On the Maximum Feasible Subsystem Problem, IISs and IIS-hypergraphs*

Edoardo Amaldi[†] Marc E. Pfetsch[‡] Leslie E. Trotter, Jr.[§]

11/5/2002

Abstract

We consider the MAX FS problem: For a given infeasible linear system $A\boldsymbol{x} \leq \boldsymbol{b}$, determine a feasible subsystem containing as many inequalities as possible. This problem, which is NP-hard and also difficult to approximate, has a number of interesting applications in a wide range of fields. In this paper we examine structural and algorithmic properties of MAX FS and of *Irreducible Infeasible Subsystems* (IISs), which are intrinsically related since one must delete at least one constraint from each IIS to attain feasibility. First we provide a new *simplex decomposition* characterization of IISs and prove that finding a smallest cardinality IIS is very difficult to approximate. Then we discuss structural properties of IIS-hypergraphs, i.e., hypergraphs in which each edge corresponds to an IIS, and show that recognizing IIS-hypergraphs subsumes the Steinitz problem for polytopes and hence is NP-hard. Finally we investigate rank facets of the *Feasible Subsystem polytope* whose vertices are incidence vectors of feasible subsystems of a given infeasible system. In particular, using the IIS-hypergraph structural result, we show that only two very specific types of rank inequalities induced by generalized antiwebs (which generalize cliques, odd holes and antiholes to general independence systems) can arise as facets.

Key words. Infeasible linear systems – feasible subsystems – Irreducible Infeasible Subsystem (IIS) – IIS-hypergraphs – independence systems – Feasible Subsystem polytope – rank facets

1 Introduction

We consider the following combinatorial optimization problem related to infeasible linear inequality systems.

MAX FS: *Given an infeasible system* $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ *with* $A \in \mathbb{R}^{m \times n}$ *and* $\boldsymbol{b} \in \mathbb{R}^m$, *find a feasible subsystem containing as many inequalities as possible.*

*Part of this work was done while the first two authors were with the School of OR&IE, Cornell University, USA. A preliminary version appeared in the Proceedings of the 10th IPCO conference [7], held in Graz, Austria, June 1999. This work was supported by NSF grant DMS-9527124.

[†]DEI, Politecnico di Milano, Piazza L. da Vinci 32, 20133 Milano, Italy,
email: amaldi@elet.polimi.it

[‡]Department of Mathematics, MA 6–2, TU Berlin, 10623 Berlin, Germany,
email: pfetsch@math.tu-berlin.de

[§]School of OR&IE, Rhodes Hall, Cornell University, Ithaca, NY 14853, USA,
email: ltrotter@cs.cornell.edu

Weighted and unweighted versions of this problem have a number of interesting applications in various fields such as operations research, computational geometry, statistical discriminant analysis and machine learning (see [2, 10, 29, 31, 34, 38, 43] and the references therein).

In linear programming (LP) it arises when the formulation phase yields infeasible models and one wishes to diagnose and resolve infeasibility by deleting as few constraints as possible, which is the complementary version of MAX FS [19, 28, 39]. In most situations this cannot be done by inspection and the need for effective algorithmic tools has become more acute with the considerable increase in model size. This type of questions was first addressed in [48]. The reader is referred to [27] for a survey on redundant and implied relations of inequality systems as well as on infeasibility issues. From the computational complexity point of view, MAX FS is NP-hard [46] even when the matrix A is totally unimodular and $\boldsymbol{b}$ is integer; it can be approximated within a factor 2 but it does not admit a polynomial-time approximation scheme, unless P = NP [4]. The above-mentioned complementary version, in which the goal is to delete as few inequalities as possible in order to achieve feasibility, is equivalent to solve to optimality but is much harder to approximate than MAX FS [5, 8].

Not surprisingly, minimal infeasible subsystems, discussed for instance in the thesis of Motzkin [37], play a key role in the study of MAX FS. An infeasible subsystem Σ' of Σ is an *Irreducible Infeasible Subsystem* (IIS) if every proper subsystem of Σ' is feasible. In order to help the modeler resolve infeasibility of large linear inequality systems, attention was first devoted to the problem of identifying IISs, with a small and possibly minimum number of inequalities [28]; see [20, 22, 47] for some heuristics and [18] for implementations in commercial solvers such as CPLEX and MINOS. Clearly, in the presence of many overlapping IISs, this does not provide enough information to repair the original system. To achieve feasibility, one must delete at least one inequality from each IIS. If all IISs were known, the complementary version of MAX FS could be formulated as the following covering problem [26].

MIN IIS COVER: *Given an infeasible system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ with $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$ and the set $\mathcal{C}$ of all its IISs, minimize $\sum_{i=1}^{m} y_i$ subject to $\sum_{i\in C} y_i \geq 1$ $\forall\, C \in \mathcal{C}$, $y_i \in \{0,1\}$, $1 \leq i \leq m$.*

Note that $|\mathcal{C}|$ can grow exponentially with m and n [17].

An exact algorithm based on a partial cover formulation is proposed in [38, 39] and several heuristics are described in [10, 19, 21, 34]; a collection of infeasible LPs is maintained in the Netlib Repository [41]. In [44, 45] the class of hypergraphs representing the IISs of infeasible systems is studied and it is shown that in some special cases MAX FS and MIN IIS COVER can be solved in polynomial time in the number of IISs.

Although MAX FS with 0-1 variables can be easily shown to admit as a special case the graphical problem of finding a maximum stable set of nodes [4], it has a different structure when the variables are real-valued. Note that, since linear system feasibility can be checked in polynomial time, MAX FS structure also differs substantially from that of the maximum satisfiability problem aimed at satisfying a maximum number of disjunctive Boolean clauses. The reader is referred to [25] for the exact definitions of these well-known problems.

Variants of the classical Agmon-Motzkin-Schoenberg relaxation method for solving linear inequality systems have also been investigated and used, among others, in machine learning as well as image and signal processing applications (see e.g. [2, 3, 6, 24]). The implicit enumeration technique described in [29] for optimizing general functions of a set of linear relations can, in principle, also be applied to the special case of MAX FS. As to more recent work on problems related to MAX FS and IISs let us mention, for instance, Håstad's breakthrough [30] which bridges the approximability gap for MAX FS on $GF(p)$, and the problems of determining minimum or minimal witnesses of infeasibility in network flows [1].

In this paper we investigate some structural and algorithmic properties of IISs, of IIS-hypergraphs in which each edge corresponds to an IIS, and of the feasible subsystem polytope defined by the convex hull of incidence vectors of feasible subsystems of a given infeasible system. In Section 2 we provide a new IIS simplex decomposition characterization and prove that finding a smallest cardinality IIS is very difficult to approximate. In Section 3 we first discuss the connection between IIS-hypergraphs and vertex-facet incidences of polyhedra which is needed in the sequel. Based on this connection we also derive that the problem of recognizing IIS-hypergraphs is NP-hard since it subsumes the well-known Steinitz problem for polytopes. In Section 4 we investigate rank facets of the feasible subsystem polytope. In particular, we focus attention on the rank inequalities arising from generalized antiwebs, which generalize cliques, odd holes and antiholes to general independence systems [33]. Finally, the appendix contains the proof of a result stated in Section 3 which completes the discussion but is not required in Section 4.

Below we denote the ith row of the matrix $A \in \mathbb{R}^{m\times n}$ by $\boldsymbol{a}^i \in \mathbb{R}^n$, $1 \leq i \leq m$; for $S \subseteq [m] := \{1, \ldots, m\}$, A_S denotes the $|S| \times n$ matrix consisting of the rows of A indexed by S. By identifying the ith inequality of the system Σ (i.e., $\boldsymbol{a}^i\boldsymbol{x} \leq b_i$) with index i itself, $[m]$ may also refer to Σ.

2 Irreducible Infeasible Subsystems

First we briefly recall the main structural results regarding IISs. For notational simplicity, we use the same A and $\boldsymbol{b}$, with $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, to denote either the original system Σ or one of its IISs.

The known characterizations of IISs are based on the following version of the Farkas Lemma: *For any linear inequality system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$, either $A\boldsymbol{x} \leq \boldsymbol{b}$ is feasible or $\exists\, \boldsymbol{y} \in \mathbb{R}^m,\ \boldsymbol{y} \geq \boldsymbol{0}$, such that $\boldsymbol{y}A = \boldsymbol{0}$ and $\boldsymbol{y}\boldsymbol{b} < 0$, but not both.*

Theorem 1 (Motzkin [37], Fan [23]). *The system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ with $A, \boldsymbol{b}$ as above is an IIS if and only if rank(A) = $m-1$ and $\exists\, \boldsymbol{y} \in \mathbb{R}^m$, $\boldsymbol{y} > 0$, such that $\boldsymbol{y}A = \boldsymbol{0}$ and $\boldsymbol{y}\boldsymbol{b} < 0$.*

The rank condition obviously implies that $m \leq n+1$.

Now let $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ be an infeasible system which is not necessarily an IIS. The following result relates the IISs of Σ to the vertices of a given *alternative polyhedron*. Recall that the *support* of a vector is the set of indices of its nonzero components.

Theorem 2 (Gleeson and Ryan [26]). *Let $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ be an infeasible system with $A, \boldsymbol{b}$ as above. Then the IISs of Σ are in one-to-one correspondence with the vertices of the polyhedron*

$$P := \{\boldsymbol{y} \in \mathbb{R}^m \,|\, \boldsymbol{y}A = \boldsymbol{0},\ \boldsymbol{y}\boldsymbol{b} = -1,\ \boldsymbol{y} \geq \boldsymbol{0}\}\,.$$

In particular, the nonzero components of any vertex of P index an IIS.

See [39] for this statement that slightly extends the original result.

Theorem 2 can also be stated in terms of rays [39] and elementary vectors [27].

