Groupoids, Hypergraphs, and Symmetries in Finite Models

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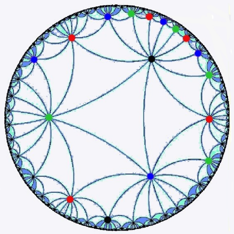


local —— global

local	global
isomorphism types of local substructures & their overlaps	realisations in (finite) relational structures
transition patterns for bisimulation types	acyclicity conditions for realisations in models
partial isomorphisms as local symmetries	global automorphisms as global symmetries

inverse semigroups **groupoids** **groups**

graphs —— hypergraphs

graphs	hypergraphs
transition systems & graph-like structures	relational structures with hypergraph of guarded sets
modal logics	guarded logics

full acyclicity in infinite tree-like unfoldings!
counterparts for finite model constructions?

local acyclicity	acyclicity of small sub-configurations
in direct products with Cayley graphs of N-acyclic groups	**in new reduced products with Cayley graphs of N-acyclic groupoids**

combinatorial questions for hypergraphs:

- **is every overlap pattern of hyperedges realisable in some finite hypergraph?**
- **does every finite hypergraph admit finite covers of any degree of acyclicity?**
- **how faithful can finite realisations/covers be w.r.t. symmetries of the specification?**

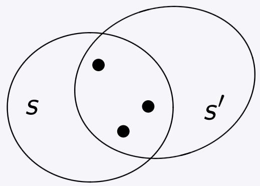
hypergraphs $\mathcal{A} = (A, S)$, $S \subseteq \mathcal{P}(A)$ occur as

- **abstractions of relational structures/data bases,**
- **specifications of clusters of variables in CSP,**
- **combinatorial patterns of structural decompositions, ...**

local overlap specifications

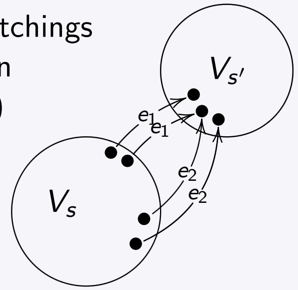
concrete specification as in given hypergraph

intersection pattern of given $\mathcal{A} = (\mathsf{A}, \mathsf{S})$
$\mathsf{I}(\mathcal{A}) = (\mathsf{S}, \mathsf{E})$, $\mathsf{E} = \{(s, s') : s \cap s' \neq \emptyset\}$



abstract specification via local matchings

disjoint patches V_s for $s \in \mathsf{S}$ with partial matchings
$\mathsf{R}_e \subseteq \mathsf{V}_s \times \mathsf{V}_{s'}$ for $e \in \mathsf{E}_{ss'}$ according to given
incidence pattern $\mathsf{I} = (\mathsf{S}, (\mathsf{E}_{ss'}))$ (multigraph)



challenge: generic construction
of finite global realisations
under acyclicity constraints

abstract overlap patterns & realisations: definitions

incidence pattern: template for sites and links

$\mathbf{I} = (\mathbf{S}, \mathbf{E})$ multigraph, $\mathsf{E} = (\mathsf{E}_{ss'} : s, s' \in \mathsf{S})$
with involution $e \in \mathsf{E}_{ss'} \longmapsto e^{-1} \in \mathsf{E}_{s's}$

I-graph: specification of local overlap pattern

$\mathbf{H} = (\mathbf{V}, (\mathbf{V}_\mathbf{s})_{\mathbf{s} \in \mathbf{S}}, (\mathbf{R}_\mathbf{e})_{\mathbf{e} \in \mathbf{E}})$

V partitioned into patches V_s, linked through
partial matchings $\mathsf{R}_e \subseteq \mathsf{V}_s \times \mathsf{V}_{s'}$ for $e \in \mathsf{E}_{ss'}$

realisation: global realisation of local specification

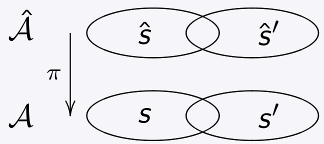
hypergraph $\hat{\mathcal{A}} = (\hat{\mathbf{A}}, \hat{\mathbf{S}})$ realising matchings as overlaps
via projection $\pi : \hat{\mathsf{S}} \longrightarrow \mathsf{S}$ and local bijections $\pi_{\hat{s}} : \hat{s} \longrightarrow \mathsf{V}_{\pi(\hat{s})}$

example: hypergraph coverings $\pi : \hat{\mathcal{A}} \rightarrow \mathcal{A}$

π a covering of $\mathcal{A} = (\mathbf{A}, \mathbf{S})$ by $\hat{\mathcal{A}} = (\hat{\mathbf{A}}, \hat{\mathbf{S}})$

