On the reverse mathematics and Weihrauch complexity of moduli of regularity and uniqueness

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(Dedicated to H. Luckhardt on the occasion of his 80th birthday)

Abstract

The notion of 'modulus of regularity', as recently studied in [19], unifies a number of different concepts used in convex optimization to establish rates of convergence for Fejér monotone iterative procedures. It generalizes the notion of 'modulus of uniqueness' to the nonunique case. In this paper, we investigate both notions in terms of reverse mathematics and calibrate their Weihrauch complexity.

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1 Introduction

In [19], the concept of modulus of regularity is introduced as a central tool to construct rates of convergence for classes of Fejér monotone sequences which appear in fixed point theory, monotone operator theory and convex optimization. The concept of modulus of regularity gives a unified account of various notions such as metric subregularity ([10, 22, 21]), Hölder regularity ([4]), error bounds ([21]) and weak sharp minima ([9]) which play a prominent role in nonlinear optimization. It is general enough to cover many equilibirium, convex feasibility, fixed point and minimization problems involving set-valued operators.

In the case where the solution in question is unique, the concept coincides with the notion of modulus of uniqueness ([13, 18, 19]) and can be understood as its generalization to the nonunique case.

While most problems in nonlinear analysis deal with classes of abstract, in general not necessarily separable, metric or normed structures, we restrict ourselves in this paper to the compact metric case. In this situation a modulus of regularity always exists for continuous functions F ([19], Proposition 3.2) and the concept has been anticipated already in [1]. We calibrate the strength of its existence in the sense of reverse mathematics as well as its Weihrauch complexity and compare this with the case of a modulus of (uniform) uniqueness in the unique case. We also consider the weaker $\forall\varepsilon\exists\delta$-version of regularity (without stating the existence of a modulus function) and show that in this form, regularity is equivalent to WKL_0 while the existence of a modulus is equivalent to ACA_0. This differs from the unique case where both the $\forall\varepsilon\exists\delta$-form as well as the modulus version of uniform uniqueness are equivalent to WKL_0. The difference also shows up in the Weihrauch complexity: the many-valued modulus-of-uniqueness operator MUNI is computable while the many-valued modulus-of-regularity operator MREG is Weihrauch equivalent to $\widehat{\mathrm{LPO}}$. Both phenomena are due to the fact that the proof already for the $\forall\varepsilon\exists\delta$-form of regularity makes substantial use of classical logic (Σ^0_1-LEM=Π^0_1-LEM+M, where M denotes the Markov principle) while in the unique case only M is used.[1]

2 Moduli of regularity and uniqueness

Definition 2.1 ([19]). *Let (X,d) be a metric space and let be $F : X \to \mathbb{R}$ a mapping. Let $zer\,F := \{x \in X : F(x) = 0\} \neq \emptyset$ and $r > 0$. We say that F is regular w.r.t. $zerF$ and $\overline{B}(z,r)$ for $z \in zerF$ if*

$$\forall n \in \mathbb{N}\,\exists k \in \mathbb{N}\,\forall x \in \overline{B}(z,r)\,\left(|F(x)| < 2^{-k} \to \exists z' \in zer\,F\,(d(x,z') < 2^{-n})\right).$$

If this holds with '$\forall x \in \overline{B}(z,r)$' replaced by '$\forall x \in X$' we say that F is regular w.r.t. $zer\,F$.
A function $f : \mathbb{N} \to \mathbb{N}$ providing given n a number $k = f(n)$ satisfying the above is called a modulus of regularity of F w.r.t. $zerF$ and $\overline{B}(z,r)$ resp. w.r.t. $zer\,F$.

Remark 2.2.

1. *In [19] the conclusion '$\exists z' \in zer\,F\,(d(x,z') < 2^{-n})$' is conveniently written using the metric distance functions as '$dist(x, zer\,F) < 2^{-n}$' but, of course, the concept does not presuppose the existence of 'dist', i.e. the locatedness of $zer\,F$.*

2. *Again for convenience and to follow the style used in analysis, the concept of a modulus of regularity in [19] is written in ε/δ-form, i.e. as a function :*

[1] The latter is already implicit in [13] and [3] since to prenex uniform uniqueness as in [13], resp. reformulating it in the strong form stated on p.714 in [3], just amounts to applying M.

$(0, \infty) \to (0, \infty)$. *All results can be, however, easily re-casted in terms of the modulus* $f : \mathbb{N} \to \mathbb{N}$ *as defined above which is more appropriate for the context of reverse mathematics and Weihrauch reducibility.*

When $zer\,F$ is not a singleton set, effective moduli of regularity can only be expected to exist in rather restricted situations (due to their strong consequences on rates of convergence for numerous iterative procedures used in nonlinear analysis). However, [19] describes important cases where such moduli can be explicitly computed.

