Controlling Cycles in Finite Hypergraphs

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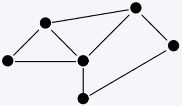
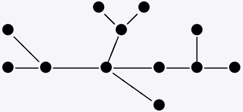
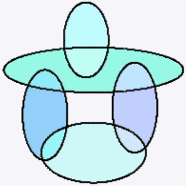
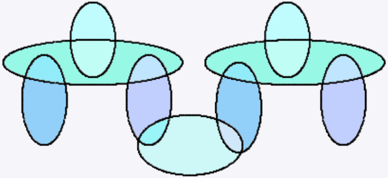
acyclicity (of graphs & hypergraphs): examples

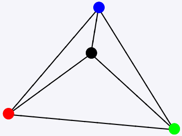








uniform width 3 hypergraph on 4 vertices
$=$ the facets of the 3-simplex



cyclic or acyclic ?

acyclicity & tree-likeness

benefits of ayclicity

- structural/combinatorial: easy enumeration & analysis, ...
- algorithmic: decomposition, divide & conquer, automata, ...
- logical/model-theoretic: all of the above

guiding ideas:

acyclicity means: local structure determines global structure

in its absence: may still have a canonical (free) unfolding of local patterns into *infinite* acyclic structure

question: *finite* approximations ?

hypergraphs & graphs: basic terminology

hypergraphs

$\mathbf{H = (V, S)}$ vertex set V
set of hyperedges $S \subseteq \mathcal{P}(V)$
width $\mathrm{w(H)} = \max\{|s| \colon s \in S\}$

graphs

width 2 hypergraphs

graph associated with hypergraph $\mathbf{H = (V, S)}$:

$\mathbf{G(H) = (V, E)}$ where $(v, v') \in E$ if $v \neq v'$ and
$v, v' \in s$ for some $s \in S$

tree unfoldings

of graphs

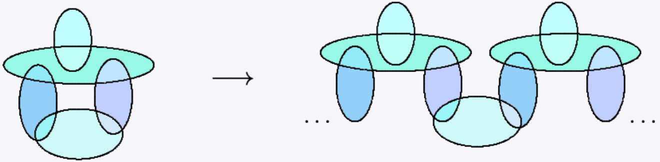
tree unfolding of $G = (V, E)$ from root node $v \in V$:

graph G^* $\begin{cases} \text{with paths/walks } e_1 \cdots e_k \text{ from } v \text{ in } G \text{ as vertices} \\ \text{with edges from } e_1 \cdots e_k \text{ to } e_1 \cdots e_k \cdot e_{k+1} \end{cases}$

of hypergraphs

unfolding of $H = (V, S)$ obtained via tree unfolding
of intersection graph $I(H) = (S, \Delta)$, $\Delta = \{(s, s') \colon s \cap s' \neq \emptyset\}$:

$$H = (V, S) \rightsquigarrow I(H) = (S, \Delta) \rightsquigarrow I^* \rightsquigarrow H^*$$

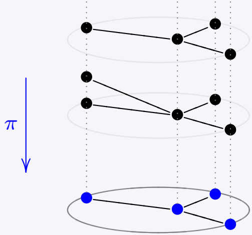


coverings (cf. topology/geometry)

roughly: homomorphic projection
(locally injective ?)

+

lifting (back-property)
(uniqueness ?)



definition

a covering $\pi\colon \hat{\mathbf{H}} = (\hat{\mathbf{V}}, \hat{\mathbf{S}}) \xrightarrow{\sim} \mathbf{H} = (\mathbf{V}, \mathbf{S})$

– π is a surjective homomorphism,
– bijective in restriction to every (hyper)edge of $\hat{s} \in \hat{S}$,
– with *back*-extensions: for $s = \pi(\hat{s})$ and $s' \in S$ exists $\hat{s}' \in \hat{S}$
s.t. $\pi(\hat{s}') = s'$ and $\pi(\hat{s} \cap \hat{s}') = s \cap s'$

