Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals

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Abstract

This paper is the first one in a sequel of papers resulting from the authors Habilitationsschrift [22] which are devoted to determine the growth in proofs of standard parts of analysis. A hierarchy $(\mathrm{G}_n\mathrm{A}^\omega)_{n\in\mathbb{N}}$ of systems of arithmetic in all finite types is introduced whose definable objects of type $1 = 0(0)$ correspond to the Grzegorczyk hierarchy of primitive recursive functions. We establish the following extraction rule for an extension of $\mathrm{G}_n\mathrm{A}^\omega$ by quantifier–free choice AC–qf and analytical axioms Γ having the form $\forall x^\delta \exists y \leq_\rho sx \forall z^\eta F_0$ (including also a 'non–standard' axiom F^- which does not hold in the full set–theoretic model but in the strongly majorizable functionals):

From a proof $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf $+ \Gamma \vdash \forall u^1, k^0 \forall v \leq_\tau tuk \exists w^0 A_0(u,k,v,w)$

one can extract a uniform bound Φ such that

$\forall u^1, k^0 \forall v \leq_\tau tuk \exists w \leq \Phi uk A_0(u,k,v,w)$ holds in the full set–theoretic type structure.

In case $n = 2$ (resp. $n = 3$) Φuk is a polynomial (resp. an elementary recursive function) in $k, u^M := \lambda x.\max(u0,\ldots,ux)$. In the present paper we show that for $n \geq 2$, $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+$F^-$ proves a generalization of the binary König's lemma yielding new conservation results since the conclusion of the above rule can be verified in $\mathrm{G}_{\max(3,n)}\mathrm{A}^\omega$ in this case.

In a subsequent paper we will show that many important ineffective analytical principles and theorems can be proved already in $\mathrm{G}_2\mathrm{A}^\omega$+AC–qf+$\Gamma$ for suitable Γ.

1 Introduction

This paper is the first one in a sequel of papers resulting from the authors Habilitationsschrift [22] which are devoted to determine the growth in proofs of standard parts of analysis.

Let U be a complete separable metric space, K a compact metric space and $A \in \Sigma^0_1$. As we have elaborated in [21] many numerically interesting theorems in analysis can be transformed into sentences having the form

(1) $\forall u \in U, k \in \mathbb{N} \forall v \in K \exists w \in \mathbb{N}\, A(u,k,v,w)$

and one is interested in a uniform bound Φuk on w which does not depend on $v \in K$, i.e.

$$\forall u \in U, k \in \mathbb{N} \forall v \in K \exists w \leq \Phi uk \, A(u,k,v,w).$$

Quite often A is monotone with respect to w, i.e.

$$A(u,k,v,w_1) \wedge w_2 \geq w_1 \rightarrow A(u,k,v,w_2)$$

and hence the bound Φuk in fact realizes '$\exists w$' (see [21] for a discussion of this phenomenon).
What do we know about the rate of growth of Φ if we know that (1) is proved using certain parts of analysis?
In [14],[15], [19],[20] we have developed a proof–theoretic method suited for the extraction of such bounds from proofs in analysis which guarantees the extractability of primitive recursive bounds for large parts of analysis. Moreover this method has been applied to concrete (ineffective) proofs in approximation theory yielding new a–priori estimates for numerically relevant data as constants of strong unicity and others which improve known estimates significantly (see [19],[20],[21]).

In analyzing these applications we developed in [21] a new monotone functional interpretation which has important advantages over the method from [15] and provides a particular perspicuous procedure of analyzing ineffective proofs in analysis.

The starting point for the investigation carried out in the present paper is the following problem: Whereas the general meta–theorems in [15], [19] and [21] only guarantee the existence of a primitive recursive bound Φ, the bounds which are actually obtained in our applications to approximation theory have a very low rate of growth which is polynomial (of degree ≤ 2) relatively to the growth of the data of the problem. Thus the problem arises to close the still large gap between polynomial and primitive recursive growth.
Before we start to discuss this question let us note that using a suitable representation of spaces like U,X and the basic notions of real analysis, sentences (1) can be formalized in the language of arithmetic in all finite types such that (1) gets (a special case of) the following logical form[1]

$$(2)\ \forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k} \exists w^0 A_0(\underline{u},\underline{k},v,w).$$

Here $\underline{u}^1 := u_1^1, \ldots, u_n^1$, $\underline{k}^0 := k_1^0, \ldots, k_m^0$, t is a closed term, τ an arbitrary finite type, $1 = 0(0)$ and $A_0(\underline{u},\underline{k},v,w)$ a quantifier–free formula containing only the free variables $\underline{u},\underline{k},v,w$. $\leq_\tau$ is defined pointwise.
By a uniform bound we now mean a functional Φ such that

$$\forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k} \exists w \leq_0 \Phi\underline{u}\,\underline{k} A_0(\underline{u},\underline{k},v,w).$$

Again the predicate 'uniform' for the bound Φ refers to the fact that Φ does **not depend on** v.
Coming back to our question from above we are interested in the determination of those parts of classical analysis, where the extractability of bounds Φ having only polynomial growth (resp. elementary recursive growth) relatively to the data is guaranteed.
In order to address this question we introduce a hierarchy $\mathrm{G}_n\mathrm{A}^\omega$ of subsystems of classical arithmetic in all finite types and investigate the rate of growth caused by various analytical principles relatively to $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf. The definable functionals $t^{1(1)}$ in $\mathrm{G}_n\mathrm{A}^\omega$ are of increasing order of growth:

[1] For the weak system $\mathrm{G}_2\mathrm{A}^\omega$ discussed below more subtle representations than those which are used in [19] are necessary. Such representations are developed in 3 of [22] and will be published in a paper under preparation.

If $n = 1$, then tf^1x^0 is bounded by a linear function in f^M, x,

if $n = 2$, then tf^1x^0 is bounded by a polynomial in f^M, x,

if $n = 3$, then tf^1x^0 is bounded by an elementary recursive (i.e. a (fixed) finitely iterated exponential) function in f^M, x,

where $f^M := \lambda x^0. \max(f0, \ldots, fx)$ and Φfx is called linear (polynomial, elementary recursive) in f, x if $\forall f^1, x^0(\Phi fx =_0 \widehat{\Phi}[f, x])$ for a term $\widehat{\Phi}[f, x]$ which is built up from $0^0, x^0, f^1, S^1, +$ (respectively $0^0, x^0, f^1, S^1, +, \cdot$ and $0^0, x^0, f^1, S^1, +, \cdot, \lambda x^0, y^0.x^y$) only. In our results the term $\widehat{\Phi}[f, x]$ can always be constructed.

Let us motivate this notion for the polynomial case:
If Φfx is a polynomial in f^1, x^0, then in particular for every polynomial $p \in \mathbb{N}[x]$ the function $\lambda x^0.\Phi px$ can be written as a polynomial in $\mathbb{N}[x]$. Moreover there exists a polynomial $q \in \mathbb{N}[x]$ (depending only on the term structure of $\widehat{\Phi}$) such that

$$\begin{cases} \text{For every polynomial } p \in \mathbb{N}[x] \text{ one can construct a polynomial } r \in \mathbb{N}[x] \text{ such that} \\ \forall f^1\big(f \leq_1 p \to \forall x^0(\Phi fx \leq_0 r(x))\big) \text{ and } deg(r) \leq q(deg(p)). \end{cases}$$

Since every closed term $t^{1(1)}$ in $\mathrm{G}_2\mathrm{A}^\omega$ is bounded by a polynomial $\Phi f^M x$ in f^M, x and $f \leq_1 p \to f^M \leq_1 p$ (since p is monotone) this also holds for tfx instead of Φfx.

In particular every closed term t^1 ($t^0 \overbrace{(0) \ldots (0)}^{k}$) of $\mathrm{G}_2\mathrm{A}^\omega$ is bounded by a polynomial $p_t \in \mathbb{N}[x]$ (resp. a polynomial $p_t \in \mathbb{N}[x_1, \ldots, x_k]$).
For general $n \in \mathbb{N}$, $n \geq 1$, every closed term t^1 of $\mathrm{G}_n\mathrm{A}^\omega$ is bounded by some function $f_t \in \mathcal{E}^n$ where $\mathcal{E}^n$ denotes the n–th level of the Grzegorczyk hierarchy.

It turns out that many basic concepts of real analysis can be defined already in $\mathrm{G}_2\mathrm{A}^\omega$: e.g. rational numbers, real numbers (with their usual arithmetical operations and inequality relations), d–tuples of real numbers (for every fixed d), sequences and series of reals, continuous functions $F : \mathbb{R}^d \to \mathbb{R}$ and uniformly continuous functions $F : [a, b]^d \to \mathbb{R}$, the supremum of $F \in C([a, b]^d, \mathbb{R})$ on $[a, b]^d$, the Riemann integral of $F \in C[a, b]$. Furthermore the trigonometric functions $\sin, \cos, \tan, \arcsin, \arccos, \arctan$ and π as well as the restriction $\exp_k$ ($\ln_k$) of the exponential function (logarithm) to $[-k, k]$ for every **fixed** number k can be introduced in $\mathrm{G}_2\mathrm{A}^\omega$ (The unrestricted functions exp and ln can be defined in $\mathrm{G}_3\mathrm{A}^\omega$).
$\mathrm{G}_2\mathrm{A}^\omega$+AC–qf proves many of the basic properties of these objects.
In this paper we determine the growth of extactable bounds Φ for $\mathrm{G}_n\mathrm{A}$+AC–qf+F^-, where F^- is a certain analytical axiom which allows (relatively to $\mathrm{G}_2\mathrm{A}^\omega$+AC–qf) very short and perspicuous proofs of fundamental theorems of analysis as e.g.

- every pointwise continuous function $f : [0, 1]^d \to \mathbb{R}$ is uniformly continuous and possesses a modulus of uniform continuity
- the attainment of the maximum value of $f \in C([0, 1]^d, \mathbb{R})$ on $[0, 1]^d$
- the sequential form of the Heine–Borel covering property for $[0, 1]^d$

- Dini's theorem together with a modulus of uniform convergence
- the existence of a uniformly continuous inverse function for every strictly increasing continuous function $f : [0,1] \to \mathbb{R}$.

In particular we show the following:
Let Δ be a set of sentences having the form $\forall x^\delta \exists y \leq_\rho sx \forall z^\tau B_0$ (B_0 quantifier–free). Then the following rule holds:

$$(*) \begin{cases} \text{From a given proof } \mathrm{G}_n\mathrm{A}^\omega\text{+AC–qf+}\Delta + F^- \ \vdash \forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k} \exists w^0 A_0(\underline{u}, \underline{k}, v, w) \\ \text{one can extract a uniform bound } \Phi \text{ such that} \\ \mathrm{G}_{\max(n,3)}\mathrm{A}_i^\omega + \Delta + \text{ b-AC } \vdash \forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k} \exists w \leq_0 \Phi \underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w), \end{cases}$$

where

$\Phi\underline{u}\,\underline{k}$ is a polynomial in $\underline{u}^M, \underline{k}$ if $n = 2$

$\Phi\underline{u}\,\underline{k}$ is an elementary recursive function in $\underline{u}^M, \underline{k}$ if $n = 3$.

Here b–AC denotes the schema

$$(\text{b–AC}^{\delta,\rho}) \ : \ \forall Z^{\rho\delta}\big(\forall x^\delta \exists y \leq_\rho Zx\, A(x,y,Z) \to \exists Y \leq_{\rho\delta} Z \forall x A(x, Yx, Z)\big), \ \text{b–AC} := \bigcup_{\delta,\rho \in \mathbf{T}} \Big\{(\text{b–AC}^{\delta,\rho})\Big\}.$$

If Δ consists of sentences B which hold in the full set–theoretic type $\mathcal{S}^\omega$ (where set–theoretic refers to say ZFC) then one can conclude that

$$\mathcal{S}^\omega \models \forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k} \exists w \leq_0 \Phi\underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w),$$

i.e. the bound Φ is verified in the full set–theoretic model although F^- is not valid in $\mathcal{S}^\omega$ but only in the model $\mathcal{M}^\omega$ of so–called strongly majorizable functionals (see 4).
(If $\Delta = \emptyset$ then we have a verification already in $\mathrm{G}_{\max(n,3)}\mathrm{A}_i^\omega$, i.e. without b–AC).
In a subsequent paper we will show that substantial parts of classical analysis can be developed in $\mathrm{G}_2\mathrm{A}^\omega$+AC–qf+$\Delta + F^-$ for suitable Δ or if the proof uses functions having e.g. exponential growth in $\mathrm{G}_3\mathrm{A}^\omega$+AC–qf+$\Delta + F^-$ (In the later case one obtains bounds which are polynomial relatively to these exponential functions. If these functions are not used iterated in the given proof one gets bounds having essentially simple exponential growth instead of being merely elementary recursive; see remark 3.2.6 for a discussion of this point), e.g. in addition to the theorems mentioned above we have

- the fundamental theorem of calculus
- Fejér's theorem on the uniform approximation of 2π–periodic continuous functions by trigonometric polynomials
- the equivalence (local and global) of ε-δ–continuity and sequential continuity of $F : \mathbb{R} \to \mathbb{R}$
- Mean value theorems for differentiation and integrals
- Cauchy–Peano existence theorem for ordinary differential equations
- Brouwer's fixed point theorem for continuous functions $F : [0,1]^d \to [0,1]^d$.

In a further paper we will consider the growth caused by single sequences of instances of principles like

- the convergence of bounded monotone sequences of real numbers
- the existence of a greatest lower bound for sequences of reals which are bounded from below
- the Bolzano–Weierstra property for bounded sequences in $\mathbb{R}^d$
- the Arzelà–Ascoli lemma.

relatively to $\mathrm{G}_{2/3}\mathrm{A}^\omega$+AC–qf+$\Delta + F^-$. Whereas the full versions of these principles are equivalent to the schema of arithmetical comprehension (provably in $\mathrm{G}_2\mathrm{A}^\omega$) and thus prove the totality of every $\alpha(<\varepsilon_0)$–recursive function, it turns out that single sequences of instances (which however may depend on the parameters of the conclusion) of these principles contribute to the growth of bounds at most by certain primitive recursive functionals (in the sense of [11],[12]). There are even important special cases where their contribution is only polynomial. In contrast to this, single instances of the principle of

- the existence of the limit superior of bounded sequences in $\mathbb{R}$

may contribute a growth of the Ackermann type.
For these results a combination of the techniques developed in this paper with a new method of eliminating Skolem functions for monotone formulas will be used.
The present paper is devoted mainly to establish $(*)$. Furthermore as a proof–theoretic application of $(*)$ we obtain (see 4 below) conservation results for a generalization WKL_{seq} of the binary König's lemma WKL to sequences of trees: We give a new formulation $\mathrm{WKL}^2_{(seq)}$ of $\mathrm{WKL}_{(seq)}$ which avoids the need of a coding functional $\Phi_{\langle\rangle} fx = \overline{f}x$ (which is not available in $\mathrm{G}_2\mathrm{A}^\omega$ but only in $\mathrm{G}_n\mathrm{A}^\omega$ for $n \geq 3$) by the use of functionals of higher type (relatively to $\mathrm{G}_3\mathrm{A}^\omega$ both formulations turn out to be equivalent). WKL^2_{seq} is provable in $\mathrm{G}_2\mathrm{A}^\omega + F^-$+$\mathrm{AC}^{1,0}$–qf+$\mathrm{AC}^{0,1}$–qf. Thus $(*)$ also applies to proofs using WKL^2_{seq} and in particular we obtain the following rule

$$\begin{cases} \text{From a proof } \mathrm{G}_2\mathrm{A}^\omega\text{+AC–qf+}\mathrm{WKL}^2_{seq} \vdash \forall u^0 \forall v \leq_\tau tu \exists w^0 A_0(u,v,w) \\ \text{one can extract constants } k, c_1, c_2 \in \mathbb{N} \text{ such that} \\ \mathrm{G}_3\mathrm{A}^\omega_i \vdash \forall u^0 \forall v \leq_\tau tu \exists w \leq_0 c_1 u^k + c_2\, A_0(u,v,w). \end{cases}$$

Finally let us emphasize that our systems based on $\mathrm{G}_2\mathrm{A}^\omega$+AC–qf must not be confused with systems of 'feasible analysis' as defined e.g. (in a second–order setting) in [6]. In $\mathrm{G}_2\mathrm{A}^\omega$ one can define for instance functionals which compute $\int_0^1 f(x)dx$ or $\sup_{x\in[0,1]} f(x)$ for uniformly continuous functions $f \in C[0,1]$ (endowed with a modulus of uniform continuity) although these notions are not (known to be) feasible (see [13]). Thus the formula A in (1) above may involve terms like $\int_0^1 f(x)dx$ or $\sup_{x\in[0,1]} f(x)$ and it is only by this fact that (1) covers many theorems in analysis. Nevertheless we obtain polynomial bounds $p \in \mathbb{N}[k]$ such that $\forall k \in \mathbb{N} \forall v \in K \exists w \leq p(k)\, A(k,v,w)$ from proofs of $\forall k \in \mathbb{N} \forall v \in K \exists w\, A(k,v,w)$ in $\mathrm{G}_2\mathrm{A}^\omega$+AC–qf+$\Delta + F^-$ (and in the presence of $u \in U$ polynomials in u^M). By monotonicity of A in w these bounds usually yield realizations for $\exists w$ (which in particular are computable in polynomial time and therefore 'feasible' since p is a polynomial!).
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2 Subsystems of primitive recursive arithmetic in all finite types

2.1 Classical and intuitionistic predicate logic PL^ω and HL^ω in the language of all finite types

The set $\mathbf{T}$ of all finite types is defined inductively by

(i) $0 \in \mathbf{T}$ and (ii) $\rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}$.

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type ρ to objects of type τ.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

(i) $0 \in \mathbf{P}$ and (ii) $\rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}$.

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write $0(00)$ instead of $0(0(0))$. Furthermore we write for short $\tau\rho_k \ldots \rho_1$ instead of $\tau(\rho_k)\ldots(\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n+1$. The types $0, 00, 0(00), 0(0(00))\ldots$ are so represented by $0, 1, 2, 3\ldots$. For arbitrary types $\rho \in \mathbf{T}$ the degree of ρ (for short $\deg(\rho)$) is defined by $\deg(0) := 0$ and $\deg(\tau(\rho)) := \max(\deg(\tau), \deg(\rho) + 1)$. For pure types the degree is just the number which represents this type. Functions having a type whose degree is > 1 are usually called functionals.
The language $\mathcal{L}(\mathrm{HL}^\omega)$ of HL^ω contains variables $x^\rho, y^\rho, z^\rho, \ldots$ for each type $\rho \in \mathbf{T}$ together with corresponding quantifiers $\forall x^\rho, \exists y^\rho$ as well as the logical constants $\wedge, \vee, \rightarrow$ and an equality relation $=_0$ between objects of type 0. Furthermore we have a propositional constant $\perp$ ('falsum'). Negation as a defined notion: $\neg A :\equiv A \rightarrow \perp$. Finally $\mathcal{L}(\mathrm{HL}^\omega)$ contains 'logical' combinators $\Pi_{\rho,\tau}$ and $\Sigma_{\delta,\rho,\tau}$ of type $\rho\tau\rho$ and $\tau\delta(\rho\delta)(\tau\rho\delta)$ for all $\rho, \tau, \delta \in \mathbf{T}$.
HL^ω has the usual axioms and rules of intuitionistic predicate logic (for all sorts of variables) plus the equality axioms for $=_0$ (e.g. see [34]). Equations $s =_\rho t$ between terms of higher type $\rho = 0\rho_k \ldots \rho_1$ are abbreviations for the formulas $\forall x_1^{\rho_1}, \ldots, x_k^{\rho_k}(sx_1 \ldots x_k =_0 tx_1 \ldots x_k)$.
$\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$ are characterized by the corresponding axioms of typed combinatory logic:

$\Pi_{\rho,\tau}x^\rho y^\tau =_\rho x$ and $\Sigma_{\delta,\rho,\tau}xyz =_\tau xz(yz)$ where $x \in \tau\rho\delta, y \in \rho\delta, z \in \delta$.