The concept of modulus of regularity generalizes that of a 'modulus of uniqueness' to the nonunique case:

Definition 2.3 ([13, 18, 19])**.** *Let* $F : X \to \mathbb{R}$ *be such that* $zer\,F = \{z\}$.

1. *We say that* $zer\,F$ *is uniformly unique w.r.t.* $\overline{B}(z, r)$ *if*

$$\forall n \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall x \in \overline{B}(z, r)\ (|F(x)| < 2^{-k} \to d(x, z) < 2^{-n}).$$

   If this holds with '$\forall x \in X$*' we say that* $zer\,F$ *is uniformly unique.*

2. $\omega : \mathbb{N} \to \mathbb{N}$ *is a modulus of uniqueness for* F *w.r.t.* $zer\,F$ *and* $\overline{B}(z, r)$ *for* $z \in zer\,F$ *if*

$$\forall n \in \mathbb{N}\, \forall x \in \overline{B}(z, r)\, (|F(x)| < 2^{-\omega(n)} \to d(x, z) < 2^{-n}).$$

   If this holds with '$\forall x \in X$*' we say that* ω *is a modulus of uniqueness for* F *w.r.t.* $zer\,F$.

The concept of modulus of uniqueness can also be considered without assuming that $zer\,F \neq \emptyset$ in the form

$$(*)\ \forall n \in \mathbb{N}\, \forall x, y \in X\ (|F(x)|, |F(y)| < 2^{-\omega(n)} \to d(x, y) < 2^{-n}).$$

Clearly, if $zer\,F = \{z\}$, then any ω with $(*)$ is a modulus of uniqueness in the sense of definition 2.3 and conversely, if ω is a modulus of uniqueness, then $\omega'(n) := \omega(n+1)$ satisfies $(*)$. Suppose that one has an algorithmic way (x_n) to construct 2^{-n}-approximate zeros x_n, i.e. $|F(x_n)| < 2^{-n}$ of F and ω satisfies $(*)$, then $(x_{\omega(2^{-n})})$ is a 2^{-n}-Cauchy sequence whose limit (for complete X and continuous F) is a zero of F. In this way, moduli of uniqueness give rates of convergence for algorithms computing approximate solutions towards the actual solution and have been used in fixed point theory to prove even new existence results (see [7] and the literature cited there). As shown in [13] (for the case of compact metric spaces), explicit moduli of uniqueness can be extracted by proof-theoretic methods from given, even nonconstructive, proofs for the uniqueness of the zero of F. This has been carried out in the context of best Chebycheff approximation in [13, 14] and best L^1-approximation in [20] (see [18] for

a comprehensive treatment of all this). Using the logical bound extraction theorems for abstract metric and normed structures (without separability or compactness assumptions) from [17, 12] such extractions of moduli of uniqueness are also possible for abstract spaces in the absence of compactness (see [18], pp. 377-381) and have been used in metric fixed point theory e.g. in [11, 7].
While the existence of a modulus of uniqueness is a uniform quantitative version of the plain uniqueness property

$$(1)\ F(x) = 0 = F(z) \to x = z$$

and can be extracted from a given proof of the latter, the existence of a modulus of regularity is a uniform quantitative version of the following trivially true (but logically more complex than (1)) property:

$$(2)\ F(x) = 0 \to \forall \varepsilon > 0\, \exists z \in zer\, F\, (d(x,z) < \varepsilon).$$

So in this generality, there is no meaningful property such that from a proof of this property a modulus of regularity can be extracted. Thus unless one is in the unique case (where the concept of a modulus of regularity coincides with that of modulus of uniqueness), one has to exploit rather specific features of the situation at hand in order to get an effective modulus of regularity (see also the comments in the introduction and [19]).

3 Reverse mathematics

In the following, RCA_0 is the usual base system used in reverse mathematics, i.e. the fragment of second order arithmetic with recursive (Δ^0_1) comprehension and Σ^0_1-induction only. WKL_0 and ACA_0 are its extension by the weak König's lemma WKL for 0/1-trees and the schema of arithmetic comprehension ACA, respectively. For details we refer to [25].
We refer to the definition of compact metric spaces X as used in reverse mathematics ([25], Definition III.2.3), where X is given as the completion $\widehat{A}$ of a countable pseudometric space A which, additionally, possesses a sequence of finite ε-nets. We recall some crucial results from [25]:

Theorem 3.1 ([25], Theorem IV.1.6)**.** *The following is provable in* WKL_0. *Let X be a compact metric space. Let $\langle\langle U_{n,k} : k \in \mathbb{N}\rangle : n \in \mathbb{N}\rangle$ be a sequence of coverings of X by open sets. Then there exists a sequence of finite subcoverings $\langle\langle U_{n,k} : k \leq l_n\rangle : n \in \mathbb{N}\rangle$.*

Theorem 3.2 ([25], Theorem IV.1.7)**.** *The following is provable in* WKL_0. *Let X be a compact metric space. Let C be a code for a closed subset in X. Then the nonemptyness of C can be expressed by a Π^0_1-formula.*

Proposition 3.3 ([25], Exercise II.6.9, [8](Lemma 1.24)). *The following is provable in* RCA_0. *Let* X, Y *be complete separable metric spaces and* $\Phi : X \to Y$ *a continuous function. Let* $V \subseteq Y$ *be (a code of) an open set, then* $\Phi^{-1}(V)$ *is open (with a code computable from a code for* V*).*

Corollary 3.4. *The following is probable in* WKL_0. *Let* X *be a compact space and* $F : X \to \mathbb{R}$ *be continuous, then the property that* F *has a zero on* X *can be expressed by a* Π^0_1*-formula.*

Proof: Clearly, RCA_0 proves that $\mathbb{R} \setminus \{0\}$ is open. Hence by Proposition 3.3, $\{x \in X : F(x) \neq 0\}$ has a code as an open set and so $zer\,F = \{x \in X : F(x) = 0\}$ has a code as a closed set. So by Theorem 3.2, provably in WKL_0, the nonemptyness of $zer\,F$ can be expressed by a Π^0_1-formula. $\square$

Remark 3.5. *The proofs of the results above establish that even if we have sequences* $\langle \Phi_n : n \in \mathbb{N}\rangle$ *and* $\langle F_n : n \in \mathbb{N}\rangle$ *uniformly given as sequences of codes that then* RCA_0 *proves that* $\langle \Phi_n^{-1}(V) : n \in \mathbb{N}\rangle$ *is a sequence of open sets and* WKL_0 *proves that the nonemptyness of* $zer\,F_n$ *can expressed as a* Π^0_1*-formula with* n *as parameter. The latter is particularly easy to see in our instances below, where the functions* Φ_n *are defined in terms of a single function* Φ *and any modulus of uniform continuity for* Φ *can be modified into one for all* Φ_n *uniformly in* n *(see also the explicit construction of the* Π^0_1*-formula in the proof of Lemma 5.5 below).*

4 Reverse mathematics of moduli of uniqueness and regularity

Theorem 4.1. 1. WKL_0 *proves that for every compact metric space* $X = \widehat{A}$ *any continuous mapping* $F : X \to \mathbb{R}$ *having at most one zero has a modulus* ω *such that*

$$\forall n \in \mathbb{N}\, \forall x, y \in X\ (|F(x)|, |F(y)| < 2^{-\omega(n)} \to d(x, y) < 2^{-n}).$$

Obviously, if $zer F = \{z\}$, *then* ω *is a modulus of uniqueness of* F *w.r.t.* $zer\,F$.

2. *Already for Lipschitz continuous functions* $F : [0, 1] \to \mathbb{R}$ *which have exactly one zero, the uniform uniqueness of the zero implies* WKL_0 *over* RCA_0.

Proof: 1) Let F possess at most one zero and define

$$U_{n,k} := \{(x, y) \in X \times X : |F(x)|, |F(y)| \leq 2^{-k} \to d(x, y) < 2^{-n}\}.$$

$\langle\langle U_{n,k} : k \in \mathbb{N}\rangle : n \in \mathbb{N}\rangle$ is a sequence of coverings of $X \times X$ (w.r.t. the product metric, [25], Example II.5.4) by open sets. Here one uses Proposition 3.3 and the fact that for the continuous functions

$$G_{n,k}(x, y) := \max\big\{\max\{|F(x)|, |F(y)|\} - 2^{-k}, 2^{-n} - d(x, y), 0\big\}$$

one has

$$(x,y) \in U_{n,k} \leftrightarrow (x,y) \notin (G_{n,k})^{-1}(\{0\}).$$

By Theorem 3.1 it follows that, provably in WKL_0, there is a sequence $\langle\langle U_{n,k} : k \leq \alpha(n)\rangle : n \in \mathbb{N}\rangle$ of finite subcoverings. Clearly, $\alpha : \mathbb{N} \to \mathbb{N}$ is a modulus of uniqueness.
2) Assume $\neg$-WKL. Then there exists an infinite 0/1-tree T with no path. Consider the subtrees T_0 and T_1 of T consisting of all finite 0/1-sequences in T which start with 0 or which start with 1 resp. One of those subtrees must be infinite as well. W.l.o.g. assume that T_0 is infinite. Define $T[1]$ as T_0 augmented by the constant-1 path. With this $T[1]$ (playing the role of T) now define as in [25] (p.129) a sequence $\langle I_n = (r_n, s_n) : n \in \mathbb{N}\rangle$ of nonempty open intervals with rational endpoints. Adapting the reasoning there to our situation (using that $b'_s \leq 2/3$ for $s \in 2^{<\mathbb{N}}$ with $\mathrm{lh}(s) \geq 1$ and $s(0) = 0$) one gets

