RATES OF CONVERGENCE AND METASTABILITY FOR ABSTRACT CAUCHY PROBLEMS GENERATED BY ACCRETIVE OPERATORS

ULRICH KOHLENBACH AND ANGELIKI KOUTSOUKOU-ARGYRAKI

Department of Mathematics
Technische Universität Darmstadt
Schlossgartenstraße 7
64289 Darmstadt, Germany

Abstract. We extract rates of convergence and rates of metastability (in the sense of Tao) for convergence results regarding abstract Cauchy problems generated by ϕ-accretive at zero operators $A : D(A)(\subseteq X) \to 2^X$ where X is a real Banach space, proved in [8], by proof-theoretic analysis of the proofs in [8] and having assumed a new, stronger notion of uniform accretivity at zero, yielding a new notion of modulus of accretivity. We compute the rate of metastability for the convergence of the solution of the abstract Cauchy problem generated by a uniformly accretive at zero operator to the unique zero of A via proof mining based on a result by the first author. Finally, we apply our results to a special class of Cauchy problems considered in [8]. This work is the first application of proof mining to the theory of partial differential equations.

1. Introduction

In this paper we establish explicit quantitative versions of the asymptotic behavior of solutions of a class of abstract Cauchy problems generated by accretive operators $A : D(A) \to 2^X$ in a real Banach space X. The operator A is assumed to satisfy the range condition, to have a zero $z \in D(A)$ i.e. $0 \in Az$, and to satisfy the condition of being ϕ-accretive at zero in the sense of [8]. This condition, in particular, implies that z is uniquely determined.

In [8] it is shown that for such A the integral solution of the problem

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty)$$

$$u(0) = x_0,$$

where $f \in L^1(0, \infty, X)$, converges to z as $t \to \infty$.

In fact, this is a corollary of a general theorem in [8] about this convergence for general so-called almost-orbits $v(t)$ of the nonexpansive semigroup $\mathcal{F} := \{S(t) :$

Date: Sep. 30, 2014
2010 Mathematics Subject Classification 03F10, 47H06, 35F25, 47H20.
Key words and phrases. proof mining, accretive operator, Cauchy problems, rate of convergence, rate of metastability, partial differential equations.

$\overline{D(A)} \to \overline{D(A)} : t \geq 0\}$ generated by $-A$. Here $v : [0, \infty) \to \overline{D(A)}$ is an almost-orbit of $\mathcal{F}$ if

$$(*)\ \lim_{s\to\infty}\big(\sup_{t\in[0,\infty)} \|v(t+s) - S(t)v(s)\|\big) = 0.$$

Analyzing the proof of the main theorem from [8] we extract an explicit computation which, in particular, eventually translates any given rate of convergence for $(*)$ into a rate of convergence of the integral solution of the Cauchy problem towards the unique zero z of A. In the case of $f \in L^1(0, \infty, X)$ such a rate of convergence on $(*)$ amounts to knowing a rate of convergence of

$$(**)\ \int_s^\infty \|f(\xi)\|d\xi \overset{s\to\infty}{\longrightarrow} 0.$$

Such a rate, however, is not possible to compute in just f and an upper bound $M \geq \int_0^\infty \|f(\xi)\|d\xi$, and even when it is computable it will strongly depend on the particulars of f.

In [22, 23], Tao introduced a so-called metastable reformulation of convergence (more precisely of the Cauchy property of a sequence) which noneffectively implies full convergence but which constructively is weaker and often easy to enhance with an explicit and highly uniform quantitative rate. In the case at hand such a 'rate of metastability' for $(**)$ which **only** depends on M but not on f itself can be easily computed, by a result of the first author in [14]. Then, however, also the conclusion has to be rephrased in metastable terms. The main result of this paper is the explicit construction for the metastable version of the convergence of the solution of our Cauchy problem towards the unique zero z of A.

Let us now explain the term 'metastable': consider a function $v : [0, \infty) \to X$ for some normed space X and suppose that for some $x \in X$

$$\lim_{t\to\infty} \|v(t) - x\| \to 0,$$

i.e.

$$(a)\ \forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall t \geq n\, (\|v(t) - x\| < 2^{-k}).$$

(a) is equivalent to

$$(b)\ \forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \in \mathbb{N}\, \forall t \in [n, n + g(n)]\, (\|v(t) - x\| < 2^{-k}).$$

Obviously, (b) follows from (a) constructively just by restricting $[n, \infty)$ to $[n, n + g(n)]$. For the other direction, however, one has to argue by contradiction and there is no way to pass from a bound on (b) to a bound on (a) in a computational way. The reformulation (b) of (a) is known in logic since the 30's as Herbrand normal form and asking for a bound $\Phi(k, g)$ on $\exists n$, i.e.

$$(c)\ \forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k, g)\, \forall t \in [n, n + g(n)]\, (\|v(t) - x\| < 2^{-k}),$$

means to solve the so-called 'no-counterexample interpretation' ([19, 20]) of (a) which is (for statements of the form $\forall\exists\forall$) a special case of the so-called Gödel functional interpretation (see [14]). We call such a Φ a rate of metastability for the convergence statement (a). Whereas in general noneffective proofs of convergence statements (a) might not provide a (uniform) computable rate of convergence, highly uniform rates of metastability can under very general conditions always be extracted using tools from mathematical logic like the aforementioned Gödel functional interpretation. During the last 15 years this has resulted in the project of 'proof mining' which uses such tools to uncover hidden computational and quantitative information from proofs (see [14] for a comprehensive treatment up to 2008). This approach has been particularly successful in the fixed point theory of nonexpansive and accretive maps (see e.g. [15, 16, 17, 18]). Our paper can also be viewed as a new case study in this program where we for the first time treat abstract Cauchy problems given by general set-valued accretive operators $A : D(A) \to 2^X$. This is both the first time set-valued accretive operators are treated within proof mining, and the first application of proof mining to partial differential equations. The uniformity of our bounds not only is witnessed by the fact that f only enters via a bound on its L^1-norm but also by the fact that our bounds do not depend on A itself but only on a 'modulus of uniform accretivity at zero' which we will introduce in the next section. In particular, such a modulus exists in the case of uniform ϕ-accretivity at zero (that we introduce in the next section) as well as in the case of ψ-strong accretivity. While uniform ϕ-accretivity at zero is more restrictive than ϕ-accretivity at zero, it is implied by m-ψ-strong accretivity (see [9] where ψ-strong accretivity is called and 'ϕ-strong accretivity') and still covers all but one of the applications to concrete Cauchy problems discussed in [8].