Furthermore we have the following quantifier–free rule of extensionality

$$\text{QF–ER}: \ \frac{A_0 \rightarrow s =_\rho t}{A_0 \rightarrow r[s] =_\tau r[t]}, \text{ where } A_0 \text{ is quantifier–free.}$$

Classical predicate logic in all finite types PL^ω results if the tertium–non–datur schema $A \vee \neg A$ is added to HL^ω. The enrichment of HL^ω (resp. PL^ω) obtained by adding the extensionality **axiom**

$(E_\rho) : \forall x^\rho, y^\rho, z^{\tau\rho}(x =_\rho y \rightarrow zx =_\tau zy)$

for every type ρ is denoted by E–HL^ω (resp. E–PL^ω).

Remark 2.1.1 *Using $\Pi_{\rho,\tau}$ and $\Sigma_{\delta,\rho,\tau}$ one defines (e.g. as in [34]) λ–terms $\lambda x^\rho.t^\tau[x]$ for each term $t^\tau[x^\rho]$ such that*
$HL^\omega \vdash \big(\lambda x^\rho.t^\tau[x]\big)s^\rho =_\tau t[s]$. In particular one can define a combinator $\Pi'_{\rho,\tau} = \lambda x^\rho, y^\tau.y$ such that $\Pi'_{\rho,\tau}x^\rho y^\tau =_\tau y$ (E.g. take $\Pi' := \Pi(\Sigma\Pi\Pi)$ for Σ, Π of suitable types).

Notational convention: Throughout this paper $A_0, B_0, C_0, \ldots$ always denote quantifier–free formulas.

2.2 Subsystems of arithmetic in all finite types corresponding to the Grzegorczyk hierarchy

In the following we extend PL^ω and HL^ω by adding certain computable functionals and universal axioms including the schema of quantifier–free induction. The following definition from [28] is a variant of a definition due to [1] and can be used for a perspicuous definition of the well–known Grzegorczyk hierarchy from [9] (see def.2.2.27).

Definition 2.2.1 *For each $n \in \mathbb{N}$ we define (by recursion on n from the outside) the n-th branch of the Ackermann function $A_n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by*

$$A_0(x,y) := y' \quad \textit{(Here and in the following } x' \textit{ stands for the successor } Sx \textit{ of } x\textit{)},$$

$$A_{n+1}(x,0) := \begin{cases} x, & \textit{if } n = 0 \\ 0, & \textit{if } n = 1 \\ 1, & \textit{if } n \geq 2, \end{cases} \qquad .$$

$$A_{n+1}(x,y') := A_n(x, A_{n+1}(x,y))$$

Remark 2.2.2 *1) $A_1(x,y) = x+y,\ A_2(x,y) = x \cdot y,\ A_3(x,y) = x^y,\ A_4(x,y) = x^{x^{\cdot^{\cdot^{\cdot x}}}}$ (y times).*

2) For each fixed $n \in \mathbb{N}$ the function A_n is primitive recursive. But: $A(x) := A_x(x,x)$ is not primitive recursive.

We now define the **Grzegorczyk arithmetic $\mathbf{G}_n\mathbf{A}^\omega$ of level $n \geq 1$ in all finite types** and their intuitionistic variant $\mathbf{G}_n\mathbf{A}^\omega_i$:

$\mathcal{L}(\mathrm{G}_n\mathrm{A}^\omega)$ is defined as the extension of $\mathcal{L}(\mathrm{PL})^\omega)$ by the addition of function constants S^{00} (successor), $\max_0^{000}, \min_0^{000}, A_0^{000}, \ldots, A_n^{000}$ and functional constants $\Phi_1^{001}, \ldots, \Phi_n^{001}, \mu_b^{001}$ (bounded μ–operator), $\tilde{R}_\rho \in \rho(\rho 0)(\rho 00)\rho 0$ (for each $\rho \in \mathbf{T}$). Furthermore we have a predicate symbol $\leq_0$.
In addition to the axioms and rules of PL^ω the theory $\mathrm{G}_n\mathrm{A}^\omega$ contains the following:

1) $\leq_0$–axioms: $x \leq_0 x,\ x \leq_0 y \vee y \leq_0 x,\ x \leq_0 y \wedge y \leq_0 z \to x \leq_0 z,\ x \leq_0 y \wedge y \leq_0 x \leftrightarrow x =_0 y$.

2) S–axioms: $Sx =_0 Sy \to x =_0 y,\ \neg 0 =_0 Sx,\ x \leq_0 Sx$.

3) (max) : $\max_0(x,y) \geq_0 x,\ \max_0(x,y) \geq_0 y,\ \max_0(x,y) =_0 x \vee \max_0(x,y) =_0 y$.

4) (min) : $\min_0(x,y) \leq_0 x,\ \min_0(x,y) \leq_0 y,\ \min_0(x,y) =_0 x \vee \min_0(x,y) =_0 y$.

5) The defining recursion equations for $A_0, \ldots, A_n$ from the definition 2.2.1 above.

6) Defining recursion equations for $\Phi_1, \ldots, \Phi_n$:

$$\begin{cases} \Phi_i f0 =_0 f0 \\ \Phi_i fx' =_0 A_{i-1}(fx', \Phi_i fx) \ \text{ for } i \geq 2 \end{cases}$$

and

$$\begin{cases} \Phi_1 f0 =_0 f0 \\ \Phi_1 fx' =_0 \max_0(fx', \Phi_1 fx). \end{cases}$$

(For $i \geq 2$, Φ_i is the iteration of the $(i-1)$-th branch A_{i-1} of the Ackermann function on the f–values $f0, \ldots, fx$ for variable x).

7) $$(\mu_b) : \begin{cases} y \leq_0 x \wedge f^{000}xy =_0 0 \to fx(\mu_b fx) =_0 0, \\ y <_0 \mu_b fx \to fxy \neq 0, \\ \mu_b fx =_0 0 \vee (fx(\mu_b fx) =_0 0 \wedge \mu_b fx \leq_0 x) \end{cases}$$

(These axioms express that $\mu_b fx = \min y \leq_0 x(fxy =_0 0)$ if such an $y \leq x$ exists and $= 0$ otherwise).

8) Defining recursion equations for $\tilde{R}_\rho$ (bounded and predicative recursion, since only type–0–values are used in the recursion):

$$\begin{cases} \tilde{R}_\rho 0yzv\underline{w} =_0 y\underline{w} \\ \tilde{R}_\rho x'yzv\underline{w} =_0 \min_0(z(\tilde{R}_\rho xyzv\underline{w})x\underline{w}, vx\underline{w}), \end{cases}$$

where $y \in \rho = 0\rho_k \ldots \rho_1,\ \underline{w} = w_1^{\rho_1} \ldots w_k^{\rho_k},\ z \in \rho 00,\ v \in \rho 0.$

9) All $\mathbb{N}, \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$–true purely universal sentences $\forall \underline{x} A_0(\underline{x})$, where $\underline{x}$ is a tuple of variables whose types have a degree ≤ 2, i.e. all such sentences which are true in the full type–structure $\mathcal{S}^\omega$ of all set–theoretic functionals, where 'set–theoretic' refers to say ZFC (The constants introduced so far have an interpretation in $\mathcal{S}^\omega$ which is uniquely determined by the axioms 1)–8). By this interpretation $\mathcal{S}^\omega$ becomes a model of the theory axiomatized by 1)–8). It is this model we refer to if we speak of 'truth' in $\mathcal{S}^\omega$).

$\mathrm{G}_n\mathrm{A}_i^\omega$ is the variant of $\mathrm{G}_n\mathrm{A}^\omega$ with intuitionistic logic only.
If we add $(E) = \bigcup_\rho \{(E_\rho)\}$ to $\mathrm{G}_n\mathrm{A}^\omega$,$\mathrm{G}_n\mathrm{A}_i^\omega$ we obtain theories which are denoted by E–$\mathrm{G}_n\mathrm{A}^\omega$, E–$\mathrm{G}_n\mathrm{A}_i^\omega$. $\mathrm{G}_n\mathrm{R}^\omega$ denotes the set of all closed terms of $\mathrm{G}_n\mathrm{A}^\omega$.

Remark 2.2.3 *1) The functionals Φ_1, Φ_2 and Φ_3 have the following meaning:*
$\Phi_1 fx = \max(f0, f1, \ldots, fx),\ \Phi_2 fx = \sum_{y=0}^x fy,\ \Phi_3 fx = \prod_{y=0}^x fy.$

2) Our definition of G_nA^ω contains some redundances (which however we want to remain for greater flexibility of our language): E.g. Φ_i $(i > 1)$ can be defined from $A_i, \tilde{R}, \min_0$ and Φ_1: With $f^M := \lambda x.\Phi_1 fx$, 2.2.18 below implies $\Phi_i fx \leq A_i(f^M x + 1, x + 1)$. Hence Φ_i can be defined by $\tilde{R}$ using $A_i(f^M x + 1, x + 1)$ as boundary function v.

3) The axiom of quantifier–free induction

$$(1)\ \forall f^1, x^0 \big(f0 =_0 0 \wedge \forall y < x(fy =_0 0 \to fy' =_0 0) \to fx =_0 0\big)$$

can be expressed as an universal sentence $\forall f^1, x^0 A_0$ by prop.2.2.6 below and thus is an axiom of $G_nA_i^\omega$. (1) implies every instance (with parameters of arbitrary type) of the schema of quantifier–free induction

$$QF\text{–}IA\ :\ \forall x^0 \big(A_0(0) \wedge \forall y < x(A_0(y) \to A_0(y')) \to A_0(x)\big)$$

since again by prop.2.2.6 there exists a term t such that $tx =_0 0 \leftrightarrow A_0(x)$: QF–IA now follows from (1) applied to $f := t$.

4) Because of the axioms in 9), our theories are not recursively enumerable. The motivation for the addition of these sentences as axioms is two–fold:
(i) As G. Kreisel has pointed out in various papers, proofs of $\mathbb{N}$–true universal lemmas have no impact on bounds extracted from proofs using such lemmas. For the methods we use for the extraction of bounds (e.g. our monotone functional interpretation) this applies even for arbitrary universal sentences $\forall x^\rho A_0$ where ρ may be an arbitrary type. Taking such sentences as axioms usually simplifies the process of the extraction of bounds enormously. The reason for our restriction to those sentences for which $\rho \leq 2$ is that on some places in this paper we deal with principles which are valid only in the type structure $\mathcal{M}^\omega$ of the so–called strongly majorizable functionals (see 4 below) but not in the full type structure $\mathcal{S}^\omega$ of all set–theoretic functionals. Since both type structures coincide up to type 1 and for the type 2 the inclusion $\mathcal{M}^\omega_2 \subset \mathcal{S}^\omega_2$ holds, the implication $\mathcal{S}^\omega \models \forall x^\rho A_0 \Rightarrow \mathcal{M}^\omega \models \forall x^\rho A_0$ holds **if** *$\rho \leq 2$. The same is true if we replace $\mathcal{M}^\omega$ by the type structure ECF of all extensional continuous functionals over $\mathbb{N}^\mathbb{N}$ (see [34] for details on ECF).*
(ii) Many important primitive recursive functions such as $sg, \overline{sg}, |x-y|$ and so on are already definable in G_1A^ω. However the usual proofs for their characteristic properties (which can be expressed as universal sentences) often make use of functions which are not definable in G_1A^ω (as e.g. $x \cdot y$). Thus we would have to carry out the boring details of a proof for these properties in G_1A^ω.

Using $\tilde{R}_0$ the following primitive recursive functions can be defined easily in $\mathrm{G_1A}^\omega$:

Definition 2.2.4

1) $$\begin{cases} prd(0) =_0 0 \\ prd(x') =_0 x \quad \textit{(predecessor)}, \end{cases}$$

2) $$\begin{cases} sg(0) =_0 0 & \overline{sg}(0) =_0 1 \quad (1 := S0) \\ sg(x') =_0 1, & \overline{sg}(x') =_0 0, \end{cases}$$

3) $$\begin{cases} x \dot{-} 0 =_0 x \\ x \dot{-} y' =_0 prd(x \dot{-} y), \end{cases}$$

4) $|x-y| =_0 \max(x \dot{-} y, y \dot{-} x)$ *(symmetrical difference),*

5) $\varepsilon(x,y) =_0 sg(|x-y|)$ *(characteristic function for* $=_0$*),*

6) $\delta(x,y) =_0 \overline{sg}(|x-y|)$ *(characteristic function for* $\neq$*).*

Remark 2.2.5 *Because of the universal axioms in 9), the theory $G_1A^\omega_i$ proves the usual properties of the functions* $\max, \min, prd, sg, \overline{sg}, \dot{-}, |x-y|, \varepsilon$ *and* δ*, e.g.*
$sg(x) = 0 \leftrightarrow x = 0,\ \overline{sg}(x) = 0 \leftrightarrow x \neq 0,\ sg(x) \leq 1,\ \overline{sg}(x) \leq 1,\ prd(x) \leq x,$
$|x-y| = 0 \leftrightarrow x = y,\ x = 0 \vee x = S(prd(x)),\ \max(x,y) = 0 \leftrightarrow x = 0 \wedge y = 0,$
$\min(x,y) = 0 \leftrightarrow x = 0 \vee y = 0,\ \max(x,y) =_0 y \leftrightarrow x \leq_0 y.$

Proposition: 2.2.6 *Let n be ≥ 1. For each formula $A \in \mathcal{L}(G_nA^\omega)$ which contains no quantifiers except for bounded quantifiers of type 0 one can construct a closed term t_A in G_nA^ω such that*

$$G_nA_i^\omega \vdash \forall x_1^{\rho_1}, \ldots, x_k^{\rho_k}\big(t_A x_1 \ldots x_k =_0 0 \leftrightarrow A(x_1, \ldots, x_k)\big),$$

where $x_1, \ldots, x_k$ are all the free variables of A.

Proof: Induction on the logical structure of A using the remark above. Bounded quantifiers are captured by μ_b:

$$\mathrm{G}_n\mathrm{A}_i^\omega \vdash \exists y \leq_0 x A(x, y, \underline{a}) \overset{(\mu_b)}{\leftrightarrow} A(x, \mu_b(\lambda x, y.t_A xy\underline{a}, x), \underline{a}).$$

Proposition: 2.2.7 *Let $n \geq 1$, $A_0(\underline{x}) \in \mathcal{L}(G_nA^\omega)$, where $\underline{x} = x_1^{\rho_1} \ldots x_k^{\rho_k}$ are all free variables of A_0, and $t_1^{0\rho_k\ldots\rho_1}, t_2^{0\rho_k\ldots\rho_1}$ are closed terms of G_nA^ω. Then there exists a closed term $\Phi^{0\rho_k\ldots\rho_1}$ in G_nA^ω such that*

$$G_nA_i^\omega \vdash \forall \underline{x}\left(\Phi\underline{x} =_0 \left\{ \begin{array}{l} t_1\underline{x},\ \textit{if } A_0(\underline{x}) \\ t_2\underline{x},\ \textit{if } \neg A_0(\underline{x}). \end{array} \right\}\right)$$

Proof: Define $t_2' := \lambda y^0, u^0.t_2$, $t_2'' := \lambda u^0.t_2$. One easily verifies that $\Phi := \lambda\underline{x}.\tilde{R}_\rho(t_{A_0}\underline{x})t_1t_2't_2''\underline{x}$ with t_{A_0} as in the previous proposition and $\rho = 0\rho_k \ldots \rho_1$ fulfils our claim.

Definition 2.2.8 (and lemma) *For* $\mathbf{n \geq 2}$ *we can define the surjective Cantor pairing function j ('diagonal counting from below') with its projections*[2] *in G_nR^ω:*

$$j(x^0, y^0) := \left\{ \begin{array}{l} \min u \leq_0 (x+y)^2 + 3x + y[2u =_0 (x+y)^2 + 3x + y]\ \textit{if existent} \\ 0^0,\ \textit{otherwise},^{3} \end{array} \right.$$

$$j_1 z := \min x \leq_0 z[\exists y \leq z(j(x, y) = z)],$$

$$j_2 z := \min y \leq_0 z[\exists x \leq z(j(x, y) = z)].$$

Using j, j_1, j_2 we can define a coding of k–tuples for every **fixed** *number k by*

$$\nu^1(x_0) := x_0,\ \nu^2(x_0, x_1) := j(x_0, x_1),\ \nu^{k+1}(x_0, \ldots, x_k) := j(x_0, \nu^k(x_1, \ldots, x_k)),$$

$$\nu_i^k(x_1, \ldots, x_k) := \left\{ \begin{array}{l} j_1 \circ (j_2)^{i-1}(x),\ \textit{if } 1 \leq i < k \\ (j_2)^{k-1}(x),\ \textit{if } 1 < i = k \end{array} \right. \quad (\textit{if } k > 1)$$

One easily verifies that $\nu_i^k(\nu^k(x_1, \ldots, x_k)) = x_i$ for $1 \leq i \leq k$ and $\nu^k(\nu_1^k(x), \ldots, \nu_k^k(x)) = x$. Finite sequences are coded (following [34]) by

$$\langle\rangle := 0,\ \langle x_0, \ldots, x_k\rangle := S(\nu^2(k, \nu^{k+1}(x_0, \ldots, x_k))).$$

Using $\tilde{R}$ one can define functions $lth, \Pi(k, y) \in G_nR^\omega$ such that for every fixed k

$$lth(\langle\rangle) = 0,\ lth(\langle x_0, \ldots, x_k\rangle) = k+1,\ \Pi(x, y) = \left\{ \begin{array}{l} x_y,\ \textit{if } y \leq k \\ 0^0,\ \textit{otherwise} \end{array} \right. \quad \textit{if } x = \langle x_0, \ldots, x_k\rangle.$$

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[2]For detailed information on this as well as various other codings see [33] and also [5] (where j is called 'Cauchy's pairing function').

[3]One easily shows that $(x+y)^2 + 3x + y$ is always even (This can be expressed as a purely universal sentence, i.e. as an axiom in $\mathrm{G}_n\mathrm{A}^\omega$). Hence the case 'otherwise' never occurs and therefore $2j(x, y) = (x+y)^2 + 3x + y$ for all x, y.

