On weak Markov's principle

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Abstract

We show that the so-called weak Markov's principle (WMP) which states that every pseudo-positive real number is positive is underivable in $\mathcal{T}^\omega$:=E-HA$^\omega$+AC. Since $\mathcal{T}^\omega$ allows one to formalize (at least large parts of) Bishop's constructive mathematics, this makes it unlikely that WMP can be proved within the framework of Bishop-style mathematics (which has been open for about 20 years). The underivability even holds if the ineffective schema of full comprehension (in all types) for negated formulas (in particular for $\exists$-free formulas) is added, which allows one to derive the law of excluded middle for such formulas.

1 Introduction

The so-called weak Markov's principle (WMP) has been first considered by Mandelkern in [14],[15] (in the former paper under the name 'almost separating principle'

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(ASP) and in the latter as 'weak limited principle of existence' (WLPE)). Under the currently common name of weak Markov's principle it has been investigated by Ishihara ([8],[9]). WMP plays a crucial role in the study of the interrelations between various continuity principles within the framework of Bishop-style constructive mathematics ([2],[3],[4]). In order to state WMP we first need the notion of 'pseudo-positivity':

Definition 1 *1) A real number $a \in \mathbb{R}$ is pseudo-positive if*

$$\forall x \in \mathbb{R}(\neg\neg(0 < x) \vee \neg\neg(x < a)).$$

2) $a \in \mathbb{R}$ is positive if $a > 0$.

Remark 2 *1) It is clear that we can without loss of generality restrict $x \in \mathbb{R}$ in the definition of pseudo-positivity to $x \in [0, 1]$.*

2) '$x > y$' for $x, y \in \mathbb{R}$ is to be read as a positive existence statement '$\exists n \in \mathbb{N}(x \geq y + \frac{1}{n+1})$' which has – constructively – to be distinguished from the negative statement '$\neg(x \leq y)$'.

Definition 3 *Weak Markov's principle is the schema*

(WMP): *Every pseudo-positive real number is positive.*

WMP follows easily from the well-known Markov's principle as well as from an appropriate continuity principle and also from the extended Church's thesis ECT_0 (see [10]). So WMP holds both in Russian constructive mathematics as well as in intuitionistic mathematics (in the sense of [4]).

Since about 20 years it has been an open problem whether WMP is derivable in Bishop-style mathematics. The problem is, of course, not completely precise as no particular formal system has been identified with Bishop-style mathematics. However, it is commonly agreed that Heyting arithmetic in all finite types HA^ω (see [17]) plus the axiom of choice AC in all types

$$\mathrm{AC}^{\rho,\tau}: \ \forall x^\rho \exists y^\tau A(x, y) \rightarrow \exists Y^{\rho\rightarrow\tau} \forall x^\rho A(x, Y(x))$$

is a framework which is quite capable of formalizing existing constructive (in the sense of Bishop) mathematics (see also [1],[6]).

In this note we show that WMP is underivable even in E-HA^ω+AC, where E-HA^ω is Heyting arithmetic in all finite types with the full axiom of extensionality (see again

[17] for a precise definition). Our proof even establishes that this underivability remains true if the (highly non-constructive) schema of full comprehension in all types for arbitrary negated formulas

$$\mathrm{CA}_{\neg}: \ \exists \Phi^{\underline{\varrho}\to 0} \forall \underline{x}^{\underline{\varrho}} (\Phi(\underline{x}) =_0 0 \leftrightarrow \neg A(\underline{x}))$$

is added to E-HA$^\omega$+AC which e.g. allows to derive the law of excluded middle for all negated formulas

$$\neg A \vee \neg\neg A.$$

Moreover, since E-HA$^\omega$ proves that every $\exists$-free formula A (i.e. A contains neither '$\exists$' nor '$\vee$') is equivalent to its double negation $\neg\neg A$,[1] we also get comprehension (and consequently the corresponding law of excluded middle) for $\exists$-free formulas

$$\mathrm{CA}_{\exists\text{-free}}: \ \exists \Phi^{\underline{\varrho}\to 0} \forall \underline{x}^{\underline{\varrho}} (\Phi(\underline{x}) =_0 0 \leftrightarrow A(\underline{x})), \ A \ \exists\text{-free},$$

which allows e.g. to derive the classical binary König's lemma WKL (and even the uniform binary König's lemma UWKL which states the existence of a functional which selects an infinite path uniformly in an infinite binary tree, see [13]).