(forth): π is a hypergraph homomorphism
(local bijections between hyperedges)

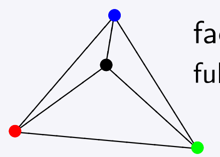
(back): π lifts overlaps $\mathsf{s} \cap \mathsf{s}'$ to all $\hat{\mathsf{s}}$ above s:



i.e., realisations of overlap specification $\mathsf{H}(\mathcal{A})$
w.r.t. $\mathsf{I}(\mathcal{A}) := (\mathsf{S}, \mathsf{E})$, $\mathsf{E} = \{(\mathsf{s}, \mathsf{s}') : \mathsf{s} \cap \mathsf{s}' \neq \emptyset\}$

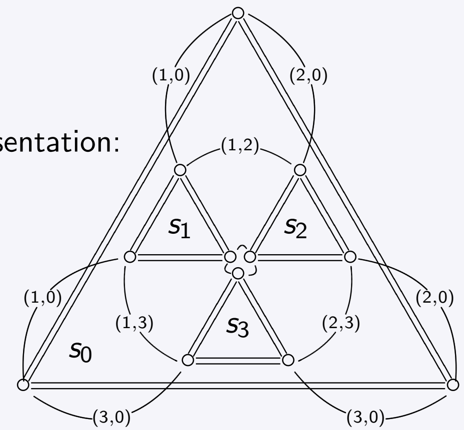
issues: degrees of acyclicity, saturation and symmetry

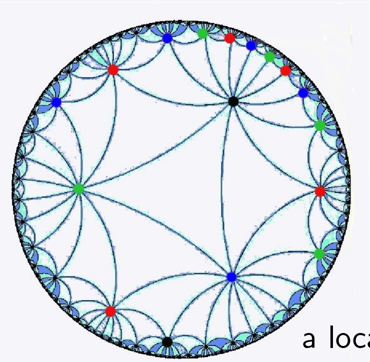
illustration: $[4]^3$



facets of the 3-simplex
full 3-uniform hypergraph on 4 vertices

I-graph representation:





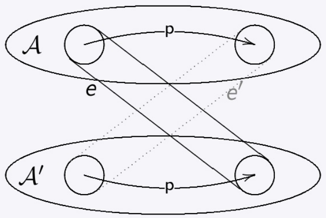
a locally finite covering

example: extension of partial isomorphisms

Hrushovski–Herwig–Lascar EPPA

for finite relational structure $\mathcal{A}$ and partial isomorphism p of $\mathcal{A}$
find *finite* extension $\hat{\mathcal{A}} \supseteq \mathcal{A}$ such that p extends to $\hat{\mathrm{p}} \in \mathrm{Aut}(\hat{\mathcal{A}})$

generic realisation of overlap pattern between copies of $\mathcal{A}$ with identifications according to p



issue: extending local to global symmetry, consistency

here now:

focus on graph/hypergraph coverings

- **degrees of acyclicity in products with groups/groupoids**
- **genericity and symmetry of these constructions**

  & a glimpse of application to Herwig–Lascar

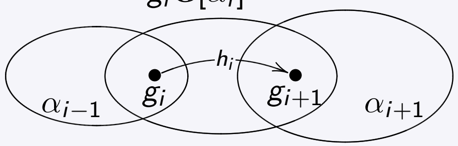
main tool: Cayley graphs of finite groups and groupoids, their degrees of acyclicity & use in products