Define

$$lth(x) := \begin{cases} 0^0, & \textit{if } x =_0 0 \\ j_1(x \dot{-} 1) + 1, & \textit{otherwise}, \end{cases}$$

$$\Pi(x, y) =_0 \begin{cases} 0^0, \textit{if } lthx \leq y \\ j_1 \circ (j_2)^{y+1}(x \dot{-} 1), & \textit{if } 0 \leq y < lthx \dot{-} 1 \\ (j_2)^{lthx}(x), & \textit{if } lthx > 0 \wedge y = lthx \dot{-} 1 \end{cases}$$

We usually write $(x)_y$ *instead of* $\Pi(x, y)$.
In order to verify that $\Pi(x, y)$ *is definable in* G_2R^ω *it suffices to show that the variable iteration* $\varphi xy = (j_2)^y(x)$ *of* j_2 *is definable in* G_2R^ω. *This however follows from the fact that* $\varphi xy \leq x$ *for all* x, y. *Thus we can define* φxy *by* $\tilde{R}$ *using* $\lambda y.x$ *as bounding function.*
For $\mathbf{n \geq 3}$ *we can code initial segments of* **variable** *length of a function* f *in* G_nA^ω, *i.e. there is a functional* $\Phi_{\langle\rangle} \in G_3R^\omega$ *such that* $\Phi_{\langle\rangle}fx = \langle f0, \ldots, f(x \dot{-} 1)\rangle$.[4]
As an intermediate step we first show the definability of

$$\begin{cases} \tilde{f}0 = f0 \\ \tilde{f}x' = \tilde{j}(\tilde{f}x, fx'), & \textit{where } \tilde{j}(x, y) := j(y, x) \end{cases}$$

in G_3R^ω: *One easily verifies (using* $j(x, x) \leq 4x^2$*) that* $\tilde{f}x \leq 4^{3^x}\left(f^Mx\right)^{2^x}$ *for all* x. *Hence the definition of* $\tilde{f}$ *can be carried out by* $\tilde{R}$ *using* $\lambda x.4^{3^{x'}}\left(f^Mx'\right)^{2^{x'}} \in G_3R^\omega$ *as bounding function.* $\tilde{f}x$ *means* $\tilde{j}(\ldots\tilde{j}(\tilde{j}(f0, f1), f2)\ldots fx)$. *Hence* $\widehat{f}x := (\lambda y.\widetilde{f(x \dot{-} y)})x$ *has the meaning* $j(f0, \ldots j(f(x-2), j(f(x-1), fx))\ldots)$. *We are now able to define* $\Phi_{\langle\rangle} \in G_3R^\omega$:

$$\Phi_{\langle\rangle}fx := \begin{cases} 0^0, & \textit{if } x = 0 \\ \widehat{(f_x)}x + 1, & \textit{otherwise}, \end{cases}$$

where

$$f_xy := \begin{cases} x, & \textit{if } y = 0 \\ f(y \dot{-} 1), & \textit{otherwise}. \end{cases}$$

We usually write $\overline{f}x$ *for* $\Phi_{\langle\rangle}fx$. *Furthermore one can define a function* $*$ *in* G_3R^ω *such that*

$$\langle x_0, \ldots, x_k\rangle * \langle y_0, \ldots, y_m\rangle = \langle x_0, \ldots, x_k, y_0, \ldots, y_m\rangle.$$

Define

$$n * m := \Phi_{\langle\rangle}(fnm)(lth(n) + lth(m)), \textit{ where}$$

[4] Of course we cannot write $\langle f0, \ldots, f(x \dot{-} 1)\rangle$ for variable x. However the meaning of $\Phi_{\langle\rangle}fx$ can be expressed via $(\Phi_{\langle\rangle}fx)_y = fy$ for all $y < x$ (and $= 0$ for $y \geq x$) and $lth(\Phi_{\langle\rangle}fx) = x$, which both are purely universal (and therefore axioms in $\mathrm{G_3A^\omega}$).

$$(fnm)(k) := \begin{cases} (n)_k, & \textit{if } k < lth(n) \\ (m)_{k \dot{-} lthn}, & \textit{otherwise.} \end{cases}$$

Note that $\Phi_{\langle\rangle}$ and $*$ are not definable in G_2R^ω since their definitions involve an iteration of the polynomial j.

Definition 2.2.9 *Between functionals of type ρ we define relations $\le_\rho$ ('less or equal') and s–maj$_\rho$ ('strongly majorizes') by induction on the type:*

$$\begin{cases} x_1 \le_0 x_2 :\equiv (x_1 \le_0 x_2), \\ x_1 \le_{\tau\rho} x_2 :\equiv \forall y^\rho (x_1 y \le_\tau x_2 y); \end{cases}$$

$$\begin{cases} x^* \textit{ s–maj}_0\; x :\equiv x^* \ge_0 x, \\ x^* \textit{ s–maj}_{\tau\rho}\; x :\equiv \forall y^{*\rho}, y^\rho (y^* \textit{ s–maj}_\rho\; y \to x^* y^* \textit{ s–maj}_\tau\; x^* y,\; xy). \end{cases}$$

Remark 2.2.10 *'s–maj' is a variant of W.A. Howard's relation 'maj' from [10] which is due to [2]. For more details see [16].*

Lemma: 2.2.11 *$G_1A_i^\omega$ proves the following facts:*

1) $\tilde{x}^* =_\rho x^* \wedge \tilde{x} =_\rho x \wedge x^* \textit{ s–maj}_\rho\; x \to \tilde{x}^* \textit{ s–maj}_\rho\; \tilde{x}.$

2) $x^* \textit{ s–maj}_\rho\; x \to x^* \textit{ s–maj}_\rho\; x^*$ ([2]).

3) $x_1 \textit{ s–maj}_\rho\; x_2 \wedge x_2 \textit{ s–maj}_\rho\; x_3 \to x_1 \textit{ s–maj}_\rho\; x_3$ ([2]).

4) $x^* \textit{ s–maj}_\rho\; x \wedge x \ge_\rho y \to x^* \textit{ s–maj}_\rho\; y.$

5) *For $\rho = \tau\rho_k \ldots \rho_1$ we have*

$$x^* \textit{ s–maj}_\rho\; x \leftrightarrow \forall y_1^*, y_1, \ldots, y_k^*, y_k \Big(\bigwedge_{i=1}^{k} (y_i^* \textit{ s–maj}_{\rho_i}\; y_i) \to x^* y_1^* \ldots y_k^* \textit{ s–maj}_\tau\; x^* y_1 \ldots y_k, x y_1 \ldots y_k \Big).$$

6) $x^* \textit{ s–maj}_1\; x \leftrightarrow x^* \text{ monotone } \wedge x^* \ge_1 x,$
where x^* is monotone iff $\forall u, v (u \le_0 v \to x^* u \le_0 x^* v)$.

7) $x^* \textit{ s–maj}_2\; x \to \lambda y^1 . x^*(\Phi_1 y) \ge_2 x.$

Proof: 1)–4) follow easily by induction on the type (in the proof of 3) one has to use 2)). 5) follows by induction on k using 2) (for details see [16]). 6) is trivial. 7) follows from $\forall y^1 (\Phi_1 y \text{ s–maj}_1 y)$.

Remark 2.2.12 *In contrast to $\ge_\rho$ the relation s–maj$_\rho$ has a nice behaviour w.r.t. substitution (see 5) of the lemma above). This makes it possible to prove results on majorization of complex terms simply by induction on the term structure. For types ≤ 2 (which are used in our applications to analysis) we can infer from a majorant to a 'real' $\ge$–bound by 6) and 7) of lemma 2.2.11.*

Next we need some basic properties of A_j which are formulated in the following lemmas (since these properties are purely universal we only have to verify their truth in order to ensure their provability in $\mathrm{G}_n\mathrm{A}_i^\omega$ for $j \le n$):

Lemma: 2.2.13 *Assume $j \ge 1$. Then $\forall x \forall y \ge 1 \big(A_j(x,y) \ge x\big)$.*

Proof: j–Induction: $j = 1: \ A_1(x,y) = x + y \ge x$.
$j \mapsto j+1: \ y$–induction: $A_{j+1}(x,1) = A_j(x, A_{j+1}(x,0)) =$

$$= \begin{cases} A_j(x,0) = x + 0 \ge x, \text{ if } j = 1 \\ A_j(x,1) \overset{j-I.H.}{\ge} x, \text{ if } j \ge 2. \end{cases}$$

$$y \mapsto y+1: \quad A_{j+1}(x,y+1) = A_j(x, \underbrace{A_{j+1}(x,y)}_{\ge x \ (y-I.H.)}) \overset{j-I.H.}{\ge} x.$$

Lemma: 2.2.14 *For all $j \in \mathbb{N}$ the following holds:*

$$\forall x, \tilde{x}, y, \tilde{y} \big(\tilde{x} \ge x \ge 1 \wedge \tilde{y} \ge y \to A_j(\tilde{x},\tilde{y}) \ge A_j(x,y)\big).$$

Proof: j–Induction. For $j = 0, 1, 2$ the lemma is trivial. $j \mapsto j+1$: To begin with we verify (for $x \ge 1$) by y–induction

$$(*) \ \forall y \big(A_{j+1}(x,y+1) \ge A_{j+1}(x,y)\big) :$$

I. $A_{j+1}(x,1) \overset{2.2.13}{\ge} x \ge 1 \overset{j \ge 2}{\equiv} A_{j+1}(x,0).)$
II. $y \mapsto y+1: \quad A_{j+1}(x,y+2) = A_j(x, \underbrace{A_{j+1}(x,y+1)}_{\substack{y-I.H. \\ \ge A_{j+1}(x,y)}}) \overset{j-I.H.}{\ge} A_j(x, A_{j+1}(x,y)) = A_{j+1}(x,y+1)$.

$(*)$ implies

$$(**) \ \forall y \forall \tilde{y} \ge y (A_{j+1}(x,\tilde{y}) \ge A_{j+1}(x,y)).$$

Again by y–induction we show (for $\tilde{x} \ge x \ge 1$):

$$(***) \ \forall y (A_{j+1}(\tilde{x},y) \ge A_{j+1}(x,y)) :$$

$y = 0: \quad A_{j+1}$–definition! $y \mapsto y+1:$

$$A_{j+1}(\tilde{x},y+1) = A_j(\tilde{x}, \quad \underbrace{A_{j+1}(\tilde{x},y)}_{\ge A_{j+1}(x,y) \ (y-I.H.)} \quad) \overset{j-I.H.}{\ge} A_j(x, A_{j+1}(x,y)) = A_{j+1}(x,y+1).$$

$(**)$ and $(***)$ yield the claim for $j+1$.

Lemma: 2.2.15 *If $j \ge 2$, then $\forall y (A_j(0,y) \le 1)$.*

Proof: j–Induction: The case $j = 2$ is clear.

$A_{j+1}(0,0) = 1, \ A_{j+1}(0,y+1) = A_j(0, A_{j+1}(0,y)) \overset{j-I.H.}{\le} 1.$

Proposition: 2.2.16 *$f^* \ge_1 1 \wedge f^*$ s–maj $f \wedge x^* \ge_0 x \to \Phi_j f^* x^* \ge_0 \Phi_j f x$.*

Proof: Assume $f^* \geq 1 \wedge f^*$ s–maj$_1 f \wedge x^* \geq_0 x$. $j = 1$:
$\Phi_1 f^* x^* = \max\limits_{y \leq x^*} f^* y \geq \max\limits_{y \leq x} fy = \Phi_1 fx$. The case $j = 2$ also is clear.
$j \geq 3$: By induction on x^* we show $\forall x^* \forall x \leq x^* (\Phi_j f^* x^* \geq_0 \Phi_j fx)$:
$x^* = 0$: $\Phi_j f^* 0 = f^* 0 \geq f0 = \Phi_j f0$.
$\Phi_j f^*(x^*+1) =$

$$A_{j-1}(f^*(x^*+1), \Phi_j f^* x^*) \begin{cases} \overset{!}{\geq} \Phi_j f0 \\ \overset{!!}{\geq} A_{j-1}(f(x+1), \Phi_j fx) = \Phi_j f(x+1) \text{ for } x+1 \leq x^*+1. \end{cases}.$$

Ad!: If $\Phi_j f^* x^* = 0$ then also $\Phi_j f0 = 0$ by induction hypothesis. If $\Phi_j f^* x^* \geq 1$ then the claim follows from 2.2.13 and $f^*(x^*+1) \geq f0 = \Phi_j f0$.
Ad!!: x^*–I.H. yields $\Phi_j f^* x^* \geq \Phi_j fx$. Because of f^* s–maj f it follows that $f^*(x^*+1) \geq f(x+1)$.
Case 1: $f(x+1) \geq 1$. Then '!' follows from 2.2.14 .
Case 2: $f(x+1) = 0$: Lemma 2.2.15 yields $A_{j-1}(f(x+1), \Phi_j fx) \leq 1$.
By lemma 2.2.13 and $f^* \geq 1$ we have $A_{j-1}(f^*(x^*+1), \Phi_j f^* x^*) \geq 1$, if $\Phi_j f^* x^* \geq 1$ (If $0 = \Phi_j f^* x^* \geq \Phi_j fx$, then $A_{j-1}(f(x+1), \Phi_j fx) \leq A_{j-1}(f^*(x^*+1), \Phi_j f^* x^*)$ follows immediately from the definition of A_{j-1}).

Lemma: 2.2.17 *For every $j \geq 1$ the following holds:*

$$\forall f \big(f \text{ monotone } \wedge f \geq 1 \rightarrow \forall x (A_j(fx, x+1) \geq_0 \Phi_j fx)\big).$$

Proof: The case $j = 1$ is trivial. Assume $j \geq 2$. We proceed by induction on x:

$$A_j(f0, 1) = A_{j-1}(f0, A_j(f0, 0)) = \begin{cases} f0 = \Phi_j f0 \text{ for } j = 2 \\ A_{j-1}(f0, 1) \overset{2.2.13}{\geq} f0 = \Phi_j f0 \text{ for } j > 2. \end{cases}$$

$$A_j(f(x+1), x+2) = A_{j-1}(f(x+1), A_j(f(x+1), x+1)) \overset{fx' \geq fx \geq 1}{\geq} A_{j-1}(f(x+1), A_j(fx, x+1))(2.2.14)$$
$$\overset{I.H., 2.2.14}{\geq} A_{j-1}(f(x+1), \Phi_j fx) = \Phi_j f(x+1).$$

Proposition: 2.2.18 *For all $j \geq 1$: $\lambda f, x. A_j(fx+1, x+1)$ s–maj Φ_j*[5] .

Proof: Assume f^* s–maj f and $x^* \geq_0 x$. By prop.2.2.16 we know $\Phi_j(f^*+1)x^* \geq_0 \Phi_j fx$.
L.2.2.11 6) yields that f^*+1 is monotone. Hence – by l.2.2.17 ,2.2.14 – $A_j(f^*(x^*)+1, x^*+1) \geq A_j(fx+1, x+1), \Phi_j(f^*+1)x^*$.

Lemma: 2.2.19 *If $A_j^*(x, y) := \max(A_j(x, y), 1)$. Then A_j^* s–maj A_j.*

Proof: For $j \leq 2$ the lemma is trivial. Assume $j \geq 3$: We have to show

$$\forall x, \tilde{x}, y, \tilde{y} \big(\tilde{x} \geq x \wedge \tilde{y} \geq y \rightarrow A_j^*(\tilde{x}, \tilde{y}) \geq A_j^*(x, y), A_j(x, y)\big) :$$

If $x \geq 1$ this follows from l.2.2.14.
Assume $x = 0$. By l.2.2.15 $\forall y (A_j^*(0, y), A_j(0, y) \leq 1)$ and therefore
$\forall \tilde{x}, \tilde{y}, y \big(A_j^*(\tilde{x}, \tilde{y}) \geq A_j^*(0, y), A_j(0, y)\big)$ (since $A_j^*(\tilde{x}, \tilde{y}) \geq 1$).

[5] For $j = 1$ the more simple functional $\lambda f, x. fx$ already majorizes Φ_1.

Definition 2.2.20 *1) The subset $G_nR^\omega_- \subset G_nR^\omega$ denotes the set of all terms which are built up from $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}, A_0, \ldots, A_n, 0^0, S, prd, \min_0$ and $\max_0$ only (i.e. without $\Phi_1, \ldots, \Phi_n, \tilde{R}_\rho$ or μ_b).*

2) $G_nR^\omega_-[\Phi_1]$ is the set of all term built up from $G_nR^\omega_-$ plus Φ_1.

Proposition: 2.2.21 *For all $n \geq 1$ the following holds: For each term $t^\rho \in G_nR^\omega$ one can construct by induction on the structure of t (without normalization) a term $t^{*\rho} \in G_nR^\omega_-$ such that*

$$G_nA^\omega_i \vdash t^* \text{ s–maj}_\rho\ t.$$

Proof: 1. Replace every occurrence of $\tilde{R}_\rho$ in t by G_ρ, where

$$G_\rho := \lambda x, y, z, v, \underline{w}. \max_0(y\underline{w}, v(prd(x), \underline{w})).$$

G_ρ is built up from Π, Σ (which are used for defining the λ–operator) and the monotone functions $\max_0$ and prd. One easily verifies that

$$(i)\ G_\rho \geq \tilde{R}_\rho \text{ and } (ii)\ G_\rho \text{ s–maj } G_\rho.$$

Together with l.2.2.11, (i) and (ii) imply G_ρ s–maj $\tilde{R}_\rho$.
2. Replace all occurrences of $\Phi_1, \ldots, \Phi_n, \mu_b$ in t by

$$\Phi^*_1 := \lambda f^1, x^0.fx,\ \Phi^*_j := \lambda f^1, x^0.A_j(fx+1, x+1) \text{ for } j \geq 2,\ \mu^*_b := \lambda f^{1(0)}, x^0.x.$$

By prop. 2.2.18 we conclude

$$\mathrm{G}_n\mathrm{A}^\omega_i \vdash \Phi^*_j \text{ s–maj } \Phi_j \wedge \mu^*_b \text{ s–maj } \mu_b.$$

3. Replace all occurrences of $A_0, \ldots, A_n$ in t by $A^*_0, \ldots, A^*_n$ from 2.2.19.
4. The constants $\Pi, \Sigma, S, prd, \min_0, \max_0$ majorize themselves and therfore need not to be replaced. The term t^* which results after having carried out 1.–3. is $\in \mathrm{G}_n\mathrm{R}^\omega_-$. t^* is constructed by replacing every constant c in t by a closed term s^*_c such that s^*_c s–maj c. Since t is built up from constants only this implies using lemma 2.2.11.1),5) that t^* s–maj t.

Corollary to the proof:
Since $\lambda x^0.x^0$ s–maj$_1$ prd and A_1 s–maj $\max_0, \min_0$, the term t^* can be constructed even without $prd, \max_0$ and $\min_0$ (One now uses $G_\rho := \lambda x, y, z, v, \underline{w}.(y\underline{w} + vx\underline{w})$ and $A^*_j(x, y) := A(x+1, y) + 1$ as majorants for $\tilde{R}_\rho$ and A_j. A^*_j s–maj A_j follows analogously to the proof of 2.2.19). However estimating $\max_0$ by A_1 may give away interesting numerical information. For the extraction of bounds from actually given proofs we may use not only max or min but any further functions which are convenient for the construction of a majorant which is numerically as sharp as possible.

The majorizing term t^* constructed in prop.2.2.21 will have (in general) a much simpler form than t since t^* does not contain any higher mathematical functional but only the 'logical' functionals Π and Σ. In the following we show that if t^* has a type ρ with $\deg(\rho) \leq 2$, than it can be simplified further by eliminating even these logical functionals. This will allow the exact calibration of the rate of growth of the definable functions of $\mathrm{G}_n\mathrm{A}^\omega$ and will be crucial also for our elimination of monotone Skolem functions in chapters 10 and 11 below.