(i) $\forall x \in [0,1)\, \exists n \in \mathbb{N}\, (x \in I_n)$, (ii) $\forall k \in \mathbb{N}\, \exists x \in [0,2/3]\, \forall i \leq k\, (x \notin I_i)$,
(iii) $\forall n \in \mathbb{N}\, (1 \notin I_n)$.

Now we define (see [28], p.309):

$$F : [0,1] \to [0,1], F(x) := \sum_{n=0}^{\infty} 2^{-n-1} F_n(x),$$

where

$$F_n(x) := \max\{0, \frac{1}{2}(s_n - r_n) - |x - \frac{1}{2}(r_n + s_n)|\}.$$

Then, clearly, $\forall x \in [0,1)\, (F(x) > 0)$ by (i), $\inf\limits_{x\in[0,2/3]} F(x) = 0$ by (ii), and $F(1) = 0$ by (iii). F is nonexpansive but obviously is not uniformly unique w.r.t. $zer\, F = \{1\}$.[2]
$\square$

Theorem 4.2. *1.* WKL_0 *proves that for every compact metric space* $X = \widehat{A}$ *any continuous mapping* $F : X \to \mathbb{R}$ *having a zero is regular w.r.t.* $zer\, F$.

2. Already for Lipschitz continuous functions $F : [0,1] \to \mathbb{R}$ *with* $zer\, F \neq \emptyset$, *the regularity of* F *w.r.t.* $zer\, F$ *implies* WKL_0 *over* RCA_0.

Proof: 1) Take $k \in \mathbb{N}$ and consider the finite cover of X by closed balls $\overline{B}(a_1, 2^{-k-1}), \ldots, \overline{B}(a_{n_k}, 2^{-k-1})$ provided by the representation of X as a compact metric space (in the sense of [25](Definition III.2.3)), where $a_1, \ldots, a_{n_k}$ are in the

[2]Alternatively, we could have modified the mapping Φ_5 in the proof of Theorem IV.2.3.5 in [25] by using '$1 - 3^{-lh(u)}$' instead of '$1 - 2^{-lh(u)}$' to achieve the Lipschitz property. However, we find the construction above more elementary. One can also adapt Specker's [27] construction or the Lipschitz functions defined in [2].

countable set A whose completion $\widehat{A}$ the space X is defined to be. Using WKL_0, the predicate

$$P(i) :\equiv \exists x \in X\, (d(a_i, x) \leq 2^{-k-1} \wedge F(x) = 0) \quad (1 \leq i \leq n_k)$$

is in Π^0_1. Here we use that $P(i)$ can be written as $\exists x \in X\, (G_i(x) = 0)$ for the continuous function $G_i : X \to \mathbb{R}$ defined by

$$G_i(x) := \max\left\{|F(x)|, \max\{2^{-k-1}, d(a_i, x)\} - 2^{-k-1}\right\} \geq 0$$

and Corollary 3.4. Hence by bounded Π^0_1-comprehension (provable in RCA_0, see [24], Theorem 1, or [25], Theorems II.3.9 and II.2.5) one gets the existence of a code σ of a finite 0/1-sequence of length n_k such that

$$\forall i\, (1 \leq i \leq n_k \to ((\sigma)_{i-1} = 0 \leftrightarrow P(i))).$$

Consider now an i with $\neg P(i)$. Then, again by WKL_0 and using [25](Theorem IV.2.2), one gets the existence of an $l_i \in \mathbb{N}$ with

$$\inf\{G_i(x) : x \in X\} > 2^{-l_i}$$

and hence

$$\forall x \in \overline{B}(a_i, 2^{-k-1})\, (|F(x)| > 2^{-l_i}).$$

In WKL_0 (needed to show the existence of a modulus of uniform continuity for F and hence for G_i) one can show that the sequence (a_i) with $a_i := \inf\{|G_i(x)| : x \in X\}$ exists. So by Σ^0_1-bounded collection (provable in RCA_0) we can prove

$$\exists l \in \mathbb{N}\, \forall i\, (1 \leq i \leq n_k \wedge (\sigma)_{i-1} = 1 \to \inf\{G_i(x) : x \in X\} > 2^{-l}).$$