2. Preliminaries

Unless specified otherwise, we follow the notation and conventions in [8]. X is a real Banach space with dual space $\tilde{X}$. The normalized duality mapping is defined by

$$J(x) := \{j \in \tilde{X} : \langle x, j\rangle = \|x\|^2,\ \|j\| = \|x\|\}.$$

Let $\langle y, x\rangle_+ := \max\{\langle y, j\rangle :\ j \in J(x)\}$. We recall the following definitions.

Definition 1. *Let $\mathcal{F} = \{S(t) :\ C \to C,\ t \geq 0\}$ be a family of self-mappings of $C \subseteq X$. $\mathcal{F}$ is said to be a nonexpansive semigroup acting on C if*

(1) *$S(0) = I$, where I is the identity mapping on C,*
(2) *$S(s+t)x = S(s)S(t)x$ for all $s, t \in [0, \infty)$ and $x \in C$,*
(3) *$\|S(t)x - S(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \in [0, \infty)$,*
(4) *$t \to S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$.*

Definition 2. *A continuous function $u : [0, \infty) \to C \subseteq X$ is said to be an almost-orbit of $\mathcal{F}$ if*

$$\lim_{s\to\infty}\Big(\sup_{t\in[0,\infty)} \|u(t+s) - S(t)u(s)\|\Big) = 0.$$

A mapping $A : X \to 2^X$ will be called an operator on X. The domain and range of A will be denoted by $D(A)$ and $R(A)$ respectively. Here $x \in D(A) :\equiv Ax \neq \emptyset$. We say that A satisfies the range condition if $\overline{D(A)} \subseteq R(I + \lambda A)$ for all $\lambda > 0$.

2.1. **Notions of Accretivity.**

Definition 3. *An operator A is said to be accretive if for all $\lambda \geq 0$, $u \in Ax$, $v \in Ay$,*

$$\|x - y + \lambda(u - v)\| \geq \|x - y\|.$$

This notion was originally introduced in 1967 independently by F.E. Browder ([3]), T. Kato ([11]) and Y. Komura ([12]). Note that A is accretive if and only if $-A$ is dissipative.

It is known (see, for instance, [5], [7]) that an operator A on X is accretive if and only if for all $(x,u),(y,v) \in A$

$$\langle u-v, x-y\rangle_+ \geq 0.$$

Definition 4. *An accretive operator A is said to be m-accretive if for all $\lambda > 0$, $R(I+\lambda A) = X$.*

Definition 5. *Let $\psi : \mathbb{R}^+ \to [0,\infty)$ be a continuous function such that $\psi(0) = 0$ and $\psi(x) > 0$ for $x \neq 0$. An accretive operator $A : D(A) \to 2^X$ on a real Banach space X is ψ-strongly accretive if*

$$\forall (x,u),(y,v) \in A \; (\langle u-v, x-y\rangle_+ \geq \psi(\|x-y\|)\|x-y\|).$$

Notice that if A is ψ-strongly accretive, then A has a unique zero $z \in D(A)$. To prove this, let $z, z' \in D(A)$ so that $Az \ni 0$, $Az' \ni 0$ and $z \neq z'$. Then

$$\langle 0, z-z'\rangle_+ \geq \psi(\|z-z'\|)\|z-z'\|$$

but by definition

$$\psi(\|z-z'\|) \leq 0 \to \|z-z'\| = 0,$$

hence $z = z'$.

We introduce a quantitative form of the above notion that we call *modulus of accretivity Θ for ψ-strong accretivity*:

Definition 6. *Given a real Banach space X and a function $\Theta_{(\cdot)}(\cdot) : \mathbb{N}\times\mathbb{N}^* \to \mathbb{N}$ we say that a ψ-strongly accretive operator $A : D(A) \to 2^X$ has a modulus of accretivity Θ if*

$$\forall k \in \mathbb{N} \; \forall K \in \mathbb{N}^* \; \forall (x,u),(y,v) \in A$$
$$(\|x-y\| \in [2^{-k}, K] \to \langle u-v, x-y\rangle_+ \geq 2^{-\Theta_K(k)}\|x-y\|).$$

Proposition 1. *Let X be a real Banach space. Every ψ-strongly accretive operator $A : D(A) \to 2^X$ has a modulus of accretivity Θ.*

Proof. For $k \in \mathbb{N}, K \in \mathbb{N}^*$ define

$$\Theta_K(k) := \min n \, (2^{-n} \leq \inf\{\psi(y) : y \in [2^{-k}, K]\})$$

which is well-defined since $\psi : \mathbb{R}^+ \to [0,\infty)$ is continuous with $\psi(x) > 0$ for $x > 0$. Then,

$$\forall k \in \mathbb{N} \; \forall K \in \mathbb{N}^* \; \forall (x,u),(y,v) \in A \; (\|x-y\| \in [2^{-k}, K] \to$$
$$\langle u-v, x-y\rangle_+ \geq \psi(\|x-y\|)\|x-y\|) \geq 2^{-\Theta_K(k)}\|x-y\|).$$

□

In [8] the notion of *ϕ-accretivity at zero* for an operator $A : D(A) \to 2^X$ is introduced:

Definition 7. *(Definition 2 in [8]) Let X be a real Banach space, let $\phi : X \to [0,\infty)$ be a continuous function such that $\phi(0) = 0$, $\phi(x) > 0$ for $x \neq 0$ so that for every sequence (x_n) in X such that $(\|x_n\|)$ is nonincreasing and $\phi(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$. An accretive operator $A : D(A) \to 2^X$ with $0 \in Az$ is said to be ϕ-accretive at zero if*

$$\forall (x,u) \in A \; (\langle u, x-z\rangle_+ \geq \phi(x-z)).$$

Note that by Theorem 8 in [9], m-ψ-strong accretivity implies ϕ-accretivity at zero.