Definition 1. *An* elementary vector *of a subspace $L \subseteq \mathbb{R}^m$ is a nonzero vector $\boldsymbol{y}$ that has minimal support (when expressed with respect to the standard basis of $\mathbb{R}^m$). In other words, if $\boldsymbol{x} \in L$ and $\operatorname{supp}(\boldsymbol{x}) \subset \operatorname{supp}(\boldsymbol{y})$ then $\boldsymbol{x} = \boldsymbol{0}$, where $\operatorname{supp}(\boldsymbol{y})$ denotes the support of $\boldsymbol{y}$.*

Corollary 1 (Greenberg [27]). *Let $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ be an infeasible system with $A, \boldsymbol{b}$ as above. Then $S \subseteq [m]$ corresponds to an IIS of Σ if and only if there exists an elementary vector $\boldsymbol{y}$ in the subspace $L := \{\boldsymbol{y} \in \mathbb{R}^m \,|\, \boldsymbol{y}A = \boldsymbol{0}\}$ with $\boldsymbol{y}\boldsymbol{b} < 0$ and $\boldsymbol{y} \geq \boldsymbol{0}$ such that $S = \operatorname{supp}(\boldsymbol{y})$.*

The following result establishes an interesting geometric property of the polyhedra obtained by deleting any inequality from an IIS.

Theorem 3 (Motzkin [37]). *Let $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ be an IIS and let $\sigma \in \Sigma$ be an arbitrary inequality of Σ. Then the polyhedron corresponding to $\Sigma \setminus \sigma$, i.e., the subsystem obtained by removal of σ, is an affine convex cone.*

2.1 IIS simplex decomposition

We provide here a new geometric characterization of IISs with at least two inequalities, that is $m \geq 2$. For $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, let $A^i := A_{[m]\setminus\{i\}}$ and $\boldsymbol{b}^i := \boldsymbol{b}_{[m]\setminus\{i\}}$ denote the $(m-1) \times n$ submatrix and, respectively, the $(m-1)$–dimensional vector obtained by removing the ith row of A and ith component of $\boldsymbol{b}$. The following result strengthens the necessity of Theorem 1.

Lemma 1. *Let $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ be an IIS. Then A^i has linearly independent rows, for all $1 \leq i \leq m$; i.e., $\operatorname{rank}(A^i) = m - 1$.*

Proof. According to Theorem 1, there exists a $\boldsymbol{y} > \boldsymbol{0}$ such that $\boldsymbol{y}A = \boldsymbol{0}$ and $\boldsymbol{y}\boldsymbol{b} = -1$ (by scaling $\boldsymbol{y}\boldsymbol{b} < 0$). Suppose some proper subset of rows is linearly dependent; i.e., there exists $\boldsymbol{z}$, such that $\boldsymbol{z}A = \boldsymbol{0}$, $\boldsymbol{z}\boldsymbol{b} \geq 0$ (without loss of generality) and some $z_k = 0$.

If some component $z_i > 0$, consider $(\boldsymbol{y} - \epsilon\boldsymbol{z})A = 0,\ (\boldsymbol{y} - \epsilon\boldsymbol{z})\boldsymbol{b} \leq -1$, where $\epsilon = \min\{y_i/z_i \,|\, 1 \leq i \leq m,\ z_i > 0\} > 0$ (and $\boldsymbol{y}$ is as above). Then $\boldsymbol{y} - \epsilon\boldsymbol{z} \geq 0$, at least one additional component of $\boldsymbol{y} - \epsilon\boldsymbol{z}$ is 0, and the Farkas Lemma contradicts minimality of the system ($\boldsymbol{y} - \epsilon\boldsymbol{z}$ fulfills the requirements).

If all $z_i \leq 0$, then $-\boldsymbol{z} \geq \boldsymbol{0},\ -\boldsymbol{z}A = \boldsymbol{0}$ and $-\boldsymbol{z}\boldsymbol{b} \leq 0$; so setting $\boldsymbol{y} = -\boldsymbol{z}$ in the Farkas Lemma leads to a contradiction of minimality, provided $-\boldsymbol{z}\boldsymbol{b} < 0$. If $-\boldsymbol{z}\boldsymbol{b} = 0$, then $(\boldsymbol{y}+\epsilon\boldsymbol{z})A = \boldsymbol{0},\ (\boldsymbol{y}+\epsilon\boldsymbol{z})\boldsymbol{b} = -1$, with $\epsilon = \min\{y_i/(-z_i) \,|\, 1 \leq i \leq m,\ -z_i > 0\}$ leads to a contradiction as above. □

It is interesting to note that this lemma together with Theorem 1 imply that an infeasible system $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ is an IIS if and only if $\operatorname{rank}(A^i) = m - 1$ for all i, $1 \leq i \leq m$.

We then have the following *simplex decomposition* result for IISs.

Theorem 4. *The system $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ is an IIS if and only if $\{A\boldsymbol{x} = \boldsymbol{b}\}$ is infeasible and $\{\boldsymbol{x} \in \mathbb{R}^n \,|\, A\boldsymbol{x} \geq \boldsymbol{b}\} = L + Q$, where L is the lineality subspace $\{\boldsymbol{x} \in \mathbb{R}^n \,|\, A\boldsymbol{x} = \boldsymbol{0}\}$ and Q is an $(m-1)$-simplex with vertices determined by maximal proper subsystems of $\{A\boldsymbol{x} = \boldsymbol{b}\}$; namely, each vertex of Q is a solution for a subsystem $\{A^i\boldsymbol{x} = \boldsymbol{b}^i\}$, $1 \leq i \leq m$.*

Proof. (⇒) The system $\{A\boldsymbol{x} = \boldsymbol{b}\}$ is obviously infeasible. To see the feasibility of $\{A\boldsymbol{x} \geq \boldsymbol{b}\}$, delete constraint $\boldsymbol{a}^i\boldsymbol{x} \geq b_i$ to get the equality system $\{A^i\boldsymbol{x} = \boldsymbol{b}^i\}$. By Lemma 1, this system has a solution, say $\boldsymbol{x}^i$, and we have $\boldsymbol{a}^i\boldsymbol{x}^i > b_i$, else $\boldsymbol{x}^i$ satisfies $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$. Applying the polyhedral resolution theorem, $P := \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \geq \boldsymbol{b}\} \neq \emptyset$ can be written as $P = K + Q$, where $K = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \geq \boldsymbol{0}\}$ is its recession cone and $Q \subseteq P$ is a polytope generated by representatives of its minimal nonempty faces.

If $\boldsymbol{x}$ satisfies $A\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{a}^i\boldsymbol{x} > 0$ for row $\boldsymbol{a}^i$ then $\boldsymbol{x}^i - \epsilon\boldsymbol{x}$ satisfies $A(\boldsymbol{x}^i - \epsilon\boldsymbol{x}) \leq \boldsymbol{b}$ for sufficiently large $\epsilon > 0$ and the original system $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ would be feasible. Therefore we must have that each $\boldsymbol{a}^i\boldsymbol{x} = 0$ for $1 \leq i \leq m$, $\boldsymbol{x} \in K$ and we get that in fact $K = L := \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{0}\}$.

For Q, minimal nonempty faces of P are given by changing a maximal set of inequalities into equalities (all but one relation). Thus the vectors $\boldsymbol{x}^i$ obtained by solving $\{A^i\boldsymbol{x} = \boldsymbol{b}^i\}$ determine Q; i.e., $Q = \operatorname{conv}(\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m\})$. For $A \in \mathbb{R}^{m\times n}$, Q is the $(m-1)$-simplex generated by the m points $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m\}$. To see that the $\boldsymbol{x}^i$ generate an $(m-1)$-simplex, we must only show that they are affinely independent. But if $\boldsymbol{x}^i$ is affinely dependent on the other $\boldsymbol{x}^j$, then $\boldsymbol{x}^i = \sum_{j\neq i} \lambda_j \boldsymbol{x}^j$ with $\sum_{j\neq i} \lambda_j = 1$. Thus we have $\boldsymbol{a}^i\boldsymbol{x}^i > b_i$, but $\boldsymbol{a}^i\boldsymbol{x}^i = \boldsymbol{a}^i(\sum_{j\neq i} \lambda_j \boldsymbol{x}^j) = \sum_{j\neq i} \lambda_j(\boldsymbol{a}^i\boldsymbol{x}^j) = \sum_{j\neq i} \lambda_j b_i = b_i$, which is a contradiction.

(⇐) If the system $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ is infeasible, then the minimality is obvious, because the simplex conditions on Q imply that every proper subsystem has an equality solution.

To show that $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ is infeasible, assume for the sake of contradiction that $\hat{\boldsymbol{x}} \in \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}\} \neq \emptyset$ and $\hat{\boldsymbol{x}}$ satisfies a maximal number of these relations at equality. Since $A\boldsymbol{x} = \boldsymbol{b}$ is assumed to be infeasible, we have $A\hat{\boldsymbol{x}} \neq \boldsymbol{b}$, i.e., there exists $i \in [m]$ with $\boldsymbol{a}^i\hat{\boldsymbol{x}} < b_i$. Let $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m$ be the vertices of Q, where $\boldsymbol{x}^i$ is a solution of $\{A^i\boldsymbol{x} = \boldsymbol{b}^i\}$ for $i = 1, \ldots, m$. Similarly, the above assumption together with the fact that $Q \subseteq \{\boldsymbol{x} \mid A\boldsymbol{x} \geq \boldsymbol{b}\}$ implies that $\boldsymbol{a}^i\boldsymbol{x}^i > b_i$. Thus we can take $\lambda = (\boldsymbol{a}^i\boldsymbol{x}^i - b_i)/(\boldsymbol{a}^i\boldsymbol{x}^i - \boldsymbol{a}^i\hat{\boldsymbol{x}})$ and have $0 < \lambda < 1$, so that $\boldsymbol{a}^i(\lambda\hat{\boldsymbol{x}} + (1-\lambda)\boldsymbol{x}^i) = b_i$.