Now assume that $|F(x)| \leq 2^{-l}$ for some $x \in X$. Then, by the definition of l, x must be in one of the balls $\overline{B}(a_i, 2^{-k-1})$ (with $1 \leq i \leq n_k$) for which $(\sigma)_{i-1} = 0$, i.e. which contains a zero z of F. Since $d(x, z) \leq 2^{-k}$, the conclusion follows.
2) is an immediate corollary to Theorem 4.1.2 as in the unique case the concepts of regularity and uniform uniqueness coincide. □

Remark 4.3. *The proof of Theorem 4.2.1 uses classical logic in the form of* Π^0_1-LEM *(implicit in the bounded* Π^0_1-CA*) and* M *(implicit in concluding*
$\forall x \in \overline{B}(a_i, 2^{-k-1})\, \exists l \in \mathbb{N}\, (G_i(x) > 2^{-l})$ *from* $\neg\exists x \in \overline{B}(a_i, 2^{-k-1})\, (G_i(x) = 0)$.

Theorem 4.4. *1.* ACA_0 *proves that if* $X = \widehat{A}$ *is a compact metric space and* $F : X \to \mathbb{R}$ *is continuous and has a zero, then* F *possesses a modulus of regularity w.r.t.* $zerF$.

2. *Over* RCA$_0$, *the statement that every Lipschitz continuous function* $F : [0,1] \to \mathbb{R}$ *with a zero has a modulus of regularity implies* ACA$_0$.

Proof: 1) Let $S := zer\,F$. By Theorem 4.2.1, already WKL$_0$ suffices to prove:

$$(1)\ \forall n \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall l \in \mathbb{N}\ \big(|F(a_l)| < 2^{-k} \to \exists p \in S(d(a_l,p) \leq 2^{-n})\big).$$

We now show (in WKL$_0$) that

$$(2)\ P(n,k,l) :\equiv \big(|F(a_l)| < 2^{-k} \to \exists p \in S(d(a_l,p) \leq 2^{-n})\big) \in \Pi^0_1.$$

Define (uniformly in n,l) continuous functions $G_{n,l} : X \to \mathbb{R}$ by

$$G_{n,l}(x) := \max\{|F(x)|, \max\{2^{-n}, d(a_l,x)\} - 2^{-n}\}.$$

Then

$$P(n,k,l) \leftrightarrow \big(|F(a_l)| < 2^{-k} \to \exists p \in X\,(G_{n,l}(p) = 0)\big).$$

Hence, by Corollary 3.4, WKL$_0$ proves (see also the proof of Lemma 5.5) that $P(n,k,l) \in (\Sigma^0_1 \to \Pi^0_1) = \Pi^0_1$.
(1) and (2) imply that

$$\forall n \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall l \in \mathbb{N}\, P(n,k,l) \text{ with } \forall l \in \mathbb{N}\, P(n,k,l) \in \Pi^0_1$$

and so by (a single use of) Π^0_1-AC$^{\mathbb{N},\mathbb{N}}$, and thus, in particular, in ACA$_0$, we get

$$\exists f : \mathbb{N} \to \mathbb{N}\, \forall n,l \in \mathbb{N}\ \big(|F(a_l)| < 2^{-f(n)} \to \exists p \in S\,(d(a_l,p) \leq 2^{-n})\big).$$

Using the continuity of F and that $\{a_l : l \in \mathbb{N}\}$ is dense in X we get

$$\exists f : \mathbb{N} \to \mathbb{N}\, \forall n \in \mathbb{N}\, \forall x \in X\ \big(|F(x)| < 2^{-f(n)} \to \exists p \in S\,(d(x,p) < 2^{-n+1})\big)$$

and so $g(n) := f(n+1)$ is a modulus of regularity for F w.r.t. $zer\,F$.
2) We use a construction from the proofs of Remarks 4.9 and 4.10 in [19] which in turn adapt a construction due to [23]. Let (a_n) be a nondecreasing sequence of rational numbers in $[0,1]$ and define

$$\begin{array}{l} f_n : [0,1] \to [0,1],\ f_n(x) := \max\{x, a_n\}, \\ T : [0,1] \to [0,1], T(x) := \frac{1}{2}(x + f(x)),\ \text{where } f(x) := \sum_{n=0}^{\infty} 2^{-n-1} f_n(x). \end{array}$$

f is nonexpansive and, therefore, T is nonexpansive too (even firmly nonexpansive) and 1 is fixed point of T. By primitive recursion one can easily show in RCA$_0$ that the sequence (x_n) defined by $x_n := T^n 0$ exists. By the comment after Corollary 1 in [16], $\alpha(n) := n+3$ is a rate of asymptotic regularity for (x_n), i.e.

