Bisimulation and Games: Model-Theoretic Aspects

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bisimulation — the quintessential back&forth

model theory, not just in classical settings, and some combinatorial challenges

organisation in two main parts

(I)

- bisimulation and back&forth games
- bisimulation as modal Ehrenfeucht–Fraïssé
- bisimulation and the (finite) model theory of modal logics

(II)

- combinatorics of finite coverings
- bisimilar coverings for graphs and hypergraphs
- bisimulation and the (finite) model theory of guarded logics

part I: bisimulation

the quintessential back&forth

on graph-like structures

Kripke structures (possible worlds/accessibility),
transition systems (states/transitions),
game graphs (positions/moves)

capture behavioural equivalence

in the sense of indistinguishability of worlds/states/positions
w.r.t. alternating sequences of accessibility/transitions/moves

core idea: dynamic b&f probing of possibilities

$\longrightarrow$ **dynamic exploration of structures that are static images of dynamic behaviour**

bisimulation game & bisimulation relations

the game: two players: **I** (challenger), **II** (defender)

play over two transition systems $\begin{cases} \mathcal{A} = (A, \mathbf{R}^{\mathcal{A}}, \mathbf{P}^{\mathcal{A}}) \\ \mathcal{B} = (B, \mathbf{R}^{\mathcal{B}}, \mathbf{P}^{\mathcal{B}}) \end{cases}$

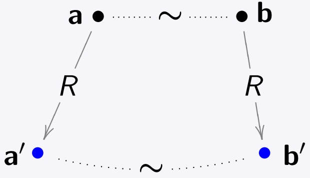
positions: pairs (a, b), correspondences between pebbled vertices

single round of challenge/response:

I moves pebble in $\mathcal{A}$ or $\mathcal{B}$ along R-edge

II must do likewise in opposite structure

effect: $(\mathbf{a}, \mathbf{b}) \rightsquigarrow (\mathbf{a}', \mathbf{b}')$



II loses in position (a, b) unless $a \sim^0 b$ (atom equiv.: $\mathbf{P}^{\mathcal{A}} \restriction a \simeq \mathbf{P}^{\mathcal{B}} \restriction b$)

either player loses when stuck

bisimulation game & bisimulation relations

winning regions for II define **bisimulation equivalences:**

$\mathcal{A}, a \sim^{\ell} \mathcal{B}, b$	**II** has a winning strategy for ℓ rounds from (a, b)
$\mathcal{A}, a \sim^{\omega} \mathcal{B}, b$	**II** has a winning strategy for any finite no. of rounds from (a, b)
$\mathcal{A}, a \sim \mathcal{B}, b$	**II** has a winning strategy for infinite game from (a, b)

winning strategies in relational formalisation:

$$\sim^{\ell} : \ (Z_m \subseteq A \times B)_{m \leqslant \ell}$$
$$\sim^{\omega} : \ (Z_m \subseteq A \times B)_{m \in \mathbb{N}}$$
$$\sim \ : \ Z \subseteq A \times B$$

stratified b&f systems, or single bisimulation relation

bisimulation game & bisimulation relations

a single bisimulation relation $Z \subseteq A \times B$ for $\sim$

with characteristic b&f requirements

(*back*) for $(a, b) \in Z$ and $(b, b') \in R^{\mathcal{B}}$ there is
$a' \in A$ s.t. $(a, a') \in R^{\mathcal{A}}$ and $(a', b') \in Z$

(*forth*) for $(a, b) \in Z$ and $(a, a') \in R^{\mathcal{A}}$ there is
$b' \in B$ s.t. $(b, b') \in R^{\mathcal{B}}$ and $(a', b') \in Z$

witnesses winning strategy for **II** in
infinite game from any $(a, b) \in Z$

b&f systems $(Z_m)_{m \leqslant \ell}$ or $(Z_m)_{m \in \mathbb{N}}$
encode winning strategies for m rounds from any $(a, b) \in Z_m$

with suitably stratified b&f conditions from $\mathbf{Z_k}$ into $\mathbf{Z_{k-1}}$

classical motif: Ehrenfeucht–Fraïssé

pebble games for FO and FO_∞

I and **II** over relational structures $\mathcal{A} = (A, \mathbf{R}^{\mathcal{A}})$ and $\mathcal{B} = (B, \mathbf{R}^{\mathcal{B}})$

positions: local isomorphisms $p\colon \mathbf{a} \mapsto \mathbf{b}$, $p\colon \mathcal{A}\restriction\mathbf{a} \simeq \mathcal{B}\restriction\mathbf{b}$