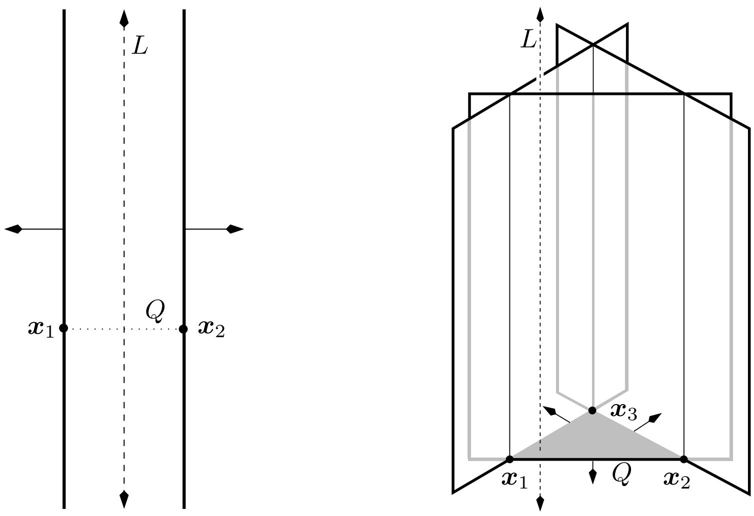


Figure 1: Illustrations of Theorem 4 in dimensions $n = 2$ and $n = 3$. The IISs corresponding to $A\boldsymbol{x} \leq \boldsymbol{b}$ are indicated by the halfspaces with arrows pointing inward. If these are turned around the resulting polyhedron can be written as the sum of a simplex Q (indicated by the dotted segment and grey area, respectively) and a lineality space L (indicated by the dashed lines).

But then at $\lambda\hat{\boldsymbol{x}} + (1-\lambda)\boldsymbol{x}^i$ more relations of $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ hold at equality than at $\hat{\boldsymbol{x}}$, contradicting the choice of $\hat{\boldsymbol{x}}$. □

According to the above proof, we can take among all possible solutions $\boldsymbol{x}^i$ of the corresponding subsystems $\{A^i\boldsymbol{x} = \boldsymbol{b}^i\}$, for $1 \leq i \leq m$, the representatives of the minimal nonempty faces of $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ that lie in the orthogonal linear subspace $L^\perp$; i.e., $Q \subset L^\perp$. By Lemma 1, we know that $\{\boldsymbol{x} \in \mathbb{R}^n \mid A^i\boldsymbol{x} = \boldsymbol{b}^i\} = \boldsymbol{x}^i + L$, where L is the lineality space of the original linear system $\{A\boldsymbol{x} \geq \boldsymbol{b}\}$. However, any choice of $\boldsymbol{x}^i$ would do (see Figure 1).

It is worth observing that Theorem 4 handles the following special cases.
– If $m = 1$, then the system $\{A^1\vec{x} \leq \vec{b}^1\}$ is empty and hence has a solution. Consider for instance $\{A\boldsymbol{x} \leq \boldsymbol{b}\} = \{\boldsymbol{0x} \leq -1\}$, then $L = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{0x} = 0\} = \mathbb{R}^n$ and $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{0x} \geq -1\} = \mathbb{R}^n + \{0\} = L + Q = L$.
– If $m = n+1$, then A has $n+1$ rows. Assuming A to be of full column rank, $L = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = 0\} = \{0\}$, $Q = \operatorname{conv}(\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n+1}\})$ is an n-simplex and $\{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \geq \boldsymbol{b}\} = \{0\} + Q$.

2.2 Minimum cardinality IISs

We now consider the complexity status of the following problem for which heuristics have been proposed in [20, 22, 38, 39].

MIN IIS: *Given an infeasible system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ as above, find a minimum cardinality IIS.*

To settle the issue left open in [20, 22, 28, 39], we prove that MIN IIS is not only NP-hard to solve optimally but also hard to approximate. If $\mathrm{DTIME}(T(m))$ denotes the class of problems solvable in deterministic time $T(m)$, the assumption NP $\not\subseteq$ $\mathrm{DTIME}(m^{\text{polylog}(m)})$ is stronger than NP $\neq$ P, but it is also believed to be extremely likely. Since polylog(m) denotes any polynomial in $\log(m)$, the assumption amounts to stating that all problems in NP cannot be solved in quasi-polynomial time. Results that hold under such an assumption are often referred to as *almost* NP-*hard*.

Theorem 5. *Assuming* P $\neq$ NP, *no polynomial-time algorithm is guaranteed to yield an IIS whose cardinality is at most c times larger than the minimum one, for any constant $c \geq 1$. Assuming* NP $\not\subseteq$ $\mathrm{DTIME}(m^{\text{polylog}(m)})$, MIN IIS *cannot be approximated within a factor $2^{\log^{1-\varepsilon}(m)}$, for any $\varepsilon > 0$, where m is the number of inequalities.*

Proof. We proceed by reduction from the following problem: Given a feasible linear system $D\boldsymbol{z} = \boldsymbol{d}$, with $D \in \mathbb{R}^{m' \times n'}$ and $\boldsymbol{d} \in \mathbb{R}^{m'}$, find a solution $\boldsymbol{z}$ satisfying all equations with as few nonzero components as possible. In [5] this problem is proved to be (almost) NP-hard to approximate within the same type of factors, but with m replaced by the number of variables n. Note that the above nonconstant factor grows faster than any polylogarithmic function, but slower than any polynomial function.

For each instance of the latter problem which has an optimal solution containing s nonzero components, we construct a particular instance of MIN IIS with a minimum cardinality IIS containing $s+1$ inequalities. Given any instance (D, $\boldsymbol{d}$), consider the system

$$\begin{bmatrix} D & -D & -\boldsymbol{d} \end{bmatrix} \begin{pmatrix} \boldsymbol{z}^+ \\ \boldsymbol{z}^- \\ z_0 \end{pmatrix} = \boldsymbol{0}, \quad \begin{bmatrix} \boldsymbol{0}^t & \boldsymbol{0}^t & -1 \end{bmatrix} \begin{pmatrix} \boldsymbol{z}^+ \\ \boldsymbol{z}^- \\ z_0 \end{pmatrix} < 0, \quad \boldsymbol{z}^+, \boldsymbol{z}^- \geq \boldsymbol{0},\ z_0 \geq 0. \quad (1)$$

Since the strict inequality implies $z_0 > 0$, the system $D\boldsymbol{z} = \boldsymbol{d}$ has a solution with s nonzero components if and only if (1) has one with $s + 1$ nonzero components. Now, applying Corollary 1, (1) has such a solution if and only if the system

$$\begin{pmatrix} D^t \\ -D^t \\ -\boldsymbol{d}^t \end{pmatrix} \boldsymbol{x} \leq \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ -1 \end{pmatrix} \tag{2}$$

has an IIS of cardinality $s + 1$. Since (2) is the alternative system of (1), the Farkas Lemma implies that exactly one of these is feasible; as (1) is feasible, (2) must be infeasible. Thus (2) is a particular instance of MIN IIS with $m = 2n' + 1$ inequalities in $n = m'$ variables.

Given that the polynomial-time reduction preserves the objective function modulo an additive unit constant, we obtain the same type of non-approximability factors for MIN IIS. □

Note that for the similar (but not directly related) problem of determining minimum witnesses of infeasibility in network flows, NP-hardness is established in [1].

3 IIS-hypergraphs

Although in the previous section the focus was on single IISs, we have seen in the introduction that the complementary version of MAX FS, in which one aims at minimizing the number of inequalities that must be deleted to make a given infeasible system feasible, can be viewed as the problem of covering all its IISs with a minimum number of inequalities. Assuming the IISs are known, the entire combinatorial structure of a MAX FS instance can thus be represented by an appropriate hypergraph containing one node per inequality and one edge for each IIS.

Let $H = (V, \mathcal{E})$ be a finite hypergraph with node set V and edge set $\mathcal{E} \subseteq 2^V$. All hypergraphs in this paper will be finite. H is called a *clutter* hypergraph, if no set of $\mathcal{E}$ contains any other set of $\mathcal{E}$, i.e., $\mathcal{E}$ is a *clutter*.

A hypergraph $H = (V, \mathcal{E})$ is *isomorphic* to a hypergraph $H' = (V', \mathcal{E}')$ if there exists a bijection $\pi : V \to V'$ and a bijection $\tau : \mathcal{E} \to \mathcal{E}'$ such that

$$\tau(E) = \{\pi(v) \,|\, v \in E\} \quad \text{for all } E \in \mathcal{E} .$$

This relation is denoted by $H \cong H'$.

In this section let K denote either the field $\mathbb{Q}$, $\mathbb{A}$, or $\mathbb{R}$. Recall that $\mathbb{A}$ denotes the real algebraic numbers, namely all real numbers that are roots of polynomials with integer coefficients.

Definition 2. *A hypergraph $H = (V, \mathcal{E})$, with $m = |V|$, is an* IIS-hypergraph *(over K) if there exists an infeasible linear system $\Sigma = \{A\boldsymbol{x} \leq \boldsymbol{b}\}$, with $A \in K^{m \times n}$ (for some n) and $\boldsymbol{b} \in K^m$, such that H is isomorphic to the clutter hypergraph $\mathcal{H}(\Sigma) := ([m], \mathcal{I})$, where the i-th inequality of Σ is identified with i and $\mathcal{I}$ is the set of IISs of Σ.*

In the above definition, infeasibility is meant with respect to $\mathbb{R}$.

