Amalgamation and Local-To-Global in the Finite with Suitable Groupoids

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Mülheim 2016

Global Finite Realisations of Local Specifications

a generic amalgamation construction

given families

$$\left.\begin{array}{rl} (\mathcal{A}_s)_{s \in S} & \text{of relational structures} \\ (\rho_e \colon \mathcal{A}_s \xrightarrow{\text{part}} \mathcal{A}_{s'})_{e \in E[s,s']} & \text{of partial isomorphisms} \end{array}\right\} (*)$$

- use the free monoidal structure I^* of walks in the multigraph $\mathrm{I} = (S, E)$, with generators $e \in E := \bigcup_{s,s' \in S} E[s, s']$
- construct a natural free amalgam of disjoint copies $(\mathcal{A}_s, w) \simeq \mathcal{A}_s$ tagged by walks w terminating in s with identifications between $(\mathcal{A}_s, w)$ and $(\mathcal{A}_{s'}, w \cdot e)$ according to ρ_e

obtain structure $\big((\mathcal{A}_s) \otimes \mathrm{I}^* \big) \,/ \approx$

"realising" the overlap pattern specified in $(*)$,
"free" in a universal algebraic sense & typically infinite

finite realisations? to be based on ...?

$\rightsquigarrow$ the infinite structure of I^* (walks in the multigraph (S, E)):
a multi-sorted monoid w.r.t. partial concatenation of walks

NB: can only concatenate $\underbrace{\text{(walks to } s) \times \text{(walks from } s)}_{\text{partiality}}$, for $s \in S$

analogies:

- I^* vs. free monoid E^* generated by E
- partial vs. global operations (and symmetries)
- groupoids (or inverse semigroups) vs. groups

realisation of $\big((\mathcal{A}_s), (\rho_e)\big)$ in general:

a relational structure $\mathcal{A}$ with an "atlas" given by superimposed hypergraph structure $(A, \tilde{S})$, $\tilde{S} \subseteq \mathcal{P}(A)$, with "charts" $\pi_{\tilde{s}} \colon \mathcal{A}\restriction\tilde{s} \simeq \mathcal{A}_{\pi(\tilde{s})}$ s.t.

- **locally, all ρ_e-overlaps are realised:**

  each $\pi_{\tilde{s}}^{-1}(\mathcal{A}_s)$ overlaps with some $\pi_{\tilde{s}'}^{-1}(\mathcal{A}_{s'})$ according to ρ_e

- **globally, no incidental overlaps occur:**

  if $\tilde{s} \cap \tilde{t} \neq \emptyset$, then this is due to $\rho_w = \rho_{e_m} \circ \cdots \circ \rho_{e_1}$
  for some *single* walk $w = e_1 \cdots e_m$ from $\pi(\tilde{s})$ to $\pi(\tilde{t})$

NB: the second, "no-nonsense" condition avoids potential relational inconsistencies for amalgams

(trivial yet instructive) example

every hypergraph (A, S) realises its **exploded view** based on disjoint $\emptyset$-structures $((s \times \{s\})_{s \in S}$ and ρ_e for $e = (s, s')$ representing non-empty intersections $s \cap s'$ in (A, S)

this realisation is obtained from disjoint sum of tagged copies of the $\mathcal{A}_s$ as quotient w.r.t. $\approx$ induced by the ρ_e

$\rightsquigarrow$ cannot work in general, but compare generic (infinite) free construction $\big((\mathcal{A}_s) \otimes \mathrm{I}^*\big) \,/\approx$

$\rightsquigarrow$ use product with suitable groupoids $\mathbb{G}$ for "local unfolding" to overcome obstructions in the finite, and more ...

$\big((\mathcal{A}_s) \otimes \mathbb{G}\big) \,/\!\approx$

groupoids (algebraic format)

to deal with $\big((\mathcal{A}_s),(\rho_e)\big)$ with "incidence pattern" $\mathrm{I} = (S, E)$, use:

I-groupoids $\mathbb{G} = \big((\mathbf{G}_{st}), \cdot, (\mathbf{1}_s), {}^{-1}\big)$**:**

generated by the $e \in E$,
with partial composition $\mathrm{G}_{st} \times \mathrm{G}_{tu} \longrightarrow \mathrm{G}_{su}$,
inverses, and neutrals $1_s \in \mathrm{G}_{ss}$

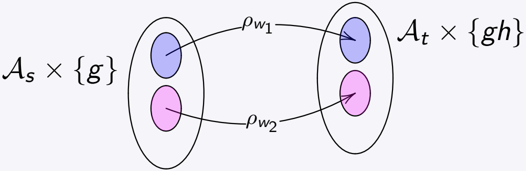
$\rightsquigarrow$ suitable for natural reduced products $\big((\mathcal{A}_s) \otimes \mathbb{G}\big) /{\approx}$

NB: can also view groupoid as category with bijective morphisms

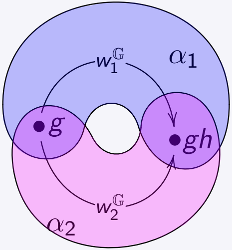
obstructions

- simple (& overcome by pre-processing):
  conflicting ρ_{e_1}, ρ_{e_2} for $e_1, e_2 \in E[s, s']$

