Back & Forth Between Malleable Finite Models

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overview

(0) back & forth in model theory

- Ehrenfeucht–Fraïssé and Karp thms

(I) modal back & forth: bisimulation

- modal Ehrenfeucht–Fraïssé and Karp thms
- bisimulation invariant FO & compactness phenomena
- controlling edge cycles in finite coverings

(II) from graphs to hypergraphs: guarded bisimulation

- guarded back & forth and guarded logic
- controlling hypergraph cycles in finite coverings
- amalgamation of controlled acyclicity & applictations

classical back & forth

model-theoretic comparison game, **Ehrenfeucht–Fraïssé game**

two players explore differences/similarities between
two locally isomorphic configurations $\mathcal{A}, \boldsymbol{a}$ and $\mathcal{B}, \boldsymbol{b}$

single round: dynamic challenge/response probing of similarity,
by extension of configuration $\mathcal{A}, \boldsymbol{a}; \mathcal{B}, \boldsymbol{b} \rightsquigarrow \mathcal{A}, \boldsymbol{a}a; \mathcal{B}, \boldsymbol{b}b$

existence of winning strategy for second player
extensionally represented by **back & forth system**
establishes notions of structural equivalence:

ℓ rounds	$\simeq^{\ell}$	ℓ-partial isomorphy
any finite no. of rounds	$\simeq^{\omega}$	finite isomorphy
unbounded play	$\simeq^{\infty}$	partial isomorphy ($\simeq_{\mathrm{part}}$)

variation: k pebbles (to be moved), controlling
configuration size as a bdd resource

Ehrenfeucht–Fraïssé and Karp theorems

for finite relational vocabularies, back & forth equivalences capture logical indistinguishability w.r.t. first-order logic FO or its infinitary variant $\mathrm{FO}_\infty = \mathcal{L}_{\infty\omega}$

one round corresponds to probing one level of $\exists$ assertions !

$\simeq^\ell$	(ℓ rounds)	$\equiv^\ell_{\mathrm{FO}}$
$\simeq^\omega$	(all finite plays)	$\equiv_{\mathrm{FO}}$
$\simeq^\infty$	(unbdd game)	$\equiv_{\mathrm{FO}_\infty}$

useful facts:

$\simeq^\ell$ has finite index with FO_ℓ-definable classes

$\simeq^\infty \rightsquigarrow \simeq$ on countble models

$\simeq^\infty \rightsquigarrow \simeq^\omega$ on ω-saturated models

(I) modal back & forth

bisimulation game and equivalence

challenge/response probing of transition systems or graphs, Kripke structures, ...

configurations: single states or vertices, worlds, ...

single round: move of single pebble along an edge (transition) in one structure needs to be matched in the other

winning strategies/back & forth systems establish equivalences:

ℓ rounds	$\sim^\ell$	ℓ-bisimulation equivalence
any finite no. of rounds	$\sim^\omega$	
unbounded play	$\sim^\infty$	bisimulation equivalence ($\sim$)

$\sim$ is for modal/process/temporal logics
what $\simeq_{\rm part}$ is for classical logic, ...and can also be seen as the mother of all back & forth equivalences

modal Ehrenfeucht–Fraïssé and Karp theorems

one round corresponds to one level of $\Diamond$ assertions !

relates to modal logic ML $\subseteq$ FO with quantification reslativised to direct edge-neighbours ($\Diamond_R$, $\Box_R$)

$$\exists y\big(Rxy \wedge \varphi(y)\big) \quad / \quad \forall y\big(Rxy \rightarrow \varphi(y)\big)$$

over transitions systems (Kripke structures) with finite vocabulary:

$\sim^\ell$	(ℓ rounds)	$\equiv^\ell_{\mathrm{ML}}$
$\sim^\omega$	(all finite plays)	$\equiv_{\mathrm{ML}}$
$\sim = \sim^\infty$	(unbdd game)	$\equiv_{\mathrm{ML}_\infty}$

useful facts:

$\sim^\ell$ has finite index with ML_ℓ-definable classes

$\sim \;\rightsquigarrow\; \sim^\omega$ on e.g. ω-saturated models

what is bisimuation invariant FO, FO/$\sim$?