Investigations of the structure of *IIS-hypergraphs* (over $\mathbb{R}$) began with [44, 45]. IIS-hypergraphs (with no trivial IISs of cardinality 1) turn out to be *bicolorable*, i.e., their nodes can be partitioned into two subsets so that neither subset contains an edge. Furthermore, IIS-hypergraphs do not share many properties with other known classes of hypergraphs generalizing bipartite graphs. See, for instance, the figure in [45] summarizing how IIS-hypergraphs fit into Berge's hierarchy. Note, however, that there is more structure for IIS-hypergraphs than simply bicolorability, as there will generally exist many different bipartitions into two feasible subsystems [27, 44].

According to hypergraph terminology, MIN IIS COVER amounts to finding a minimum cardinality *transversal*, i.e., a subset of nodes having nonempty intersection with every edge. Clearly, the problem can also be viewed as that of finding a maximum *stable* set in IIS-hypergraphs. The special structure of IIS-hypergraphs accounts for the fact a minimum transversal (maximum stable set) can be found in polynomial time in the size of the hypergraph if the corresponding alternative polyhedron is nondegenerate (a subclass of uniform hypergraphs) [45], while the problem is NP-hard even for simple graphs, i.e., for 2-uniform hypergraphs.

In this section we first introduce some terminology and discuss a property of IIS-hypergraphs which is needed in Section 4 to investigate facets of the feasible subsystem polytope. In Subsection 3.2, the same property is used to settle the complexity status of the problem of recognizing whether a given hypergraph is an IIS-hypergraph.

3.1 Connection between IIS-hypergraphs and vertex-facet incidences of polyhedra

Theorem 2 provides a connection between the combinatorial structure of the IISs of any given infeasible system (i.e., its IIS-hypergraph) and the vertex-facet incidences of its alternative polyhedron. To formalize this connection, we need the following concepts related to finite hypergraphs.

Let $H = (V, \mathcal{E})$ be a hypergraph. For $E \in \mathcal{E}$ define $\overline{E} := V \setminus E$ to obtain the *complement hypergraph* $\overline{H} := (V, \overline{\mathcal{E}})$, where $\overline{\mathcal{E}} = \{\overline{E} \mid E \in \mathcal{E}\}$.

Definition 3 (see [11]). *For each node $v \in V$, the set $S_v := \{E \in \mathcal{E} \mid v \in E\}$ denotes the set of all edges of H which contain v. Then $H^* := \{\mathcal{E}, \mathcal{E}^*\}$, with the edges of H as nodes and $\mathcal{E}^* := \{S_v \mid v \in V\}$ as edges, is the* dual hypergraph *of H.*

It is easily verified that $H^{**} \cong H$ and $(\overline{E})^* \cong \overline{(E^*)}$ for every edge E of H.

Definition 4. *Let P be a pointed polyhedron with vertex set V_P. Let $F_1, \dots, F_m$ be the facets of P and let $\mathcal{F}_i := \{v \in V_P \mid v \in F_i\}$ be the vertex set of facet F_i, for $1 \le i \le m$. Then define $\mathcal{H}(P) := (V_P, \{\mathcal{F}_1, \dots, \mathcal{F}_m\})$. A hypergraph $H = (V, \mathcal{E})$ is a* vertex-facet incidence hypergraph *of P if H is isomorphic to $\mathcal{H}(P)$.*

Now we have the following relation:

Lemma 2. *Let $H = (V, \mathcal{E})$ be a finite IIS-hypergraph (over K) and $\overline{H^*}$ be a clutter hypergraph. Let $\Sigma : A\boldsymbol{x} \le \boldsymbol{b}$, with $A \in K^{m \times n}$ and $\boldsymbol{b} \in K^m$, be any infeasible system such that $\mathcal{H}(\Sigma) \cong H$. Then $\overline{H^*}$ is a vertex-facet incidence hypergraph of the alternative polyhedron corresponding to Σ.*

Proof. Denote by $\mathcal{I}$ the set of IISs of the given Σ. According to Theorem 2, the elements of $\mathcal{I}$ are in one-to-one correspondence with the supports of the vertices of the alternative polyhedron

$$P = \{\boldsymbol{y} \in \mathbb{R}^m \mid A^t\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{b}^t\boldsymbol{y} = -1,\ \boldsymbol{y} \ge \boldsymbol{0}\}.$$

Identify V with $[m]$ (the set of inequalities of Σ) so that $\mathcal{E} = \mathcal{I}$. Let $E \in \mathcal{E}$ correspond to an IIS and $\boldsymbol{v}$ be the vertex of P associated with E. The complement of the support of $\boldsymbol{v}$ is $\overline{E}$, and it determines which faces defined by $y_j = 0$, $1 \leq j \leq m$, are satisfied by $\boldsymbol{v}$ with equality, i.e., which of these faces contain $\boldsymbol{v}$. This means that each set $\overline{E} \in \overline{\mathcal{E}}$ gives the set of all faces containing a specific vertex.

By definition, each set in $\overline{\mathcal{E}^*}$ coincides with the vertex set of a face defined by $y_j = 0$ for some $1 \leq j \leq m$. Furthermore, each facet of P must be defined by $y_j = 0$ for some $1 \leq j \leq m$. Since $\overline{\mathcal{E}^*}$ is a clutter, no vertex set of the faces defined by $y_j = 0$ contains another. Altogether this implies that each $y_j = 0$ defines a facet of P. Thus, $\overline{H^*}$ is a vertex-facet incidence hypergraph of P. ☐

It is worth noting that the reverse direction of the previous lemma also holds.

Lemma 3. *Let $H = (V, \mathcal{E})$ be a vertex-facet incidence hypergraph of a polyhedron P (with a description over K) which is not a cone. Then $\overline{H^*}$ is an IIS-hypergraph (over K).*

For completeness the proof is given in the Appendix.

Note the slight asymmetry between the assumptions of Lemma 2 and Lemma 3, which is due to the fact that vertex-facet incidences cannot capture all information about the face lattice of unbounded polyhedra (see the comments at the end of Section 3). Restricting attention to hypergraphs H such that $\overline{H^*}$ is a clutter hypergraph yields the following result.

Corollary 2. *Let $H = (V, \mathcal{E})$ be a finite hypergraph and $\overline{H^*}$ be a clutter hypergraph. Then H is an IIS-hypergraph if and only if $\overline{H^*}$ is a vertex-facet incidence hypergraph of a polyhedron.*

Proof. For IIS-hypergraphs, Lemma 2 guarantees the "if"-direction. If $\overline{H^*}$ is a vertex-facet incidence hypergraph of a polyhedron P and it is a clutter hypergraph then P cannot be a cone. Thus by Lemma 3, H is an IIS-hypergraph. ☐

3.2 IIS-hypergraph recognition

In this subsection we address the interesting problem of recognizing IIS-hypergraphs.

IIS-hypergraph Recognition problem *over K: Given a hypergraph H, is H an IIS-hypergraph over K?*

The *face lattice* of a polytope P is its set of faces, ordered by inclusion, with the meet defined by intersection. It is well-known (see, e.g., [49]) that the face lattice of P has a rank function $r(\cdot)$, satisfying $r(F) = \dim(F) + 1$ for every face F, and is both atomic and coatomic. Two polytopes $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ are *affinely equivalent* (denoted by $P \cong Q$) if there exists an affine map $\phi : \mathbb{R}^p \to \mathbb{R}^q$, which establishes a one-to-one correspondence between points in P and Q. Two polytopes with isomorphic face lattices are *combinatorially equivalent*. For the definitions of poset and (face) lattice we again refer the reader to [49].

We prove NP-hardness of IIS-hypergraph recognition by polynomial-time reduction from the following decision problem.

Steinitz problem *over K: Given a lattice $\mathcal{L}$, does there exist a polytope $P \subset \mathbb{R}^d$ (for some d) with vertices in K^d whose face lattice is isomorphic to $\mathcal{L}$?*

If the answer is affirmative, $\mathcal{L}$ is *realizable* as a polytope. In this case d can be assumed to be the dimension of $\mathcal{L}$. See [15] for related material. We need a special lattice construction arising from hypergraphs.

Let $H = (V, \mathcal{E})$ be a hypergraph. Define the poset $\mathcal{L}(H)$ as the set of all intersections of sets in $\mathcal{E}$, ordered by set inclusion. Furthermore, adjoin a maximal element $\hat{1}$. Clearly, $\mathcal{L}(H)$ is bounded and has a meet (defined by intersection); hence it is a lattice. Note that the size of $\mathcal{L}(H)$ can be exponential in the size of H. If H is a vertex-facet incidence hypergraph of a polytope P then $\mathcal{L}$ is isomorphic to the face lattice of P. This follows from the fact that all faces are determined by their vertex sets or by the facets they are contained in.

Conversely, let $\mathcal{L}$ be an arbitrary ranked, atomic, and coatomic lattice. Let V be the set of atoms of $\mathcal{L}$. For each coatom F, let $E_F := \{v \in V \mid v \text{ is below } F \text{ in } \mathcal{L}\}$. Then define the hypergraph $\mathcal{H}(\mathcal{L}) := (V, \{E_F \mid F \text{ coatom of } \mathcal{L}\})$. Note that, since $\mathcal{L}$ is atomic, $\mathcal{H}(\mathcal{L})$ is a clutter hypergraph by construction. If $\mathcal{L}$ is the face lattice of a polytope, then $\mathcal{H}(\mathcal{L})$ is a vertex-facet incidence hypergraph.

Theorem 6. *For $K \in \{\mathbb{Q}, \mathbb{A}\}$, there is a polynomial-time reduction from the Steinitz problem (over K) to the IIS-hypergraph Recognition problem (over K).*

Proof. We show that for any instance of the Steinitz problem, given by an arbitrary lattice $\mathcal{L}$, we can construct in polynomial time a special instance of the latter problem, given by a clutter hypergraph H, such that the answer to the first instance is affirmative if and only if the answer to the second instance is affirmative.