– *faithful* (locally simple) in case *back*-extensions are unique

(1) graphs and locally acyclic covers

- acyclic covers of cyclic graphs are necessarily infinite
- no short cycles = no cycles locally
  no cycles of length up to $2N+1 \Leftrightarrow$ *N-local acyclicity*
- (local) acyclicity compatible with $\begin{cases} \text{passage to subgraphs} \\ \text{direct products} \end{cases}$

$\rightsquigarrow$ uniform & canonical constructions of
faithful N-locally acyclic graph coverings via Cayley groups

thm (O_ 01)

for every finite graph $\mathbf{G}$, for every $N \in \mathbb{N}$
there is a faithful covering $\pi\colon \hat{\mathbf{G}} \xrightarrow{\sim} \mathbf{G}$
by a finite N-locally acyclic graph $\hat{\mathbf{G}}$

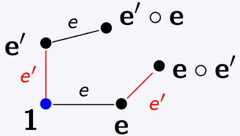
Cayley graphs

- NB: naive truncated tree-unfoldings with cut-off edges re-directed to vicinity of root do not work (why not?)
- instead: products with Cayley groups of large girth

Cayley groups and graphs

group G with involutive generators $e \in E$
gives rise to a highly symmetric graph:

vertices $g \in \mathrm{G}$
e-coloured edges between g and $g \circ e$



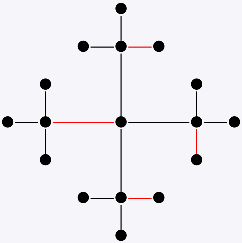
girth of a Cayley group/graph:
minimal length of a cycle (at 1)

Cayley graphs of large girth

elementary construction of Cayley group G of girth $> N$
with set E of involutive generators:

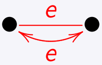
on regularly E-edge-coloured

tree $\mathbf{T}, \boldsymbol{\lambda}$ of depth N,



let $e \in E$ operate through

swaps of nodes in e-edges:

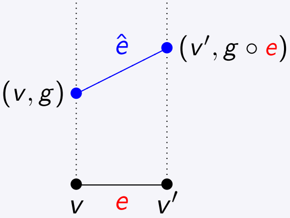


$\mathbf{G} := \langle \mathbf{E} \rangle^{\mathrm{Sym}(\mathbf{T})} \subseteq \mathrm{Sym}(\mathbf{T})$
subgroup generated by the permutations $e \in E$

no short cycles: $e_1 \circ e_2 \circ \cdots \circ e_k \neq 1$ at least for $k \leqslant N$

faithful locally acyclic covers (prf of the thm)

consider product of graph (V, E)
with Cayley graph G generated by E



$$(V,E) \otimes G := (V \times G, E \otimes G)$$

$$E \otimes G := \{((v,g),(v',g')) \colon (v,v') = e \in E, g' = g \circ e\}$$

- $\pi : \begin{array}{ccc} (V,E) \otimes \mathrm{G} & \xrightarrow{\sim} & (V,E) \\ (v,g) & \longmapsto & v \end{array}$ is a faithful cover

- $\mathrm{girth}(\mathrm{G}) > N \;\Rightarrow\; (V,E) \otimes \mathrm{G}$ has no cycles of length $n \leqslant N$

(2) hypergraphs

recall terminology

hypergraph $\mathbf{H} = (V, S)$ set of nodes V
set of hyperedges $S \subseteq \mathcal{P}(V)$

associated graph: $\mathbf{G}(\mathbf{H}) = (V, E)$ where $(v, v') \in E$ if $v \neq v'$
and $v, v' \in s$ for some $s \in S$

hypergraph cover $\pi\colon \hat{\mathbf{H}} = (\hat{\mathbf{V}}, \hat{\mathbf{S}}) \xrightarrow{\sim} \mathbf{H} = (\mathbf{V}, \mathbf{S})$