a question in the spirit of classical "preservation" theorems
NB: $\sim$-invariance is *not* an elementary notion ($\sim \, = \, \sim^\infty$)

theorem (van Benthem 83): FO/$\sim \, \equiv$ ML

interestingly (& easily) equivalent to this
compactness phenomenon for $\varphi(x) \in$ FO:

$$\boxed{\sim\text{-invariance} \;\Rightarrow\; \sim^\ell\text{-invariance for some finite } \ell} \qquad (*)$$

bridging the gap between finite & infinitary equivalence

in the following: **two (almost orthogonal) proofs**
with rather different perspectives & potential

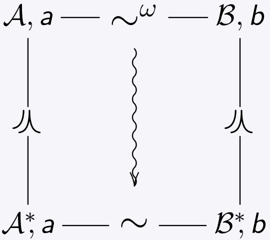
(A) classical proof

two applications of FO-compactness in chain

$\sim$-invariance $\overset{(ii)}{\Longrightarrow}$ $\sim^\omega$-invariance $\overset{(i)}{\Longrightarrow}$ $\sim^\ell$-invariance for suff. large ℓ

(i) standard compactness argument for contrapositive

(ii) saturation argument for upgrading $\sim^\omega \rightsquigarrow \sim$:



passage to ω-saturated elementary extensions

NB: smooth & elegant; lose track of $\ell = \ell(\varphi)$; **no chance for fmt**

(B) compactness without compactness

try orthogonal upgrading according to

$$\begin{array}{ccccc} \mathcal{A}, a & \text{—} & \sim^{\ell} & \text{—} & \mathcal{B}, b \\ | & & \Big\downarrow\!\!\wr & & | \\ \sim & & & & \sim \\ | & & & & | \\ \mathcal{A}^{*}, a & \text{—} & \equiv^{\mathsf{FO}}_{q} & \text{—} & \mathcal{B}^{*}, b \end{array}$$

for suitable $\ell = \ell(q)$

possible obstructions to $\sim^{\ell} \rightsquigarrow \simeq^{q} / \equiv^{\mathsf{FO}}_{q}$ **:**

differences w.r.t. (i) **small multiplicties**

(ii) **short cycles**

need to avoid both (no level of $\sim$ controls either feature)

use structural upgrades in products/coverings

(i) to avoid all multiplicties less than q,
pass to direct product with q-clique K_q

$$\begin{array}{ccccc} \mathcal{A} & \text{———} & \sim^\ell & \text{———} & \mathcal{B} \\ | & & & & | \\ \wr & & & & \wr \\ | & & & & | \\ \mathcal{A} \otimes K_q & \text{——} & \sim^\ell & \text{——} & \mathcal{B} \otimes K_q \end{array}$$

(ii) to avoid all short cycles, can pass to
product with Cayley graph of large girth (*)

both structural upgrades are available in fmt (*), and support van Benthem and Rosen thms: $\text{FO}/{\sim} \equiv \text{ML}$ classically and fmt

slightly different argument yields optimal value $\ell(q) = 2^q - 1$

(*) finite Cayley groups of large girth available from simple combinatorial group action on finite coloured trees (Biggs 89)

variations & extensions

argument (B) allows for many variations

dealing with

- enriched modal logics & notions of bisimulation eq.
- other (non-elementary) classes of structures of interest

$\rightarrow$ Dawar–O_09 for plenty of examples

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so far – so good (but all about graph-like structures)
in part II deal with richer relational formats:

graphs	$\rightsquigarrow$	hypergraphs
edge traversal	$\rightsquigarrow$	non-trivial overlaps/amalgamation
modal	$\rightsquigarrow$	*guarded*

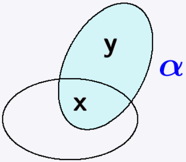
(II) guarded back & forth

guarded fragment GF $\subseteq$ FO (Andréka–van Benthem–Németi 97)

FO-quantification relativised to *guarded* subsets/tuples contained in some $[\boldsymbol{a}] = \{a \colon a \text{ in } \boldsymbol{a}\}$ for $\boldsymbol{a} \in R^{\mathcal{A}}$

$$\exists \boldsymbol{y}\big(\alpha(\boldsymbol{xy}) \wedge \varphi(\boldsymbol{xy})\big) \quad / \quad \forall \boldsymbol{y}\big(\alpha(\boldsymbol{xy}) \rightarrow \varphi(\boldsymbol{xy})\big)$$

with *guard atom* α covering all the free variables of φ

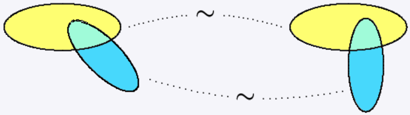


associated back & forth game limits configurations to guarded subsets/tuples, challenge/response must preserve local isomorphism type and respect overlaps