Proposition: 2.2.22 *Assume $n \geq 1$, deg(ρ) ≤ 2 (i.e. $\rho = 0\rho_k \dots \rho_1$ where deg(ρ_i) ≤ 1 for $i = 1, \dots, k$) and $t^\rho \in G_nR^\omega_-$. Then one can construct (by 'logical' normalization, i.e. by carrying out all possible Π, Σ–reductions) a term $\widehat{t}[x_1^{\rho_1}, \dots, x_k^{\rho_k}]$ such that*

1) *$\widehat{t}[x_1, \dots, x_k]$ contains at most $x_1 \dots, x_k$ as free variables,*

2) *$\widehat{t}[x_1, \dots, x_k]$ is built up only from $x_1, \dots, x_k, A_0, \dots, A_n, S^1, 0^0, prd, \min_0, \max_0$,*

3) *$G_nA_i^\omega \vdash \forall x_1^{\rho_1}, \dots, x_k^{\rho_k}(\widehat{t}[x_1, \dots, x_k] =_0 tx_1 \dots x_k)$.*

Proof: We carry out reductions $\Pi st \rightsquigarrow s$ and $\Sigma str \rightsquigarrow sr(tr)$ in $tx_1 \dots x_k$ as long as no further such reduction is possible and denote the resulting term by $\widehat{t}[x_1, \dots, x_k]$. The well–known strong normalization theorem for typed combinatory logic ensures that this situation will always occur after a finite number of reduction steps. Since $\Pi xy = x$ and $\Sigma xyz = xz(yz)$ are axioms of $\mathrm{G}_n\mathrm{A}_i^\omega$ the quantifier–free rule of extensionality yields

$$\mathrm{G}_n\mathrm{A}_i^\omega \vdash \forall x_1^{\rho_1}, \dots, x_k^{\rho_k}(\widehat{t}[x_1, \dots, x_k] =_0 tx_1 \dots x_k).$$

It remains to show that $\widehat{t}[x_1, \dots, x_k]$ does not contain the combinators Π, Σ anymore:
Assume that $\widehat{t}[x_1, \dots, x_k]$ contains an occurrence of Σ (resp. Π). Then Σ (Π) must occur in the form $\Sigma, \Sigma t_1$ or $\Sigma t_1 t_2$ ($\Pi, \Pi t_1$) but not in the form $\Sigma t_1 t_2 t_3$ (resp. $\Pi t_1 t_2$) since in the later case we could have carried out the reduction $\Sigma t_1 t_2 t_3 \rightsquigarrow t_1 t_3(t_2 t_3)$ (resp. $\Pi t_1 t_2 \rightsquigarrow t_1$) contradicting the construction of $\widehat{t}$. All the terms $s = \Sigma, \Sigma t_1, \Sigma t_1 t_2, \Pi, \Pi t_1$ have a type whose degree is ≥ 1. Hence s can occur in $\widehat{t}$ only in the form $r(s)$, where $r = \Sigma, \Sigma t_4, \Sigma t_4 t_5, \Pi$ or Πt_4 since these terms are the only reduced ones requiring an argument of type ≥ 1, which can be built up from $x_1^{\rho_1}, \dots, x_k^{\rho_k}, \Sigma, \Pi, A_i, S^1, 0^0$ and $\max_0$ (because of deg(ρ_i) ≤ 1). Now the cases $r = \Sigma t_4 t_5$ and $r = \Pi t_4$ can not occur since otherwise r(s) would allow a reduction of Σ resp. Π. Hence $r(s)$ is again a Π, Σ–term having a type of degree ≥ 1 and therefore has to occur within a term r' for which the same reasoning as for r applies etc. .... Thus we obtain a contradiction to the finite structure of $\widehat{t}$.

Remark 2.2.23 *Proposition2.2.22 becomes false if deg(ρ) $= 3$: Define $\rho := 0(0(000))$ and $t^\rho := \lambda x^{0(000)}.x(\Pi_{0,0})$. Then $tx =_0 x(\Pi_{0,0})$ contains Π but no Π–reduction applies.*

Corollary 2.2.24 *Assume $n \geq 1$, deg(ρ) ≤ 2 (i.e. $\rho = 0\rho_k \dots \rho_1$ where deg(ρ_i) ≤ 1 for $i = 1, \dots, k$) and $t^\rho \in G_nR^\omega$. Then one can construct (by majorization and subsequent 'logical' normalization) a term $t^*[x_1^{\rho_1}, \dots, x_k^{\rho_k}]$ such that*

1) *$t^*[x_1, \dots, x_k]$ contains at most $x_1 \dots, x_k$ as free variables,*

2) *$t^*[x_1, \dots, x_k]$ is built up only from $x_1, \dots, x_k, A_0, \dots, A_n, S^1, 0^0, prd, \min_0, \max_0$,*

3) *$G_nA_i^\omega \vdash \lambda x_1, \dots, x_k.t^*[x_1, \dots, x_k]$ s–maj t.*

   In particular: $\forall x_1^*, x_1, \dots, x_k^*, x_k\big(\bigwedge_{i=1}^{k}(x_i^* \text{ s–maj}_{\rho_i} x_i \rightarrow t^*[x_1^*, \dots, x_k^*] \geq_0 tx_1 \dots x_k\big)$.

Proof: The corollary follows immediately from prop.2.2.21 and prop.2.2.22 (using lemma 2.2.11 (1)).

Remark 2.2.25 *As before, 2) can be strengthened in that $t^*[x_1, \dots, x_k]$ is built up only from $x_1, \dots, x_k, A_0, \dots, A_n, 0^0$.*

The use of the concept of majorization combined with logical normalization has enabled us to majorize a term t of type ≤ 2 by a term t^* which does not contain any functionals of type > 1. This allows the calibration of the rate of growth of the functions given by $t^1 \in \mathrm{G}_n\mathrm{R}^\omega$ in usual mathematical terms **without any computation of recursor terms** (which would require the reduction of closed number terms to numerals):

Definition 2.2.26 ([9] ,[28]) *The function $f(\underline{x},y)$ is defined from $g(\underline{x}), h(\underline{x},y,z)$ and $b(\underline{x},y)$ by* **limited recursion** *if*

$$\begin{cases} f(\underline{x},0) =_0 g(\underline{x}) \\ f(\underline{x},y+1) =_0 h(\underline{x},y,f(\underline{x},y)) \\ f(\underline{x},y) \leq_0 b(\underline{x},y). \end{cases}$$

Definition 2.2.27 (n-th level of the Grzegorczyk hierarchy) *For each $n \geq 0$, $\mathcal{E}^n$ is defined to be the smallest class of functions containing the successor function S, the constant–zero function, the projections $U_i^n(x_1,\ldots,x_n) = x_i$, and $A_n(x,y)$ which is closed under substitutions and limited recursion.*

Remark 2.2.28 *Grzegorczyk's original definition of $\mathcal{E}^n$ uses somewhat different functions $g_n(x,y)$ instead of $A_n(x,y)$. Ritchie ([28]) showed that the same class of $\mathcal{E}^n$ of functions results if the g_n are replaced by the (more natural) A_n (which are denoted by f_n in [28]). See also [5] for a proof of this result.*

Proposition: 2.2.29 *Assume $n \geq 1$ and $t^1 \in G_nR^\omega$. Then one can construct a function $f_t \in \mathcal{E}^n$ such that $\forall x^0(tx \leq_0 f_tx)$ and every function $f \in \mathcal{E}^n$ can be defined in G_nR^ω, i.e. there is a term $t_f^1 \in G_nR^\omega$ such that $\forall x^0(fx = tx)$.*
In particular for $n = 1,2,3$ the following holds:

$$\begin{cases} t^1 \in G_1R^\omega \Rightarrow \exists c_1, c_2 \in \mathbb{N}: \; G_1A_i^\omega \vdash \forall x^0(tx \leq_0 c_1x + c_2) \text{ (linear growth)}, \\ t^1 \in G_2R^\omega \Rightarrow \exists k, c_1, c_2 \in \mathbb{N}: \; G_2A_i^\omega \vdash \forall x^0(tx \leq_0 c_1x^k + c_2) \text{ (polynomial groth)}, \\ t^1 \in G_3R^\omega \Rightarrow \exists k, c \in \mathbb{N}: \; G_3A_i^\omega \vdash \forall x^0(tx \leq_0 2_k^{cx}), \text{ where } 2_0^a = a, 2_{k'}^a = 2^{2_k^a} \\ \qquad \text{(finitely iterated exponential growth)}. \end{cases}$$

More generally, if t^ρ (where $\rho = 0\underbrace{(0)\ldots(0)}_{m-times}$), defines an m–ary function:

$$\begin{cases} t^\rho \in G_1R^\omega \Rightarrow \exists c_1,\ldots,c_{m+1} \in \mathbb{N}: \; G_1A_i^\omega \vdash \forall x_1^0,\ldots,x_m^0(t\underline{x} \leq_0 c_1x_1 + \ldots + c_mx_m + c_{m+1}), \\ t^\rho \in G_2R^\omega \Rightarrow \exists p \in \mathbb{N}[x_1,\ldots,x_m]: \; G_2A_i^\omega \vdash \forall \underline{x}(t\underline{x} \leq_0 p\underline{x}), \\ t^\rho \in G_3R^\omega \Rightarrow \exists k, c_1,\ldots,x_m \in \mathbb{N}: \; G_3A_i^\omega \vdash \forall \underline{x}(t\underline{x} \leq_0 2_k^{c_1x_1+\ldots+c_mx_m}). \end{cases}$$

The constants $c_i, k \in \mathbb{N}$ and the polynomial $p \in \mathbb{N}[x_1,\ldots,x_m]$ can be effectively written down for each given term t.

Proof: To t^1 we construct $\widehat{t}[x]$ (according to cor.2.2.24 and the corollary to the proof of 2.2.21) such that $\widehat{t}[x]$ is built up from $x^0, 0^0$ and $A_0,\ldots,A_n$, and $\lambda x.\widehat{t}[x]$ s–maj$_1$ t. The later property implies $\forall x^0(\widehat{t}[x] \geq_0 tx)$. By [28] (p. 1037) we know that $A_0,\ldots,A_n \in \mathcal{E}^n$. Since $\mathcal{E}^n$ is closed under

substitution it follows that $f_t := \lambda x.\widehat{t}[x] \in \mathcal{E}^n$.
For the other direction assume $f \in \mathcal{E}^n$. Since $\mathrm{G}_n\mathrm{R}^\omega$ contains $S, \lambda x.0^0$, the projections U_i^k and A_n, and it is closed under substitution (because λ–abstraction is available) and limited recursion (because of $\tilde{R}$) it follows that f is definable in $\mathrm{G}_n\mathrm{R}^\omega$.
We now consider the special cases $n = 1, 2, 3$:
$n = 1$: Assume $t^\rho \in \mathrm{G}_1\mathrm{R}^\omega$ where $\rho = 0\underbrace{(0)\ldots(0)}_{m}$. $\widehat{t}[x_1^0, \ldots, x_m^0]$ is built up from $x_1^0, \ldots, x_m^0, 0^0, A_0$ and A_1 only. Both $A_0(x_1, x_2) = 0 \cdot x_1 + 1 \cdot x_2 + 1$ and $A_1(x_1, x_2) = 1 \cdot x_1 + 1 \cdot x_2 + 0$ are functions having the form $c_1x_1 + c_2x_2 + c_3$ or – more generally – $c_1x_1 + \ldots + c_kx_k + c_{k+1}$. Since substitution of such functions again yields a function which can be written in this form it follows that $\widehat{t}[x_1, \ldots, x_m] = c_1x_1 + \ldots + c_mx_m + c_{m+1}$ for suitable constants $c_1, \ldots, c_{m+1}$.
$n = 2$: Assume $t^\rho \in \mathrm{G}_2\mathrm{R}^\omega$. $\widehat{t}[x_1^0, \ldots, x_m^0]$ is built up from $x_1^0, \ldots, x_m^0, 0^0, A_0, A_1, A_2$. Since A_0, A_1 and A_2 are polynomials (in two variables) and substitution of polynomials in several variables yields a function which can be written again as a polynomial, it is clear that $\widehat{t}[x_1, \ldots, x_m] = p(x_1, \ldots, x_m)$ for a suitable polynomial in $\mathbb{N}[x_1, \ldots, x_m]$. In the case $m = 1$, $p(x)$ can be bounded by $c_1x^k + c_2$ for suitable numbers c_1, c_2.
$n = 3$: Assume $t^\rho \in \mathrm{G}_3\mathrm{R}^\omega$. For $\tilde{A}_3(x, y) := A_3(\max_0(x, 2), \max_0(y, 2))$ the following holds:
$(*)$ $\tilde{A}_3$ s–maj A_0, A_1, A_2, A_3. Replace in $\widehat{t}[x_1, \ldots, x_m]$ all occurrences of A_i with $i \leq 3$ by $\tilde{A}_3$ and denote the resulting term by $\tilde{t}[x_1, \ldots, x_m]$. $(*)$ yields

$$\forall x_1, \ldots, x_m \big(\tilde{t}[x_1, \ldots, x_m] \geq \widehat{t}[x_1, \ldots, x_m] \geq tx_1 \ldots x_m\big).$$

Let k be the number of $\tilde{A}_3$–occurrences in $\tilde{t}[x_1, \ldots, x_m]$. Then $\tilde{t}[x_1, \ldots, x_m]$ can be bounded by y_k, where $y_0 := 0$, $y_{k'} := y^{y_k}$ and $y := \max(x_1, \ldots, x_m, 2)$ and hence $\forall \underline{x}\big(2^{\underline{x}}_{\tilde{k}} \geq t\underline{x}\big)$ for a suitable $\tilde{k} \geq k$, where $2^{\underline{x}}_0 := x_1 + \ldots + x_m$ and $2^{\underline{x}}_{k'} = 2^{2^{\underline{x}}_k}$.

Remark 2.2.30 *This proposition provides a quite perspicuous characterization of the rate of growth of the functions which are definable in G_nA^ω. Of course for concrete terms t the bounds given for $n = 1, 2, 3$ may be to rough. To obtain better estimates one will use combinations of any convenient functions like e.g.* $\max, \min$ *(instead of replacing them by $x + y$) and (for $n = 3$) the growth of t will be expressed using* $\max, \min, A_0, A_1, A_2$ *and* A_3 *and not* A_3 *allone. Thus one can treat also all intermediate levels between e.g. polynomial and iterated exponential growth.*

By cor.2.2.24 and the remark on it, the estimates for $n = 1, 2, 3$ generalize to function parameters as follows: Let $t^{1(1)} \in \mathrm{G}_n\mathrm{R}^\omega$, then tf^1 can be bounded by a linear (polynomial resp. elementary recursive) function **in** f^* where f^* s–maj f (for f^* we may take e.g. f^M). By **'$\mathbf{tf^1x^0}$ is linear (polynomial, elementary recursive) in $\mathbf{f, x}$'** we mean that $tfx =_0 \tilde{t}[x, f]$ for all x, f, where $\tilde{t}[x, f]$ is a term which is built up only from $x, f, 0^0, S^1, +$ ($x, f, 0^0, S^1, +, \cdot$ resp. $x, f, 0^0, S^1, +, \cdot, (\cdot)^{(\cdot)}$).[6] In particular this implies that if f^* is a linear (polynomial, elementary recursive) function then tf^* can be written again as a linear (polynomial, elementary recursive) function. This holds even uniformly in the following sense (which we formulate here explicitly only for the most interesting polynomial case):

[6] In our results $\tilde{t}[x, f]$ can always be constructed by majorization and 'logical' normalization.

Proposition: 2.2.31 *Let $t^{1(1)} \in G_2R^\omega$. Then one can construct a polynomial $q \in \mathbb{N}[x]$ such that*

$$\begin{cases} \textit{For every polynomial } p \in \mathbb{N}[x] \\ \textit{one can construct a polynomial } r \in \mathbb{N}[x] \textit{ such that} \\ \forall f^1\big(f \le_1 p \to \forall x^0(tfx \le_0 r(x))\big) \textit{ and } deg(r) \le q(deg(p)) \end{cases}$$

This extends to the case where t has tuples $f_1^1, \ldots, f_k^1, x_1^0, \ldots, x_l^0$ of arguments with $f_1, \ldots, f_k \le_1 p$ and $r \in \mathbb{N}[x_1, \ldots, x_l]$.

Proof: Let $p \in \mathbb{N}[x]$. Since p is monotone, $f \le p$ implies p s–maj f. By the corollary to the proof of prop.2.2.21 one can construct a term $t^* \in \mathrm{G_2R}^\omega_-$ (without $prd, \min_0, \max_0$) such that t^* s–maj t. Let $\widehat{t}[f,x]$ be constructed to t^*fx according to prop.2.2.22. Then $\widehat{t}[p,x] \ge_0 tfx$ for all $f \le_1 p$ and $\widehat{t}[p,x]$ is built up from $x, 0^0, A_0, A_1$ and p only. As in the proof of prop.2.2.29 one concludes that $\widehat{t}[p,x]$ can be written as a polynomial r in x. The existence of the polynomial q bounding the degree of r in the degree of p follows from the fact that the degree of a polynomial $p_1 \in \mathbb{N}[x_1, \ldots, x_m]$ obtained by substitution of a polynomial p_2 for one variable in a polynomial p_3 is $\le \deg(p_2) \cdot \deg(p_3)$ and that $\deg(p_2 + p_3)$, $\deg(p_2 \cdot p_3) \le \deg(p_2) + \deg(p_3)$.

2.3 Extensions of $\mathbf{G}_n\mathbf{A}^\omega$

Definition 2.3.1 *1) Let $G_\infty A^\omega$ denote the union of the theories G_nA^ω for all $n \ge 1$ and $G_\infty A_i^\omega$ its intuitionistic variant.*
E–$G_\infty A^\omega$ and E–$G_\infty A_i^\omega$ are the corresponding theories with full extensionality.
$G_\infty R^\omega$ is the set of all closed terms of these theories, i.e. $G_\infty R^\omega := \bigcup\limits_{n \in \mathbb{N}} G_nR^\omega$.

2) PRA^ω is the theory obtained from $G_\infty A^\omega$ by adding the Kleene–recursor operators $\widehat{R}_\rho$ (on which S. Feferman's theory $\widehat{PA}^\omega\!\restriction$ is based on; see [4]):

$$\begin{cases} \widehat{R}_\rho 0yz\underline{v} =_0 y\underline{v} \\ \widehat{R}_\rho(Sx)yz\underline{v} =_0 z(\widehat{R}_\rho xyz\underline{v})x\underline{v}, \end{cases}$$

where $y \in \rho, z \in \rho00$ and $\underline{v} = v_1^{\rho_1} \ldots v_k^{\rho_k}$ are such that $y\underline{v}$ is of type 0.
Correspondingly we have the theories PRA_i^ω, E–PRA^ω and E–PRA_i^ω.
The set of all closed terms of PRA^ω is denoted by $\widehat{PR}^\omega$.