single round: challenge/response for
extension by one new pebble pair
$(p\colon \mathbf{a} \mapsto \mathbf{b}) \rightsquigarrow (p'\colon \mathbf{a}a' \mapsto \mathbf{b}b')$

winning regions:
b&f equivalences

$$\begin{cases} \mathcal{A}, \mathbf{a} \simeq^{\ell} \mathcal{B}, \mathbf{b} & \ell \text{ rounds} \\ \mathcal{A}, \mathbf{a} \simeq^{\omega} \mathcal{B}, \mathbf{b} & \text{any finite no. of rounds} \\ \mathcal{A}, \mathbf{a} \simeq^{\infty} \mathcal{B}, \mathbf{b} & \text{infinite game} \end{cases}$$

$\simeq^\infty$ **classically known as partial isomorphy**

Ehrenfeucht–Fraïssé

linking game equivalence to equivalence w.r.t. FO and FO_∞

Ehrenfeucht–Fraïssé/Karp thms

$\mathcal{A},\mathbf{a} \simeq^{\ell} \mathcal{B},\mathbf{b}$	$\Leftrightarrow$	$\mathcal{A},\mathbf{a} \equiv^{\ell}_{\mathrm{FO}} \mathcal{B},\mathbf{b}$ *	FO-equiv. to qfr-depth ℓ
$\mathcal{A},\mathbf{a} \simeq^{\omega} \mathcal{B},\mathbf{b}$	$\Leftrightarrow$	$\mathcal{A},\mathbf{a} \equiv_{\mathrm{FO}} \mathcal{B},\mathbf{b}$ *	full FO-equiv.
$\mathcal{A},\mathbf{a} \simeq^{\infty} \mathcal{B},\mathbf{b}$	$\Leftrightarrow$	$\mathcal{A},\mathbf{a} \equiv^{\infty}_{\mathrm{FO}} \mathcal{B},\mathbf{b}$	FO_∞-equiv.

observations/proof ingredients:

- the sets $Z_m := \big\{(p\colon \mathbf{a} \mapsto \mathbf{b})\colon \mathcal{A},\mathbf{a} \equiv^{m}_{\mathrm{FO}} \mathcal{B},\mathbf{b}\big\}$ satisfy b&f conditions
- **I** can force $\mathcal{A},\mathbf{a} \not\equiv^{m}_{\mathrm{FO}} \mathcal{B},\mathbf{b} \rightsquigarrow \mathcal{A},\mathbf{a}a' \not\equiv^{m-1}_{\mathrm{FO}} \mathcal{B},\mathbf{b}b'$
- equivalence classes $[\mathcal{A},\mathbf{a}]/{\simeq^{\ell}}$ are FO-definable at qfr-depth ℓ *

∗ for finite relational vocabularies

bisimulation & basic modal logic ML

on graph-like structures

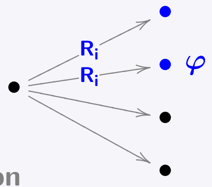
with binary (transition) relations $\mathbf{R} = (R_1, \dots)$ $\rightsquigarrow$ modalities $\Diamond_i / \Box_i$
and unary (state) predicates $\mathbf{P} = (P_1, \dots)$ $\rightsquigarrow$ basic propositions p_i

atomic formulae: $\bot, \top$ and p_i
booleans connectives: $\wedge, \vee, \neg$
modal quantification:

$$\Diamond_i \varphi \equiv \exists y \big(R_i xy \wedge \varphi(y) \big)$$

$$\Box_i \varphi \equiv \forall y \big(R_i xy \rightarrow \varphi(y) \big)$$

relativised FO quantification



observation

- atomic bisimulation condition ($\sim^0$) matches atomic equiv. $\equiv^0_{\mathrm{ML}}$
- bisimulation b&f matches modal quantification pattern

bisimulation — modal Ehrenfeucht–Fraïssé

modal Ehrenfeucht–Fraïssé/Karp thms

$\mathcal{A}, a \sim^{\ell} \mathcal{B}, b$	$\Leftrightarrow$	$\mathcal{A}, a \equiv^{\ell}_{\mathrm{ML}} \mathcal{B}, b$ *	ML-equiv. to depth ℓ
$\mathcal{A}, a \sim^{\omega} \mathcal{B}, b$	$\Leftrightarrow$	$\mathcal{A}, a \equiv_{\mathrm{ML}} \mathcal{B}, b$ *	full ML-equiv.
$\mathcal{A}, a \sim^{\infty} \mathcal{B}, b$	$\Leftrightarrow$	$\mathcal{A}, a \equiv^{\infty}_{\mathrm{ML}} \mathcal{B}, b$	ML_∞-equiv.