Cayley groups & graphs without short cycles

Cayley group & graph:
group $\mathbb{G} = \mathbb{G}[\mathrm{E}] = (\mathrm{G}, \cdot, 1)$ with generators $e \in \mathrm{E}$
$\rightsquigarrow$ edge relations $\mathrm{R}_e = \{(g, g \cdot e) \colon g \in \mathrm{G}\}$ for $e \in \mathrm{E}$

generator cycles & large girth:
minimal length of generator cycle $e_1 \cdots e_n = 1$
$\rightsquigarrow$ large girth in Cayley graphs from permutation group action on suitably acyclic graphs

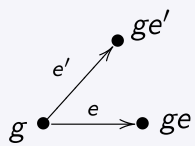
coset cycles & more than large girth:
in $\mathbb{G} = \mathbb{G}[\mathrm{E}]$ consider components w.r.t. $\alpha \subseteq E$
$\rightsquigarrow$ avoid even short *coset cycles* in Cayley graph



α-component of g
$\leftrightsquigarrow$ coset $\mathrm{g}\mathbb{G}[\alpha]$

groups & groupoids

group $\mathbb{G}$[E] generated by E:
each transition $e \in E$ applicable everywhere

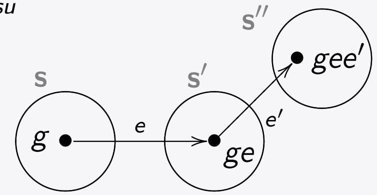


incidence pattern $\mathrm{I} = (\mathrm{S}, \mathrm{E}) = (\mathrm{S}, (\mathrm{E}_{\mathrm{ss}'}))$
specifies *which* transitions are required locally
$\rightsquigarrow$ **and a group may be too homogeneous**

I-groupoid $\mathbb{G}[\mathrm{E}] = (\mathrm{G}, (\mathrm{G}_{\mathrm{st}}), \cdot, (\lambda_{\mathrm{s}}), {}^{-1})$ generated by E

with partial operation $\mathrm{G}_{st} \times \mathrm{G}_{tu} \longrightarrow \mathrm{G}_{su}$
modelled on the path monoid of I

$\mathrm{e} \in \mathrm{E}_{\mathrm{ss}'}$ applicable in s-nodes,
$\mathrm{e}' \in \mathrm{E}_{\mathrm{s}'\mathrm{s}''}$ in s′-nodes



combinatorial paths to good finite group(oid)s

A (Alon–Biggs 1989)

finite Cayley groups without short generator cycles

permutation group action on acyclic graphs

B (O_2010)

finite Cayley groups without short coset cycles

permutation group action on amalgamated sub-groups/graphs

C (new here)

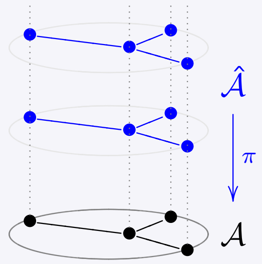
finite Cayley groupoids without short coset cycles

amalgamation technique (B) lifted to finite groupoids

groups for graph coverings

$\pi\colon \hat{\mathcal{A}} \to \mathcal{A}$ a graph covering:

π a homomorphism (forth)
with lifting property (back)



thm (O_2002)

finite graphs $\mathcal{A} = (\mathsf{A}, \mathsf{E})$ admit, for $\mathsf{N} \in \mathbb{N}$, finite coverings $\pi\colon \mathcal{A} \otimes \mathbb{G} \longrightarrow \mathcal{A}$ by direct products with Cayley graphs of group $\mathbb{G}[\mathsf{E}]$ of large girth s.t. $\mathcal{A} \otimes \mathbb{G}$ is N-locally acyclic

unbranched coverings, fully symmetric by construction

from graphs to hypergraphs

hypergraph acyclicity

equivalent criteria for finite $\mathcal{A} = (\mathsf{A}, \mathsf{S})$
in terms of $\mathcal{A}$ and its Gaifman graph $\mathsf{G}(\mathcal{A})$:

- $\mathcal{A}$ has tree decomposition with bag set S
- $\mathcal{A}$ trivialises through retractions
- **$\mathcal{A}$ conformal & chordal**

conformal: every clique in $\mathsf{G}(\mathcal{A})$ induced by individual $s \in \mathsf{S}$