Notice that, again, for $z \in D(A)$ so that $Az \ni 0$, assuming that there exists a $D(A) \ni z' \neq z$ so that $Az' \ni 0$, for $(z', 0) \in A$ we obtain

$$\langle 0, z' - z \rangle_+ \geq \phi(z' - z)$$

therefore

$$\phi(z' - z) = 0,$$

and because $z' - z \neq 0 \to \phi(z' - z) > 0$,

$$z = z'.$$

However, here there exists no uniform notion of a modulus of accretivity as the distance that $\phi(x)$ has from 0 not only depends on the distance that $\|x\| > 0$ has from 0 but on $x \neq 0$ itself.

In our proof-theoretic analysis of the proofs in [8] we consider operators $A : D(A) \to 2^X$ that have a well-defined modulus of accretivity. This is the case, for instance, when A has the -more restrictive- accretivity property that we introduce below.

Definition 8. *We say that a ϕ-accretive at zero operator $A : D(A) \to 2^X$, where X is a real Banach space, is uniformly ϕ-accretive at zero if $\phi : X \to [0, \infty)$ is in particular of the form*

$$\phi(x) = g(\|x\|)$$

where $g : \mathbb{R}^+ \to [0, \infty)$ is a continuous function such that $g(0) = 0$ and $g(\alpha) > 0$ for $\alpha \neq 0$.

The motivation for this choice is the possibility to define again a uniform notion of modulus of accretivity Θ for uniformly ϕ-accretive at zero operators in the following sense:

Definition 9. *Given a real Banach space X and a function $\Theta_{(\cdot)}(\cdot) : \mathbb{N} \times \mathbb{N}^* \to \mathbb{N}$, we say that a uniformly ϕ-accretive at zero operator $A : D(A) \to 2^X$ with $Az \ni 0$ has a modulus of accretivity Θ if*

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}^*\ \forall (x, u) \in A\ (\|x - z\| \in [2^{-k}, K] \to \langle u, x - z \rangle_+ \geq 2^{-\Theta_K(k)}).$$

Proposition 2. *Let X be a real Banach space. Every uniformly ϕ-accretive at zero operator $A : D(A) \to 2^X$ with $Az \ni 0$ has a modulus of accretivity Θ.*

Proof. By assumption

$$\forall (x, u) \in A\ (\langle u, x - z \rangle_+ \geq \phi(x - z) = g(\|x - z\|)).$$

In a similar spirit as in the proof of Proposition 1, we have

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}^*\ \forall x \in D(A)\ (\|x - z\| \in [2^{-k}, K]\ \to g(\|x - z\|) \geq 2^{-\Theta_K(k)})$$

where we have defined

$$\Theta_K(k) := \min\ n(2^{-n} \leq \inf\{g(\alpha) : \alpha \in [2^{-k}, K]\}).$$

□

Remark 1. *For a uniformly ϕ-accretive at zero operator, in the case where the function g is nondecreasing, the modulus of accretivity Θ does not depend on K, as in this case, clearly,*

$$\inf\{g(\alpha) \ : \ \alpha \in [2^{-k}, K]\} = g(2^{-k}).$$

This is usually the case in many applications and, in particular, in the application we discuss in Section 4. Clearly, the analogous conclusion holds for ψ-strongly accretive operators.

Note that given a ψ-strongly accretive operator A, if A is also uniformly ϕ-accretive at zero with $0 \in Az$, given a modulus of ψ-strong accretivity $\Theta^{\psi}_K(k)$, we easily obtain a modulus of uniform ϕ-accretivity at zero $\Theta^{\phi}_K(k)$ by noticing that

$$\forall k \in \mathbb{N} \ \forall K \in \mathbb{N}^* \ \forall (x,u) \in A$$

$$(\|x-z\| \in [2^{-k}, K] \to (\langle u, x-z\rangle_+ \geq 2^{-\Theta^{\psi}_K(k)}\|x-z\| \geq 2^{-\Theta^{\psi}_K(k)} \cdot 2^{-k}))$$

which gives

$$\Theta^{\phi}_K(k) := \Theta^{\psi}_K(k) + k$$

as a modulus of uniform ϕ-accretivity at zero.

At this point we introduce in higher generality the property of *uniform accretivity at zero* for an operator $A : D(A) \to 2^X$ with $0 \in Az$ as follows:

Definition 10. *Let X be a real Banach space. An accretive operator $A : D(A) \to 2^X$ with $0 \in Az$ is called uniformly accretive at zero if*

$$\forall k \in \mathbb{N} \ \forall K \in \mathbb{N}^* \ \exists m \in \mathbb{N} \ \forall (x,u) \in A$$

$$(\|x-z\| \in [2^{-k}, K] \to \langle u, x-z\rangle_+ \geq 2^{-m})(*).$$

Any function $\Theta_{(\cdot)}(\cdot) : \mathbb{N} \times \mathbb{N}^* \to \mathbb{N}$ is called a modulus of accretivity at zero for A if $m := \Theta_K(k)$ satisfies $(*)$.