in full analogy with classical picture:

$\mathcal{A}, \mathbf{a} \simeq^{\ell} \mathcal{B}, \mathbf{b}$	$\Leftrightarrow$	$\mathcal{A}, \mathbf{a} \equiv^{\ell}_{\mathrm{FO}} \mathcal{B}, \mathbf{b}$	FO-equiv. to qfr-depth ℓ
$\mathcal{A}, \mathbf{a} \simeq^{\omega} \mathcal{B}, \mathbf{b}$	$\Leftrightarrow$	$\mathcal{A}, \mathbf{a} \equiv_{\mathrm{FO}} \mathcal{B}, \mathbf{b}$	full FO-equiv.
$\mathcal{A}, \mathbf{a} \simeq^{\infty} \mathcal{B}, \mathbf{b}$	$\Leftrightarrow$	$\mathcal{A}, \mathbf{a} \equiv^{\infty}_{\mathrm{FO}} \mathcal{B}, \mathbf{b}$	FO_∞-equiv.

corollary

- the semantics of ML and ML_∞ is invariant under bisimulation
- the semantics of ML-formulae of depth ℓ is invariant under $\sim^{\ell}$

variations & the quintessential nature of bisimulation

- **bisimulation in game graphs for other logics**

  states: admissible assignments
  transitions: quantification patterns

  all Ehrenfeucht–Fraïssé games are bisimulation games

close to original (basic modal) bisimulation:

- **two-way and global bisimulation $\approx$**
  with extended challenge/response options
  (backward moves & jumps) for corresponding modalities

- **hypergraph/guarded bisimulation $\rightarrow$ part II**

bisimulation — modal Ehrenfeucht–Fraïssé

typical example of a bisimulation issue and classical counterpart:

when does $\equiv_{\mathsf{ML}}$ ($\sim^{\omega}$) coincide with full bisimulation $\sim$?

when does $\equiv_{\mathsf{FO}}$ ($\simeq^{\omega}$) coincide with partial isomorphy $\simeq^{\infty}$?

Hennessy–Milner thm (the modal answer)

over suitably saturated models, $\sim^{\omega}$ ($\equiv_{\mathsf{ML}}$) coincides with $\sim$ ($\equiv^{\infty}_{\mathsf{ML}}$)

- **finitely branching**
- **modally or ω-saturated** (ω-saturation is good also for $\simeq^{\omega}/\simeq^{\infty}$)
- **recursively saturated pairs** (also good for $\simeq^{\omega}/\simeq^{\infty}$)

crucial in classical model-theoretic arguments for modal logics

model theory of modal logics

modal model theory = bisimulation invariant model theory

here briefly look at:

- tree model property
- finite model property
- **expressive completeness** **(classical and fmt)**

tree unfoldings

tree unfolding $\mathcal{A}$ into $\mathcal{A}_a^*$

based on the set of labelled directed paths w rooted at a in $\mathcal{A}$ with natural projection onto the endpoints as a homomorphism

$$\begin{array}{rcl} \pi\colon \mathcal{A}_a^* & \longrightarrow & \mathcal{A} \\ w & \longmapsto & \pi(w) \end{array}$$

that induces a bisimulation $\mathcal{A}_a^*, a \sim \mathcal{A}, a$

$\pi\colon \mathcal{A}_a^* \longrightarrow \mathcal{A}$ is an example of a bisimilar covering:

- **π is a homomorphism:** the forth-property for its graph
- **π has lifting property:** the back-property for its graph

inducing a bisimulation relation $\{(w, \pi(w)) \colon w \in A_a^*\}$

tree unfoldings and tree model property

$$\left.\begin{array}{r}\text{bisimilar unfoldings into tree structures}\\ \text{preservation under bisimulation}\end{array}\right\} \Rightarrow \text{ tree model property}$$

tree model property

for all $\sim$-invariant logics $\mathsf{ML}, \ldots, \mathsf{L}_\mu, \ldots \mathsf{ML}_\infty$:
every satisfiable formula has a tree model

for $\approx$-invariant logics analogously: a forest model property

of great importance: can employ good model theoretic and algorithmic properties of trees, MSO on trees, tree automata, ... for robust decidability and complexity results for modal logics

finite (tree) model property

for basic modal logic ML (and some close relatives)
even get finite tree models, hence the

finite model property:

every satisfiable formula of ML has a finite (tree) model

ad-hoc method: for $\varphi \in$ ML of depth ℓ,
truncate tree model at depth ℓ (preserving $\sim^\ell$)
and prune $\sim^\ell$-equivalent siblings (finite index!)