Many equivalent formulations of WMP have been found meanwhile. One of those, due to Ishihara [8], is particularly interesting and reads as follows

Every mapping of a complete metric space into a metric space is strongly extensional

where $f : X \to Y$ is strongly extensional if

$$\forall x_1, x_2 \in X (d_Y(f(x_1), f(x_2)) > 0 \to d_X(x_1, x_1) > 0).$$

In this formulation, the underivability of WMP is particularly easy to prove as we indicate at the end of this paper. However, to conclude from there the underivability of the usual formulation of WMP in E-HA$^\omega$+AC we would have to undertake the tedious task of verifying that Ishihara's equivalence proof can be formalized in E-HA$^\omega$+AC. Richman [16] has shown that the proof that WMP implies Ishihara's strong extensionality statement requires a weak form of countable choice and fails in certain sheaf models.

[1] Note that E-HA$^\omega$ only has prime formulas of the form $s =_0 t$ which are decidable and therefore stable.

2 The independence result

Definition 4 (Independence-of-premise for negated formulas)
$\mathrm{IP}_\neg$: $(\neg A \to \exists x^\rho B) \to \exists x^\rho(\neg A \to B)$, where x does not occur free in A.

The following theorem was proved in [12](thm.3.3)

Theorem 5 ([12]) Let δ, ρ, γ be arbitrary types and G be a sentence of the form

$$G \equiv \forall x^\delta (A \to \exists y \leq_\rho sx \neg B(x,y))$$

and

$$\tilde{G} :\equiv \exists Y \leq s \forall x (A \to \neg B(x, Y(x))),$$

where s is a closed term of E-HA$^\omega$ and $x_1 \leq_\rho x_2$ is pointwise defined as $\forall \underline{v}(x_1\underline{v} \leq_0 x_2\underline{v})$ for a suitable tuple $\underline{v}$ of variables.
Let $C(u,v), D(u,v,w)$ only contain u,v resp. u,v,w as free variables and let t be a closed term. Then

$$\begin{cases} \text{E-HA}^\omega\text{+AC+IP}_\neg + G \vdash \forall u^1 \forall v \leq_\gamma tu(\neg C(u,v) \to \exists w^0 D(u,v,w)) \\ \Rightarrow \text{there exists a closed term } \Phi \text{ of E-HA}^\omega \text{ s.t.}^2 \\ \text{E-HA}^\omega\text{+AC+IP}_\neg + \tilde{G} \vdash \forall u^1 \forall v \leq_\gamma tu \exists w \leq_0 \Phi(u)(\neg C \to D(w)). \end{cases}$$

Note that the bound $\Phi(u)$ does not depend on v.

Corollary 6

$$\begin{cases} \text{E-HA}^\omega\text{+AC+IP}_\neg\text{+CA}_\neg \vdash \forall u^1 \forall v \leq_\gamma tu(\neg C(u,v) \to \exists w^0 D(u,v,w)) \\ \Rightarrow \text{there exists a closed term } \Phi \text{ of E-HA}^\omega \text{ s.t.} \\ \text{E-HA}^\omega\text{+AC+IP}_\neg\text{+CA}_\neg \vdash \forall u^1 \forall v \leq_\gamma tu \exists w \leq_0 \Phi(u)(\neg C \to D(w)). \end{cases}$$

Proof: The corollary follows from the previous theorem by observing that:
(i) we can without loss of generality assume that all instances of $\mathrm{CA}_\neg$ are parameter-free since parameters $\underline{a}$ in A can be taken into the comprehension together with $\underline{x}$,
(ii) every parameter-free instance of $\mathrm{CA}_\neg$ is – relative to E-HA$^\omega$ – equivalent to a