– π is a surjective homomorphism,
– bijective in restriction to every $\hat{s} \in \hat{S}$,
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hypergraph acyclicity

acyclicity of $\mathbf{H} = (\mathbf{V}, \mathbf{S})$ three equivalent characterisations:

- **H** admits reduction $\mathbf{H} \rightsquigarrow \emptyset$

  via decomposition steps: $\begin{cases} \text{delete } v \text{ if } |\{s \in S \colon v \in s\}| \leqslant 1 \\ \text{delete } s \text{ if } s \subsetneq s' \in S \end{cases}$

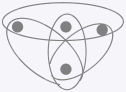
- **H** has a tree decomposition $\delta \colon \mathcal{T} \to S$ with bags from S

- **H** is **conformal** & **chordal**

conformality:

every clique in $G(H)$ contained in hyperedge

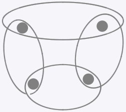
no



chordality:

every cycle of length $\geqslant 4$ has a chord

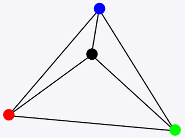
no



examples & limitations

the facets of the 3-simplex

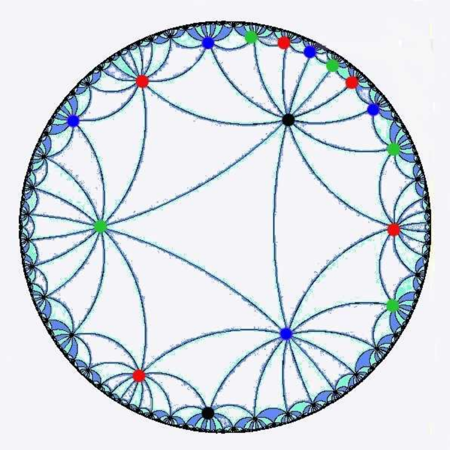
uniform width 3 hypergraph on 4 vertices



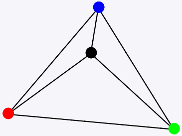
- chordal but not conformal
- finite coverings cannot be 1-locally acyclic
- admits locally finite covers without short chordless cycles
- also admits finite cover, in which every induced sub-hypergraph of up to 5 vertices is acyclic

example ctd

a locally finite cover



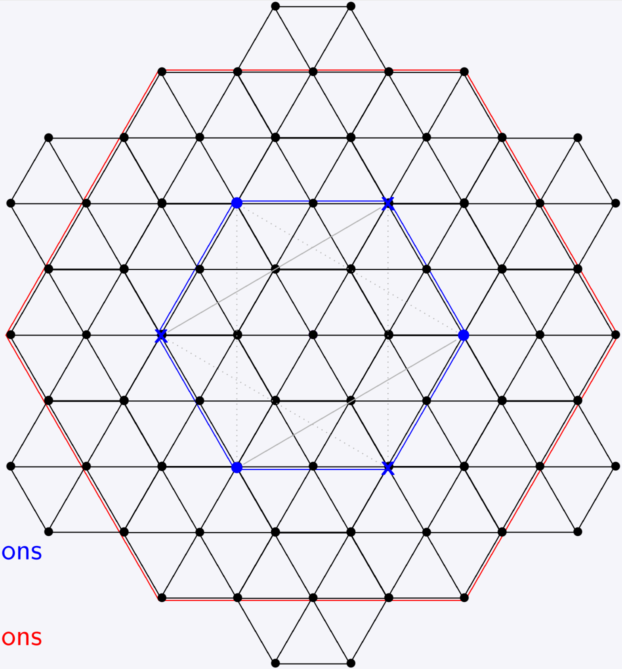
covering



conformal; shortest chordless cycles have length 12

here by isometric tesselation in hyperbolic geometry

finite covers ?