Thus PRA^ω is equivalent to $\widehat{PA}^\omega\!\restriction$+all true $\forall x^\rho A_0$–sentences for $\rho \le 2$. We now show that the same theory results if we only add the (unrestricted) iteration functional Φ_{it} together with the axioms

$$\begin{cases} \Phi_{it}0yf =_0 y \\ \Phi_{it}x'yf =_0 f(\Phi_{it}xyf) \quad i.e. \Phi_{it}xyf = f^xy \end{cases}$$

instead of the constants $\widehat{R}_\rho$:
We define $\widehat{R}_\rho$ through one intermediate step:

Firstly we show that $\widehat{R}_\rho$ can be defined from $\tilde{\Phi}$ $(= \widehat{R}_0)$, where

$$\begin{cases} \tilde{\Phi}0yf =_0 y \\ \tilde{\Phi}x'yf =_0 f(\tilde{\Phi}xyf)x \quad (f \in 0(0)(0)). \end{cases}$$

One easily verifies that $\widehat{R}_\rho$ can be defined as $\widehat{R}_\rho := \lambda x, y, z, \underline{v}.\tilde{\Phi}x(y\underline{v})(\lambda x_1^0, x_2^0.zx_1x_2\underline{v})$.
$\tilde{\Phi}$ in turn is definable using Φ_{it}: This follows from the fact that for $\tilde{f}x := \max(\Phi_1(\lambda y_1.\Phi_1(\lambda y_2.fy_1y_2)x)x, x') (= \max\limits_{y_1,y_2 \leq_0 x}(fy_1y_2, x'))$ one has $\Phi_{it}xy\tilde{f} \geq_0 \tilde{\Phi}xyf$ for all x, y, f.
Thus using Φ_{it} as a bound in the recursion one can define $\tilde{\Phi}$ by the bounded recursor operator $\tilde{R}$.
Put together we have shown that $\widehat{R}_\rho$ is definable in PRA^ω. Since on the other hand Φ_{it} is trivially definable using $\widehat{R}$ our claim follows.
On the level of type 1 the theories PRA^ω and $\mathrm{G}_\infty\mathrm{A}^\omega$ coincide: The functions given by the closed terms of type level 1 of both theories are just the primitive recursive ones: For PRA^ω this follows from [4]. Since $\mathrm{G}_\infty\mathrm{A}^\omega$ is a subtheory of PRA^ω it suffices to verify that all primitive recursive functions are definable in it. This however follows immediately from prop.2.2.29 and the well–know fact (due to Grzegorczyk) that the class of all primitive recursive functions is just the union of all $\mathcal{E}^n$.
In contrast to this, both theories differ already on the type–2–level:

Proposition: 2.3.2 *The functional Φ_{it} is not definable in $G_\infty A^\omega$, i.e. there is no term $t \in G_\infty R^\omega$ such that t satifies the defining equations of Φ_{it}.*

Proof: Assume that Φ_{it} is definable in $\mathrm{G}_\infty\mathrm{A}^\omega$. Then there exists an n such that Φ_{it} is already definable in $\mathrm{G}_n\mathrm{A}^\omega$. On the hand from the proof above we know that within $\mathrm{G}_n\mathrm{A}^\omega + \Phi_{it}$ the unbounded recursors $\widehat{R}_\rho$ and therefore all primitive recursive functions (in particular A_{n+1}) are definable. Hence A_{n+1} could be defined in $\mathrm{G}_n\mathrm{A}^\omega$ contradicting prop.2.2.29, since A_{n+1} cannot be bounded by a function from $\mathcal{E}^n$ (see [28]).
Finally we introduce the theory PA^ω which results from PRA^ω if

1) $\widehat{R}_\rho$ is replaced by the Gödel–recursor operators R_ρ characterized by

$$\begin{cases} R_\rho 0yz =_\rho y \\ R_\rho x'yz =_\rho z(R_\rho xyz)x, \quad \text{where } y \in \rho, z \in \rho 0 \rho, \end{cases}$$

2) the schema of full induction

$$(\mathrm{IA}) \; : \; A(0) \wedge \forall x(A(x) \to A(x')) \to \forall x A(x)$$

for arbitrary formulas $A \in \mathcal{L}(\mathrm{PA}^\omega)$ is added.
The set of all closed terms of PA^ω is denoted by T (following Gödel).
PA^ω_i is the intuitionistic variant of PA^ω. E–PA^ω, E–PA^ω_i are the corresponding theories with full extensionality (E).

$\mathrm{G}_2\mathrm{A}^\omega, \ldots, \mathrm{PRA}^\omega$ of subsystems of arithmetic in all finite types PA^ω. Furthermore we have determined the growth of the functionals $t^{1(1)}$ which are definable in these theories. In particular for

$n \leq 3$ it turned out that t can be majorized by a term t^* of type $1(1)$ such that

$t^* f^1 x^0$ is a linear function in f, x, if $n = 1$,

$t^* f^1 x^0$ is a polynomial function in f, x, if $n = 2$,

$t^* f^1 x^0$ is an elementary recursive function in f, x, if $n = 3$,

and in the case $n = 2$, for every polynomial p^1 there is a polynomial r^1 such that $t^* fx \leq_0 rx$ for all $f \leq_1 p$.

3 Monotone functional interpretation of $\mathrm{G}_n\mathrm{A}^\omega$, PRA^ω ,PA^ω and their extensions by analytical axioms: the rate of growth of provable function(al)s

3.1 Gödel functional interpretation

Definition 3.1.1 *The schema of the quantifier–free axiom of choice is given by*

$$AC^{\rho,\tau}\text{–}qf \ : \ \forall x^\rho \exists y^\tau A_0(x,y) \to \exists Y^{\tau\rho} \forall x^\rho A_0(x, Yx),$$

where A_0 is a quantifier–free formula of the respective theory.

$$AC\text{–}qf := \bigcup_{\rho,\tau \in \mathbf{T}} \{AC^{\rho,\tau}\text{–}qf\}.$$

If

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \forall x^\rho \exists y^\tau A_0(x,y),$$

then

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{\rho,\tau}\text{–qf} \ \vdash \exists Y^{\tau\rho} \forall x^\rho A_0(x, Yx).$$

In order to determine the growth which is implicit in the functional dependency '$\forall x^\rho \exists y^\tau$' we have to determine the rate of growth of a functional term which realizes (or bounds) '$\exists Y^{\tau\rho}$'. Let A' denote one of the well–known negative translations of A (see [25] for a systematical treatment) and A^D be the Gödel functional interpretation of A (as defined in [25] or [34]).
A^D has the logical form

$$\exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y}, \underline{a}),$$

where A_D is quantifier–free, $\underline{x}, \underline{y}$ are tuples of variables of finite type and $\underline{a}$ is the tuple of all free variables of A. For our theories this functional interpretation holds:

Theorem 3.1.2 *Let Γ be a set of purely universal sentences $H \equiv \forall u^\gamma H_0(u) \in \mathcal{L}(G_nA^\omega)$ and $n \in \mathbb{N} \cup \{\infty\}$ $(n \geq 1)$. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Gamma + AC\text{–}qf \ \vdash A \Rightarrow \exists \underline{t} \in G_nR^\omega \ \textit{such that} \\ G_nA_i^\omega + \Gamma \vdash \forall \underline{y} \left((A')_D(\underline{ta}, \underline{y}, \underline{a}) \right). \end{cases}$$

$\underline{t}$ can be extracted from a given proof
(An analogous result holds if $G_nA^\omega, G_nR^\omega, G_nA_i^\omega$ are replaced by PRA^ω, $\widehat{PR}^\omega$, PRA_i^ω or PA^ω, T, PA_i^ω).

Proof: For PA^ω the proof is given e.g. in [34]. The interpretation of the logical axioms and rules only requires the closure under λ–abstraction, definition by cases and the existence of characteristic functionals for the prime formulas. All this holds in $\mathrm{G}_n\mathrm{R}^\omega$ and $\widehat{PR}^\omega$. The interpretation of the universal axioms is trivial.

Corollary 3.1.3 *Let Γ be as above and $A_0(\underline{x},\underline{y})$ is a quantifier–free formula which has only $\underline{x},\underline{y}$ as free variables. Then*

$$\begin{cases} G_nA^\omega + \Gamma + AC\text{–}qf \vdash \forall \underline{x}\exists \underline{y} A_0(\underline{x},\underline{y}) \Rightarrow \exists \underline{t} \in \; G_nR^\omega \textit{ such that}: \\ G_nA_i^\omega + \Gamma \vdash \forall \underline{x} A_0(\underline{x}, \underline{tx}) \end{cases}$$

(Analogously for PRA^ω and PA^ω).

By the well–known elimination procedure for the extensionality axiom (E) one may replace $\mathrm{G}_n\mathrm{A}^\omega$ by E–$\mathrm{G}_n\mathrm{A}^\omega$ if the types of $\underline{x}$ are ≤ 1 and the types in AC–qf are somewhat restricted:

Corollary 3.1.4 *Assume that $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$, and $\underline{x} = x_1^{\rho_1},\ldots,x_k^{\rho_k}$ where $\rho_i \leq 1$ for $i = 1,\ldots,k$. Then*

$$\begin{cases} E\text{–}G_nA^\omega + \Gamma + AC^{\alpha,\beta}\text{–}qf \vdash \forall \underline{x}\exists \underline{y} A_0(\underline{x},\underline{y}) \Rightarrow \exists \underline{t} \in G_nR^\omega \textit{ such that}: \\ G_nA_i^\omega + \Gamma \vdash \forall \underline{x} A_0(\underline{x}, \underline{tx}) \end{cases}$$

(Analogously for E–PRA^ω and E–PA^ω).

Proof: The corollary follows from the previous corollary using the elimination of extensionality procedure as carried out in [25] and observing the following facts:

1) The hereditary extensionality of $\tilde{R}_\rho$ (i.e. $\mathrm{Ex}(\tilde{R})$ in the notation of [25]) can be proved by (QF–IA). Similarly for Φ_i. The heriditary extensionality of μ_b follows easily from the axioms μ_b.

2) $(\mathrm{AC}^{1,0}\text{–qf})_e$ is provable by bounded search using μ_b and prop. 2.2.6 .

3) For $H \in \Gamma$ the implication $H \to H_e$ holds logically.

3.2 Monotone functional interpretation

In [21] we introduced a new **monotone functional interpretation** which extracts instead of a realizing term t for $\exists y$ in cor.3.1.3 a 'bound' t^* for t (in the sense of s–maj, which for types ≤ 2 provides a $\geq$–bound by lemma 2.2.11.7). This is sufficient in order to estimate the rate of growth of t. The construction of t^* does not cause any rate of growth in addition to that actually involved in a given proof since besides the terms from the proof only the functionals $\max_\rho$[7] and Φ_1 are used (For the theories $\mathrm{G}_n\mathrm{A}^\omega$ even Φ_1 is not necessary for the construction of t^* but only for the very simple transformation of t^* into a $\geq$–bound for type ≤ 2 by lemma 2.2.11). This has been confirmed in applications to concrete proofs in approximation theory where t^* could be used to improve known estimates significantly (see [19] ,[20] ,[21]). In most applications in analysis the formula $\forall x \exists y A(x,y) \quad (A \in \Sigma^0_1)$ will be monotone w.r.t. y, i.e.

$$\forall x, y_1, y_2 \big(y_2 \geq y_1 \wedge A(x,y_1) \to A(x,y_2)\big),$$

[7] $\max_{\tau\rho}(x_1^{\tau\rho}, x_2^{\tau\rho}) := \lambda y^\rho . \max_\tau(x_1 y, x_2 y)$.

and thus the bound t^* in fact also realizes '$\exists y$' (This phenomenon is discussed in [21]).
The monotone functional interpretation has various properties which are important for the following but do not hold for the usual functional interpretation:

1) The extraction of t^* by monotone functional interpretation from a given proof is much easier than the extraction of t provided by the usual functional interpretation: E.g. no decision of prime formulas and no functionals defined by cases are needed for the construction of t^* (but only for its verification) since the logical axioms $A \to A \wedge A$ and $A \vee A \to A$ have a simple **monotone** functional interpretation (whereas these axioms are the difficult ones for the usual functional interpretation). Because of this also the structure of the term t^* is more simple than that of t, in particular $t^* \in \mathrm{G}_n\mathrm{R}^\omega_{-}$ whereas $t \in \mathrm{G}_n\mathrm{R}^\omega$.

2) The bound t^* obtained by monotone functional interpretation for $\exists z^\tau$ in sentences $\forall x^1 \forall y \leq_\rho sx \exists z^\tau A_0(x,y,z)$ **does not depend on** y, i.e. $\forall x^1 \forall y \leq_\rho sx \exists z \leq_\tau t^* x\ A_0(x,y,z)$ (Here $\tau \leq 2$ and s is a closed term).
The **most important property of our monotone functional interpretation** however is the following

3) Sentences of the form

$$(*)\ \forall x^\gamma \exists y \leq_\delta sx \forall z^\eta A_0(x,y,z)$$

have a simple monotone functional interpretation which is fulfilled by any term s^* such that s^* s–maj s (see [21]). This means that sentences $(*)$ although covering many strong non–constructive analytical theorems which usually do not have a functional interpretation in the usual sense not even in T (as we will see in 4 below) do not contribute to the growth of the bound t^* by their proofs but only by the term s and therefore can be treated simply as axioms.

Definition 3.2.1 (bounded choice) *The schema of 'bounded' choice is defined as*

$$(b\text{–}AC^{\delta,\rho})\ :\ \forall Z^{\rho\delta}\big(\forall x^\delta \exists y \leq_\rho Zx\ A(x,y,Z) \to \exists Y \leq_{\rho\delta} Z \forall x A(x,Yx,Z)\big),$$

$$b\text{–}AC\ :=\bigcup_{\delta,\rho\in\mathbf{T}}\Big\{(b\text{–}AC^{\delta,\rho})\Big\}.$$

(a discussion of this principle can be found in [16]).

Theorem 3.2.2 *Let $n \geq 1$ and Δ be a set of sentences having the form $\forall u^\gamma \exists v \leq_\delta tu \forall w^\eta H_0(u,v,w)$, where $t \in G_nR^\omega$. Then the following rule holds*

$$\begin{cases} \textit{From a proof } G_nA^\omega + \Delta + AC\text{–}qf \vdash A \\ \textit{one can extract by neg. transl. and monotone functional interpretation a tuple } \underline{\Psi} \in G_nR^\omega_{-}\textit{:} \\ G_nA^\omega_i + \Delta + b\text{-}AC \vdash (\underline{\Psi} \textit{ satisfies the monotone functional interpretation of } (A)'), \end{cases}$$

where $(A)'$ denotes the negative translation of A.
In particular for $A_0(x,y,z)$ containing only x,y,z free and $s \in G_nR^\omega$ the following rule holds for $\tau \leq 2$:

$$\begin{cases} \textit{From a proof } G_nA^\omega + \Delta + AC\text{–}qf \vdash \forall x^1 \forall y \leq_\rho sx \exists z^\tau A_0(x,y,z) \\ \textit{by monotone functional interpretation one can extract a } \Psi \in G_nR^\omega_{-}[\Phi_1] \textit{ such that} \\ G_nA^\omega_i + \Delta + b\text{–}AC \vdash \forall x^1 \forall y \leq_\rho sx \exists z \leq_\tau \Psi x\ A_0(x,y,z). \end{cases}$$

Ψ is built up from $0^0, 1^0, \max_\rho, \Phi_1$ and majorizing terms[8] *(for terms t occurring in those quantifier axioms $\forall x Gx \to Gt$ and $Gt \to \exists x Gx$ which are used in the given proof) by use of λ–abstraction and substitution. If $\tau \leq 1$ (resp. $\tau = 2$) then Ψ has the form $\Psi \equiv \lambda x^1.\Psi_0 x^M$ (resp. $\Psi \equiv \lambda x^1, y^1.\Psi_0 x^M y^M$), where $x^M := \Phi_1 x$ and Ψ_0 does not contain Φ_1 (An analogous result holds for PRA^ω,$\widehat{PR}^\omega$, PRA_i^ω resp. PA^ω,T,PA_i^ω).*

Corollary 3.2.3 *For $1 \leq n \leq 3$ the following holds (for $A_0(x^0, y^\rho, z^0)$ containing only x, y, z free)*

$$G_nA^\omega + \Delta + AC\text{–}qf \vdash \forall x^0 \forall y \leq_\rho sx \exists z^0 A_0(x,y,z) \Rightarrow$$

$$\begin{cases} \exists c_1, c_2 \in \mathbb{N} : G_1A_i^\omega + \Delta + b\text{–}AC \vdash \forall x^0 \forall y \leq_\rho sx \exists z \leq_0 c_1 x + c_2 \ A_0(x,y,z), \ \text{if } n = 1 \\ \exists k, c_1, c_2 \in \mathbb{N} : G_2A_i^\omega + \Delta + b\text{–}AC \vdash \forall x^0 \forall y \leq_\rho sx \exists z \leq_0 c_1 x^k + c_2 \ A_0(x,y,z), \ \text{if } n = 2 \\ \exists k, c \in \mathbb{N} : G_3A_i^\omega + \Delta + b\text{–}AC \vdash \forall x^0 \forall y \leq_\rho sx \exists z \leq_0 2_k^{cx} A_0(x,y,z), \ \text{if } n = 3. \end{cases}$$

This generalizes to the case $\forall x^0, \tilde{x}^1 \forall y \leq_\rho sx\tilde{x} \exists z^0 A_0$: One obtains a bound which is linear (polynomial, elementary recursive) in $x^0, \tilde{x}^M$ in the sense of chapter 1 for $n = 1$ ($n = 2$,$n = 3$) and for $n = 2$ prop.2.2.31 applies.

Remark 3.2.4 *1) For $\delta, \rho \leq 1$ the theory G_nA^ω may be strengthened to $E\text{–}G_nA^\omega$ in thm.3.2.2 and cor.3.2.3 if AC–qf is restricted as in 3.1.4 .*

2) Theorem 3.2.2 and cor.3.2.3 generalize immediately to tuples $\underline{x}, \underline{y}, \underline{z}$ of variables instead of x, y, z, if b–AC is formulated for tuples. Furthermore instead of $\exists w^\tau A_0$ we may also have $\exists z^\tau \exists z' A_0$ where z' is of arbitrary type: It still is possible to bound $\exists z^\tau$.

Remark 3.2.5 *Cor.3.2.3 is a considerable generalization of a theorem due to Parikh ([27]): Parikh shows for a subsystem (called PB) of the first order fragment of G_2A^ω: If PB$\vdash \forall x \exists y A(x,y)$ (where A contains only bounded quantifiers and only x, y as free variables) then there is a polynomial p such that PB$\vdash \forall x \exists y \leq p(x) \ A(x,y)$.*

***Proof of thm.*3.2.2** : For PA$^\omega$ the theorem is proved in [21] . We only recall the treatment of Δ: The negative translation $\neg\neg\forall u^\gamma \neg\neg\exists v \leq_\delta tu \forall w^\eta \neg\neg H_0$ of $H :\equiv \forall u \exists v \leq tu \forall w \ H_0$ is intuitionistically implied by H. The functional interpretation transforms H into
$H^D :\equiv \exists V \leq t \forall u, w \ H_0(u, Vu, w)$. Let t^* be such that t^* s–maj t. Then (by lemma2.2.11.4) $V \leq t \to t^*$ s–maj V. Hence t^* satisfies the monotone functional interpretation of H (provable by H^D and thus in the presence of b-AC by H). The same proof applies to PRA$^\omega$. For G$_n$A$^\omega$ one has to use prop.2.2.21 to show that the majorizing terms for the terms occuring in the quantifier axioms can be choosen in G$_n$R$^\omega_{\preceq}$ (and not only in G$_n$R$^\omega$).
***Proof of cor.*3.2.3** : The corollary follows immediately from thm.3.2.2 and prop.2.2.29 using the embedding $x^0 \mapsto \lambda y^0.x^0$ of type 0 into type 1. The assertion for the case $\forall x^0, \tilde{x}^1 \forall y \leq_\rho sx\tilde{x} \exists z^0 A_0$ follows using prop.2.2.22,remark 2.2.25 and the fact that $\tilde{x}^M$ s–maj$_1\tilde{x}$.