[2] I.e. Φ is a primitive recursive functional in the sense of Gödel's T.

sentence G as considered in the theorem (with $A :\equiv 0 = 0$, x a dummy variable and $s := \lambda x.1$)[3]

$$\exists \Phi \leq 1 \neg\neg \forall \underline{x}(\Phi(\underline{x}) =_0 0 \leftrightarrow \neg A(\underline{x})),$$

with $\tilde{G} \equiv G$. $\Box$

Remark 7 In the presence of $\mathrm{CA}_\neg$ one can actually derive $\mathrm{IP}_\neg$.

In order to formalize WMP in the language of E-HA$^\omega$ we have to represent real numbers in $[0,1]$. Furthermore, we show that we can arrange the representation such that the names of reals in $[0,1]$ used belong to a compact subspace of the Baire space. We represent real numbers (as in Bishop's original treatment) as Cauchy sequences with a fixed rate of convergence (the latter is inessential in a setting as ours which contains AC since – using only $\mathrm{AC}^{0,0}$ from integers to integers – we can replace an arbitrary Cauchy sequence by one with any fixed rate of convergence – e.g. 2^{-n} – which has the same limit).

Rational numbers in $[0,1]$ are represented as codes $j(n,m)$ of pairs (n,m) of natural numbers n,m.
$j(n,m)$ represents the rational number $\frac{n}{m+1}$, if $n \leq m+1$ and 0, otherwise.
Here j e.g. is Cantor's surjective pairing function $j(x,y) := \frac{1}{2}((x+y)^2 + 3x + y)$. On the codes of $\mathbb{Q} \cap [0,1]$, i.e. on $\mathbb{N}$, we have an equivalence relation by

$$n_1 =_{\mathbb{Q}} n_2 :\equiv \; n_1, n_2 \text{ represent the same rational number.}$$

On $\mathbb{N}$ one easily defines a primitive recursive function $|\cdot -_{\mathbb{Q}} \cdot|$ representing the usual distance function on $[0,1] \cap \mathbb{Q}$ and relations $<_{\mathbb{Q}}, \leq_{\mathbb{Q}}$ which represent the relations on $\mathbb{Q} \cap [0,1]$. For convenience we will write $\frac{1}{k+1}$ instead of its code $j(1,k)$ in $\mathbb{N}$.

By the coding of rational numbers in $[0,1]$ as natural numbers, **sequences of such rationals** are just functions f^1 (and every function f^1 can be conceived as a sequence of rational numbers in $[0,1]$ in a unique way). So real numbers in $[0,1]$ can be represented by functions f^1 modulo this coding. We now show that **every** function can be conceived as a representative of a uniquely determined Cauchy sequence of rationals in $[0,1]$ with modulus $1/(k+1)$ and therefore can be conceived as an representative of a uniquely determined real number. Finally we show, that we can restrict ourselves to codes $f \leq_1 M$ for some primitive recursive function M.

[3] Here $1^\rho := \lambda \underline{v}.1^0$ for a suitable tuple of variables $\underline{v}$.

Definition 8 *Primitive recursively in f we define*

$$\widehat{f}n := \begin{cases} fn, \ \mathit{if}\ \forall k, m, \tilde{m} \leq_0 n(m, \tilde{m} \geq_0 k \to |fm -_{\mathbb{Q}} f\tilde{m}| \leq_{\mathbb{Q}} \frac{1}{k+1}) \\ f(n_0 - 1) \ \mathit{for}\ n_0 := \min l \leq_0 n \ \mathit{such\ that} \\ \quad [\exists k, m, \tilde{m} \leq_0 l(m, \tilde{m} \geq_0 k \wedge |fm -_{\mathbb{Q}} f\tilde{m}| >_{\mathbb{Q}} \frac{1}{k+1})], \\ \mathit{otherwise}. \end{cases}$$

1) if f^1 represents a Cauchy sequence of rational numbers in $[0,1]$ with modulus $1/(k+1)$, then $\forall n^0(fn =_0 \widehat{f}n)$,

2) for every f^1 the function $\widehat{f}$ represents a Cauchy sequence of rational numbers in $[0,1]$ with modulus $1/(k+1)$.