— 5-acyclic,
locally finite

— just 3-acyclic
after identifications

— 5-acyclic
after identifications

finite conformal covers

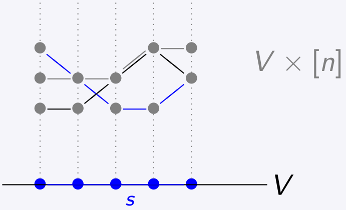
thm (Hodkinson–O_ 03)

for every finite hypergraph $\mathbf{H} = (V, S)$ there is

a covering $\pi\colon \hat{\mathbf{H}} \xrightarrow{\sim} \mathbf{H}$

by a finite conformal hypergraph $\hat{\mathbf{H}}$

method: suitable restriction of 'free' cover in $V \times [n]$
with graphs of functions $\rho\colon s \rightarrow [n]$
as hyperedges above $s \in S$



finite conformal covers: an application

extension properties for partial automorphisms (EPPA)

Hrushovski (graphs), Herwig (hypergraphs), Herwig–Lascar

for finite $\mathbf{H}$ and partial isomorphism $\rho \in \mathrm{Part}(\mathbf{H}, \mathbf{H})$,
there is a finite extension $\mathbf{H}' \supseteq \mathbf{H}$
s.t. $\rho \subseteq \hat{\rho} \in \mathrm{Aut}(\mathbf{H}')$

conservative versions:

- $S' := [S]^{\mathrm{Aut}(\mathbf{H}')}$ yields solution without 'new' hyperedges
- **question:** how about avoiding 'new' cliques ?

lifting Hrushovski–Herwig–Lascar

thm (Hodkinson–O_ 03)

- EPPA can be made conservative w.r.t. cliques
- the class of finite conformal hypergraphs has EPPA
- the class of finite K_n-free graphs has EPPA

method: HHL-extension + conformal cover

lift generic extension $\mathbf{H}' = (V', S') \supseteq \mathbf{H} = (V, S)$
to conformal cover of the induced hypergraph

$$\mathbf{H}'' = (V', S'') \text{ with } S'' = \{g(V) \colon g \in \mathrm{Aut}(\mathbf{H}')\}$$

crux: homogeneity of our cover construction!

N-acyclic hypergraph covers

definition

$\mathbf{H}$ is N-acyclic if every induced sub-hypergraph $\mathbf{H}' \subseteq \mathbf{H}$
of up to N vertices is acyclic (tree-decomposable)

example of periodic 2-unfolding of tetrahedron above: 5-acyclic

thm (O_ 10)

for every finite hypergraph $\mathbf{H} = (V, S)$ and every $N \in \mathbb{N}$,

there is a covering $\boldsymbol{\pi} : \hat{\mathbf{H}} \xrightarrow{\sim} \mathbf{H}$

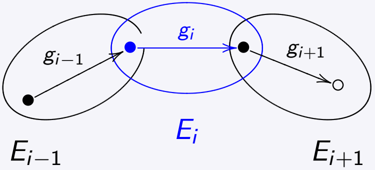
by a finite N-acyclic hypergraph $\hat{\mathbf{H}}$

method: some ingredients in the following

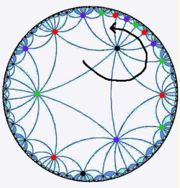
Cayley groups again, but stronger

- recall Cayley groups G with generators $e \in E$ of large girth:
  $e_1 \circ e_2 \circ \cdots \circ e_k \neq 1$ for small k
- now use much stronger notion of acyclicity:
  $g_1 \circ g_2 \circ \cdots \circ g_k \neq 1$ for small k

  where $g_k \in G[E_k]$ for $E_k \subsetneq E$ are such that corresponding cosets *locally* overlap *without shortcuts*



motivation:



N-acyclic Cayley groups: no such cycles for $k \leqslant N$

N-acyclic Cayley groups and hypergraphs

thm

for very finite set E and $N \in \mathbb{N}$ there is
an N-acylic Cayley group with generator set E

N-acyclicity: $g_1 \circ g_2 \circ \cdots \circ g_k \neq 1$ for $k \leqslant N$, where $g_k \in G[E_k]$ ...