Remark 3.2.6 *The size of the numbers k, c_1, c_2, c in the cor.3.2.3 above depends on the depth of nestings of the functions $+, \cdot$ resp. x^y occuring in the given proof. Such nestings may occur explicitly by the formation of terms like $(x \cdot (x \cdot (\ldots)))$ by substitution or are logically circumscribed. In the*

[8]Here $t^*[\,\underline{a}\,]$ is called a majorizing term if $\lambda\underline{a}.t^*$ s–maj $\lambda\underline{a}.t$, where $\underline{a}$ are all free variables of t.

later case they are made explicit by the (logical) normalization of the bound extracted by monotone functional interpretation. The process of normalization may increase the term depth enormously (In fact by an example due to [29] even non–elementary recursively in the type degree of the term). This corresponds to the fact that there are proofs of $\exists x^0 A_0(x)$–sentences such that the term complexity of a realizing term for $\exists x^0$ is not elementary recursive in the size of the proof (see [36]). However such a tremendous term complexity is very unlikely to occur in concrete proofs from mathematical practice: Firstly the parameter which is crucial for this complexity (the quantifier–complexity resp. the type degree of the modus ponens formulas) is very small in practice, lets say ≤ 3. Secondly even complex modus ponens formulas are able to cause an explosion of the term complexity only under very special circumstances which describe logically the iteration of a substitution process as in the example from [36] (we intend to discuss this matter in detail in another paper). Hence if a given proof does not involve such an iterated substitution process the degree of the polynomial bound in cor.3.2.3 will essentially be of the order of the degrees of the polynomials occuring in the proof and if the proof uses the exponential function 2^x (without applying it to itself) it will be a polynomial in 2^x. Hence the results of this paper which establish that substantial parts of analysis can be developed in a system whose provable growth is polynomial bounded also apply in a relativised form to proofs using e.g. the exponential function.

¿From the proof of thm.3.2.2 it follows that b–AC is needed only to derive $\tilde{F} :\equiv \exists V \leq_{\delta\gamma} t \forall u^\gamma, w^\eta F_0(u, Vu, w)$ from $F :\equiv \forall u^\gamma \exists v \leq_\delta tu \forall w^\eta F_0(u, v, w)$.[9] Hence if in the conclusion Δ is replaced by $\tilde{\Delta} := \left\{ \tilde{F} : F \in \Delta \right\}$ then b–AC can be omitted. In particular this is the case if each $F \in \Delta$ has the form $\exists v \leq t \forall w\ F_0(v, w)$ since $\tilde{F} \equiv F$ for such sentences.
Combining the proof of thm.3.2.2 with the proof of thm.2.9 from [15] one can strengthen the theorem by weakening b–AC($-\forall$) to b–AC–qf, i.e. b–AC restricted to quantifier–free formulas:
As in the proof of thm.2.9 in [15] one shows that

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC\text{–}qf} \vdash \forall u^\gamma, W^{\eta\delta} \exists v \leq_\delta tu\ F_0(u, v, Wv) \to \forall u^\gamma \exists v \leq_\delta tu \forall w^\eta F_0.$$

Thus Δ can be replaced by $\widehat{\Delta} := \{\forall u, W \exists v \leq tu\ F_0 : F \in \Delta\}$ without weakening of the theory. Since the implication

$$\forall u, W \exists v \leq tu F_0(u, v, Wv) \to \exists V \leq \lambda u, W.tu \forall u, W F_0(u, VuW, W(VuW))$$

can be proved by b–AC–qf (u, W can be coded into a single variable in $\mathrm{G}_n\mathrm{A}^\omega$ for $n \geq 2$)[10] the proof of the conclusion of thm.3.2.2 can be carried out in

$$\mathrm{G}_n\mathrm{A}_i^\omega + \widehat{\Delta} + \mathrm{b\text{–}AC\text{–}qf}$$

and thus a fortiori in

$$\mathrm{G}_n\mathrm{A}_i^\omega + \Delta + \mathrm{b\text{–}AC\text{–}qf}.$$

However replacing Δ by $\widehat{\Delta}$ may make the extraction of a bound more complicated since it causes a raising of the types involved. Since we are interested in an extraction method which is as practical as possible and yields bounds which are numerically as good as possible but not (primarily) in the proof–theoretic strength of the theory used to verify these bounds we prefer the more simple extraction from thm.3.2.2 . Similarly to thm. 2.12 in [15] we have the following generalization of thm.3.2.2 to a larger class of formulas:

[9] Thus in particular only b–AC restricted to universal formulas (b–AC–$\forall$) is used.
[10] For $n = 1$ one has to formulate b–AC–qf for tuples of variables.

Theorem 3.2.7 *Let Δ be as in thm.3.2.2 , $n \geq 1$, $\rho_1, \rho_2 \in$ **T** arbitrary types, $\tau_1, \tau_2 \leq 2$, $A_0(x,y,z,a,b)$ a quantifier–free formula containing at most x,y,z,a,b free and $s,r \in G_nR^\omega$. Then the following rule holds:*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \ \vdash \forall x^1 \forall y \leq_{\rho_1} sx \exists z^{\tau_1} \forall a \leq_{\rho_2} rxz \exists b^{\tau_2} A_0(x,y,z,a,b) \\ \Rightarrow \ \text{by monotone functional interpretation } \exists \Psi_1, \Psi_2 \in \ G_nR^\omega_-[\Phi_1]: \\ E\text{–}G_nA^\omega + \Delta + b\text{–}AC \ \vdash \forall x^1 \forall y \leq_{\rho_1} sx \exists z \leq_{\tau_1} \Psi_1 x \forall a \leq_{\rho_2} rxz \exists b \leq_{\tau_2} \Psi_2 x \ A_0(x,y,z,a,b). \end{cases}$$

Ψ_1, Ψ_2 are built up as Ψ in thm.3.2.2 . (An analogous result holds for PRA^ω and PA^ω).

Proof: Since the implication

$$\forall x^1 \forall y \leq_{\rho_1} sx \exists z^{\tau_1} \forall a \leq_{\rho_2} rxz \exists b^{\tau_2} A_0(x,y,z,a,b) \rightarrow$$
$$\forall x^1 \forall y \leq_{\rho_1} sx \forall A \leq_{\rho_2 \tau_1} rx \exists z^{\tau_1}, b^{\tau_2} A_0(x,y,z,Az,b)$$

holds logically the assumption of the theorem implies

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC\text{–}qf} \ \vdash \forall x^1 \forall y \leq_{\rho_1} sx \forall A \leq_{\rho_2 \tau_1} rx \exists z^{\tau_1}, b^{\tau_2} A_0(x,y,z,Az,b).$$

By thm.3.2.2 and remark 3.2.4 2) one can extract (by monotone functional interpretation) terms $\Psi_1, \Psi_2 \in \mathrm{G}_n\mathrm{R}^\omega_-[\Phi_1]$ such that

$$\forall x^1 \forall y \leq_{\rho_1} sx \forall A \leq_{\rho_2 \tau_1} rx \exists z \leq_{\tau_1} \Psi_1 x \exists b \leq_{\tau_2} \Psi_2 x \ A_0(x,y,z,Az,b).$$

As in the proof of 2.12 in [15] (using the fact that lemma 2.11 from [15] also holds for E–$\mathrm{G}_n\mathrm{A}^\omega_i$+ b–AC) one concludes the assertion of the theorem.

Theorem 3.2.8 *All of our results on G_nA^ω ($G_nA^\omega_i$, E–G_nA^ω, E–$G_nA^\omega_i$) and G_nR^ω remain valid if these theories are replaced by $G_nA^\omega[\underline{\chi}]$ ($G_nA^\omega_i[\underline{\chi}]$, E–$G_nA^\omega[\underline{\chi}]$, E–$G_nA^\omega_i[\underline{\chi}]$) and $G_nR^\omega[\underline{\chi}]$, where for a theory $\mathcal{T}$, $\mathcal{T}[\underline{\chi}]$ is defined as the extension obtained by adding a tuple $\underline{\chi}$ of function symbols $\chi_i^{\rho_i}$ with deg(ρ_i) ≤ 1 together with*

(1) *arbitrary purely universal axioms $\forall x^\tau A_0(x)$ on $\underline{\chi}$, where $\tau \leq 2$ and only x is free in $A_0(x)$*

plus axioms having the form

(2) *$\underline{\chi}^*$ s–maj $\underline{\chi}$ for $\underline{\chi}^* \in G_nR^\omega_-$,*

where (1),(2) are valid in the full type structure $\mathcal{S}^\omega$ under a suitable interpretation of $\underline{\chi}$ ($G_nR^\omega[\underline{\chi}]$ denotes the set of all closed terms of the extended theories).
In particular the bounds extracted in thm.3.2.2, 3.2.7 and cor.3.2.3 are still $\in G_nR^\omega_-[\Phi_1]$.

Proof: The theorem follows immediately from the proofs above (observing that also (2) is purely universal) if one extends the construction of t^* in the proof of prop.2.2.21 by the clause
'Replace all occurrences of χ_i in t by χ_i^*'. Since the majorizing terms χ_i^* are $\in \mathrm{G}_n\mathrm{R}^\omega_-$ this also holds for t^*.

Remark 3.2.9 *The reason for the restriction to deg(ρ_i) ≤ 1 in the theorem above is that the addition of symbols for higher type functionals χ in general destroys the possibility of elimination of extensionality since Ex(χ) may not be provable (and cannot be added simply as an axiom since it is not purely universal). Also* (2) *is no longer purely universal if deg(ρ_i) ≥ 2.*

By theorem 3.2.8 the extension by symbols for majorizable functions has no impact on the bounds extracted from a proof. This is the reason why we may make free use of such extensions (e.g. in a subsequent paper we will add new function symbols for sin and cos etc., see also [22]).
By cor.3.1.3 and thm.3.2.2 we can extract realizing functionals respectively uniform bounds for $\forall\exists A_0$–sentences (in the later case even for the more general sentences from thm.3.2.7). Since the theories $\mathrm{G}_n\mathrm{A}^\omega$ are based on classical logic it is in general not possible to extract computable realizations or bounds for $\forall\exists\forall A_0$–sentences: Let us consider e.g.

$$(+)\ \forall x^0\exists y^0\forall z^0(Pxy \vee \neg Pxz),$$

which holds by classical logic. If $Pxy :\equiv Txxy$, where T is the Kleene T–predicate, then any upper bound f on y, i.e.

$$\forall x^0\exists y \leq_0 fx\forall z^0(Pxy \vee \neg Pxz)$$

can be used to decide the halting–problem (and therefore must be ineffective): For h which is defined primitive recursively in f such that

$$hx := \begin{cases} 0, \text{ if } \exists y \leq fx(Txxy) \\ 1 \text{ otherwise} \end{cases}$$

one has $hx = 0 \leftrightarrow \exists yTxxy$ for all x. T is elementary recursive and therefore can be defined already in $\mathrm{G}_3\mathrm{A}^\omega$.
If one generalizes (+) to tuples of number variables then – by Matijacevic's result on Hilbert's 1oth problem– there is a polynomial $P\underline{x}\,\underline{y}$ whith coefficients in $\mathbb{N}$ such that there is no tuple $t_1, \ldots, t_k$ of recursive functions (for $\underline{y} = y_1 \ldots y_k$) with

$$\forall\underline{x}\exists y_1 \leq t_1\underline{x} \ldots \exists y_k \leq t_k\underline{x}\forall\underline{z}(P\underline{x}\underline{y} = 0 \vee \neg P\underline{x}\,\underline{z} = 0).$$

Since $P \in \mathrm{G}_2\mathrm{R}^\omega$ and $\mathrm{G}_2\mathrm{R}^\omega$ allows the coding of finite tuples of natural numbers one can define already in $\mathrm{G}_2\mathrm{R}^\omega$ a predicate P such that there is no recursive bound on y in (+).
The use of non–constructive $\forall\exists$–dependencies as in (+) is a characteristic feature of classical logic. If **intuitionistic** logic is used the situation changes completely: In chapter 8 of [22] it is shown that even in the presence of a large class of non–constructive analytical axioms (including as a special case arbitrary $\forall u^\delta\exists v \leq_\rho su\forall w^\tau A_0$–sentences) one can extract uniform bounds $\Psi \in \mathrm{G}_n\mathrm{R}^\omega$ on z in sentences $\forall x^1\forall y \leq_\gamma tx\exists z\ B(x, y, z)$, which are proved in $\mathrm{G}_n\mathrm{A}^\omega_i$ from such non–constructive axioms, where B **is an arbitrary formula** (containing only x, y, z free). This extraction, which is achieved by a new monotone version of modified realizability, will be dveloped in a subsequent paper (see also [23]).
Although in the case of theories based on classical logic it is not always possible to extract effective bounds for $\forall x\exists yA(x, y)$–sentences when A is not purely existential, one may obtain **relative bounds**: By $\mathrm{AC}^{0,0}$–qf and classical logic

$$(1)\ \forall x^0\exists y^0\forall z^0(Pxy \vee \neg Pxz)$$

is equivalent to

$$(2)\ \forall x, f^1\exists y(Pxy \vee \neg Px(fy))$$

and a bound on y in (2) is given by $\Psi xf := \max_0(0, f0) = f0$ since[11]

$$(Px0 \vee \neg Px(f0)) \vee (Px(f0) \vee \neg Px(ff0)).$$

For a more complex situation let us consider

$$F :\equiv \big(\forall x^0 \exists y^0 \forall z^0 A_0(x,y,z) \to \forall u^0 \exists v^0 B_0(u,v)\big),$$

which is –by $\mathrm{AC}^{0,0}$–$\forall$ and prenexing– equivalent to

$$\tilde{F} :\equiv \forall f^1, u \exists x, z, v \big(A_0(x, fx, z) \to B_0(u,v)\big).$$

The implication $F \to \tilde{F}$ holds logically. $\tilde{F}$ is a $\forall\exists F_0$–sentence. Thus v (and also x,z) can be bounded by a functional Ψuf in u, f with $\Psi \in \mathrm{G}_n\mathrm{R}^\omega$ if F is proved in $\mathrm{G}_n\mathrm{A}^\omega + \Delta$+AC–qf. Ψ is an effective bound **relatively to the oracle** f.
By raising the types one can replace $\tilde{F}$ by a different (and more complex) $\forall\exists F_0$–sentence $\widehat{F}$ which is more closely related to F in that the equivalence of F and $\widehat{F}$ can be proved using only $\mathrm{AC}^{1,0}$–qf:

$$\begin{aligned} F &\leftrightarrow \big(\exists \Phi^2 \forall x^0, f^1 A_0(x, \Phi xf, f(\Phi xf)) \to \forall u \exists v B_0(u,v)\big) \\ &\leftrightarrow \forall \Phi, u \exists x, f, v \big(A_0(x, \Phi xf, f(\Phi xf)) \to B_0(u,v)\big) \equiv: \widehat{F}. \end{aligned}$$

If F and therefore $\widehat{F}$ is proved in $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf, then one can extract from this proof a term $t \in \mathrm{G}_n\mathrm{R}^\omega$ such that $t\Phi u$ realizes '$\exists v$'. If $\widehat{F}$ is proved in $\mathrm{G}_n\mathrm{A}^\omega + \Delta$+AC–qf one obtains (using monotone functional interpretation) a term $t^* \in \mathrm{G}_n\mathrm{R}^\omega$ such that for every Φ^* which majorizes Φ, $t^*\Phi^* u$ is a bound for v:

$$\Phi^* \text{ s–maj } \Phi \to (\forall x, f A_0(x, \Phi xf, f(\Phi xf)) \to \forall u \exists v \leq t^*\Phi^* u\ B_0(u,v)).$$

4 The axiom F and the principle of uniform boundedness

In [21] we introduced the following axiom:[12]

$$\boldsymbol{F_0} :\equiv \forall \Phi^2, y^1 \exists y_0 \leq_1 y \forall z \leq_1 y (\Phi z \leq_0 \Phi y_0).$$

F_0 states that every functional Φ^2 assumes its maximum value on the fan $\{z^1 : z \leq_1 y\}$ for each y^1. This is an indirect way of expressing that Φ is bounded on $\{z^1 : z \leq_1 y\}$:

$$\boldsymbol{B_0} :\equiv \forall \Phi^2, y^1 \exists x^0 \forall z \leq_1 y (\Phi z \leq_0 x).$$

F_0 immediately implies B_0: Put $x := \Phi y_0$. The proof of the implication '$B_0 \to F_0$' uses the least number principle and classical logic:
If x is a bound for Φz on $\{z^1 : z \leq_1 y\}$ then there exists a minimal bound x_0 and therefore a z_0 such that $z_0 \leq_1 y \wedge \Phi z_0 =_0 x_0$ (since otherwise $\sup\limits_{\{z^1 : z \leq_1 y\}} \Phi z < x_0$, contradicting the minimality of x_0).
Our motivation for expressing B_0 via F_0 is that F_0 –in contrast to B_0– has (almost) the logical form $\forall x \exists y \leq sx \forall z A_0$ of an axiom $\in \Delta$ in theorems 3.2.2,3.2.7, 3.2.8 and cor.3.2.3 . This is the case because F_0 contains instead of the unbounded quantifier '$\exists x^0$' only the bounded quantifier '$\exists y_0 \leq_1 y$' (of

[11] More generally fz is an upper bound where z is a variable.

[12] In [21] this axiom is denoted by F instead of F_0. In this paper we reserve the name F for a generalization of this axiom which will be introduced below.

higher type). The reservation 'almost' refers to the fact that there is still an unbounded existential quantifier in F_0 hidden in the negative occurrence of '$z \leq_1 y$'. However this quantifier can be eliminated by the use of the extensionality axiom (E). By (E), F_0 is equivalent to

$$\tilde{F}_0 :\equiv \forall \Phi^2, y^1 \exists y_0 \leq_1 y \forall z^1 \big(\Phi(\min_1(z,y)) \leq_0 \Phi y_0\big) \text{ (see lemma 4.8 below).}$$

This use of extensionality does not cause problems for our monotone functional interpretation since the elimination of extensionality procedure applies: Because of the type–structure of F_0 the implication '$F_0 \rightarrow (F_0)_e$' is trivial.
F_0 is not true in the full type structure $\mathcal{S}^\omega$ of all set–theoretic functionals:

Definition 4.1

$$\begin{cases} \mathcal{S}_0 := \omega, \\ \mathcal{S}_{\tau(\rho)} := \{\textit{all set–theoretic functions } x : \mathcal{S}_\rho \rightarrow \mathcal{S}_\tau\}, \\ \mathcal{S}^\omega := \bigcup\limits_{\rho \in \mathbf{T}} \mathcal{S}_\rho, \end{cases}$$

where 'set–theoretic' is meant in the sense of ZFC.[13]

Proposition: 4.2 $\mathcal{S}^\omega \not\models F_0$.