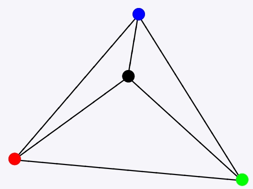
chordal: every cycle in $\mathsf{G}(\mathcal{A})$ of length > 3 has a chord

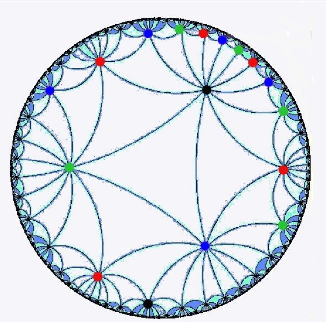
approximation: N-acyclicity (N-conformality & N-chordality)

guarantees acyclicity of sub-hypergraphs of size up to N

hypergraph coverings must be different

- non-trivial covers cannot preserve incidence degrees
  $\longrightarrow$ no unbranched coverings
- non-trivial locality phenomena
  $\longrightarrow$ need to control coset cycles
  $\longrightarrow$ groupoids rather than groups





example $[4]^3$

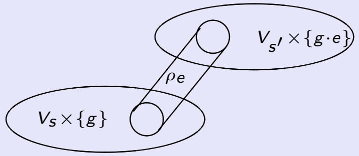
reduced products $\mathcal{A} \otimes \mathbb{G}$, $\mathsf{H} \otimes \mathbb{G}$, ...

simple generic idea:

use disjoint union of copies of
patches associated with $s \in \mathsf{S}$ } a direct product
indexed by elements $g \in \mathsf{G}_{*s} \subseteq \mathbb{G}$

and factor out equivalence $\approx$ induced by
specified local overlaps (R_e, matchings ρ_e)

e.g., in $\mathsf{H} \otimes \mathbb{G}$ for $e \in \mathsf{E}_{ss'}$ and
R_e/ρ_e of $\mathsf{H} = (\mathsf{V}, (\mathsf{V}_s), (\mathsf{R}_e))$:
$(v, g) \approx (v', g \cdot e)$ if $\rho_e(v) = v'$



in $\mathcal{A} \otimes \mathbb{G}$: a in layer g is identified with a in layer g′
precisely if $g' \in g\mathbb{G}[\alpha_a]$ for $\alpha_a = \{(s, s') : a \in s \cap s'\}$
— this is why coset cycles matter for acyclicity

acyclicity in reduced products

covering lemma:

for hypergraph $\mathcal{A}$ and suitable I-groupoid $\mathbb{G} = \mathbb{G}(\mathsf{E})$
based on the overlap pattern $\mathsf{I} = (\mathsf{S}, \mathsf{E}) = \mathsf{I}(\mathcal{A})$ of $\mathcal{A}$:

- $\pi \colon \mathcal{A} \otimes \mathbb{G} \longrightarrow \mathcal{A}$ is a covering;
- $\mathcal{A} \otimes \mathbb{G}$ respects all symmetries of $\mathbb{G}$ and $\mathcal{A}$;
- if $\mathbb{G}$ has no coset cycles of length up to N,
  then $\mathcal{A} \otimes \mathbb{G}$ is an N-acyclic hypergraph

further applications:

- generic, symmetry-preserving, N-acyclic realisations $\mathsf{H} \otimes \mathbb{G}$
  for any abstract overlap specification H
- new combinatorial proof of Herwig–Lascar extension theorem

extending local to global symmetry

for classes $\mathcal{C}$ of relational structures defined in terms
of finitely many forbidden homomorphisms:

theorem (Herwig–Lascar 1999)

if finite $\mathcal{A} \in \mathcal{C}$ has extension $\hat{\mathcal{A}} \supseteq \mathcal{A}$ in $\mathcal{C}$ that extends
the partial isomorphism p of $\mathcal{A}$ to automorphism of $\hat{\mathcal{A}}$,
then there is also a *finite* $\hat{\mathcal{A}}$ with these properties

new route: find $\hat{\mathcal{A}}$ as sufficiently acyclic,
symmetric finite realisation of p-overlaps
between copies of $\mathcal{A}$