more generic method: passage to $\sim^\ell$-quotient of any
model yields a finite model (usually not a tree model)

generalises to extensions preserved under levels of $\approx$

expressive completeness of modal logics

. . . relative to FO, a classical theme of FO model theory

FO$/\!\sim$ $\left\{ \begin{array}{l} \text{the classes of } \sim\text{-invariant FO-properties of} \\ \text{(just finite, or all) relational structures} \end{array} \right.$

semantic classes

corresponding to the undecidable classes of those $\varphi(x) \in$ FO that satisfy $\mathcal{A}, a \sim \mathcal{B}, b \;\Rightarrow\; \big(\mathcal{A}, a \models \varphi \Leftrightarrow \mathcal{B}, b \models \varphi \big)$

classical 'preservation thms', too, respond to the quest for syntactic representation — mostly without asking the question

in this case, the answer to the unasked question is 'yes', twice:

FO$/\!\sim\, \equiv$ **ML** **classically, van Benthem**

FO$/\!\sim\, \equiv$ **ML** **in fmt, Rosen**

expressive completeness: FO/$\sim$ $\equiv$ ML

it suffices to show, for $\varphi(x) \in$ FO:

$\sim$-invariance implies $\sim^{\ell}$-invariance for some finite level $\ell \in \mathbb{N}$

a compactness property (!)

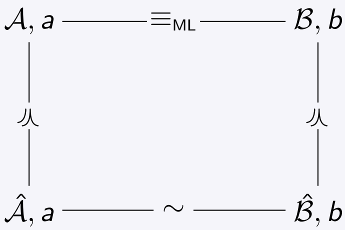
then $\varphi \equiv \varphi' \in$ ML by Ehrenfeucht–Fraïssé:
ML-definability of $\sim^{\ell}$-classes & finite index

NB: two, a priori independent, readings: classical & fmt

expressive completeness: generic classical approach

$\sim$-invariance $\Rightarrow$ $\sim^{\ell}$-invariance for some ℓ $\quad(*)$

classical compactness argument with upgrading along $\equiv_{\mathsf{FO}}$-axis
through Hennessy–Milner property for ω-saturated structures



ω-saturated extns

elegant and smooth, but no information regarding target ℓ

and not an option for fmt version

expressive completeness: a constructive approach

$\sim$-invariance $\Rightarrow$ $\sim^{\ell}$-invariance for some ℓ

upgrading along $\sim$-axis of $\sim^{\ell(q)}$ $(\equiv^{\ell}_{\mathrm{ML}})$ to $\simeq^{q}$ $(\equiv^{q}_{\mathrm{FO}})$
through bisimulation preserving model transformations

$$\begin{array}{ccc} \mathcal{A}, a & \text{———} \sim^{\ell(q)} \text{———} & \mathcal{B}, b \\ \wr & & \wr \\ \hat{\mathcal{A}}, a & \text{———} \equiv^{q}_{\mathrm{FO}} \text{———} & \hat{\mathcal{B}}, b \end{array}$$

bisimilar companions

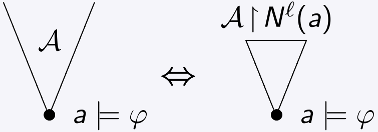
more constructive, potentially suitable for fmt,
also yielding information regarding ℓ(q)

FO/$\sim$ $\equiv$ ML: an elementary proof with added value

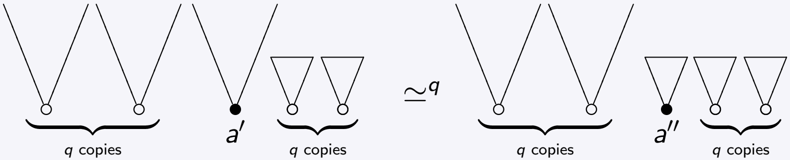
$\sim$-invariance $\Rightarrow$ $\sim^\ell$-invariance for $\ell = 2^q - 1$

simple, ad-hoc argument (good classically & fmt)
using the locality of FO/$\sim$ & Ehrenfeucht–Fraïssé:

show that



in q-round FO game on:



back to generic constructive approach

upgrading in

$$\begin{array}{ccc} \mathcal{A} & \text{———} \sim^{\ell(q)} \text{———} & \mathcal{B} \\ | & & | \\ \wr & & \wr \\ | & & | \\ \hat{\mathcal{A}} & \text{———} \equiv^{q}_{\mathrm{FO}} \text{———} & \hat{\mathcal{B}} \end{array}$$

requires (finite) model transformations $\mathcal{A}/\mathcal{B} \longmapsto \hat{\mathcal{A}}/\hat{\mathcal{B}}$ that are