If $\mathcal{L}$ is ranked, atomic, and coatomic, take $H = \overline{\mathcal{H}(\mathcal{L})^*}$. Note that these properties of $\mathcal{L}$ can be checked (Test 1) and H can be constructed in polynomial time in the size of $\mathcal{L}$, namely the number of elements. If any of these properties fail, let H be any hypergraph which is not an IIS-hypergraph, e.g., take $H = (\{1,2,3\}, \{\{1,2\}, \{2,3\}, \{1,3\}\})$.

In [32] it is proved that, if H is a vertex-facet incidence hypergraph of a polyhedron P of dimension d, then there exists a number $\widetilde{\chi} = \widetilde{\chi}(H) \in \mathbb{Z}$, namely the *reduced Euler characteristic* of the order complex of $\mathcal{L}(H)$ (see e.g. [12]) such that $\widetilde{\chi} = (-1)^{d-1}$ if P is bounded while $\widetilde{\chi} = 0$ if P is unbounded. Moroever, $\widetilde{\chi}$ can be computed in polynomial time in the size of $\mathcal{L}(H)$. Note that this result implies that no unbounded polyhedron and polytope can have isomorphic vertex-facet incidence hypergraphs.

Since $\widetilde{\chi}(\overline{H^*})$ can be computed in polynomial time in the size of $\mathcal{L}(\overline{H^*})$, which equals the size of $\mathcal{L}$. If $\widetilde{\chi}(\overline{H^*}) = 0$ (Test 2), then replace H by any hypergraph which is not an IIS-hypergraph.

The resulting H is the input to the IIS-hypergraph Recognition problem. Assume that the answer to the IIS-hypergraph Recognition of H is affirmative, i.e., H is an IIS-hypergraph. As noted above, the atomicity of $\mathcal{L}$ implies that $\overline{H^*}$ is a clutter hypergraph. By Lemma 2, $\overline{H^*}$ is a vertex-facet incidence hypergraph of some polyhedron P.

First assume that P is a polytope. By construction, $\mathcal{L}$ is isomorphic to $\mathcal{L}(\overline{H^*}) = \mathcal{L}(\mathcal{H}(\mathcal{L}))$. Since P is a polytope, $\mathcal{L}(\overline{H^*})$ is isomorphic to the face lattice of P and hence so is $\mathcal{L}$, i.e., the answer to the Steinitz problem for $\mathcal{L}$ is affirmative.

Now assume P is an unbounded polyhedron. Then $\overline{H^*}$ is a vertex-facet incidence hypergraph of an unbounded polyhedron and, according to the above-mentioned result, we have $\widetilde{\chi}(\overline{H^*}) = 0$. But in this case we replaced the input by an instance which is not an IIS-hypergraph; this is a contradiction.

Conversely assume that the answer to the Steinitz problem for $\mathcal{L}$ is affirmative. Then there exists a polytope P such that $\mathcal{L}$ is isomorphic to the face lattice of P and hence, by construction, $\overline{H^*}$ is a vertex-facet incidence hypergraph of P. Now P is not a cone unless $P = \{0\}$, a case which can be easily identified and discarded. By applying Lemma 3 to $\overline{H^*}$, it follows that H is an IIS-hypergraph.

Note that since $\mathcal{L}$ is ranked, atomic, and coatomic, it has necessarily passed Test 1. Furthermore, by the above-mentioned result we have $\widetilde{\chi}(\overline{H^*}) = \pm 1$, which implies that it also passed Test 2. Thus, the answer to the IIS-hypergraph Recognition question for H is affirmative. □

Given polynomials $f_1, \ldots, f_r, g_1, \ldots, g_s, h_1, \ldots, h_t \in \mathbb{Z}[x_1, \ldots, x_l]$, the problem to decide whether the polynomial system $f_1 = 0, \ldots, f_r = 0$, $g_1 \geq 0, \ldots, g_s \geq 0$, $h_1 > 0, \ldots, h_t > 0$ has a solution in $K^l = \mathbb{A}^l$ is called the *Existential theory of the reals* (ETR). ETR is polynomial-time equivalent to the Steinitz problem for 4-polytopes over $\mathbb{A}$ [42]. All polytopes realizable over $\mathbb{R}$, are realizable over $\mathbb{A}$. Moreover, ETR is polynomial-time equivalent to the Steinitz problem for d-Polytopes with $d+4$ vertices over $\mathbb{A}$ [36]. Since ETR is easily verified to be NP-hard [13], the same is valid for the general Steinitz problem (over $\mathbb{A}$) and for the IIS-hypergraph recognition problem.

According to Theorem 2.7 of [15], for $K = \mathbb{Q}$ or $\mathbb{A}$, deciding whether an arbitrary polynomial $f \in \mathbb{Z}[x_1, \ldots, x_l]$ has zeros in K^l, where l is a positive integer, is equivalent to solving the Steinitz problem for K. For $K = \mathbb{Q}$, it is not even clear whether the Steinitz problem (and therefore the IIS-hypergraph Recognition) is decidable, since finding roots in $K = \mathbb{Q}$ of a single polynomial $f \in \mathbb{Z}[x_1, \ldots, x_l]$ is the unsolved rational version of Hilbert's 10th problem. By the quantifier elimination result of Tarski, the problem is decidable for $K = \mathbb{A}$. Note that, unlike $\mathbb{R}$, $\mathbb{A}$ admits a finite representation. For $K = \mathbb{A}$, it is unknown whether the Steinitz problem is in NP. See [14, 35] and references therein for this and related issues.

Finally it is worth noting that to establish the reverse direction of Theorem 6 one would need to provide an appropriate input (a lattice) to the Steinitz problem. This task appears to be difficult to achieve because we need to consider the case of unbounded polyhedra. In fact, as shown in [32], it is in general impossible to reconstruct the face lattice of an unbounded polyhedron P given a vertex-facet incidence hypergraph H of P, even when H is a clutter hypergraph.

4 Feasible Subsystem (FS) Polytope

An *independence system* $(E, \mathcal{I})$ is defined by a finite ground set E and a collection of subsets $\mathcal{I} \subseteq 2^E$ such that $I \in \mathcal{I}$ and $J \subset I$ imply $J \in \mathcal{I}$. The subsets of E that (do not) belong to $\mathcal{I}$ are the so-called *independent* (*dependent*) sets. An independence system can be defined by its collection of *independent sets* $\mathcal{I}$ or, equivalently, by the collection $\mathcal{C}$ of all minimal dependent subsets of E; i.e., any dependent subset each of whose proper subsets are independent. To any independence system $(E, \mathcal{I})$ with the collection of *circuits* $\mathcal{C}$, we can associate the polytope

$$P(\mathcal{I}) = \text{conv}(\{\boldsymbol{y} \in \{0,1\}^{|E|} \mid \boldsymbol{y} \text{ is the incidence vector of an } I \in \mathcal{I}\},$$

which will also be denoted by $P(\mathcal{C})$.

Now consider an infeasible system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ with no single inequality that is trivially infeasible. Let $[m] = \{1, \ldots, m\}$ be the set of indices of the inequalities in Σ. If $\mathcal{I}$ denotes the set of all feasible subsystems of Σ, $([m], \mathcal{I})$ is clearly an independence system and its set of circuits $\mathcal{C}$ corresponds to the set of all IISs. We denote by $P_{FS}(\Sigma)$ the *Feasible Subsystem polytope*, defined as the convex hull of all the incidence vectors of feasible subsystems.

Before investigating this polytope, let us recall some definitions and facts regarding general independence system polytopes. The *rank function* is defined by $r(S) =$

$\max\{|I| \;\; | I \subseteq S,\;\; I \in \mathcal{I}\}$ for all $S \subseteq E$. For any $S \subseteq E$, the *rank inequality* for S is $\sum_{e \in S} y_e \leq r(S)$, which is clearly valid for $P(\mathcal{I})$. A subset $S \subseteq E$ is *closed* if $r(S \cup \{t\}) \geq r(S)+1$ for all $t \in E-S$ and *nonseparable* if $r(S) < r(T)+r(S-T)$ for all $T \subset S,\; T \neq \emptyset$. For any set $S \subseteq E$, S must be closed and nonseparable for the corresponding rank inequality to define a facet of $P(\mathcal{I})$. These conditions generally are only necessary, but sufficient conditions can be stated using the following concept [33]. For $S \subseteq E$, the *critical graph* $G_S(\mathcal{I}) = (S, F)$ is defined as follows: $(e, e') \in F$, for $e,\, e' \in S$, if and only if there exists an independent set I such that $I \subseteq S$, $|I| = r(S)$ and $e \in I$, $e' \notin I$, $I - e + e' \in \mathcal{I}$. It is shown in [33] that if S is a closed subset of E and the critical graph $G_S(\mathcal{I})$ of $\mathcal{I}$ on S is connected, then the corresponding rank inequality induces a facet of the polytope $P(\mathcal{I})$. (See references in [16].)

We now turn to the feasible subsystem polytope. According to well-known facts about independence system polytopes, $P_{FS}(\Sigma)$ is full-dimensional if and only if there are no trivially infeasible inequalities in Σ. Moreover, the inequalities $y_i \geq 0$ are facet defining for all $1 \leq i \leq m$, and it is easy to verify that for each i the inequality $y_i \leq 1$ defines a facet of $P_{FS}(\Sigma)$ if and only if there is no IIS of cardinality 2 that includes the ith inequality of Σ.

4.1 Rank facets arising from IISs

In fact, Parker [38] began an investigation of the polytope associated to the MIN IIS COVER problem, considering it as a special case of the general set covering polytope (see also references in [16]). Since there is a simple correspondence between set covering polytopes and the associated independence system polytopes [33], the results in [38] can be translated so that they apply to $P_{FS}(\Sigma)$.