In the following we will assume that $\Theta_K(k)$ is nondecreasing in K. Note that this assumption is possible without loss of generality, as for any $\Theta_K(k)$ we may define a nondecreasing

$$\Theta^{\mathcal{M}}_K(k) := \max\{\Theta_i(k) \ : \ i \leq K\}.$$

As discussed in the introduction, our approach is based on a logical metatheorem which in turn uses a proof-theoretic extraction algorithm due to the first author ('monotone functional interpretation', see [14]) to get uniform effective bounds. This algorithm keeps track of uniform bounding information by recursion over the given proof, starting from the axioms/assumptions used and proceeding to the conclusion of the proof. That is why these axioms/assumptions have to have the right uniformity and, if they do not, the interpretation will automatically strengthen them accordingly. In the case at hand this strengthening (imposing sufficient uniformity) is precisely what corresponds to the concept of 'modulus of accretivity'. In fact, the proof-theoretic interpretation validates a nonstandard (in the sense of not being valid in the standard set-theoretic model) uniform boundedness principle Σ^0_1-UB^X which allows one to infer that every ϕ-accretive at zero operator has such a modulus of (uniform) accretivity (see [10]).

Remark 2. *It is interesting to investigate how a modulus of accretivity for an operator A with $0 \in Az$ is associated with a modulus of uniqueness (see* [13, 14]*) for z. We distinguish the following three cases.*

- *For a ψ-strongly accretive operator A, any modulus of accretivity Θ yields a 'modulus of uniqueness for the zero z of A' in the following sense: Let $Az \ni 0$ and suppose $\exists v \in Az'$ with $\|v\| \leq 2^{-\delta}$, where $\delta \in \mathbb{N}$. Then (the first inequality following from* [1]*, p. 12 (1.4)) for all $k \in \mathbb{N},\ K \in \mathbb{N}^*$*
$$\|z - z'\| \in [2^{-k}, K] \to \|v\|\|z - z'\| \geq \langle v, z - z'\rangle_+ \geq 2^{-\Theta_K(k)}\|z - z'\|$$
and so
$$\|z - z'\| \in [2^{-k}, K] \to 2^{-\delta} \geq 2^{-\Theta_K(k)}.$$
Let us take
$$\delta = \delta_K(k) := \Theta_K(k) + 1 > \Theta_K(k).$$
Then
$$2^{-\delta_K(k)} < 2^{-\Theta_K(k)}$$
and therefore
$$\|z - z'\| \leq K \wedge \exists v \in Az'(\|v\| \leq 2^{-\delta_K(k)}) \to \|z - z'\| < 2^{-k}.$$
In the special case where $\psi(\cdot)$ is nondecreasing, the K dependence for the modulus of uniqueness disappears and the condition $\|z - z'\| \leq K$ is not needed (see Remark 1).
- *For a ϕ-accretive at zero operator A, as it has been already stressed, there exists no well-defined modulus of accretivity, thus we cannot associate a modulus of uniqueness for the zero z of A with a modulus of accretivity for A.*
- *For a uniformly accretive at zero operator A, any modulus of accretivity Θ also yields a modulus of uniqueness for the zero z of A as follows: Let $Az \ni 0$ and suppose $\exists v \in Az'$ with $\|v\| \leq 2^{-\delta}$, where $\delta \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, $K \in \mathbb{N}^*$*
$$\|z - z'\| \in [2^{-k}, K] \to 2^{-\delta} \cdot K \geq \|v\|\|z - z'\| \geq \langle v, z - z'\rangle_+ \geq 2^{-\Theta_K(k)}.$$
Let us take
$$\delta = \delta_K(k) := \log_2 K + \Theta_K(k) + 1 > \log_2 K + \Theta_K(k).$$
Then
$$2^{-\delta_K(k)} < \frac{1}{K \cdot 2^{\Theta_K(k)}}$$
and so
$$\|z - z'\| \leq K \wedge \exists v \in Az'(\|v\| \leq 2^{-\delta_K(k)}) \to \|z - z'\| < 2^{-k}.$$
Here, even in the special case where A is uniformly ϕ-accretive at zero with $g(\cdot)$ nondecreasing thus making the K dependence for the modulus of accretivity disappear (Remark 1), the K dependence for the modulus of uniqueness does not disappear as we still have the term $\log_2 K$.

2.2. **Setting up the framework.** In the following, $A : D(A) \to 2^X$ is an accretive operator with the range condition. Consider the following initial value problem

Problem 1.

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty)$$
$$u(0) = x$$

where $f \in L^1(0, \infty, X)$.

Definition 11. *A continuous function* $u : [0, \infty) \to X$ *is an integral solution of Problem 1 if* $u(0) = x$ *and for* $s \in [0, t]$ *and* $(w, y) \in A$

$$\|u(t) - w\|^2 - \|u(s) - w\|^2 \leq 2 \int_s^t \langle f(\tau) - y, u(\tau) - w\rangle_+ d\tau.$$

It is known (for instance see [1], Chapter III.2) that for each $x \in \overline{D(A)}$ Problem 1 has a unique integral solution u so that $u(t) \in \overline{D(A)}$ for all t. Moreover, it is known ([6]) that for $x_0 \in \overline{D(A)}$ the following initial value problem :

Problem 2.

$$u'(t) + A(u(t)) \ni 0, t \in [0, \infty)$$
$$u(0) = x_0,$$

has a unique integral solution given by the Crandall-Liggett formula :

$$u(t) := S(t)(x_0) = \lim_{n \to \infty} (I + \frac{t}{n} A)^{-n}(x_0).$$

Definition 12. *A continuous function* $u : [0, \infty) \to X$ *is said to be a strong solution of Problem 2 if it is Lipschitz on every bounded subinterval of* $[0, \infty)$*, almost everywhere differentiable on* $\mathbb{R}^+$*,* $u(t) \in D(A)$ *almost everywhere,* $u(0) = x_0$ *and* $u'(t) + A(u(t)) \ni 0$ *for almost every* $t \in \mathbb{R}^+$.