- compatible with bisimulation:
  ideally want $\approx$ coverings (for symmetry & homogeneity)

- suitable to eliminate all obstacles for $\simeq^q$ ($\equiv^q_{\mathrm{FO}}$)
  that are *not controlled* by any level of $\sim^\ell$:

  want to avoid short cycles & small multiplicities

part II: the combinatorics of finite coverings

in this part **(shortened)**:

- **bisimilar graph coverings**:
  graph acyclicity in finite direct products
  with Cayley graphs of large girth

- **bisimilar hypergraph coverings**
  hypergraph acyclicity in finite reduced products
  with Cayley graphs of groups & groupoids
  of more than just large girth

- **hypergraph bisimulation & guarded bisimulation**
  for guarded logics & other applications

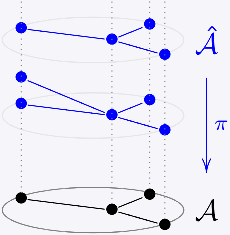
graph coverings

definition: $\approx$-bisimilar coverings

$\pi\colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ a *covering* of $\mathcal{A} = (A, E)$ by $\hat{\mathcal{A}} = (\hat{A}, \hat{E})$:

(*forth*) $\pi\colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ surjective homomorphism

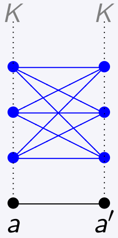
(*back*) π lifts edges/paths from $a \in \mathcal{A}$ to any $\hat{a}$ in its fibre



coverings by products

- **boost multiplicities**
  in products with large cliques K:

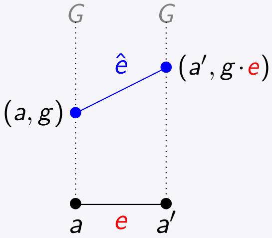
  put K-fibre $K \times \{a\}$ for every a



- **avoid short cycles**
  in products with Cayley graphs of large girth:

  for $\mathcal{A} = (A, E)$ use Cayley group/graph G with generators e for $e \in E$

  $\hat{\mathcal{A}} = \mathcal{A} \otimes G = (A \times G, \hat{E})$
  $\hat{E} = \{((a, g), (a', g \cdot e)) \colon e = (a, a') \in E\}$



these are (finite) $\approx$-bisimilar coverings!

avoiding short cycles in finite coverings

in products with Cayley groups of large girth

Cayley groups/graphs:

- group $G = (G, \cdot, 1)$ with generators $e \in E$
- associated Cayley graph has e-coloured edges from g to $g \cdot e$

highly symmetric, regular & homogeneous objects

Cayley groups/graphs of girth $> \mathbf{N}$:

no non-trivial generator cycles $e_1 \cdot e_2 \cdots e_n = 1$ for $n \leqslant N$

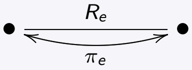
products $\mathcal{A} \otimes \mathbf{G}$ with such G are N-acyclic coverings
useful for upgrading $\sim^{\ell}$ to $\simeq^{\mathbf{q}}$

Cayley graphs of large girth

goal: no non-trivial generator cycles $e_1 \cdot e_2 \;\cdots\; e_n = 1$ for small n

aside on construction (after Biggs)

find G as subgroup $G = \langle \pi_e \colon e \in E \rangle \subseteq \mathrm{Sym}(V)$
generated by permutations π_e of undirected
deterministically E-coloured graph $(V, (R_e))$



lemma

if $H = (V, (R_e))$ is deterministically E-coloured s.t.
every colour sequence $w = e_1 \cdots e_n$ labels some non-cyclic path

$$v_0 \stackrel{e_1}{\longrightarrow} v_1 \quad \ldots \quad v_{n-1} \stackrel{e_n}{\longrightarrow} v_n \neq v_0 \quad \text{in } H,$$

then $\pi_{e_1} \;\cdots\; \pi_{e_n} \neq 1$
so that $G = \langle \pi_e \colon e \in E \rangle \subseteq \mathrm{Sym}(V)$ has girth $> N$

locally acyclic graph coverings

thm (APAL 04)

every finite graph admits, for every $N \in \mathbb{N}$,
simple/unbranched N-acyclic finite coverings
by products with Cayley graphs of large girth