$$\forall n \in \mathbb{N}\, \forall k \geq n+3\, (|x_k - Tx_k| < 2^{-n}).$$

All this can easily be established in RCA_0. Suppose now that the 2-Lipschitz function $F : [0,1] \to \mathbb{R}$ $F(x) := |x - Tx|$ has a modulus of regularity $g : \mathbb{N} \to \mathbb{N}$ (note that $1 \in zer\, F \neq \emptyset$). Then (reasoning as in the proof of Theorem 4.1 in [19]; note that (x_n) obviously is Fejér monotone w.r.t. $zer\, F = Fix(T)$ since T is nonexpansive) one can easily show in RCA_0 that $\rho(n) := \alpha(g(n+1)) = g(n+1)+3$ is a rate of convergence for (x_n). So $z := \lim\limits_{n\to\infty} x_n$ can be shown in RCA_0 to exist and is a fixed point of T, i.e. a zero of F. Since $f(z) = z$, it is clear that $\forall n \in \mathbb{N}\, (a_n \leq z)$. Suppose that there would exist a $k \in \mathbb{N}$ with $a_n + 2^{-k} \leq z$ for all $n \in \mathbb{N}$. Then by Π^0_1-induction (and hence in RCA_0) also $x_n \leq z - 2^{-k}$ for all $n \in \mathbb{N}$ in contradiction to $\lim x_n = z$: clearly $x_0 = 0 \leq a_n \leq z - 2^{-k}$. Assume that $x_n \leq z - 2^{-k}$. Then

$$\begin{array}{rl} x_{n+1} = \frac{1}{2}(x_n + f(x_n)) = & \frac{1}{2}\left(x_n + \sum_{l=0}^{\infty} 2^{-l-1} \max\{x_n, a_l\}\right) \\ \overset{\text{I.H., assumption}}{\leq} & \frac{1}{2}\left(z - 2^{-k} + \sum_{l=0}^{\infty} 2^{-l-1}(z - 2^{-k})\right) = z - 2^{-k}. \end{array}$$

Hence $z = \lim a_n$. The claim now follows from the well-known fact that the convergence of increasing sequences of rational numbers in $[0,1]$ is equivalent to ACA_0 over RCA_0 ([25], Theorem III.2.2, and note that the sequence (c_n) constructed in the relevant part of the proof of this theorem is an increasing sequence of rational numbers in $[0,2]$ so that we may take $a_n := c_n/2$; alternatively one can also adapt Specker's [26] construction). $\square$

5 Weihrauch complexity of moduli of uniqueness and regularity

We recall the standard concepts used in the notion of Weihrauch reducibility which can be found e.g. in [5].

Definition 5.1. *A represented space is a pair (X, δ_X), where X is a set and $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is a partial surjective function.*

Definition 5.2. *Let $(X, \delta_X), (Y, \delta_Y)$ be represented spaces and let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function. Then $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a realizer of f if*

$$\forall p \in dom(f\delta_X)\ (\delta_Y(F(p)) \in f(\delta_X(p))).$$

Definition 5.3. *Let f, g be multi-functions on represented spaces. Then f is said to be Weihrauch reducible to g, in symbols $f \leq_W g$, if there are computable functions $H, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $H\langle id, GK\rangle$ is a realizer of f whenever G is a realizer of g. We say that f and g are Weihrauch equivalent, in symbols $f \equiv_W g$, if both $f \leq_W g$ and $g \leq_W f$.*

The parallelization $\widehat{\mathrm{LPO}}$ of the 'omniscience principle' LPO (i.e. Σ^0_1-LEM) is defined as[3]

$$\widehat{\mathrm{LPO}} : \mathbb{N}^{\mathbb{N}\times\mathbb{N}} \to \mathbb{N}^\mathbb{N},\ \widehat{\mathrm{LPO}}(q)(n) = \begin{cases} 0, & \text{if } \exists k \in \mathbb{N}\,(q(n,k) = 0) \\ 1, & \text{otherwise.} \end{cases}$$