3. Proof-theoretic analysis of strong asymptotic behaviour

We extract rates of convergence and metastability (in the sense of Tao [22, 23]) by logically analyzing the proofs as given in [8]. We point out again that in our analysis we assume uniform accretivity at zero for the operator $A : D(A) \to 2^X$ which thus comes with a modulus of accretivity Θ that we will make use of.
We logically analyze via proof mining the proof of the following theorem by García-Falset in [8] :

Theorem 1. *(Theorem 8 in* [8]*) Let* X *be a real Banach space. If* A *is an operator on* X *with the range condition that is* ϕ*-accretive at zero and such that Problem 2 has a strong solution for each* $x_0 \in D(A)$ *and* $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\}$ *is the nonexpansive semigroup generated by* $-A$ *via the Crandall-Liggett formula, then every almost-orbit* $u : [0, \infty) \to \overline{D(A)}$ *of* $\mathcal{F}$ *is strongly convergent to the zero* z *of* A.

and we show the following :

Theorem 2. *Let* X *be a real Banach space. Let* A *be an operator on* X *with the range condition that is uniformly accretive at zero with a modulus of accretivity* Θ*, and such that Problem 2 has a strong solution for each* $x_0 \in D(A)$ *and* $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0 \}$ *is the nonexpansive semigroup generated by* $-A$

via the Crandall-Liggett formula. Then every $u : [0,\infty) \to \overline{D(A)}$ that fulfills the condition[1]*:*

$$\exists \Phi : \mathbb{N} \to \mathbb{N}\ \forall k \in \mathbb{N}\ \exists s \in [0, \Phi(k)]\ (\sup_{t\geq 0} \|u(s+t) - S(t)u(s)\| \leq 2^{-k}),$$

is strongly convergent to the zero z of A i.e.

$$\forall k \in \mathbb{N}\ \forall x \geq \Psi(k, B, \Phi, \Theta)\ (\|u(x) - z\| < 2^{-k})$$

with rate of convergence

$$\Psi(k, B, \Phi, \Theta) = (B(\Phi(k+1)) + 2) \cdot 2^{\Theta_{K(\Phi(k+1))}(k+2)+1} + \Phi(k+1)$$

where [2]

$$K(s) := \ulcorner \sqrt{2(B(s)+1)} \urcorner$$

and $B(s)$ is any nondecreasing upper bound on $\frac{1}{2}\|u(s) - z\|^2$.

Proof. The proof is based on performing proof mining on the proof of Theorem 1 (i.e. Theorem 8 in [8]).

Let $u : [0,\infty) \to \overline{D(A)}$ be as in the assumption of the theorem. Let $s \geq 0$ be fixed.

Case 1. Assume that $u(s) \in D(A)$.

Consider the following initial value problem, which is of the form of Problem 2:

Problem 3.

$$w_s'(t) + A(w_s(t)) \ni 0,$$
$$w_s(0) = u(s).$$

Problem 3 has a unique solution

$$w_s(t) = S(t)u(s)$$

which is a strong solution by assumption. Thus the derivative $w_s'(t)$ is defined almost everywhere and $-w_s'(t) \in Aw_s(t)$ almost everywhere, i.e.

$$\exists \mathcal{S} \subset [0,\infty)(\mu(\mathcal{S}) = 0\ \wedge\ \forall t \in [0,\infty) \setminus \mathcal{S}\ w_s'(t) \downarrow),$$
$$\exists \mathcal{S}' \subset [0,\infty)(\mu(\mathcal{S}') = 0\ \wedge\ \forall t \in [0,\infty) \setminus \mathcal{S}'\ - w_s'(t) \in Aw_s(t)),$$

where $\mu(\cdot)$ denotes the Lebesgue measure. There exists $j(t) \in J(w_s(t) - z)$, where $J(\cdot)$ is the normalized duality mapping as defined in Section 2, so that, for all $t \in [0,\infty) \setminus \mathcal{S}$:

$$\langle -w_s'(t), w_s(t) - z\rangle_+ = \langle -w_s'(t), j(t)\rangle = \langle -\frac{1}{h}(w_s(t) - w_s(t-h)) + \xi(t,h), j(t)\rangle$$
$$= \frac{1}{h}\langle w_s(t-h) - w_s(t), j(t)\rangle + \langle \xi(t,h), j(t)\rangle \ (1)$$

[1] clearly this is a weakening of the assumption of $u : [0,\infty) \to \overline{D(A)}$ being an almost-orbit of $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\}$ i.e.

$$\exists \Phi : \mathbb{N} \to \mathbb{N}\ \forall k \in \mathbb{N}\ \forall s \geq \Phi(k)\ (\sup_{t\geq 0} \|u(s+t) - S(t)u(s)\| \leq 2^{-k}).$$

[2] note that $\ulcorner x \urcorner \in \mathbb{N}$ denotes the smallest natural number exceeding $x \in \mathbb{R}$.

where $\lim_{h\to 0}\xi(t,h)=0$.

(Notice: $\langle w_s(t)-z, j(t)\rangle = \|w_s(t)-z\|^2 = \|j(t)\|^2$.) Now :

$$\begin{aligned}\langle w_s(t-h)-w_s(t), j(t)\rangle &= \langle w_s(t-h)-w_s(t)+z-z, j(t)\rangle\\ &= \langle w_s(t-h)-z, j(t)\rangle + \langle z-w_s(t), j(t)\rangle\\ &= \langle w_s(t-h)-z, j(t)\rangle - \langle w_s(t)-z, j(t)\rangle\\ &= \langle w_s(t-h)-z, j(t)\rangle - \|w_s(t)-z\|^2 \ (2).\end{aligned}$$

Notice that by the properties of the duality mapping (see [1] page 12 (1.4))

$$\begin{aligned}&\langle w_s(t-h)-z, j(t)\rangle\\ &\le \|w_s(t-h)-z\|\|j(t)\|\\ &\le \frac{1}{2}\|w_s(t-h)-z\|^2 + \frac{1}{2}\|j(t)\|^2\\ &= \frac{1}{2}\|w_s(t-h)-z\|^2 + \frac{1}{2}\|w_s(t)-z\|^2 (3)\end{aligned}$$