Hence every function f gives a uniquely determined real number in $[0,1]$, namely that number which is represented by $\widehat{f}$.

Definition 9 *1)* $f_1 =_{\mathbb{R}} f_2 :\equiv \forall k^0(|\widehat{f}_1(k) -_{\mathbb{Q}} \widehat{f}_2(k)| \leq_{\mathbb{Q}} \frac{3}{k+1})$;

2) $f_1 <_{\mathbb{R}} f_2 :\equiv \exists k^0(\widehat{f}_2(k) - \widehat{f}_1(k) >_{\mathbb{Q}} \frac{3}{k+1})$;

3) $f_1 \leq_{\mathbb{R}} f_2 :\equiv \neg(f_2 <_{\mathbb{R}} f_1)$;

We can restrict the set of representing functions for [0,1] to the compact (in the sense of the Baire space) set $\{f : f \leq_1 M\}$, where $M(n) := j(3(n+1), 3(n+1)-1)$: Each fraction r having the form $\frac{i}{3(n+1)}$ (with $i \leq 3(n+1)$) is represented by a number $k \leq M(n)$, i.e. $k \leq M(n) \wedge k$ codes r. Thus $\{k : k \leq M(n)\}$ contains (modulo this coding) an $\frac{1}{3(n+1)}$–net for [0,1]. Primitive recursively in f we define

$$\tilde{f}(k) = \mu i \leq_0 M(k)[\forall j \leq_0 M(k)(|\widehat{f}(3(k+1)) -_{\mathbb{Q}} j| \geq_{\mathbb{Q}} |\widehat{f}(3(k+1)) -_{\mathbb{Q}} i|)].$$

$\tilde{f}$ has (provably in E-HA$^\omega$) the following properties (using that $\widehat{\tilde{f}} =_1 \tilde{f}$):

1) $\forall f^1(\tilde{f} \leq_1 M)$.

2) $\forall f^1(f =_{\mathbb{R}} \tilde{f})$.

Using this construction we can reduce e.g. quantification $\forall x \in [0,1]\ A(x)$ to quantification of the form $\forall f \leq_1 M\ A(f)$ and equivalently $\forall f^1 A(f)$ for properties A which are $=_{\mathbb{R}}$–extensional.

Theorem 10 E-HA$^\omega$+AC+IP$_\neg$+CA$_\neg \nvdash$ WMP.

Proof: Let $\mathcal{T}$:=E-HA$^\omega$+AC+IP$_\neg$+CA$_\neg$ and assume that $\mathcal{T} \vdash$ WMP. Then (restricting w.l.g. a, x to $[0,1]$)

$$\mathcal{T} \vdash \forall a \in [0,1](\forall x \in [0,1](\neg\neg(0 < x) \vee \neg\neg(x < a)) \rightarrow \exists k^0(a > \frac{1}{k+1}))$$

which can – modulo our representation of $x, a \in [0,1]$ in E-HA$^\omega$ – be written in the form

$$\mathcal{T} \vdash \forall a \leq_1 M(\forall x^1(\neg\neg(0 <_{\mathbb{R}} x) \vee \neg\neg(x <_{\mathbb{R}} a)) \rightarrow \exists k^0(a >_{\mathbb{R}} \frac{1}{k+1})).$$

This is equivalent to

$$\mathcal{T} \vdash \forall a \leq_1 M(\forall x^1 \exists n \leq_0 1[(n = 0 \rightarrow \neg\neg(0 <_{\mathbb{R}} x)) \wedge (n \neq 0 \rightarrow \neg\neg(x <_{\mathbb{R}} a))] \rightarrow \exists k^0(a >_{\mathbb{R}} \tfrac{1}{k+1})),$$

which implies

$$\mathcal{T} \vdash \forall a \leq_1 M\, \forall N \leq_2 1$$
$$(\forall x^1[(N(x) = 0 \rightarrow \neg\neg(0 <_{\mathbb{R}} x)) \wedge (N(x) \neq 0 \rightarrow \neg\neg(x <_{\mathbb{R}} a))] \rightarrow \exists k^0(a >_{\mathbb{R}} \tfrac{1}{k+1})).$$