Proof: Define

$$\Phi^2 y^1 := \begin{cases} \text{the least } n \text{ such that } yn =_0 0, \text{ if it exists} \\ 0^0, \text{ otherwise.} \end{cases}$$

Φ is not bounded on $\{z^1 : z \leq_1 \lambda x^0.1^0\}$ since $\Phi(\overline{1,x}) =_0 x$, where

$$(\overline{1,x})(k) := \begin{cases} 1^0, \text{ if } k <_0 x \\ 0^0, \text{ otherwise.} \end{cases}$$

On the other hand F_0 is true in the type structure $\mathcal{M}^\omega$ of all strongly majorizable set–theoretic functionals, which was introduced in [2] :

Definition 4.3

$$\begin{aligned} &\mathcal{M}_0 := \omega, \ x^* \ s\text{–}maj_0 \ x :\equiv x^*, x \in \omega \wedge x^* \geq x; \\ &x^* \ s\text{–}maj_{\tau(\rho)} \ x :\equiv x^*, x \in \mathcal{M}_\tau^{\mathcal{M}_\rho} \wedge \forall y^*, y \in \mathcal{M}_\rho (y^* \ s\text{–}maj_\rho \ y \rightarrow x^* y^* \ s\text{–}maj_\tau \ x^* y, xy), \\ &\mathcal{M}_{\tau(\rho)} := \Big\{ x \in \mathcal{M}_\tau^{\mathcal{M}_\rho} : \exists x^* \in \mathcal{M}_\tau^{\mathcal{M}_\rho} (x^* \ s\text{–}maj_{\tau(\rho)} \ x) \Big\}; \\ &\mathcal{M}^\omega := \bigcup_{\rho \in \mathbf{T}} \mathcal{M}_\rho \end{aligned}$$

(Here $\mathcal{M}_\tau^{\mathcal{M}_\rho}$ denotes the set of all set–theoretic functions: $\mathcal{M}_\rho \rightarrow \mathcal{M}_\tau$).

Proposition: 4.4 $\mathcal{M}^\omega \models F_0$.

Proof: It suffices to show that $\mathcal{M}^\omega \models B_0$: $\Phi \in \mathcal{M}_2$ implies the existence of a functional $\Phi^* \in \mathcal{M}_2$ such that Φ^* s–maj$_2$ Φ. Hence $\Phi^* y^M \geq_0 \Phi z$ for all y^1, z^1 such that $y \geq_1 z$ ($y^M x^0 := \max\limits_{i \leq x}(yi)$).

[13]The following proposition also holds if we omit the axiom of choice since only comprehension is used for the refutation of F_0.

For our applications in this and subsequent papers we also need a strengthening F of F_0, which generalizes F_0 to sequences of functionals and still holds in $\mathcal{M}^\omega$:

Definition 4.5
$$\boldsymbol{F} :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0 \forall z \leq_1 yk \big(\Phi kz \leq_0 \Phi k(y_0 k)\big).$$

Using AC on the meta-level and $M_{\rho 0} = M_\rho^{M_0}$ (see [2]) prop.4.4 yields

Proposition: 4.6 $\mathcal{M}^\omega \models F$.

F implies the existence of a sequence of bounds for a sequence $\Phi^{2(0)}$ of type–2–functionals on a sequence of fan's:

Proposition: 4.7 $G_1 A_i^\omega \vdash F \rightarrow \forall \Phi^{2(0)}, y^{1(0)} \exists \chi^1 \forall k^0 \forall z \leq_1 yk(\Phi kz \leq_0 \chi k)$.

Proof: Put $\chi k := \Phi(y_0 k)k$ for y_0 from F.
Similarly to F_0 also F can be transformed into a sentence $\tilde{F}$ having the logical form $\forall x \exists y \leq sx \forall z\ A_0$:

Lemma: 4.8
$$E\text{–}G_1 A_i^\omega \vdash F \leftrightarrow \widetilde{\boldsymbol{F}} :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0, z^1 \big(\Phi k(\min_1(z, yk)) \leq_0 \Phi k(y_0 k)\big).$$

Proof: '$\rightarrow$' is trivial. '$\leftarrow$' follows from $z \leq_1 yk \rightarrow \min_1(z, yk) =_1 z$ by the use of (E).
Because of this lemma we can treat F as an axiom $\in \Delta$ in the presence of (E). In order to apply our monotone functional interpretation we firstly have to eliminate (E) from the proof. This can be done as in cor.3.1.4 and remark 3.2.4 since $F \rightarrow (F)_e$.

Theorem 4.9 *Assume that $n \geq 1$. Let Δ be a set of sentences having the form $\forall u^\gamma \exists v \leq_\delta tu \forall w^\eta\ B_0$, where $t \in G_n R^\omega$ and $\gamma, \eta \leq 2$, $\delta \leq 1$ such that $\mathcal{S}^\omega \models \Delta$. Furthermore let $s \in G_n R^\omega$ and $A_0 \in \mathcal{L}(G_n A^\omega)$ be a quantifier–free formula containing only x, y, z free and let $\alpha, \beta \in \mathbf{T}$ such that $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$, and $\tau \leq 2$. Then the following rule holds:*

$$\begin{cases} E\text{–}G_n A^\omega + F + \Delta + AC^{\alpha,\beta}\text{–}qf\ \vdash \forall x^1 \forall y \leq_1 sx \exists z^\tau\ A_0(x,y,z) \\ \Rightarrow \text{ by elimination of (E), neg. transl. and monotone functional interpretation } \exists \Psi \in G_n R_-^\omega[\Phi_1]: \\ G_n A_i^\omega + \tilde{F} + \Delta + b\text{–}AC\ \vdash \forall x^1 \forall y \leq_1 sx \exists z \leq_\tau \Psi x\ A_0(x,y,z) \textit{ and therfore} \\ \mathcal{M}^\omega, \mathcal{S}^\omega \models \forall x^1 \forall y \leq_1 sx \exists z \leq_\tau \Psi x\ A_0(x,y,z).^{14} \end{cases}$$

Ψ is built up from $0^0, 1^0, \max_\rho, \Phi_1$ and majorizing terms[15] *for the terms t occurring in the quantifier axioms $\forall x Gx \rightarrow Gt$ and $Gt \rightarrow \exists x Gx$ which are used in the given proof by use of λ–abstraction and substitution. If $\tau \leq 1$ then Ψ has the form $\Psi \equiv \lambda x^1.\Psi_0 x^M$, where $x^M := \Phi_1 x$ and Ψ_0 does not contain Φ_1 (An analogous result holds for E–PRA^ω,E–PA^ω with $\Psi \in \widehat{PR}^\omega$ resp. $\Psi \in T$).*

Proof: By lemma 4.8 and elimination of extensionality the assumption yields

$$\mathrm{G}_n\mathrm{A}^\omega + \tilde{F} + \Delta + \mathrm{AC}^{\alpha,\beta}\text{–qf}\ \vdash \forall x^1 \forall y \leq_1 sx \exists z^\tau\ A_0(x,y,z).$$

[14] Note that the conclusion holds in $\mathcal{S}^\omega$ although $\mathcal{S}^\omega \not\models \tilde{F}$.
[15] Here $t^*[\underline{a}]$ is called a majorizing term if $\lambda\underline{a}.t^*$ s–maj $\lambda\underline{a}.t$, where $\underline{a}$ are all free variables of t.

By thm.3.2.2 there exists a $\Psi \in \mathrm{G}_n\mathrm{R}^\omega_-[\Phi_1]$ satisfying the properties of the theorem such that

$$\mathrm{G}_n\mathrm{A}^\omega_i + \tilde{F} + \Delta + \text{b–AC} \ \vdash \forall x^1 \forall y \leq_1 sx \exists z \leq_\tau \Psi x \ A_0(x, y, z).$$

¿From [16] and prop.4.6 we know that $\mathcal{M}^\omega \models$ PA$^\omega$ + $\tilde{F}$+b–AC and therefore $\mathcal{M}^\omega \models$ G$_n$A$^\omega$ + $\tilde{F}$+b–AC. Note that every $\mathcal{S}^\omega$–true universal sentence $\forall x^\rho A_0(x)$ with $\deg(\rho) \leq 2$ as well as every sentence from Δ is also true in $\mathcal{M}^\omega$. This follows from $\mathcal{S}_0 = \mathcal{M}_0, \mathcal{S}_1 = \mathcal{M}_1$ and $\mathcal{S}_2 \supset \mathcal{M}_2$. Hence $\mathcal{M}^\omega \models$ G$_n$A$^\omega$ + $\tilde{F}$ + Δ+b–AC
and therefore

$$\mathcal{M}^\omega \models \forall x^1 \forall y \leq_1 sx \exists z \leq_\tau \Psi x A_0(x, y, z).$$

Since $\tau \leq 2$ this implies

$$\mathcal{S}^\omega \models \forall x^1 \forall y \leq_1 sx \exists z \leq_\tau \Psi x A_0(x, y, z).$$

Remark 4.10 *It is the need of the (E)–elimination that prevents us from dealing with stronger forms of F, where y_0 may be given as a functional in Φ and y, since for such a strengthened version the interpretation $(F)_e$ would not follow from F (without using (E) already). The same obstacle arises when F is generalized to higher types $\rho > 1$:*

$$F_\rho :\equiv \forall \Phi^{0\rho 0}, y^{\rho 0} \exists y_0 \leq_{\rho 0} y \forall k^0 \forall z \leq_\rho yk\big(\Phi kz \leq_0 \Phi k(y_0 k)\big).$$

F_ρ, which still is true in $\mathcal{M}^\omega$, will be used in the intuitionistic context studied in chapter 8 below.

In our applications of F we actually make use of the following consequence of F+AC^{1,0}–qf:

Definition 4.11 *The schema of* **uniform Σ^0_1–boundedness** *is defined as*

$$\mathbf{\Sigma^0_1\text{–UB}} : \begin{cases} \forall y^{1(0)}\big(\forall k^0 \forall x \leq_1 yk \exists z^0 \ A(x, y, k, z) \\ \qquad \qquad \to \exists \chi^1 \forall k^0 \forall x \leq_1 yk \exists z \leq_0 \chi k \ A(x, y, k, z)\big), \end{cases}$$

where $A \equiv \exists \underline{l} A_0(\underline{l})$ and $\underline{l}$ is a tuple of variables of type 0 and A_0 is a quantifier–free formula (which may contain parameters of arbitrary types).

Proposition: 4.12 *Assume that $n \geq 2$.*
G_nA^ω+$AC^{1,0}$–qf $\vdash F \to \Sigma^0_1$–UB.

Proof: $\forall k^0 \forall x^1 \leq_1 yk \exists z^0 \ A(x, y, k, z)$ implies
$\forall k^0 \forall x^1 \exists z^0, v^0\big(xv \leq_0 ykv \to A(x, y, k, z)\big)$. Thus using the fact that k, x as well as $z, v, \underline{l}$ can be coded together in G$_2$A$^\omega$, one obtains by AC^{1,0}–qf the existence of a functional $\Phi^{2(0)}$ such that $\forall k^0 \forall x \leq_1 yk \ A(x, y, k, \Phi kx)$. Proposition 4.7 yields
$\exists \chi^1 \forall k^0 \forall x \leq_1 yk(\chi k \geq_0 \Phi kx)$.

Remark 4.13 *In the proof above we have made use of classical logic for the shift of the quantifier on v as an existential quantifier in front of the implication. Nevertheless one can make use of the principle of uniform boundedness (and even generalizations of this principle) in intuitionistic theories (as will be shown in a subsequent paper). This is possible since instead of classical logic we could have used also (E) to derive $\forall k, x \exists z \ A(\min_1(x, yk), y, k, z)$ and (E) does not cause any problems intuitionistically.*

Σ^0_1–UB together with classical logic implies the existence of a modulus of uniform continuity for each extensional $\Phi^{1(1)}$ on $\{z^1 : z \leq_1 y\}$ (where 'continuity' refers to the usual metric on the Baire space $\mathbb{N}^{\mathbb{N}}$):

Proposition: 4.14 *For $n \geq 2$ the following holds*

$$G_nA^\omega + \Sigma^0_1\text{–}UB \vdash$$
$$\forall \Phi^{1(1)}\big(ext(\Phi) \to \forall y^1 \exists \chi^1 \forall k^0 \forall z_1, z_2 \leq_1 y\big(\bigwedge_{i \leq_0 \chi k}(z_1 i =_0 z_2 i) \to \bigwedge_{j \leq_0 k}(\Phi z_1 j =_0 \Phi z_2 j)\big)\big),$$

where $ext(\Phi) :\equiv \forall z_1^1, z_2^1(z_1 =_1 z_2 \to \Phi z_1 =_1 \Phi z_2)$.

Proof: $\forall z_1, z_2 \leq_1 y(z_1 =_1 z_2 \to \Phi z_1 =_1 \Phi z_2)$ implies

$$\forall z_1, z_2 \leq_1 y \forall k^0 \exists n^0\big(\bigwedge_{i \leq_0 n}(z_1 i =_0 z_2 i) \to \bigwedge_{j \leq_0 k}(\Phi z_1 j =_0 \Phi z_2 j)\big).$$

By Σ^0_1–UB (using the coding of z_1, z_2 into a single variable) we conclude

$$\exists \chi^1 \forall k^0 \forall z_1, z_2 \leq_1 y\big(\bigwedge_{i \leq_0 \chi k}(z_1 i =_0 z_2 i) \to \bigwedge_{j \leq_0 k}(\Phi z_1 j =_0 \Phi z_2 j)\big).$$

Remark 4.15 *The weaker axiom F_0 instead of F proves Σ^0_1–UB only in a weaker version which asserts instead of the bounding function χ^1 only the existence of a bound n^0 for every k^0. This is sufficient to prove that every $\Phi^{1(1)}$ is uniformly continuous but not to show the existence of a modulus of uniform continuity.*

For many applications a weaker version F^- of F is sufficient which we will study now for the following reasons:

1) F^- has already the logical form $\forall x \exists y \leq sx \forall z A_0$ of an axiom $\in \Delta$ and needs (in contrast to F) no further transformation. This simplifies the extraction of bounds and allows the generalization to higher types (see thm.4.21 below).

2) F^- can be eliminated from the proof for the verification of the bound extracted in a simple purely syntactical way (see thm.4.21) yielding a verification in $\mathrm{G}_{\max(3,n)}\mathrm{A}^\omega_i$. In particular no relativation to $\mathcal{M}^\omega$ is needed. For F such an elimination uses much more complicated tools and gives a verification only in HA^ω and only for $\tau \leq 1$ and $\Delta = \emptyset$ in thm.4.9 (see [21]).

Definition 4.16 $F^- :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0, z^1, n^0\big(\bigwedge_{i <_0 n}(zi \leq_0 yki) \to \Phi k(\overline{z,n}) \leq_0 \Phi k(y_0 k)\big)$, *where, for $z^{\rho 0}$, $(\overline{z,n})(k^0) :=_\rho zk$, if $k <_0 n$ and $:= 0^\rho$, otherwise (It is clear that $\lambda z, n.(\overline{z,n}) \in G_2R^\omega$).*

Remark 4.17 *Since F^- is a weakening of F (to finite initial sequences) it is also true in $\mathcal{M}^\omega$. By the proof of prop.4.2 F^- does not hold in $\mathcal{S}^\omega$.*

Lemma: 4.18 $G_1A^\omega_i \vdash F^- \to \forall \Phi^{2(0)}, y^{1(0)} \exists \chi^{1(0)} \forall k^0, z^1, n^0\big(\bigwedge_{i <_0 n}(zi \leq_0 yki) \to \Phi k(\overline{z,n}) \leq_0 \chi k\big).$

Definition 4.19 *The schema Σ^0_1–UB^- is defined as the following weakening of Σ^0_1–UB:*

$$\mathbf{\Sigma^0_1\text{–}UB^-}: \begin{cases} \forall y^{1(0)}\big(\forall k^0 \forall x \leq_1 yk \exists z^0\ A(x,y,k,z) \to \exists \chi^1 \forall k^0, x^1, n^0 \\ \qquad \big(\bigwedge\limits_{i<_0 n} (xi \leq_0 yki) \to \exists z \leq_0 \chi k\ A((\overline{x,n}), y, k, z)\big)\big), \end{cases}$$

where $A \in \Sigma^0_1$.

Proposition: 4.20 *For each $n \geq 2$ we have G_nA^ω+$AC^{1,0}$–$qf \vdash F^- \to \Sigma^0_1$–$UB^-$.*

Proof: Analogously to the proof of prop.4.12 using lemma 4.18 instead of prop.4.7.

Theorem 4.21 *Assume $n \geq 1$, $\tau \leq 2$, $s \in G_nR^\omega$. Let $A_0(x,y,z) \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula containing only x,y,z as free variables. Then the following rule holds:*

$$\begin{cases} G_nA^\omega \oplus AC\text{–}qf \oplus F^- \vdash \forall x^1 \forall y \leq_\rho sx \exists z^\tau\ A_0(x,y,z) \\ \Rightarrow \text{ by neg. transl. and monotone functional interpretation } \exists \Psi \in G_nR^\omega_-[\Phi_1] \text{ such that} \\ G_{\max(3,n)}A^\omega_i \vdash \forall x^1 \forall y \leq_\rho sx \exists z \leq_\tau \Psi x\ A_0(x,y,z). \end{cases}$$

Ψ is built up from $0^0, 1^0, \max_\rho, \Phi_1$ and majorizing terms for the terms t occurring in the quantifier axioms $\forall x Gx \to Gt$ and $Gt \to \exists x Gx$ which are used in the given proof by use of λ–abstraction and substitution.[16]
If $\tau \leq 1$ then Ψ has the form $\Psi \equiv \lambda x^1.\Psi_0 x^M$, where $x^M := \Phi_1 x$ and Ψ_0 does not contain Φ_1.
For $\rho \leq 1$, $G_nA^\omega \oplus AC$–$qf \oplus F^-$ can be replaced by E–G_nA^ω+$AC^{\alpha,\beta}$–qf+F^-, where α, β are as in thm.4.9 . A remark analogous to 3.2.4 applies. Furthermore on may add axioms Δ (having the form as in thm. 3.2.2) to $G_nA^\omega \oplus AC$–$qf \oplus F^-$. Then the conclusion holds in $G_{\max(3,n)}A^\omega_i + \Delta + b$–$AC$.
An analogous result holds for PRA^ω and PA^ω with $\Psi \in \widehat{PR}^\omega$ resp. $\in T$ and verification in PRA^ω_i resp. PA^ω_i.

Proof: The assumption implies

$$\begin{aligned} &\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC\text{–}qf}\ \vdash \big(\exists Y \leq \lambda \Phi^{2(0)}, y^{1(0)}.y \forall \Phi, \tilde{y}^{1(0)}, k^0, \tilde{z}^1, n^0 \\ &\quad \big(\bigwedge_{i<n} (\tilde{z}i \leq \tilde{y}ki) \to \Phi k(\overline{\tilde{z},n}) \leq_0 \Phi k(Y\Phi\tilde{y}k)\big) \to \forall x^1 \forall y \leq_\rho sx \exists z^\tau A_0(x,y,z)\big), \end{aligned}$$

and therefore

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC\text{–}qf}\ \vdash \forall Y \leq \lambda \Phi, y.y \forall x^1 \forall y \leq_\rho sx \exists \Phi, \tilde{y}, k, \tilde{z}, n, z(\ldots).$$

By theorem 3.2.2 and a remark on it we can extract $\Psi_1, \Psi_2 \in \mathrm{G}_n\mathrm{R}^\omega_-[\Phi_1]$ such that

$$\mathrm{G}_n\mathrm{A}^\omega_i \vdash \forall Y \leq \lambda \Phi, y.y \forall x^1 \forall y \leq_\rho sx \exists \Phi, \tilde{y}, k, \tilde{z} \exists n \leq_0 \Psi_1 x \exists z \leq_\tau \Psi_2 x(\ldots).$$

Hence

$$\begin{aligned} &\mathrm{G}_n\mathrm{A}^\omega_i \vdash \forall x\big(\exists Y \leq \lambda \Phi^{2(0)}, y^{1(0)}.y \forall \Phi, \tilde{y}^{1(0)}, k^0, \tilde{z}^1 \forall n \leq_0 \Psi_1 x \\ &\quad \big(\bigwedge_{i<n} (\tilde{z}i \leq \tilde{y}ki) \to \Phi k(\overline{\tilde{z},n}) \leq \Phi k(Y\Phi\tilde{y}k)\big) \to \forall y \leq_\rho sx \exists z \leq_\tau \Psi_2 x A_0(x,y,z)\big). \end{aligned}$$

[16] Here $\oplus$ means that F^- and AC–qf must not be used in the proof of the premise of an application of the quantifier–free rule of extensionality QF–ER. $\mathrm{G}_n\mathrm{A}^\omega$ satisfies the deduction theorem w.r.t $\oplus$ but not w.r.t $+$. In fact the theorem also holds for $(\mathrm{G}_n\mathrm{A}^\omega$+AC–qf$)\oplus F^-$ since the deduction property is used in the proof only for F^-.