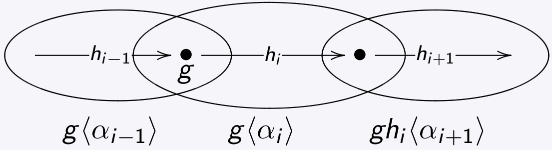
- **uniform construction, which preserves all symmetries**
- **adaptable to many special frame classes** (→ APAL 09)
  FO/$\sim$ $\equiv$ ML on many natural (finite) frame classes

construction idea for Cayley graphs extends to much stronger notions of acyclicity in groups and in groupoids that are useful towards hypergraph constructions

more than just large girth

avoid not just short generator cycles but even short coset cycles

coset cycles:
steps in a coset chain are based on cosets $g_i\langle\alpha_i\rangle$
w.r.t. generator subsets $\alpha_i \subseteq E$ in $G = \langle E\rangle$



G is N-c-acyclic if it has no coset cycles of length up to N

N-c-acyclic Cayley groups

G is N-c-acyclic if it has no coset cycles of length up to N

and such objects do exist!

thm (JACM 10)

can find finite N-c-acyclic Cayley groups
for every finite set E of generators and $N \in \mathbb{N}$

$\rightsquigarrow$ **extend bisimilar unfolding idea from graphs to hypergraphs and, in logical terms, from modal to guarded scenarios**

construction uses intricate interleaving of amalgamations and group actions

from graphs to hypergraphs

hypergraphs: structures $\mathcal{A} = (A, S)$ with vertex set A, and set of hyperedges $S \subseteq \mathcal{P}(A)$

idea: clusters and their link structure

example: hypergraph of guarded subsets
of a relational structure $\mathcal{A} = (A, \mathbf{R}^{\mathcal{A}})$

$$\mathbf{H}(\mathcal{A}) = (\mathbf{A}, \mathbf{S}[\mathcal{A}])$$

with hyperedges generated by subsets
$[\mathbf{a}] \subseteq A$ for $\mathbf{a} \in R^{\mathcal{A}}$, $R \in \mathbf{R}$
closed under subsets & singleton sets

relational structure	$=$	**hypergraph link structure (topology)**
		+ local relational content

the logical motivation: from modal to guarded logics

the guarded fragment GF (Andréka–van Benthem–Németi 98)

key idea: relativise quantification to guarded clusters

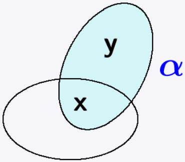
recall hypergraph $H(\mathcal{A}) = (A, S[\mathcal{A}])$ of
guarded subsets generated by $[\mathbf{a}]$ for $\mathbf{a} \in R^{\mathcal{A}}$

guarded quantification:

$\exists \mathbf{y}\big(\alpha(\mathbf{xy}) \wedge \varphi(\mathbf{xy})\big)$

$\forall \mathbf{y}\big(\alpha(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy})\big)$

guard atom α: $\mathrm{free}(\varphi) \subseteq \mathrm{var}(\alpha)$



quantification relativised
to guarded tuples

ML $\subsetneq$ GF $\subsetneq$ FO

model-theoretic motivation: reflection on ML $\subseteq$ FO in extension
from graph-like structures to general relational format

the logical motivation: GF and guarded bisimulation

guarded bisimulation $\sim_{\mathbf{g}}^{\ell}$ / $\sim_{\mathbf{g}}^{\omega}$ / $\sim_{\mathbf{g}}$

- bisimulations of hypergraphs of guarded subsets that locally respect relational content ($\sim_{\mathrm{g}}^{0}$: $\mathcal{A}\restriction\mathbf{a} \simeq \mathcal{B}\restriction\mathbf{b}$)
- FO pebble game restricted to guarded pebble configurations

the guarded Ehrenfeucht–Fraïssé/Karp thms

$\mathcal{A},\mathbf{a} \;\sim_{\mathrm{g}}^{\ell}\; \mathcal{A}',\mathbf{a}'$	$\Leftrightarrow$	$\mathcal{A},\mathbf{a} \equiv_{\mathsf{GF}}^{\ell} \mathcal{A}',\mathbf{a}'$	GF_{ℓ}-equiv. to depth ℓ
$\mathcal{A},\mathbf{a} \;\sim_{\mathrm{g}}^{\omega}\; \mathcal{A}',\mathbf{a}'$	$\Leftrightarrow$	$\mathcal{A},\mathbf{a} \equiv_{\mathsf{GF}} \mathcal{A}',\mathbf{a}'$	full GF-equiv.
$\mathcal{A},\mathbf{a} \;\sim_{\mathrm{g}}\; \mathcal{A}',\mathbf{a}'$	$\Leftrightarrow$	$\mathcal{A},\mathbf{a} \equiv_{\mathsf{GF}}^{\infty} \mathcal{A}',\mathbf{a}'$	GF_{∞}-equiv.