alternative characterisation

Cayley group G with generator set E is N-acyclic if
this (dual) coset hypergraph is N-acyclic:

$$\mathbf{H}(G) := (\{gG[E']\colon E' \subseteq E\}, \{[g]\colon g \in G\})$$

$$[g] = \{gG[E']\colon E' \subseteq E\}$$

N-acyclic Cayley groups

inductive construction:

combinatorial group action & amalgamation of Cayley graphs
on E-coloured graphs $G[E']$ for smaller $E' \subseteq E$

to avoid short cycles in $G[E']$ for increasingly large $E' \subseteq E$

use towards hypergraph covers:

reduced product of hypergraph $\mathbf{H} = (V, S)$ with
Cayley group G with generator set $S_0 \subseteq S$:

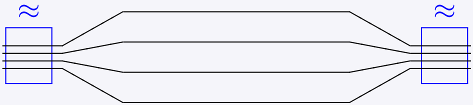
$$\mathbf{H} \otimes G : \begin{cases} \text{quotient } (\mathbf{H} \times G)/\!\approx \\ (v, g) \approx (v, g') \text{ if } \; g \circ (g')^{-1} \in G[E'] \\ \qquad\qquad\qquad\quad \text{for } E' = \{ s \in S_0 \colon v \in s \} \end{cases}$$

fact: reduced products with N-acyclic G preserve N-acyclicity

from locally finite to finite

local-to-global construction

use reduced products $\hat{\mathbf{H}} \otimes G$
to glue many layers of truncated locally finite cover $\hat{\mathbf{H}}$
along its boundary



surplus layers of good interior region
can be used to repair defects near boundary

related results & applications: relational structures

N-acyclic covers

for $N \in \mathbb{N}$, every finite relational structure $\mathcal{A}$ admits a covering by an N-acyclic finite structure $\hat{\mathcal{A}}$

$\rightarrow$ $\hat{\mathcal{A}}$ avoids small cyclic substructures

analysis of N-acyclic hypergraphs/structures

for $N \gg w, \ell, n$:

the class of all N-acyclic hypergraphs of width w supports a notion of bounded convex hulls:

closures of sets of $\leqslant n$ vertices
under chordless paths of lengths $\leqslant \ell$
are of uniformly bounded size (hence acyclic)

related results & applications: finite model theory

suitably enriched N-acyclic covers
and above structural analysis yield:

FO-similarity with tree unfoldings

for finite $\mathcal{A}$ find
- infinite acyclic tree unfolding $\mathcal{A}^*$
- finite N-acyclic cover $\hat{\mathcal{A}} \xrightarrow{\sim} \mathcal{A}$

such that $\hat{\mathcal{A}} \equiv_q \mathcal{A}^*$

$\hat{\mathcal{A}}$: N-acyclic, finite

$\mathcal{A}^*$: acyclic, infinite

FO-indistinguishable
up to quantifier depth q

application: finite model theory of guarded logics

related results: graphs/transition systems

N-acyclic group covers

for $N \in \mathbb{N}$, every finite transition system $\mathcal{A}$ admits
a covering by a substructure of a finite N-acyclic Cayley group

without short cycles even w.r.t. certain transitions between clusters of transition labels

potential applications:
graphs used in knowledge representation,
analysis of shared knowledge

related results: relational structures/databases

weakly N-acyclic covers (Barany–Gottlob–O_ 10)

by way of further relaxation of acyclicity:

. . . finite covers that are conformal and *N-chordal in projection*

in canonical & homogeneous construction
of feasible complexity (!)

→ avoid *homomorphic images* of small cyclic structures

application: *finite controllability* of interactions
between certain DB constraints and queries

summary and open questions

controlling cycles in graphs & hypergraphs

- **graphs and graph coverings:**

  canonical, efficient constructions
  comparatively well understood

- **hypergraph coverings:**

  some variability even w.r.t. definitions
  combinatorially interesting

  - limits for canonical and efficient constructions: *open*
  - links with established discrete mathematics: *to be explored*

    branched coverings of simplicial complexes, expanders, . . .
  - further applications (!?)