The premise '$\forall x^1[(N(x) = 0 \rightarrow \neg\neg(0 <_{\mathbb{R}} x)) \wedge (N(x) \neq 0 \rightarrow \neg\neg(x <_{\mathbb{R}} a))]$' can easily be proved (relative to E-HA$^\omega$) to be equivalent to its double negation and hence is equivalent to a negated formula $\neg B$. So by corollary 6 we get a closed number term t^0 which can be reduced to a numeral $\overline{m}$ s.t.

$$\mathcal{T} \vdash \forall a \leq_1 M\, \forall N \leq_2 1$$
$$(\forall x^1[(N(x) = 0 \rightarrow \neg\neg(0 <_{\mathbb{R}} x)) \wedge (N(x) \neq 0 \rightarrow \neg\neg(x <_{\mathbb{R}} a))] \rightarrow a >_{\mathbb{R}} \tfrac{1}{\overline{m}+1}).$$

Using AC this yields

$$\mathcal{T} \vdash \forall a \leq_1 M$$
$$(\forall x^1 \exists n \leq_0 1[(n = 0 \rightarrow \neg\neg(0 <_{\mathbb{R}} x)) \wedge (n \neq 0 \rightarrow \neg\neg(x <_{\mathbb{R}} a))] \rightarrow a >_{\mathbb{R}} \tfrac{1}{\overline{m}+1})$$

and hence

$$\mathcal{T} \vdash \forall a \in [0,1](a \text{ pseudo-positive } \to a > \frac{1}{\overline{m}+1})$$

which obviously is absurd. □

Final Remark:

1) As mentioned in the introduction, the statement that every mapping $f : X \to Y$ from a complete metric space into a metric space is strongly extensional, which was shown by Ishihara [8] to be equivalent to WMP, can be seen to be underivable in the systems considered in this paper particularly easy: choose as X the Baire space with the usual metric and as Y the discrete space $\mathbb{N}$. In E-HA$^\omega$ the extensionality axiom implies that every functional F^2 is a function : $\mathbb{N}^\mathbb{N} \to \mathbb{N}$. If the above mentioned principle were derivable in E-HA$^\omega$+AC then one could derive that every F^2 is strongly extensional which in this particular case is equivalent to the statement

$$\forall f^1, g^1(F(f) \neq_0 F(g) \to \exists x^0(f(x) \neq_0 g(x)).$$

By modified realizability (which is sound for E-HA$^\omega$+AC, see [17],[18]) one could extract a closed term t of E-HA$^\omega$ witnessing '$\exists x$' uniformly in F, f, g. However, such a term would satisfy the Gödel functional ('Dialectica') interpretation of the extensionality axiom for functionals of type 2. As shown by Howard [7] such a term does not exist in E-HA$^\omega$ as it would not be majorizable whereas all closed terms in E-HA$^\omega$ are.[4] Using our theorem 5 instead we can also extend this proof to the situation where CA$_\neg$ is added to E-HA$^\omega$+AC. Note that the proof of theorem 5 in [12] uses modified realizability and Howard's concept of majorizability too. Our proof for the underivability of the usual formulation of WMP is more direct as the argument sketched in this remark relies on Ishihara's proof of the fact that WMP implies the principle mentioned above which, moreover, would have to be proved to be formalizable in E-HA$^\omega$+AC first.

2) Our proof of the underivability of WMP in E-HA$^\omega$+AC+CA$_\neg$ generalizes to extensions by any further principles whose monotone modified realizability interpretation (in the sense of [12]) is realizable by closed terms of E-HA$^\omega$.

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[4] In this way one can also show that E-HA$^\omega$+AC (as well as E-HA$^\omega$+AC+IP$_\neg$) is not closed under Markov's rule (see [11] for more information on this).

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