It remains to show that

$$\mathrm{G_3A}^\omega_i \vdash \forall n_0 \exists Y \le \lambda \Phi^{2(0)}, y^{1(0)}.y \forall \Phi, \tilde{y}^{1(0)}, k^0, \tilde{z}^1 \forall n \le_0 n_0$$
$$\Big(\bigwedge_{i<n} (\tilde{z}i \le \tilde{y}ki) \to \Phi k(\overline{\tilde{z}, n}) \le \Phi k(Y\Phi\tilde{y}k)\Big) :$$

Define[17]

$$\tilde{Y} := \lambda \Phi, \tilde{y}, k, n_0. \max_{j \le_0 (\overline{\tilde{y}k})n_0} \Phi k\big(\overline{\min_1(\lambda i.(j)_i, \tilde{y}k), n_0}\big).$$

One easily shows (using the fact that $\Phi_{\langle\cdot\rangle} \in \mathrm{G_3R}^\omega$) that $\tilde{Y}$ is definable in $\mathrm{G_3A}^\omega_i$. In the same way we can define (using μ_b)

$$\widehat{Y} := \lambda \Phi, \tilde{y}, k, n_0. \min_{j \le_0 (\overline{\tilde{y}k})n_0} \Big[\Phi k\big(\overline{\min_1(\lambda i.(j)_i, \tilde{y}k), n_0}\big) =_0 \tilde{Y}\Phi\tilde{y}kn_0\Big].$$

For every n_0 we now put

$$Y := \lambda \Phi, \tilde{y}, k.\big(\overline{\min_1\big(\lambda i.(\widehat{Y}\Phi\tilde{y}kn_0)_i, \tilde{y}k), n_0}\big).$$

Analogously to prop.4.14 one shows

Proposition: 4.22 *For $n \ge 2$ the following holds*

$$G_nA^\omega \oplus \Sigma^0_1\text{–}UB^- \;\vdash \forall \Phi^{1(1)}\big(ext(\Phi) \wedge \Phi \textit{ pointwise continuous } \to$$
$$\forall y^1 \exists \chi^1 \forall k^0 \forall z_1, z_2 \le_1 y\big(\bigwedge_{i \le_0 \chi k} (z_1 i =_0 z_2 i) \to \bigwedge_{j \le_0 k} (\Phi z_1 j =_0 \Phi z_2 j)\big)\big).$$

We now show that F^- implies (relatively to $\mathrm{G_2A}^\omega$+$\mathrm{AC}^{1,0}$–qf) a generalization of the binary ('weak') König's lemma WKL:

Definition 4.23 (Troelstra(74)) *WKL*$:\equiv \forall f^1\big(T(f) \wedge \forall x^0 \exists n^0(lth\ n =_0 x \wedge fn =_0 0) \to \exists b \le_1 \lambda k.1 \forall x^0(f(\bar{b}x) =_0 0)\big)$,
where $Tf :\equiv \forall n^0, m^0(f(n * m) =_0 0 \to fn =_0 0) \wedge \forall n^0, x^0(f(n * \langle x\rangle) =_0 0 \to x \le_0 1)$ *(i.e. $T(f)$ asserts that f represents a 0,1–tree).*

In the following we generalize WKL to a sequential version WKL_{seq} which states that for every sequence of infinite 0,1–trees there exists a sequence of infinite branches:

Definition 4.24

$$WKL_{seq} :\equiv \begin{cases} \forall f^{1(0)}\big(\forall k^0(T(fk) \wedge \forall x^0 \exists n^0(lth\ n =_0 x \wedge fkn =_0 0)) \\ \qquad\qquad \to \exists b \le_{1(0)} \lambda k^0, i^0.1 \forall k^0, x^0(fk((\overline{bk})x) =_0 0)\big). \end{cases}$$

This formulation of WKL (which is used e.g. in [35] and [30],[31],[32] and in a similar way in the system RCA_0 considered in the context of 'reverse mathematics' with set variables instead of function variables) and WKL_{seq} uses the functional $\Phi_{\langle\cdot\rangle}bx = \bar{b}x$ which is definable in $\mathrm{G}_n\mathrm{A}^\omega_i$ only for $n \ge 3$ and causes exponential growth. Since we are mostly interested in polynomial growth and therefore in systems based on $\mathrm{G_2A}^\omega$ we introduce a different formulation WKL^2_{seq} of WKL_{seq} which avoids the coding of finite sequences (of variable length) as numbers and can be used in $\mathrm{G_2A}^\omega$ and is equivalent to WKL_{seq} in the presence of the functional $\Phi_{\langle\cdot\rangle}$. This is achieved by expressing trees as higher type (≥ 2) functionals which are available in our finite type theories:

[17]Note that our definition of $\overline{f}x$ implies that $\bigwedge_{i<n} (\tilde{z}i \le_0 \tilde{y}ki) \to \overline{\tilde{z}}n \le_0 (\overline{\tilde{y}k})n_0$ for $n \le_0 n_0$.

Definition 4.25

$$WKL^2_{seq} :\equiv \begin{cases} \forall \Phi^{0010}\big(\forall k^0, x^0 \exists b \leq_1 \lambda n^0.1^0 \bigwedge\limits_{i=0}^{x} (\Phi k(\overline{b,i})i =_0 0) \\ \qquad\qquad\qquad \to \exists b \leq_{1(0)} \lambda k^0, n^0.1 \forall k^0, x^0 (\Phi k(\overline{bk,x})x =_0 0)\big). \end{cases}$$

Proposition: 4.26 $G_3A^\omega_i \vdash WKL^2_{seq} \leftrightarrow WKL_{seq}$.

Proof: '$\to$': Define $\Phi k^0 b^1 x^0 := fk(\overline{b}x)$ and assume $\forall k^0 T(fk)$ and $(+)\ \forall k, x \exists n(lth\ n = x \wedge fkn = 0)$. It follows that

$$\forall k, x \exists b \leq \lambda n.1 \bigwedge_{i=0}^{x} (\Phi k(\overline{b,i})i =_0 0)$$

(Put $b := \lambda i.(n)_i$ for n as in $(+)$).
Hence WKL^2_{seq} yields

$$\exists b \leq \lambda k, n.1 \forall k, x(\Phi k(\overline{bk,x})x =_0 0),$$

i.e.

$$\exists b \leq \lambda k, n.1 \forall k, x\big(fk((\overline{bk})x) =_0 0\big).$$

'$\leftarrow$': Define

$$fkn := \begin{cases} \Phi k(\lambda i.(n)_i)(lth\ n), \text{ if } \forall j \leq lth\ n\big((\Phi k(\overline{\lambda i.(n)_i, j})j =_0 0\big) \wedge (n)_j \leq 1\big) \\ 1^0, \quad \text{otherwise.} \end{cases}$$

The assumption $\forall k, x \exists b \leq_1 \lambda n^0.1^0 \bigwedge\limits_{i=0}^{x} \big(\Phi k(\overline{b,i})i =_0 0\big)$ implies $\forall k, x \exists n(lth\ n = x \wedge fkn = 0)$. Since furthermore $T(fk)$ for all k (by f–definition), WKL_{seq} yields

$$\exists b \leq_{1(0)} \lambda k, n.1 \forall k^0, x^0\big(fk((\overline{bk})x) =_0 0\big),$$

i.e.

$$\exists b \leq \lambda k, n.1 \forall k, x(\Phi k(\overline{bk,x})x =_0 0).$$

Theorem 4.27 $G_2A^\omega + AC^{0,1}\text{–}qf \vdash \Sigma^0_1\text{–}UB^- \to WKL^2_{seq}$.

Proof: Assume that

$$\forall b \leq_{1(0)} \lambda k^0, i^0.1 \exists k^0, x^0\big(\Phi k(\overline{bk,x})x \neq_0 0\big).$$

By Σ^0_1–UB^- it follows that (since the type $1(0)$ can be coded in type 1):

$$(*)\ \exists x_0 \forall b \leq_{1(0)} \lambda k, i.1 \exists k, x \leq_0 x_0\big(\Phi k(\underbrace{\overline{(\overline{bk,x_0}), x}}_{=_1 \overline{bk,x}})x \neq_0 0\big).$$

Assume $\forall k^0, x^0 \exists b^1\big(\bigwedge\limits_{i=0}^{x} (bi \leq_0 1 \wedge \Phi k(\overline{b,i})i =_0 0)\big)$. $\mathrm{AC}^{0,1}$–qf yields

$$\forall x^0 \exists b^{1(0)} \forall k^0\big(\bigwedge_{i=0}^{x} (bki \leq_0 1 \wedge \Phi k(\overline{bk,i})i =_0 0)\big)$$

Since $\overline{bk,i} =_1 \overline{(\overline{bk,x}),i}$ for $i \le x$ and $\overline{bk,x} \le_1 \lambda i.1$ if $\bigwedge\limits_{i=0}^{x} (bki \le_0 1)$ this implies

$$\forall x^0 \exists b \le_{1(0)} \lambda k, i.1 \forall k \bigwedge_{i=0}^{x} \big(\Phi k(\overline{bk,i})i = 0\big),$$

which contradicts $(*)$.

Together with propositions 2.2.29,4.20 this theorem implies the following

Corollary 4.28 *Let $n \ge 2$. Then*[18]

$$(G_nA^\omega + AC^{1,0}\text{–}qf + AC^{0,1}\text{–}qf) \oplus F^- \vdash WKL^2_{seq}.$$

Hence theorem 4.9 and theorem 4.21 capture proofs using WKL^2_{seq}. In particular (combined with cor.3.2.3) we have the following rule

$$\begin{cases} E\text{–}G_2A^\omega + AC^{\alpha,\beta}\text{–qf} + \text{WKL}^2_{seq} \vdash \forall x^0 \forall y \le_1 sx \exists z^0 A_0(x,y,z) \\ \Rightarrow \exists (eff.) k, c_1, c_2 \in \mathbb{N} \textit{ such that} \\ G_3A_i^\omega \vdash \forall x^0 \forall y \le_1 sx \exists z \le_0 c_1x^k + c_2 \ A_0(x,y,z), \end{cases}$$

where $s \in G_2R^\omega$ and A_0 is a quantifier–free formula of G_2A^ω which contains only x, y, z as free variables and $(\alpha = 0 \wedge \beta \le 1)$ or $(\alpha = 1 \wedge \beta = 0)$. For G_nA^ω and $\oplus$ instead of $E\text{–}G_nA^\omega, +$ this result holds for full AC–qf and $y \le_\rho sx$ where ρ is an arbitrary type.

Remark 4.29 *WKL^2_{seq} does not imply (relative to say PA^ω+AC) F^- since $\mathcal{S}^\omega \models WKL^2_{seq}$, but $\mathcal{S}^\omega \not\models F^-$.*

Remark 4.30 Π^0_2–conservation of WKL over a second–order fragment RCA_0 of $\widehat{PA}^\omega\!\upharpoonright$+AC–qf was proved at first model–theoretically by H. Friedman in an unpublished paper. In [30] a proof–theoretic treatment (using cut–elimination) is given. For the finite type systems PA^ω+AC–qf+WKL (where $\text{PA}^\omega := \text{WE–HA}^\omega$ with WE–HA^ω as in [34]) and $\widehat{PA}^\omega\!\upharpoonright$+AC–qf+WKL conservation results for Π^0_2–sentences and even for $\forall x^1 \forall y \le_\rho sx \exists z^\tau A_0(x,y,z)$–sentences were obtained in [14],[18], [15] using functional interpretation. A new and more simple proof using (a weaker version of) our axiom F^- and monotone functional interpretation is given in [21]. It is this proof which we have adapted in this paper for the weak systems based on $\text{G}_n\text{A}^\omega$. In an unpublished paper L. Harrington gave a model–theoretic proof for Π^1_1–conservation of RCA_0+WKL over RAC_0 (see also [3]; In [6] also a model–theoretic proof for Π^1_1–conservation of WKL relatively to a second–order system of 'feasible' arithmetic is given).
In [31],[32] a proof-theoretic treatment of this result is formulated (also for a second–order system based on elementary recursive functions only) which however makes incorrect use of Herbrand normal forms and establishes only conservation for $\forall f^1 \exists x^0 A_0$–sentences (see [17] for a discussion of this point).
In [26],[30] proofs for Π^0_2–conservation over PRA for certain second–order systems based on WKL, Π^0_1–comprehension **without** function parameters and Π^0_2–induction rule **without** function parameters are presented. However the resulting theories (even without WKL) prove the totality of the Ackermann function as was observed in [22] (see also [24]).

[18] The proofs of 4.20 and 4.27 also yield $\text{G}_n\text{A}^\omega \oplus \text{AC}^{1,0}\text{–qf} \oplus \text{AC}^{0,1}\text{–qf} \oplus F^- \vdash \text{WKL}^2_{seq}$.

References

[1] Ackermann, W., Zum Aufbau der rellen Zahlen. Math. Annalen **99** pp.118–133 (1928).

[2] Bezem, M.A., Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals. J. Symb. Logic **50** pp. 652–660 (1985).

[3] Clote, P., Hájek, P., Paris, J., On some formalized conservation results in arithmetic. Arch. Math. Logic **30**, pp. 201–218 (1990).

[4] Feferman, S., Theories of finite type related to mathematical practice. In: Barwise, J. (ed.), Handbook of Mathematical Logic, North–Holland, Amsterdam, pp. 913–972 (1977).

[5] Felscher, W., Berechenbarkeit: Rekursive und programmierbare Funktionen. Springer–Lehrbuch, Berlin Heidelberg New York 1993.

[6] Ferreira, F., A feasible theory for analysis. J. Smybolic Logic **59**, pp. 1001–1011 (1994).

[7] Gödel, K., Zur intuitionistischen Arithmetik und Zahlentheorie. Ergebnisse eines Mathematischen Kolloquiums, vol. 4 pp. 34–38 (1933).

[8] Gödel, K., Über eine bisher noch nicht benutzte Erweiterung des finiten Standpunktes. Dialectica **12**, pp. 280–287 (1958).

[9] Grzegorczyk, A., Some classes of recursive functions. Rozprawny Matematyczne, 46 pp., Warsaw (1953).

[10] Howard, W.A., Hereditarily majorizable functionals of finite type. In: Troelstra (1973).

[11] Kleene, S.C., Introduction to Metamathematics. North–Holland (Amsterdam), Noordhoff (Groningen), New–York (Van Nostrand) (1952).

[12] Kleene, S.C., Recursive functionals and quantifiers of finite types, I. Trans. A.M.S. **91**, pp.1–52 (1959).

[13] Ko, K.–I., Complexity theory of real functions. Birkhäuser; Boston, Basel, Berlin (1991).

[14] Kohlenbach, U., Theorie der majorisierbaren und stetigen Funktionale und ihre Anwendung bei der Extraktion von Schranken aus inkonstruktiven Beweisen: Effektive Eindeutigkeitsmodule bei besten Approximationen aus ineffektiven Eindeutigkeitsbeweisen. Dissertation, Frankfurt/Main, pp. xxii+278 (1990).

[15] Kohlenbach, U., Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. J. Symbolic Logic **57**, pp. 1239–1273 (1992).

[16] Kohlenbach, U., Pointwise hereditary majorization and some applications. Arch. Math. Logic **31**, pp.227–241 (1992).

[17] Kohlenbach, U., Remarks on Herbrand normal forms and Herbrand realizations. Arch. Math. Logic **31**, pp.305–317 (1992).

[18] Kohlenbach, U., Constructions in classical analysis by unwinding proofs using majorizability (abstract, Logic Colloquium 89, Berlin). J. Symbolic Logic **57**, p.307 (1992).

[19] Kohlenbach, U., Effective moduli from ineffective uniqueness proofs. An unwinding of de La Vallée Poussin's proof for Chebycheff approximation. Ann. Pure Appl. Logic **64**, pp. 27–94 (1993).

[20] Kohlenbach, U., New effective moduli of uniqueness and uniform a–priori estimates for constants of strong unicity by logical analysis of known proofs in best approximation theory. Numer. Funct. Anal. and Optimiz. **14**, pp. 581–606 (1993).

[21] Kohlenbach, U., Analyzing proofs in analysis. Preprint 25 p. To appear in: Proceedings Logic Colloquium'93 (Keele), Oxford University Press.

[22] Kohlenbach, U., Real growth in standard parts of analysis. Habilitationsschrift, pp. xv+166, Frankfurt (1995).

[23] Kohlenbach, U., Exploiting partial constructivity relatively to non–constructive lemmas in given proofs (abstract). The Bulletin of Symbolic Logic **1**, pp. 243-244 (1995).

[24] Kohlenbach, U., A note on the Π^0_2–induction rule. To appear in: Arch. Math. Logic.

[25] Luckhardt, H., Extensional Gödel functional interpretation. A consistency proof of classical analysis. Springer Lecture Notes in Mathematics **306** (1973).

[26] Mints, G.E., What can be done with PRA. J. Soviet Math. **14**, pp.1487–1492, 1980 (Translation from: Zapiski Nauchuyh Seminarov, LOMI, vol. 60 (1976), pp. 93–102).

[27] Parikh, R.J. Existence and feasibility in arithmetic. J. Symbolic Logic **36**, pp.494–508 (1971).

[28] Ritchie, R.W., Classes of recursive functions based on Ackermann's function. Pacific J. Math. **15**, pp.1027– 1044 (1965).

[29] Schwichtenberg, H., Complexity of normalization in the pure lambda–calculus. In: Troelstra, A.S.–van Dalen, D. (eds.) L.E.J. Brouwer Centenary Symposium, pp. 453–458. North–Holland (1982).

[30] Sieg, W., Fragments of arithmetic. Ann. Pure Appl. Logic **28**,pp. 33–71 (1985).

[31] Sieg, W., Reduction theories for analysis. In: Dorn, Weingartner (eds.), Foundations of Logic and Linguistics, pp. 199–231, New York, Plenum Press (1985).

[32] Sieg, W., Herbrand analyses. Arch. Math. Logic **30**, pp. 409–441 (1991).

[33] Smorynski, C. Logical number theory I. Springer Universitext, Berlin Heidelberg (1991).

[34] Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes in Mathematics **344** (1973).

[35] Troelstra, A.S., Note on the fan theorem. J. Symbolic Logic **39**, pp. 584–596 (1974).

[36] Zhang, W., Cut elimination and automatic proof procedures. Theoretical Computer Sccience **91**, pp. 265–284 (1991).