By (2) and (3)

$$\langle w_s(t-h)-w_s(t), j(t)\rangle \le \frac{1}{2}\|w_s(t-h)-z\|^2 - \frac{1}{2}\|w_s(t)-z\|^2 \ (4).$$

Define $q_s(t) := \langle -w_s'(t), j(t)\rangle = \langle -w_s'(t), w_s(t)-z\rangle_+$. Note that $q_s(t)$ is defined almost everywhere, as $w_s'(t)$ is defined almost everywhere. Now $-w_s'(t)\in Aw_s(t)$ almost everywhere. By the accretivity of A, the condition $q_s(t)\ge 0$ holds for all $t\in[0,\infty)\setminus(\mathcal{S}\cup\mathcal{S}')$. By (4), for all $t\in[0,\infty)\setminus(\mathcal{S}\cup\mathcal{S}')$, (1) gives :

$$0\le q_s(t) = \langle -w_s'(t), j(t)\rangle \le \frac{1}{h}\frac{1}{2}(\|w_s(t-h)-z\|^2 - \|w_s(t)-z\|^2) + \langle \xi(t,h), j(t)\rangle$$

thus

$$0\le q_s(t)\le -\frac{1}{2}\frac{d}{dt}\|w_s(t)-z\|^2 \text{ a.e. } (5)$$

as the derivative of $\|w_s(t)-z\|$ is defined almost everywhere i.e.

$$\exists \mathcal{S}''\subset[0,\infty)(\mu(\mathcal{S}'')=0 \ \wedge\ \forall t\in[0,\infty)\setminus\mathcal{S}'' \ \frac{d}{dt}\|w_s(t)-z\|\downarrow)$$

since $t\to\|w_s(t)-z\|$ is Lipschitzian, because by assumption $w_s(t)$ is Lipschitzian.

By (5) we deduce that $\frac{d}{dt}\|w_s(t)-z\|^2\le 0$ almost everywhere. Therefore $\|w_s(t)-z\|^2$ is nonincreasing in t (see [4], p.120 and note that $\|w_s(t)-z\|^2$ is Lipschitz on bounded intervals thus absolutely continuous). For all $t\in[0,\infty)$ by (5) we have

$$0\le\int_0^t q_s(t)dt \le -\frac{1}{2}\int_0^t \frac{d}{dt}\|w_s(t)-z\|^2 dt$$

$$= -\frac{1}{2}\|w_s(t)-z\|^2 + \frac{1}{2}\|w_s(0)-z\|^2 \le \frac{1}{2}\|w_s(0)-z\|^2.$$

Thus $q_s(t)$ is Lebesgue integrable on $[0,\infty)$. Therefore

$$\lim_{t\to\infty}\inf q_s(t)=0$$

and

$$\forall k\in\mathbb{N}\ \exists t\in[0,\infty)\setminus(\mathcal{S}\cup\mathcal{S}'\cup\mathcal{S}'')\ (q_s(t)\le 2^{-k}).$$

We now construct an upper bound $T(k,s)$ on t as follows. Let $B(s)$ be a nondecreasing upper bound on

$$\frac{1}{2}\|w_s(0)-z\|^2.$$

For instance let

$$B(s)=\frac{1}{2}\max\{\|w_r(0)-z\|^2\ :\ r\le s\}.$$

We set

$$T(k,s):=(B(s)+1)\cdot 2^k.$$

We claim that

$$\forall k\in\mathbb{N}\ \exists t\in[0,T(k,s)]\setminus(\mathcal{S}\cup\mathcal{S}'\cup\mathcal{S}'')\ (q_s(t)\le 2^{-k}).$$

Assume the contrary, i.e. assume that

$$\exists k\in\mathbb{N}\ \forall t\in[0,T(k,s)]\setminus(\mathcal{S}\cup\mathcal{S}'\cup\mathcal{S}'')\ (q_s(t)>2^{-k});$$

then, by the monotonicity property of the Lebesgue integral

$$\int_0^{T(k,s)}2^{-k}dt\le\int_0^{T(k,s)}q_s(t)dt$$

therefore

$$T(k,s)\cdot 2^{-k}=B(s)+1\le\int_0^{T(k,s)}q_s(t)dt$$

which is a contradiction.
Hence we have for each k a $t_k\le T(k,s)$ with $t_k\notin(\mathcal{S}\cup\mathcal{S}'\cup\mathcal{S}'')$ and $q_s(t_k)\le 2^{-k}$.
By the definition of the modulus of accretivity Θ we have

$$\forall n,k\in\mathbb{N}\ \forall K\in\mathbb{N}^*\ (\|w_s(t_k)-z\|\in[2^{-n},K]\to 2^{-\Theta_K(n)}\le q_s(t_k))$$

therefore

$$\forall n,k\in\mathbb{N}\ \forall K\in\mathbb{N}^*\ (\|w_s(t_k)-z\|\in[2^{-n},K]\to 2^{-\Theta_K(n)}\le 2^{-k}).$$

We set $k=\Theta_K(n)+1$ and thus obtain :

$$\forall n\in\mathbb{N}\ \forall K\in\mathbb{N}^*\ (\|w_s(t_{\Theta_K(n)+1})-z\|\in[2^{-n},K]\to 2^{-\Theta_K(n)}\le 2^{-\Theta_K(n)-1})$$

whose conclusion is obviously false. Thus the premise is false. Since for all $t\ge 0$ (recall that $\|w_s(t)-z\|$ is nonincreasing in t)

$$K=K_0(s):=\ulcorner\sqrt{2B(s)}\urcorner\ge\|w_s(0)-z\|\ge\|w_s(t)-z\|$$

we, therefore, have

$$\forall n\in\mathbb{N}\ (\|w_s(t_{\Theta_{K_0(s)}(n)+1})-z\|<2^{-n})$$

where

$$t_{\Theta_{K_0(s)}(n)+1}\le(B(s)+1)\cdot 2^{\Theta_{K_0(s)}(n)+1}$$

and so, using again the fact that $\|w_s(t)-z\|$ is nonincreasing in t,

$$\forall n\in\mathbb{N}\ \forall t\ge(B(s)+1)\cdot 2^{\Theta_{K_0(s)}(n)+1}\ (\|w_s(t)-z\|<2^{-n})\ (6).$$