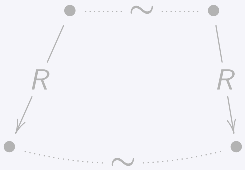
graph game $\rightsquigarrow$ hypergraph game

transitions $\rightsquigarrow$ overlap/amalgamation

challenges in hypergraph games/coverings



challenge/response



compare: modal

crucial difference: moves no longer forgetful!
some “history” persits in elements carried through chains of overlaps/amalgamation steps

look to control (short) hypergraph cycles:

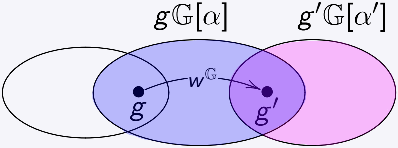
establish degrees of qualified hypergraph acyclicity in finite coverings that guarantee acyclicity (tree-decomposability) in small induced sub-configurations

a notion of coset acyclicity

want: local acyclicity in natural (reduced) products with $\mathbb{G}$
need: more than large girth (no short generator cycles) in $\mathbb{G}$

no short generator cycles	$\rightsquigarrow$	no short coset cycles
large girth		(coset) N-acyclicity

links in a coset cycle:

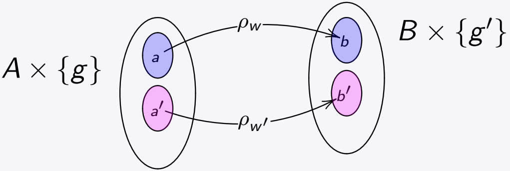


useful facts:

- (coset!) 2-acyclicity guarantees relational consistency in amalgamation chains based on $\mathbb{G}$
- higher levels of acyclicity gurantee "local tree-likeness" of amalgams, i.e., local freeness and universality w.r.t. homs

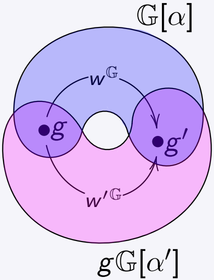
why coset 2-acyclicity matters: a sketch

in reduced product



$w^{\mathbb{G}} = g^{-1}g = w'^{\mathbb{G}}$

in $\mathbb{G}$



NB: the disjoint union of two partial isomorphisms
need not be a partial isomorphism

if $\mathbb{G}[\alpha]$ carries a and $\mathbb{G}[\alpha']$ carries a', $w^{\mathbb{G}} = w'^{\mathbb{G}}$ may constitute a bad cycle in $\mathbb{G}$ unless also $w = w' \in \mathbb{G}[\alpha \cap \alpha']$ (carrying both)

— i.e., unless the 2-cycle degerates

core results

theorem (JACM12/arXiv15)

for every finite generator set and N can find finite coset N-acyclic groups/groupoids $\mathbb{G}$

idea: in an inductive construction generate $\mathbb{G}$ from (semi)group action on amalgamation chains that unfold short coset cycles

cf. constructions of acyclic Cayley graphs (Alon, Biggs) here lifted to more intricate adaptation for coset cycles (and for groupoids!)

natural (reduced) products with these yield

(∗) finite N-acyclic coverings of hypergraphs/relational structures

(∗∗) finite N-acyclic realisations of any amalgamation pattern over finite families of finite structures & partial isos

some applications/uses

(reduced) products with suitable acyclic groups/groupoids yield:

(∗) finite N-acyclic coverings of hypergraphs/relational structures

$\rightarrow$ local tree-decomposability is key to upgrading $\sim_{\mathrm{g}}^{\ell} \rightsquigarrow \simeq^{q}$ to show $\mathrm{FO}/{\sim_{\mathrm{g}}} \equiv \mathrm{GF}$ (fmt and classically)

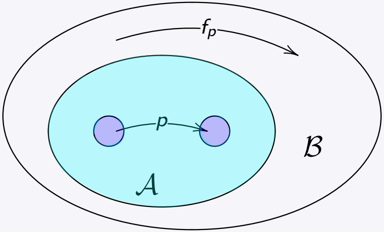
(∗∗) finite N-acyclic realisations of any amalgamation pattern over finite families of finite structures & partial isos