- substantial (& pointing to non-trvial acyclicity requirements):
  non-confluent ρ_{w_1}, ρ_{w_2} for $w_1, w_2 \in \mathrm{I}^*[s, t]$
  violating "no-nonsense" condition



$$w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}}$$

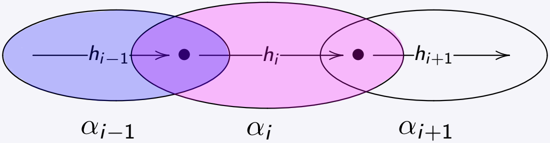


suitable groupoids: coset acyclicity

theorem 1

for every $\mathsf{N} \in \mathbb{N}$ and incidence pattern $\mathrm{I} = (S, E)$ there are finite I-groupoids $\mathbb{G}$ without *coset cycles* of length up to N

3 steps in a coset cycle:



idea: in an inductive construction generate $\mathbb{G}$ from (semi)group action on amalgamation chains that unfold short cosets cycles

cf. constructions of acyclic Cayley graphs (Alon, Biggs) here lifted to more intricate adaptation for coset cycles $\rightarrow$ O_10 (JACM 13) for groups

any degree of acyclicity in symmetric realisations

theorem 2

for any overlap specification $\big((\mathcal{A}_s),(\rho_e)\big)$, obtain realisations of the form $\big((\mathcal{A}_s)\otimes\mathbb{G}\big)\;/\approx$ for suitable finite groupoids $\mathbb{G}$, that

- have any desired degree of (local/size-bdd) acyclicity
- admit transitive automorphisms in the second factor
- respect all symmetries of the specification $\big((\mathcal{A}_s),(\rho_e)\big)$

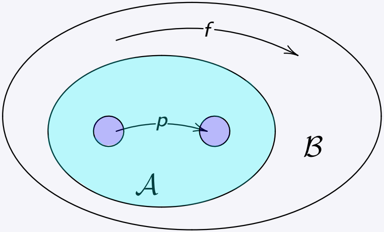
symmetric realisations

EPPA: from local to global symmetries

extension property for partial automorphisms (EPPA):
how to extend local symmetries to global symmetries

theorem (Herwig 98, extending Hrushovski 92 for graphs)

every finite relational structure $\mathcal{A}$ admits a finite extension $\mathcal{B} \supseteq \mathcal{A}$ s.t. every partial isomorphism in $\mathcal{A}$ lifts to a full automorphism of $\mathcal{B}$



theorem (Herwig–Lascar 00)

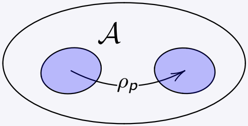
same, as a *finite model property* over any class $\mathcal{C}$
defined by finitely many forbidden homomorphisms

new proof of Herwig–Lascar EPPA

through groupoidal realisations of an overlap specification
for $\mathcal{A} = (A, R)$ and $P \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{A})$

(i) **the incidence pattern $\mathbf{I}(\mathcal{A}, P)$:**
multigraph on singleton vertex
with a loop $e_p \in E$ for each $p \in P$

(ii) **the overlap specification $\big((\mathcal{A}), (\rho_p)\big)$:**
after pre-processing, $\big((\mathcal{A}_s), (\rho_e)\big)$
turns non-trivially groupoidal (!)



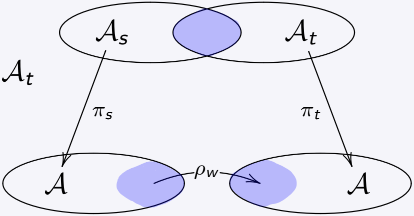
(iii) **symmetric realisations of $\big((\mathcal{A}), (\rho_p)\big)$**
are EPPA extensions

(iv) ***N*-acyclic EPPA extensions are *N*-free:**
admit N-local homomorphisms into every (finite or infinite)
EPPA extension due to their N-local tree-decomposability

the EPPA extensions we get for $(\mathcal{A}, P)$:

$\mathcal{B} \supseteq \mathcal{A}$
with superimposed hypergraph structure (B, S), $S \subseteq \mathcal{P}(B)$,
and projections $(\pi_s)_{s \in S}$ such that:

- $\mathcal{B} = \bigcup_{s \in S} \mathcal{A}_s$ where $\mathcal{A}_s := \mathcal{B} \restriction s$
- $\big(\pi_s \colon \mathcal{A}_s \simeq \mathcal{A}\big)_{s \in S}$ an "atlas" for $\mathcal{B}$
- overlaps between "charts" $\mathcal{A}_s$ and $\mathcal{A}_t$ induced by compositions $w \in \mathrm{P}^*$



- up to any desired threshold N, each $\bigcup_{i=1}^{N} \mathcal{A}_{s_i} \subseteq \mathcal{B}$ is a free amalgam (and acyclic)

further applications

applying theorem 2 (finite realisations of any degree of acyclicity) to overlap specification of a given hypergraph (its exploded view):

corollary

every finite hypergraph admits, for $N \in \mathbb{N}$, finite coverings that

- are N-acyclic in the sense that every induced sub-hypergraph on up to N vertices is acyclic (tree decomposable)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

with further applications in guarded logics:

- **expressive completeness results**
  classical & in finite model theory
- **finite model properties**
  also linked to Herwig–Lascar EPPA

some related references

Bárány–Gottlob–O_(LMCS 2014, arXiv:1309.5822): Querying the guarded fragment

Bárány–ten Cate–O_(VLDB 2012, arXiv:1203.0077): Queries with guarded negation

Grädel–O_(2014): The freedoms of (guarded) bisimulation

Hodkinson–O_(BSL 2003): Finite conformal hypergraph covers and Gaifman cliques in finite structures

Herwig–Lascar(Transactions of the AMS 2000): Extending partial isomorphisms and the profinite topology on free groups

O_(Journal of the ACM 2012): Highly acyclic groups, hypergraph covers and the guarded fragment

O_(arXiv:1404.4599, 2015): Finite groupoids, finite coverings and symmetries in finite structures