From now on, we assume that all IISs are nontrivial, i.e., they are of cardinality greater or equal to two. Let S be an arbitrary IIS of Σ, with $A_S \boldsymbol{x} \leq \boldsymbol{b}_S$ its associated subsystem. Then the rank inequality

$$\sum_{i \in S} y_i \leq r(S) = |S| - 1$$

is called an *IIS-inequality*. Since the corresponding covering inequality $\sum_{i \in S} y_i \geq 1$ is proved to be facet defining in [38], we have:

Theorem 7. *Every IIS-inequality defines a (rank) facet of $P_{FS}(\Sigma)$.*

We give a geometric proof (based on the above-mentioned sufficient conditions [33] and our IIS simplex decomposition result) in the following, which is simpler than that of [38] and which provides additional insight into the IIS structure.

Proof. It is easy to verify that IIS-inequalities are valid for $P_{FS}(\Sigma)$. Since the critical graph corresponding to any IIS is clearly connected (in fact, a complete graph), we just need to show that the index set of every IIS is closed.
a) First consider the case of maximal IISs defined by subset $S \subseteq E$, i.e., with $|S| = n+1$, where E is the index set of the entire system Σ.

For each $i \in S$, consider the unique $\boldsymbol{x}^i = A^{-1}_{S\setminus\{i\}} \boldsymbol{b}_{S\setminus\{i\}}$. By the proof of Theorem 4, we know that $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{n+1}$ are affinely independent. If we define $\boldsymbol{d}_i := (\boldsymbol{x}^i - \hat{\boldsymbol{x}})$ for all i, $1 \leq i \leq n+1$, where $\hat{\boldsymbol{x}} := \frac{1}{n+1} \sum_{i=1}^{n+1} \boldsymbol{x}^i$ is the barycenter of the $\boldsymbol{x}^i$'s, then $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{n+1}$ are also affinely independent. Clearly $\sum_{i=1}^{n+1} \boldsymbol{d}_i = \boldsymbol{0}$ and the $\boldsymbol{d}_i$'s generate $\mathbb{R}^n$. Since each $\boldsymbol{x}^i$ satisfies exactly n of the $n+1$ inequalities in

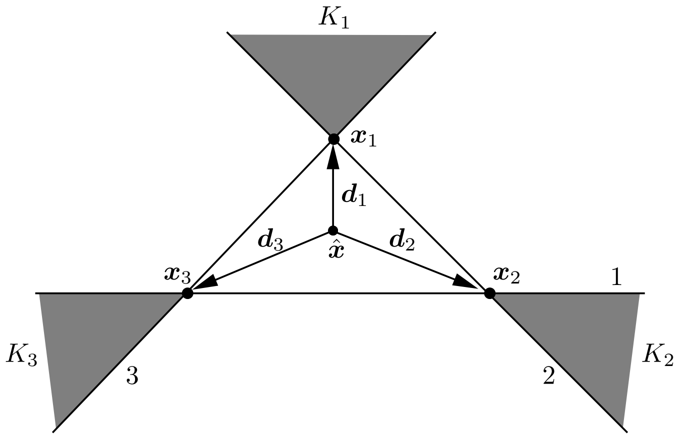


Figure 2: Illustration of the proof of Theorem 7.

$A_S\boldsymbol{x} \leq \boldsymbol{b}$ with equality and for the ith one $\boldsymbol{a}^i\boldsymbol{x}^i > b_i$ (otherwise S would be feasible), we have $\hat{\boldsymbol{x}} \in \{\boldsymbol{x} \in \mathbb{R}^n \mid A_S\boldsymbol{x} \geq \boldsymbol{b}_S\ \}$. In other words, $\hat{\boldsymbol{x}}$ satisfies the reversed inequalities of the IIS. In fact, $\hat{\boldsymbol{x}}$ is an interior point of the above "reversed" polyhedron.

According to Theorem 3, deleting any inequality from an IIS yields a feasible subsystem that defines an affine cone. For maximal IISs, we have $n+1$ affine cones $K_i := \boldsymbol{x}^i + K_i'$, where $K_i' = \{\boldsymbol{x} \in \mathbb{R}^n \mid A_{S\setminus\{i\}}\boldsymbol{x} \leq \boldsymbol{0}\}$ for $1 \leq i \leq n+1$. Note that the ray generated by $\boldsymbol{d}_i$ passing through $\boldsymbol{x}^i$, i.e., $R_i := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{x}^i + \alpha\boldsymbol{d}_i,\ \alpha \geq 0\}$, is contained in K_i because we have

$$A_{S\setminus\{i\}}(\alpha\boldsymbol{d}_i) = \alpha A_{S\setminus\{i\}}(\boldsymbol{x}^i - \hat{\boldsymbol{x}}) = \alpha(\boldsymbol{b}_{S\setminus\{i\}} - A_{S\setminus\{i\}}\hat{\boldsymbol{x}}) \leq \boldsymbol{0},$$

where we used the fact that $A_{S\setminus\{i\}}\hat{\boldsymbol{x}} \geq \boldsymbol{b}_{S\setminus\{i\}}$. To show that the maximal IIS defined by S is closed, we consider an arbitrary inequality $\tilde{\boldsymbol{a}}\boldsymbol{x} \leq \tilde{b}$ with $\tilde{\boldsymbol{a}} \neq \boldsymbol{0}$ and verify that $H := \{\boldsymbol{x} \in \mathbb{R}^n \mid \tilde{\boldsymbol{a}}\boldsymbol{x} \leq \tilde{b}\}$ has a nonempty intersection with at least one of the K_i's, $1 \leq i \leq n+1$. This implies, in particular, that for any inequality index $t \in E - S$ we have rank$(S \cup \{t\})$ = rank$(S) + 1 = n+1$, which means that the IIS under consideration is closed.

Since $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{n+1}$ generate $\mathbb{R}^n$ and $\sum_{i=1}^{n+1} \boldsymbol{d}_i = \boldsymbol{0}$, we have

$$\sum_{i=1}^{n+1} \tilde{\boldsymbol{a}}\boldsymbol{d}_i = \tilde{\boldsymbol{a}}(\sum_{i=1}^{n+1} \boldsymbol{d}_i) = 0$$

and therefore $\tilde{\boldsymbol{a}} \neq \boldsymbol{0}$ implies that we cannot have $\tilde{\boldsymbol{a}}\boldsymbol{d}_i = 0\ \forall i,\ 1 \leq i \leq n+1$. Thus there exists at least one i, such that $\tilde{\boldsymbol{a}}\boldsymbol{d}_i < 0$. But this implies that $R_i \cap H \neq \emptyset$. In other words, $K_i \cap H \neq \emptyset$ and this proves the theorem for maximal IISs.

b) The result can be easily extended to non-maximal IISs, i.e., with $|S| < n+1$. From Theorem 4 we know that $P := \{\boldsymbol{x} \in \mathbb{R}^n \mid A_S\boldsymbol{x} \geq \boldsymbol{b}_S\} = L + Q$ with $Q \subseteq L^\perp$. Since P is full-dimensional (the barycenter of Q is an interior point), $n = \dim(P) = \dim(L) + \dim(Q)$ and $\dim(Q) = \operatorname{rank}(A_S) = |S| - 1 < n$ imply that $\dim(L) \geq 1$.

Two cases can arise:

i) If the above-mentioned $\tilde{\boldsymbol{a}}$ belongs to the linear hull of the rows of A_S denoted by $\operatorname{lin}(\{\boldsymbol{a}^i \mid i \in S\}) = L^\perp$, then since $\dim(L^\perp) = \dim(Q)$, we can apply the above result to $L^\perp$.

ii) If $\tilde{\boldsymbol{a}} \notin \text{lin}(\{\boldsymbol{a}^i \mid i \in S\}) = L^\perp$, then the projection of $H^= := \{\boldsymbol{x} \in \mathbb{R}^n \mid \tilde{\boldsymbol{a}}\boldsymbol{x} = \tilde{b}\}$ onto $L^\perp$ yields all of $L^\perp$ and therefore $H = \{\boldsymbol{x} \in \mathbb{R}^n \mid \tilde{\boldsymbol{a}}\boldsymbol{x} \leq \tilde{b}\}$ must have a nonempty intersection with all the cones corresponding to the maximal consistent subsystems of $\{A_S\boldsymbol{x} \leq \boldsymbol{b}_S\}$. □

It is worth emphasizing that closedness of every IIS makes the feasible subsystem polytope quite special among all independence system polyhedra, since the circuits of a general independence system need not be closed. For example, consider the independent system defined by stable sets of nodes in a simple graph; here the circuits correspond to the edges of the graph and it is clear that these circuits are not necessarily closed (it suffices to consider any K_3 in the graph).

We now turn to the **IIS-inequality Separation problem**, which is defined as follows: *Given an infeasible system Σ and an arbitrary vector $\boldsymbol{y} \in \mathbb{R}^m$, show that $\boldsymbol{y}$ satisfies all IIS-inequalities or find at least one violated by $\boldsymbol{y}$.*

In view of the trivial valid inequalities, we can assume that $\boldsymbol{y} \in [0,1]^m$. Moreover, we may assume with no loss of generality, that the nonzero components of $\boldsymbol{y}$ correspond to an infeasible subsystem of Σ.

Proposition 1. *The separation problem for IIS-inequalities is* NP*-hard.*

Proof. We proceed by polynomial-time reduction from the decision version of the MIN IIS problem, which is NP-hard according to Theorem 5. Given an infeasible system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ with m inequalities, n variables, and an integer K with $1 \leq K \leq n+1$, does it have an IIS of cardinality at most K?

Let $(A, \boldsymbol{b})$ and K define an arbitrary instance of the above decision problem. Consider the particular instance of the separation problem given by the same infeasible system together with the vector $\boldsymbol{y}$ such that $y_i = 1 - 1/(K+1)$ for all i, $1 \leq i \leq m$.