$\rightarrow$ generic finite N-acyclic hypergraph coverings

$\rightarrow$ local-to-global lifting arguments for (partial) symmetries in finite structures (aka EPPA results)

plus finite model properties and chracterisation theorems or finite controllability results for extensions like guarded negation fragment

EPPA results: from local to global symmetries



given: finite relational $\mathcal{A} \in \mathcal{C}$ and $P \subseteq \mathrm{PartIso}(\mathcal{A})$

want: finite extenison $\mathcal{B} \in \mathcal{C}$ extending $p \in P$ to $f_p \in \mathrm{Aut}(\mathcal{B})$
(provided there is at least an infinite such $\mathcal{B}$ in $\mathcal{C}$)

Hrushovski 92	for finite graphs
Herwig 95	for finite relational structures
Herwig–Lascar 00	for finite relational structures that omit finitely many finite homomorphisms

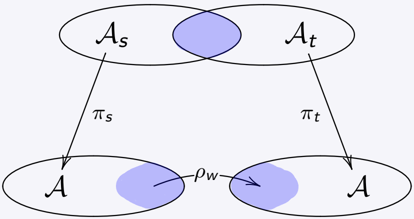
EPPA as an application of (∗∗)

get finite EPPA extension structure $\mathcal{B} \supseteq \mathcal{A}$ for $(\mathcal{A}, P)$,
with hypergraph structure (B, S) and projections $(\pi_s)_{s \in S}$ s.t.:

- $\mathcal{B} = \bigcup_{s \in S} \mathcal{A}_s$ where $\mathcal{A}_s := \mathcal{B} \restriction s$

- $\big(\pi_s \colon \mathcal{A}_s \simeq \mathcal{A}\big)_{s \in S}$ an atlas for $\mathcal{B}$

- all overlaps between charts induced by compositions $w \in \mathrm{P}^*$

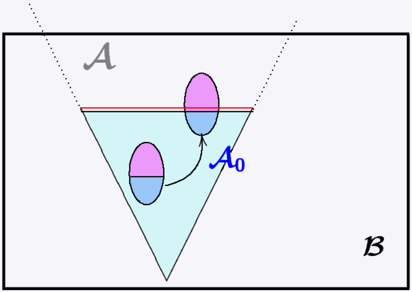


- up to any desired size bound, every small substructure of $\mathcal{B}$ is acyclic and covered by $\mathcal{A}$-charts that form a free amalgam

- hence "N-locally free" and universal w.r.t. bdd size homs

finite model property for GF
based on Herwig's EPPA (Grädel 99)

after relational Skolemisation use EPPA to obtain finite model as finite closure of suitable finite substructure of infinite model w.r.t. guarded ∀∃-requirements



$\mathcal{B} \supseteq \mathcal{A}_0$ EPPA extension

. . . and more applications/extensions

Grädel's argument for fmp for GF extends,
via Herwig–Lascar EPPA and acyclic coverings, to

- fmp relative to classes with forbidden homomorphisms
- finite controllability of UCQ w.r.t. guarded constraints
- similar results for the richer guarded negation fragment
  GNF $\supseteq$ GF, $\exists^*\mathrm{pos}$FO   (Bárány–ten Cate–Segoufin 11)

N-acyclic Cayley groups can be used for characterisations of

- GNF $\subseteq$ FO, and
- the common knowledge extension of modal logic:

  ML[CK] $\equiv$ FO$^*/\!\sim$ (classically and fmt)

  work in progress with Felix Canavoi

some related references

Bárány–Gottlob–O (LMCS 2014, arXiv:1309.5822): Querying the guarded fragment

Bárány–ten Cate–O (VLDB 2012, arXiv:1203.0077): Queries with guarded negation

Grädel–O (2014): The freedoms of (guarded) bisimulation

Hodkinson–O (BSL 2003): Finite conformal hypergraph covers and Gaifman cliques in finite structures

Herwig–Lascar(Transactions of the AMS 2000): Extending partial isomorphisms and the profinite topology on free groups

O (Journal of the ACM 2012): Highly acyclic groups, hypergraph covers and the guarded fragment

O (arXiv:1404.4599): Finite groupoids, finite coverings and symmetries in finite structures

($\rightarrow$ `http://www.mathematik.tu-darmstadt.de/~otto/`)