A$, never define facets.

Proposition 3.2. *The cycle constraints (2) define facets of P_M and can be separated in polynomial time.*

Some of the features of our problem resemble the acyclic subgraph problem (ASP), studied by Grötschel, Jünger, and Reinelt [4]. The separation algorithm referred to in Proposition 3.2, however, is more complicated than the one for ASP, since the usual affine transformation trick ($\boldsymbol{x}' = \mathbb{1} - \boldsymbol{x}$) to turn the separation problem into a shortest cycle problem does not work in our case.

One can strengthen the LP relaxation considerably by adding so called *Morse inequalities*, which say that the number of critical faces of dimension i is at least the Betti number β_i, see Forman [3]. This translates to the inequality $\sum_{F \in \mathcal{F}_i} \boldsymbol{x}(\delta(F)) \leq f_i - \beta_i$.

4 Computational Results

We performed preliminary computational experiments with a branch-and-cut code for the above integer programming formulation. The algorithm was implemented using the branch-and-cut-and-price framework SCIP, developed by Tobias Achterberg at the Zuse Institute Berlin. We computed Morse matchings for the smaller problems in a collection of simplicial complexes maintained by Hachimori [5]. As a primal heuristic we used a simple greedy algorithm. Whenever possible we branched as follows: for a face $F \in \mathcal{F}$, we branch on the following three constraints: $\boldsymbol{x}(\delta^-(F)) = 1$, $\boldsymbol{x}(\delta^+(F)) = 1$, $\boldsymbol{x}(\delta(F)) = 0$; this seems to work very well. Additionally, we added Gomory cuts.

Computing optimal Morse matchings in practice appears to be hard for relatively small problems. The reason seems to be the high symmetry of the problems and the weakness of the LP relaxation. One has a good chance, however, if the absolute difference between the optimal value and the bounds implied by the Morse inequalities is small. In fact, for many of the problems in Hachimori's collection this difference is 0 and the algorithm "only" has to find the optimal primal solution, which it usually finds fast. Summarizing, our code can solve all 10 problems in the collection with up to 160 arcs in the Hasse diagram (and two larger ones) in about an hour; about 50% of these problems are solved in a few seconds.

It is clear, that there are still many things to investigate. Our plan for the future is to find other (facet defining) inequalities for P_A that can help to improve the dual bound. Furthermore, it seems interesting to check whether local search methods can help to improve primal solutions.

References

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