Suppose that Σ has an IIS of cardinality at most K which is indexed by the set S. Then the corresponding IIS-inequality $\sum_{i \in S} y_i \leq |S| - 1$ is violated by the vector $\boldsymbol{y}$ because

$$\sum_{i \in S} y_i = \sum_{i \in S} (1 - \frac{1}{K+1}) = |S| - \frac{|S|}{K+1} > |S| - 1,$$

where the strict inequality is implied by $|S| \leq K$. Thus the vector $\boldsymbol{y}$ can be separated from $P_{FS}(\Sigma)$.

Conversely, if there exists an IIS-inequality violated by $\boldsymbol{y}$, then

$$\sum_{i \in S} y_i = |S| - \frac{|S|}{(K+1)} > |S| - 1$$

implies that the cardinality of the IIS defined by S is at most K.

Therefore, the original infeasible system Σ has an IIS of cardinality at most K if and only if some IIS-inequality is violated by the given vector $\boldsymbol{y}$. □

4.2 Rank facets arising from generalized antiwebs

In [33] the concept of generalized antiwebs, which generalize cliques, odd holes and antiholes to independence systems, is introduced. Necessary and sufficient conditions are also established for the corresponding rank inequalities to define facets of the associated independence system polytope.

Let m, t, q be integers such that $2 \leq q \leq t \leq m$, let $E = \{e_0, \dots, e_{m-1}\}$ be a finite set, and define for each $i \in M := \{0, \dots, m-1\}$ the subset $E^i = \{e_i, \dots, e_{i+t-1}\}$ (where the indices are taken modulo m) formed by t consecutive elements of E. An (m,t,q)-*generalized antiweb* on E is the independence system having the following family of subsets of E as circuits:

$$\mathcal{AW}(m,t,q) = \{C \subseteq E \,|\, C \subseteq E^i \text{ for some } i \in M, \ |C| = q\}.$$

Define $P(\mathcal{AW}(m,t,q))$ to be the polytope of the independence system defined by $\mathcal{AW}(m,t,q)$ and $\mathcal{AW}(m,t) := \mathcal{AW}(m,t,t)$. Note that the case $t = q = 1$ would correspond to m trivially infeasible inequalities, e.g., $\mathbf{0}\,\boldsymbol{x} \leq -1$.

As observed in [33], $\mathcal{AW}(m,t,q)$ corresponds to *generalized cliques* when $m = t$, to *generalized odd holes* when $q = t$ and t does not divide m, and to *generalized antiholes* when $m = qt + 1$.

In this section we determine under which circumstances generalized antiwebs give rise to rank facets of the form $\sum_{i \in S} y_i \leq r(S)$ of $P_{FS}(\Sigma)$. Defining the hypergraph $\mathcal{H}(\mathcal{AW}(m,t,q)) := (E, \mathcal{AW}(m,t,q))$, the first question is: for which values of m, t, and q is $\mathcal{H}(\mathcal{AW}(m,t,q))$ an IIS-hypergraph?

Lemma 4. *If $\mathcal{H}(\mathcal{AW}(m,t,q))$ is an IIS-hypergraph then $t = q$.*

Proof. Suppose that $q < t$ and consider E^1, an arbitrary circuit $C \in \mathcal{AW}(m,t,q)$ with $C \subseteq E^1$, and an arbitrary element $e \in E^1 \setminus C$. By definition of $\mathcal{AW}(m,t,q)$, any cardinality q subset of E^1 is a circuit. This must be true in particular for all subsets containing e and $q-1$ elements of C. But then C cannot be closed because $r(C \cup \{e\}) = r(C)$ and thus we have a contradiction to the fact that all IISs are closed (consequence of Theorem 7). □

To provide a characterization of IIS-hypergraphs arising from generalized antiwebs, we need the following result that is proved using topological arguments.

Proposition 2 (Joswig, Kaibel, Pfetsch, Ziegler [32]). *Let $1 < k < m$ be integers. Then $\mathcal{H}(\mathcal{AW}(m,k))$ is a vertex-facet incidence hypergraph of a polyhedron P if and only if P is a simplex or a polygon.*

Together with Lemma 2 and Lemma 4 we obtain:

Proposition 3. *$\mathcal{H}(\mathcal{AW}(m,t,q))$ is an IIS-hypergraph if and only if $t = q$ and*

i. $t = m$ *or*
ii. $t = m - 2$.

Proof. Lemma 4 implies that necessarily $t = q$. Now assume $H := \mathcal{H}(\mathcal{AW}(m,t))$ is an IIS-hypergraph. If $t = m$, we have a single IIS of size m. Therefore assume $t < m$.

Since $t < m$, $\overline{H^*}$ is a clutter hypergraph and hence, by Lemma 2, $\overline{H^*}$ is a vertex-facet incidence hypergraph of a polyhedron P. Now $\overline{\mathcal{AW}(m,t)} \cong \mathcal{AW}(m,k)$ with $k := m - t > 0$ and $\mathcal{H}(\mathcal{AW}(m,k))^* \cong \mathcal{H}(\mathcal{AW}(m,k))$. Hence $\mathcal{H}(\mathcal{AW}(m,k))$ is a vertex-facet incidence hypergraph of P. Since $2 \leq t < m$ we have $0 < k < m-1$. Furthermore $k > 1$ because $\mathcal{H}(\mathcal{AW}(m,1))$ can only be a vertex-facet hypergraph if $m = k = 1$, and this case is excluded by $1 < t < m$.

By Proposition 2, P is a polygon; i.e., $k = 2$ ($t = m - 2$). Note that the case of a simplex ($k = m - 1$) cannot arise. Clearly, examples of infeasible inequality systems exist for all possible values of the above parameters. This proves sufficiency. □

This proposition implies that only two types of generalized antiwebs can arise as induced hypergraph of IIS-hypergraphs. In particular, the only generalized cliques that can occur are those with $t = m$, namely those corresponding to single IISs. For generalized odd holes the only cases that can arise are those with $t = m - 2$. Finally, all generalized antiholes are ruled out since $m = tq + 1 \Leftrightarrow m = (m-2)^2 + 1$, which is never satisfied.

To determine in which cases facets arise from generalized antiwebs, we need the two following results.

Lemma 5 (Laurent [33]). *The valid inequality $\sum_{e \in E} y_e \leq \lfloor m(q-1)/t \rfloor$ (rank inequality) arising from a generalized antiweb defines a facet of the independence system polytope $P(\mathcal{AW}(m,t,q))$ if and only if $t = m$ or t does not divide $m(q-1)$.*

Note that the right hand side of the above inequality is the rank of the independence system defined by $\mathcal{AW}(m,t,q)$ (see [33]).

Let $\mathcal{C}$ be the set of circuits of an independence system $\mathcal{I}$ over the ground set $[m]$. For any $S \subseteq [m]$, let $\mathcal{C}_S = \{C \in \mathcal{C} \mid C \subseteq S\}$ denote the family of circuits of $\mathcal{I}$ induced on S.

Lemma 6 (Laurent [33]). *The rank inequality $\sum_{e \in S} y_e \leq r(S)$ induces a facet of $P(\mathcal{C})$ if and only if S is closed and it induces a facet of $P(\mathcal{C}_S)$.*

Altogether we obtain the following characterization of the rank facets of $P_{FS}(\Sigma)$ that can be induced by generalized antiwebs.

Theorem 8. *Let Σ be an infeasible inequality system with m inequalities and $\mathcal{C}$ be the IISs of Σ. Let $S \subseteq [m]$ and assume $\mathcal{C}_S = \mathcal{AW}(|S|, t)$ for some $2 \leq t \leq |S|$. The rank inequality*

$$\sum_{e \in S} y_e \leq \left\lfloor \frac{|S|(q-1)}{t} \right\rfloor \tag{3}$$

defines a facet of $P_{FS}(\Sigma)$ if and only if $t = q$ and one of the following holds

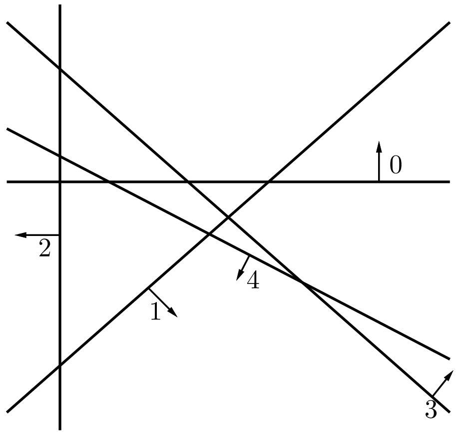
i. $t = |S|$ (IIS-inequality)
ii. S is closed, $t = |S| - 2$ and $t \neq 2$.