The formulation of the convergence principle for bounded monotone sequences of reals is formulated in the framework of Weihrauch complexity as follows

$$\mathrm{MCT} :\subseteq \mathbb{R}^\mathbb{N} \to \mathbb{R},\ (x_n) \mapsto \sup_{n\in\mathbb{N}} x_n$$

with dom(MCT)$= \{(x_n) : \forall n \in \mathbb{N}\,(x_n \leq x_{n+1})$ and (x_n) bounded$\}$. $\mathrm{MCT}_{\mathbb{Q}\cap[0,1]}$ is the restriction of MCT to $\{(r_n) \in \mathbb{Q}^\mathbb{N} : \forall n \in \mathbb{N}\,(0 \leq r_n \leq r_{n+1} \leq 1)\}$.
It is well-known (see e.g. [6], Facts 3.5 and 11.26; the result essentially is also already in [15], Proposition 5.5) that

$$\mathrm{MCT}_{\mathbb{Q}\cap[0,1]} \equiv_W \mathrm{MCT} \equiv_W \widehat{\mathrm{LPO}}.$$

Let us define

$$\mathrm{MREG}_{[0,1]} :\subseteq C[0,1] \rightrightarrows \mathbb{N}^\mathbb{N},\ F \mapsto \{f \in \mathbb{N}^\mathbb{N} : f \text{ modulus of regularity of } F \text{ w.r.t. } zer\,F\},$$

with $\mathrm{dom}(\mathrm{MREG}_{[0,1]}) := \{F \in C[0,1] : zer\,F \neq \emptyset\}$, and

$$\mathrm{MUNI}_{[0,1]} :\subseteq C[0,1] \rightrightarrows \mathbb{N}^\mathbb{N},$$
$$F \mapsto \{f \in \mathbb{N}^\mathbb{N} : f \text{ modulus of uniqueness } (*) \text{ of } F \text{ w.r.t. } zer\,F\},$$

with $\mathrm{dom}(\mathrm{MUNI}_{[0,1]}) := \{F \in C[0,1] : F$ has at most one zero$\}$.
We will show that $\mathrm{MUNI}_{[0,1]}$ is computable and so this a fortiori holds for its restriction to those F which have exactly one zero.
In the proofs below we refer to the standard representations of $\mathbb{R}$ and $C[0,1]$ but suppress explicitly mentioning them.

Lemma 5.4. $\mathrm{MCT} \leq_W \mathrm{MREG}_{[0,1]}$.

[3] The official definition is slightly different but modulo Currying trivially equivalent to this, see [5] Lemma 6.3, where our formulation is called C.

Proof: By the comments above, it suffices to show that

$$\mathrm{MCT}_{\mathbb{Q}\cap[0,1]} \leq_{\mathrm{W}} \mathrm{MREG}_{[0,1]}.$$

As the proof of Theorem 4.4.2 shows, uniformly in a given increasing sequence (a_n) of rational numbers in $[0,1]$ one can compute a (2-Lipschitz) function $F := K((a_n)) \in \mathrm{dom}(\mathrm{MREG}_{[0,1]})$ such that if $g \in \mathrm{MREG}_{[0,1]}(F)$, then $a := \lim a_n = \sup a_n$ can be computed uniformly as a functional $H((a_n), g)$ using that $g(n+1)+3$ is a rate of convergence for $(T^n 0)$ whose limit is a (note that in (a_n), the mapping T, and hence the sequence $(T^n 0)$, is computable). $\square$

Lemma 5.5. $\mathrm{MREG}_{[0,1]} \leq_{\mathrm{W}} \widehat{\mathrm{LPO}}$.

Proof: Let $F \in C[0,1]$ with $zer\, F \neq \emptyset$. From a name of F in the usual standard representation of $C[0,1]$ one can compute a modulus $\omega : \mathbb{N} \to \mathbb{N}$ of uniform continuity for F, i.e.

$$\forall k \in \mathbb{N}\, \forall x, y \in [0,1]\, (|x-y| < 2^{-\omega(k)} \to |F(x) - F(y)| < 2^{-k}),$$

and from this a (common) modulus $\tilde{\omega}(k) := \max\{\omega(k), k\}$ of uniform continuity for the function $G_{n,l}$ from the proof of Theorem 4.4.1 for all $n, l \in \mathbb{N}$. Using compactness (WKL) the statement

$$\exists p \in [0,1]\, (G_{n,l}(p) = 0)$$

can be written equivalently as

$$\forall m\, \exists i \leq 2^{\tilde{\omega}(m)}\, \left(\left| G_{n,l}(i/2^{\tilde{\omega}(m)})(m) \right| <_{\mathbb{Q}} 2^{-m+1} \right) \in \Pi^0_1,$$

where for (a name of) $x \in \mathbb{R}$, '$x(m)$' denotes the 2^{-m}-rational approximation to x provided by that name. Note that WKL is only needed to verify the above equivalence but not to construct the Π^0_1-formula from a name of F.