hypergraphs

issues in logic & combinatorics:

- **degrees of acyclicity** and their algorithmic and model-theoretic relevance for guarded logics
- **hypergraph coverings:** reproduce link structure locally; smooth out global link structure (e.g., regarding cycles)

3 equivalent definitions of hypergraph acyclicity:

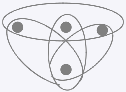
- **tree-decomposable with hyperedges as bags**
- **decomposable via elementary deletion steps (Graham)**
- **conformality and chordality** (of associated Gaifman graph)

hypergraph acyclicity

- **conformality and chordality:**

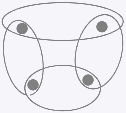
conformality: every Gaifman clique is contained in some $s \in S$

no



chordality: every Gaifman cycle of length > 3 has a chord

no



N-acyclicity: sub-configurations up to size N are acyclic

conformality & chordality just up to size N

hypergraph bisimulation & coverings

definition: bisimilar coverings

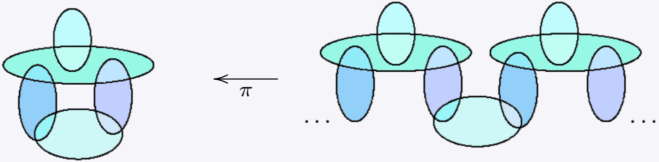
$\pi \colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ a *covering* of $\mathcal{A} = (A, S)$ by $\hat{\mathcal{A}} = (\hat{A}, \hat{S})$:

(*forth*) $\pi \colon \hat{\mathcal{A}} \longrightarrow \mathcal{A}$ homomorphism
i.e., $\pi \restriction \hat{s} \colon \hat{s} \to \pi(\hat{s}) = s \in S$ bijective for all $\hat{s} \in \hat{S}$

(*back*) π lifts overlaps $s \cap s' \neq \emptyset$ from $\mathcal{A}$ to any $\hat{s} \in \hat{S}$ above s

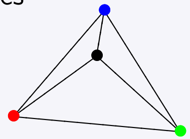
examples of natural hypergraph coverings:

- tree- and forest-like unfoldings (typically infinite)
- reduced products with suitable groups/groupoids (more below)



the combinatorial challenge: an example

the facets of the 3-simplex/tetrahedron
the uniform width 3 hypergraph on 4 vertices

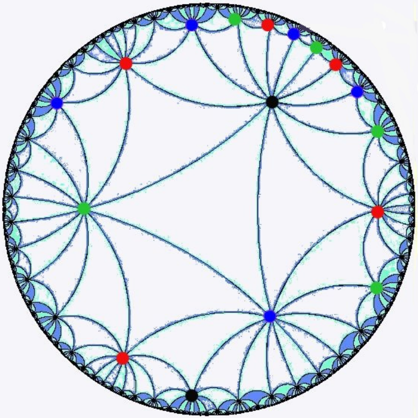


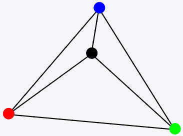
- chordal but not conformal
- finite coverings cannot be 1-locally acyclic
- admits locally finite coverings without short chordless cycles

Question: can extend ideas from graph coverings?

the combinatorial challenge: an example

a locally finite covering **of the tetrahedron**





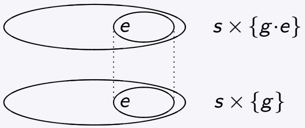
conformal; shortest chordless cycles have length 12
here by regular triangulation of the hyperbolic plane

reduced products with Cayley groups

plain reduced product $\mathcal{A} \otimes \mathbf{G}$

between hypergraph $\mathcal{A} = (A, S)$ and group G with generators e associated with subsets $e \subseteq s \in S$

$\mathcal{A} \otimes \mathbf{G}$: $\begin{cases} \text{quotient of } \mathcal{A} \times G \text{ w.r.t. glueing} \\ \text{layer}(g) \text{ and layer}(g \cdot e) \text{ in } e \subseteq s \end{cases}$