Proof. By Proposition 3, there are only two cases in which $\mathcal{AW}(|S|, t)$ can arise as an induced hypergraph of an IIS-hypergraph (in both of them necessarily $t = q$).

i) Case $t = |S|$: $\mathcal{AW}(|S|, t)$ consists of a single circuit (IIS). Since Theorem 7 implies that S is closed, this gives (together with Lemma 6) another proof that the rank facets arising from IISs define facets.

ii) Case $t = |S| - 2$: By Lemma 5, inequality (3) defines a facet for $P(\mathcal{AW}(|S|, t))$ if and only if t does not divide $|S|(t-1) = (t+2)(t-1) = t^2 + t - 2$. Clearly this can only be the case if $t = 1$ (which is not feasible) or $t = 2$. Therefore by Lemma 6, inequality (3) defines a facet of $P_{FS}(\Sigma)$ if and only if S is closed and $t \neq 2$. □

Example. Figure 3 gives an example of an infeasible system with $m = 5$ inequalities in dimension $n = 2$ (see also [40]). Its IISs form an $\mathcal{AW}(5,3)$. The inequalities are indexed by $0, 1, 2, 3, 4$. In the corresponding $P_{FS}(\Sigma)$ polytope the variables are numbered likewise. Its full description is given by the following facets:



	e_0	e_1	e_2	e_3	e_4
E_0	1	1	1		
E_1		1	1	1	
E_2			1	1	1
E_3	1			1	1
E_4	1	1			1

	E_0	E_1	E_2	E_3	E_4
S_0	1			1	1
S_1	1	1			1
S_2	1	1	1		
S_3		1	1	1	
S_4			1	1	1

Figure 3: Left: an infeasible linear inequality system, whose IISs $\{0,1,2\}$, $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,0\}$, and $\{4,0,1\}$ form a generalized antiweb $\mathcal{AW}(5,3)$. **Top right**: incidence matrix of $\mathcal{H}(\mathcal{AW}(5,3))$ according to the notation of Section 3. **Bottom right**: incidence matrix of the dual hypergraph $\mathcal{H}(\mathcal{AW}(5,3))^*$. This matrix is the transpose of the above matrix. Clearly, the incidence matrix of the complement hypergraph is a vertex-facet incidence matrix of a polygon.

- Trivial bounds: $0 \le y_i \le 1$ for $0 \le i \le 4$.
- The IIS-inequalities: $\sum_{i \in S} y_i \le 2$ for S = $\{0,1,2\}$, $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,0\}$, $\{4,0,1\}$.
- The rank inequality $y_0 + y_1 + y_2 + y_3 + y_4 \le 3$ arising from the unique generalized antiweb.

5 Concluding Remarks

A question that naturally arises is whether our results are also valid for more general (mixed) linear systems with equality as well as inequality relations. Since any equation $\boldsymbol{ax} = b$ can be substituted by the pair of inequalities $\boldsymbol{ax} \le b$ and $-\boldsymbol{ax} \le -b$, any generalized MAX FS instance I with m_1 equations and m_2 inequalities can obviously be reduced to a usual MAX FS instance I' with $2m_1 + m_2$ inequalities, in which one aims at maximizing the number of such pairs of inequalities that can be simultaneously satisfied. Clearly, since any vector $\boldsymbol{x}$ satisfies at least one inequality out of each pair, an optimal solution of I contains m^* linear relations if and only if an optimal solution of I' contains $m^* + m_1$ inequalities. Thus, from a computational point of view, generalized instances of MAX FS with mixed systems can be dealt with a polyhedral approach based, among others, on the facet-defining inequalities discussed in this paper. Not all of the above results, however, can be easily generalized to mixed systems. In particular, it is still open whether the simplex decomposition characterization (Theorem 4) can be extended. On the other hand, the complexity results regarding MIN IIS (Theorem 5) and the IIS-hypergraph Recognition problem (Theorem 6) obviously hold for this generalized class of instances. Note also that generalized versions of the alternative polyhedron result (Theorem 2) for general mixed systems or mixed systems (LPs) where all inequalities are nonnegativity constraints are given in [39].

In this paper we have investigated structural and algorithmic properties of IISs, IIS-hypergraphs, and of the feasible subsystem polytope $P_{FS}(\Sigma)$. On the structural and geometric side, we have: provided a new characterization of IISs, given a new proof of the fact that all IISs are closed, and shown that only two very specific types of generalized antiwebs (generalized cliques and odd holes) can arise as induced hypergraphs of an IIS-hypergraph. In particular, the only generalized cliques that can occur are those corresponding to single IISs. The above results imply that the feasible subsystem polytope $P_{FS}(\Sigma)$ admits only a very limited type of rank facets induced by generalized antiwebs. This is in sharp contrast with other known independence system polytopes related to graphical problems, such as the maximum cardinality stable set problem in a graph, for which a wealth of such rank facets have been extensively studied. On the algorithmic side, we have established that: finding smallest cardinality IISs is very hard to approximate, IIS-hypergraph recognition is NP-hard and IIS rank facets cannot be separated in polynomial time, unless P = NP.

Interesting open questions include: What is the computational complexity of separating inequalities arising from generalized antiwebs? Do other P_{FS}-specific rank facets exist? Does the polytope P_{FS} admit higher order facets besides the ones studied in [9] with $0, 1, 2$ coefficients?

Acknowledgements. The authors would like to thank Michael Joswig, Volker Kaibel and Günter M. Ziegler for helpful discussions. We are also grateful to the anonymous referees for useful comments that improved the paper.

Appendix

To prove Lemma 3 of Section 3.1, we first need to verify the following.

Claim. *Let P be a d-dimensional pointed polyhedron which has a description over K and is not a polyhedral cone. Let m be the number of facets. Then there exists a polyhedron*

$$P' = \left\{ \boldsymbol{y} \in \mathbb{R}^m \mid A^t \boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{b}^t \boldsymbol{y} = -1,\ \boldsymbol{y} \geq \boldsymbol{0} \right\},$$

where $A \in K^{m \times (m-d-1)}$ and all inequalities $y_j \geq 0$, $1 \leq j \leq m$, define facets, which is affinely (and hence combinatorially) equivalent to P.

Proof. By projection onto the affine hull of P we can assume, w. l. o. g., that P is full-dimensional. Moreover, it can be represented as $P = \{\boldsymbol{x} \in \mathbb{R}^d \mid C\boldsymbol{x} \leq \boldsymbol{c}\}$. Since P has a minimal description over K, $C \in K^{m \times d}$ and each inequality defines a facet. The resulting polyhedron is affinely equivalent to P and can be represented as:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \boldsymbol{x} \leq \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{pmatrix} \right\},$$

where C_1 is a full-rank $d \times d$ matrix (P is pointed), C_2 is an $(m-d) \times d$ matrix, $\boldsymbol{c}_1 \in K^d$, and $\boldsymbol{c}_2 \in K^{m-d}$. Now apply the (bijective) affine transformation $\boldsymbol{x} \mapsto C_1^{-1}(\boldsymbol{c}_1 - \boldsymbol{u})$, where $\boldsymbol{u} := \boldsymbol{c}_1 - C_1\boldsymbol{x} \in \mathbb{R}^d$ and get:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} C_1^{-1}(\boldsymbol{c}_1 - \boldsymbol{u}) \leq \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{pmatrix} \;\Leftrightarrow\; \begin{pmatrix} -I \\ -C_2C_1^{-1} \end{pmatrix} \boldsymbol{u} \leq \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{c}_2 - C_2C_1^{-1}\boldsymbol{c}_1 \end{pmatrix}.$$

Setting $\boldsymbol{c}' := \boldsymbol{c}_2 - C_2C_1^{-1}\boldsymbol{c}_1$ and $C' := -C_2C_1^{-1} \in K^{(m-d)\times d}$ gives

$$P \cong \{\boldsymbol{u} \in \mathbb{R}^d \mid C'\boldsymbol{u} \leq \boldsymbol{c}',\ \boldsymbol{u} \geq \boldsymbol{0}\}.$$

Clearly, all inequalities define facets. The introduction of slack variables $\boldsymbol{s} \in \mathbb{R}^{m-d}$ yields

$$P \cong \left\{ (\boldsymbol{u}, \boldsymbol{s}) \in \mathbb{R}^d \times \mathbb{R}^{m-d} \,|\, C'\boldsymbol{u} + I\boldsymbol{s} = \boldsymbol{c}',\ \boldsymbol{u} \geq \boldsymbol{0},\ \boldsymbol{s} \geq \boldsymbol{0} \right\} ,$$

where all inequalities still define facets and the matrix $[C'\ I]$ has size $(m-d) \times m$.

Since P is not a cone, we must have $\boldsymbol{c}' \neq \boldsymbol{0}$. Therefore $\boldsymbol{c}'$ has at least one nonzero component; assume it is the last one. By adding multiples of the last row to the other rows of $[C'\ I \,|\, \boldsymbol{c}']$, we can eliminate all other nonzero components of $\boldsymbol{c}'$. The resulting system with matrix $[A'\ A'']$ and right hand side $(0, \ldots, 0, \alpha)^t$, with $\alpha \neq 0$, is clearly affinely equivalent. We denote by A^t the matrix $[A'\ A'']$ without the last row and by $\boldsymbol{b}^t$ the last row of $[A'\ A'']$ divided by $-\alpha$ (in order to scale the right hand side to -1). Then $A \in K^{m \times (m-d-1)}$, $\boldsymbol{b} \in K^m$ and

$$P \cong P' := \left\{ \boldsymbol{y} \in \mathbb{R}^m \,|\, A^t\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{b}^t\boldsymbol{y} = -1,\ \boldsymbol{y} \geq \boldsymbol{0} \right\},$$

where each inequality $\boldsymbol{y}_j \geq 0$ defines a facet for $j = 1, \ldots, m$. Since only affine transformations were applied, P' is affinely equivalent to P. □

Proof of Lemma 3. According to the claim, there exists a polyhedron P' affinely equivalent to P, where $P' = \left\{ \boldsymbol{y} \in \mathbb{R}^m \,|\, A^t\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{b}^t\boldsymbol{y} = -1,\ \boldsymbol{y} \geq \boldsymbol{0} \right\}$. Each face of P' defined by $y_j = 0$ is a facet, $1 \leq j \leq m$. Now V corresponds to the vertices of P and hence P'. If one identifies V with the set of vertices of P', then each set of $\mathcal{E}$ is the vertex set of a facet of P'. Moreover, each set $E^* \in \mathcal{E}^*$ is the set of facets which contain a specific vertex $\boldsymbol{v}$ of P'. If we identify $[m]$ with the set of facets, $\overline{E^*}$ is the support of $\boldsymbol{v}$. Thus, by Theorem 2, $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ is an infeasible system whose IISs correspond bijectively to the sets in $\overline{\mathcal{E}^*}$. □

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