With (a_l) being some standard enumeration of the dyadic rational numbers in $[0,1]$, the property $\forall l \in \mathbb{N}\, P(n,k,l)$ in the proof of Theorem 4.4.1 can now be written (coding three universal quantifiers into a single one) as

$$\forall l \in \mathbb{N}\, (q(\langle n, k\rangle, l) \neq 0)$$

for a function $q \in \mathbb{N}^{\mathbb{N}}$ which can be uniformly computed as a function $q := K(F)$ in (a name of) F. Now let $p = \widehat{\mathrm{LPO}}(q)$. Then the statement

$$\forall n \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall l \in \mathbb{N}\, P(n,k,l)$$

established in the proof of Theorem 4.4.1 has the form

$$\forall n \in \mathbb{N}\, \exists k \in \mathbb{N}\, (p(\langle n, k\rangle) = 1)$$

so that an $f \in \mathbb{N}^{\mathbb{N}}$ with

$$\forall n \in \mathbb{N}\, \forall l \in \mathbb{N}\, P(n, f(n), l)$$

can be uniformly computed in p as

$$f := H(p) := \lambda n. \min k\, [p(\langle n, k\rangle) = 1].$$

As in the proof of Theorem 4.4.1 it follows that $g(n) := f(n+1)$ is a modulus of regularity for F w.r.t. $zer\, F$, i.e. $g \in \mathrm{MREG}_{[0,1]}(F)$.

$\square$

Corollary 5.6.

$$\mathrm{MREG}_{[0,1]} \equiv_{\mathrm{W}} \widehat{\mathrm{LPO}}.$$

In contrast to this, we have that $\mathrm{MUNI}_{[0,1]}$ is computable (in fact this holds for every computably compact computable Polish space K instead of $[0,1]$ but for the sake of simplicity we treat here only the case $[0,1]$):

Proposition 5.7. $\mathrm{MUNI}_{[0,1]}$ *is computable.*

Proof: We use the representation of $[0,1]$ from [18] (Chapter 4) by which

- each $f \in \mathbb{N}^{\mathbb{N}}$ represents a unique element in $[0,1]$,
- primitive recursively in $f \in \mathbb{N}^{\mathbb{N}}$ one can define $\tilde{f} \leq N := \lambda n.j(2^{n+3}, 2^{n+2}-1)$ such that $\tilde{f}$ represents the same real in $[0,1]$ as f does (here $f \leq g :\equiv \forall n \in \mathbb{N}\, (f(n) \leq g(n))$).

On the level of names f, g for $x, y \in [0,1]$ and a given name $\widehat{F} \in \mathbb{N}^{\mathbb{N}}$ for $F \in C[0,1]$ one can express

$$(x,y) \in U_{n,k} :\equiv \big(|F(x)|, |F(y)| \leq 2^{-k} \to |x-y| < 2^{-n}\big)$$

as a Σ^0_1-formula

$$\exists l \in \mathbb{N}\, (\Phi(\widehat{F}, f, g, n, k, l) = 0),$$

where Φ is a (primitive recursively) computable functional : $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^3 \to \mathbb{N}$. If F has at most one zero, then

$$\tilde{\Psi}(f, g, n) := \min m. [\Phi(\widehat{F}, f, g, n, (m)_0, (m)_1) = 0]$$

defines (computably in $\widehat{F}$) a total function : $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$. With $\tilde{\Psi}$ also $\Psi(f, g, n) := (\tilde{\Psi}(f, g, n))_0$ is computable in $\widehat{F}$. Hence the restriction of $\lambda f, g.\Psi(f, g, n)$

to functions $f, g \leq N$ has (uniformly in n) a modulus of uniform continuity $\omega(n,k)$ which is computable in $\widehat{F}$, i.e.

$$\forall f_1, f_2, g_1, g_2 \leq N\, \forall k, n \in \mathbb{N}$$
$$\big(\overline{f}_1(\omega(n,k)) = \overline{f}_2(\omega(n,k)) \wedge \overline{g}_1(\omega(n,k)) = \overline{g}_2(\omega(n,k)) \to \Psi(f_1, g_1, n) = \Psi(f_2, g_2, n)\big)\,.$$

Using ω one can compute (uniformly in $\widehat{F}$)

$$\alpha(n) := \sup\{\Psi(f,g,n) : f, g \leq N\}\,.$$

Clearly, α is a modulus of uniform uniqueness in the form $(*)$ for *zer* F. $\square$

The proof above uses an unbounded search which terminates by the assumption of the uniqueness of the solution and which does not provide any complexity information. This is in contrast to the situation where one has a **proof** (even if that is prima facie noneffective) for the uniqueness from which - as discussed briefly in section 2 - one can then extract a modulus of uniqueness which reflects the numerical content of that uniqueness proof.

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