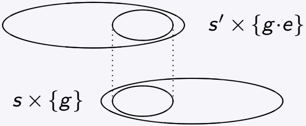
e-transitions in G glue layers of $\mathcal{A} \times \mathbf{G}$ through identification in e

reduced products with Cayley groups

unfolded reduced product $\mathcal{A}^! \otimes \mathbf{G}$

of exploded view $\mathcal{A}^!$ of $\mathcal{A} = (A, S)$ and group G with generators e associated with non-trivial intersections $e = s \cap s'$

$$\mathcal{A}^! \otimes \mathbf{G}\colon \begin{cases} \text{quotient of } \dot{\bigcup} S \times G \text{ w.r.t. glueing} \\ \text{layer}(g) \text{ and layer}(g \cdot e) \text{ to overlap just in } s \cap s' \end{cases}$$



e-transitions in G for e = (s, s′) glue copies of s and s′ in e-related layers

extending the scope: groupoids vs. groups

groupoids: like ‘many-sorted’ groups with
sort-sensitive partial operation

$$\mathbf{G} = \big(G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S}, {}^{-1}\big)$$

with operation $G_{st} \times G_{tu} \stackrel{\cdot}{\longrightarrow} G_{su}$

examples: **bijective morphisms in a category,**
changes of co-ordinates in manifolds

why groupoids are more suitable in hypergraph constructions:

- overlaps of hyperedges (in exploded view)
  behave like local changes of co-ordinates
- (reduced) products with groupoids can offer
  just the right transitions at the right place

. . . **unlike the graph/group situation**

extending the scope: products with groups/groupoids

main results

- **plain reduced products with N-c-acyclic Cayley groups preserve N-acyclicity of $\mathcal{A}$**

⇝ local–global construction of finite N-acyclic coverings from locally finite N-acyclic coverings (JACM 12)

- **unfolded reduced products with N-c-acyclic Cayley groupoids produce N-acyclic coverings of $\mathcal{A}$**

⇝ direct construction of finite N-acyclic coverings (arXiv 15)

- **N-c-acyclic groupoids can be constructed by similar group action & amalgamation ideas**

back to the (finite) model theory of guarded logics

in striking analogy with ML find, for instance:

- **generalised tree model property**
- **finite model property**
- **expressive completeness: $\mathsf{FO}/{\sim_{\mathrm{g}}} \equiv \mathsf{GF}$ (classical and fmt)**

GF and guarded bisimulation/coverings

in striking analogy with modal model theory, based on
invariance/preservation under guarded bisimulation:

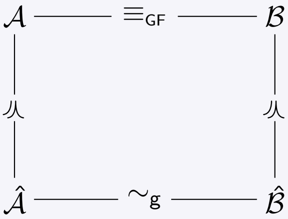
- **generalised tree model property**
  tree/forest unfoldings (Grädel 99):
  acyclic hypergraph coverings
- **finite model properties (and decidability)**
  via Herwig extensions (Grädel 99), **and small models via succinct coverings** (Bárány–Gottlob–O_LMCS 13)
- **classical/fmt expressive completeness results**
  compactness&saturation (Andréka–van Benthem–Németi 98)
  upgrading in coverings (O_JACM 12)
- **also: new proof of Herwig–Lascar EPPA theorem**
  based on realisations of overlaps between copies of $\mathcal{A}$
  groupoidal products & coverings (O_arXiv 15)

expressive completeness: $\mathrm{FO}/{\sim_{\mathrm{g}}} \equiv \mathrm{GF}$

crux (as in modal case): compactness property

$\varphi \in \mathrm{FO}$ $\sim_{\mathrm{g}}$-invariant $\;\Rightarrow\;$ $\sim_{\mathrm{g}}^{\ell}$-invariance for some ℓ

- classical compactness argument allows upgrading along $\equiv_{\mathrm{FO}}$-axis, by use of ω-saturated elementary extensions



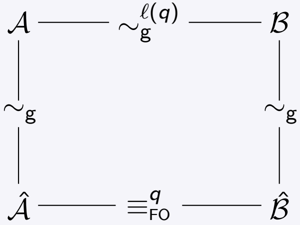
ω-saturated extns.

expressive completeness: FO$/\sim_{\mathrm{g}} \equiv$ GF

crux (as in modal case): compactness property

$\varphi \in \mathrm{FO}\ \sim_{\mathrm{g}}$-invariant $\Rightarrow\ \sim_{\mathrm{g}}^{\ell}$-invariance for some ℓ

- constructive upgrading along $\sim_{\mathrm{g}}$-axis
  uses rich N-acyclic (finite) coverings



bisimilar companions

summary: how far do bisimulation analogies carry?

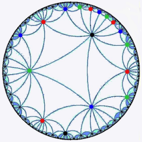
- infinite tree unfoldings as fully acyclic coverings:
  a complete analogy, good for most classical purposes
  analogy with freeness & richness of ω-saturated extns

- finite coverings meet different combinatorial challenges
  w.r.t. control of cycles and local-global-distinctions

- gain considerable extensions of the analogies between
  graphs/hypergraphs & modal/guarded logics

- especially through new hypergraph constructions
  via reduced products with suitable groupoids

the end



some pointers

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