Case 2. Now assume that $u(s)\in\overline{D(A)}$. Then there exists a sequence $(x_k(s))\subseteq D(A)$ such that $x_k(s)\to u(s)$. Let

$$\tilde{w}_{k,s}(t):=S(t)x_k(s)\subseteq D(A).$$

We want to show that

$$\lim_{t\to\infty}\|w_s(t)-z\|=0$$

where
$$w_s(t) := S(t)u(s).$$
By the triangle inequality:
$$|\|\tilde{w}_{k,s}(t) - z\| - \|w_s(t) - z\|| \leq \|\tilde{w}_{k,s}(t) - w_s(t)\|.$$
Notice that
$$\|\tilde{w}_{k,s}(t) - w_s(t)\| = \|S(t)x_k(s) - S(t)u(s)\| \leq \|x_k(s) - u(s)\|,$$
the above inequality following from Definition 1(3). Thus
$$|\,\|\tilde{w}_{k,s}(t) - z\| - \|w_s(t) - z\|| \leq \|x_k(s) - u(s)\|.$$
We assume, without loss of generality, that we can make an appropriate choice of $x_{\tilde{k}}(s) \in D(A)$ so that
$$\|x_{\tilde{k}}(s) - u(s)\| \leq 2^{-\tilde{k}}$$
thus
$$\Big|\|\tilde{w}_{\tilde{k},s}(t) - z\| - \|w_s(t) - z\|\Big| \leq 2^{-\tilde{k}}.$$
In particular this gives
$$\begin{aligned}\frac{1}{2}\|\tilde{w}_{\tilde{k},s}(t) - z\|^2 &\leq \frac{1}{2}(\|w_s(t) - z\|^2 + 2 \cdot 2^{-\tilde{k}}\|w_s(t) - z\| + (2^{-\tilde{k}})^2) \\ &\leq \frac{1}{2}\|w_s(0) - z\|^2 + 2^{-\tilde{k}}\|w_s(t) - z\| + \frac{1}{2}(2^{-\tilde{k}})^2 \\ &\leq B(s) + 2^{-\tilde{k}}\|w_s(t) - z\| + \frac{1}{2}(2^{-\tilde{k}})^2.\end{aligned}$$
Now for $k \in \mathbb{N}$, take $\mathbb{N} \ni \tilde{k} > k$ such that:
$$2^{-\tilde{k}}\|w_s(0) - z\| + \frac{1}{2}(2^{-\tilde{k}})^2 \leq 1.$$
Then (by evaluating the previous estimate at $t = 0$) an upper bound on $\frac{1}{2}\|\tilde{w}_{\tilde{k},s}(0) - z\|^2$, denoted by $\tilde{B}(s, \tilde{k})$, can be taken as $\tilde{B}(s, \tilde{k}) := B(s) + 1$, where $B(s)$ is a non-decreasing upper bound on $\frac{1}{2}\|w_s(0) - z\|^2$.

By (6) of Case 1 applied to $\|\tilde{w}_{\tilde{k},s}(t) - z\|$ here:
$$\forall t \geq (\tilde{B}(s, \tilde{k}) + 1) \cdot 2^{\Theta_{K(s)}(k)+1} \ (\|\tilde{w}_{\tilde{k},s}(t) - z\| < 2^{-k})$$
i.e.
$$\forall t \geq (B(s) + 2) \cdot 2^{\Theta_{K(s)}(k)+1} \ (\|\tilde{w}_{\tilde{k},s}(t) - z\| < 2^{-k})$$
and thus
$$\forall t \geq (B(s) + 2) \cdot 2^{\Theta_{K(s)}(k)+1} \ (\|w_s(t) - z\| < 2^{-\tilde{k}} + 2^{-k} < 2 \cdot 2^{-k}).$$

Since $k \in \mathbb{N}$ and $s \geq 0$ were arbitrary we thus have
$$\forall k \in \mathbb{N} \ \forall s \geq 0 \ \forall t \geq (B(s) + 2) \cdot 2^{\Theta_{K(s)}(k+1)+1} \ (\|w_s(t) - z\| < 2^{-k}) \ (7).$$
Note that here we have taken
$$K(s) := \ulcorner\sqrt{2(B(s) + 1)}\urcorner,$$
as
$$\tilde{B}(s, \tilde{k}) = B(s) + 1 \geq \frac{1}{2}\|\tilde{w}_{\tilde{k},s}(0) - z\|^2.$$

Now, by assumption, $u : [0,\infty) \to \overline{D(A)}$ fulfills the condition:

$$\exists \Phi : \mathbb{N} \to \mathbb{N}\ \forall k \in \mathbb{N}\ \exists s \in [0, \Phi(k)]\ (\varphi(s) \leq 2^{-k})\ (8)$$

where we have set

$$\varphi(s) := \sup_{t \geq 0} \|u(s+t) - S(t)u(s)\|.$$

The triangle inequality gives, for all $t \geq 0$,

$$\begin{aligned}\|u(t+s) - z\| &\leq \|u(t+s) - S(t)u(s)\| + \|S(t)u(s) - z\| \\ &= \|u(t+s) - S(t)u(s)\| + \|w_s(t) - z\| \\ &\leq \varphi(s) + \|w_s(t) - z\|.\end{aligned}$$

By (7)

$$\forall k \in \mathbb{N}\ \forall s \geq 0\ \forall t \geq (B(s)+2) \cdot 2^{\Theta_{K(s)}(k+1)+1}\ (\|u(t+s) - z\| < \varphi(s) + 2^{-k}).$$

For Φ as in (8) the above gives

$$\forall k \in \mathbb{N}\ \exists s \in [0, \Phi(k)]\ \forall t \geq (B(s)+2) \cdot 2^{\Theta_{K(s)}(k+1)+1}\ (\|u(t+s) - z\| < 2 \cdot 2^{-k})$$

which, using that $(B(\cdot)+2) \cdot 2^{\Theta_{K(\cdot)}(k+1)+1}$ is nondecreasing, implies

$$\forall k \in \mathbb{N}\ \forall x \geq (B(\Phi(k))+2) \cdot 2^{\Theta_{K(\Phi(k))}(k+1)+1} + \Phi(k)\ (\|u(x) - z\| < 2 \cdot 2^{-k}),$$

that is,

$$\forall k \in \mathbb{N}\ \forall x \geq (B(\Phi(k+1))+2) \cdot 2^{\Theta_{K(\Phi(k+1))}(k+2)+1} + \Phi(k+1)\ (\|u(x) - z\| < 2^{-k}).$$

□

Remark 3. *Our logical analysis shows that Theorem 1 (Theorem 8 in* [8]*) is true not only under the assumption that the continuous function $u(\cdot) : [0,\infty) \to \overline{D(A)}$ is an almost-orbit, i.e.*

$$\forall k \in \mathbb{N}\ \exists s_0 \geq 0\ \forall s \geq s_0\ (\sup\{\|u(t+s) - S(t)u(s)\|\ :\ t \in [0,\infty)\} \leq 2^{-k}),$$

i.e., equivalently,

$$\forall k \in \mathbb{N}\ \exists s_0 \geq 0\ \forall s \geq s_0\ \forall m \in \mathbb{N}\ (\sup\{\|u(t+s) - S(t)u(s)\|\ :\ t \leq m\} \leq 2^{-k})\ (*),$$

but also under the weaker assumption

$$\forall k \in \mathbb{N}\ \exists s_0 \geq 0\ \forall m \in \mathbb{N}\ (\sup\{\|u(t+s_0) - S(t)u(s_0)\|\ :\ t \leq m\} \leq 2^{-k})\ (**).$$

Notice, moreover, that $(*)$ implies

$$\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \forall m \in \mathbb{N}\ (\sup\{\|u(t+n) - S(t)u(n)\|\ :\ t \leq m\} \leq 2^{-k}),$$

whose noneffectively equivalent metastable form is

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \in \mathbb{N}\ (\sup\{\|u(t+n) - S(t)u(n)\|\ :\ t \leq g(n)\} \leq 2^{-k})\ (+),$$

while $(**)$ implies

$$\forall k \in \mathbb{N}\ \exists q \in \mathbb{Q}^+\ \forall m \in \mathbb{N}\ (\sup\{\|u(t+q) - S(t)u(q)\|\ :\ t \leq m\} \leq 2^{-k}),$$

whose noneffectively equivalent metastable form is

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{Q}^+ \to \mathbb{N}\ \exists q \in \mathbb{Q}^+\ (\sup\{\|u(t+q) - S(t)u(q)\|\ :\ t \leq g(q)\} \leq 2^{-k})\ (++).$$

Moreover, note that instead of using metastability in the form of (+) one could also work with the still weaker form (++) which, however, makes things more complicated without any apparent benefit.

In the following theorem we show a metastable (in the sense of Tao) version of Theorem 2 above, namely a version where the statement referring to the (weakening of the condition of the) almost-orbit is replaced by a metastable statement in the form (+), thus giving a metastable version for the convergence of the result. That is, the rate of convergence $\Phi(k)$ that appeared in Theorem 2 so that

$$\forall k \in \mathbb{N}\ \exists s \in [0, \Phi(k)]\ (\sup_{t \geq 0} \|u(s+t) - S(t)u(s)\| \leq 2^{-k})$$

is substituted with a rate of metastability $\Phi : \mathbb{N} \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ in Theorem 3 below so that

$$\begin{gathered}\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g)\ \forall t \in [0, g(n)] \\ (\|u(t+n) - S(t)u(n)\| \leq 2^{-k}).\end{gathered}$$

Clearly, the conclusion of Theorem 3, being metastable, is weaker than that of Theorem 2, however it will serve a useful purpose; it will illustrate the pattern of metastability, which will be realized in Theorem 4 later on. We will later see that Theorem 4 can be regarded as a corollary of Theorem 3 below, where in particular the quantity $\Phi(k, g, \ldots)$ corresponding to the metastability information relating to the almost-orbit in Theorem 3 (i.e. $\Phi(k, g)$), can be computed by applying a logical metatheorem by the first author given in [14], thus providing, as we will see, a computable rate of metastability for the strong convergence of the solution of the abstract Cauchy problem generated by a uniformly accretive at zero operator A to the zero z of A, which is the central result of this paper.

Theorem 3. *Let X be a real Banach space. Let A be an operator on X with the range condition that is uniformly accretive at zero with modulus of accretivity Θ and such that Problem 2 has a strong solution for each $x_0 \in D(A)$ and $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\}$ is the nonexpansive semigroup generated by $-A$ via the Crandall-Liggett formula. Then every almost-orbit $u : [0, \infty) \to \overline{D(A)}$ of $\mathcal{F}$ is strongly convergent to the zero z of A with rate of metastability $\Psi(k, \bar{g}, B, \Phi, \Theta)$ so that*

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Psi(k, \bar{g}, B, \Phi, \Theta)\ \forall x \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ (\|u(x) - z\| < 2^{-k}),$$

where

$$\Psi(k, \bar{g}, B, \Phi, \Theta) = \Phi(k+1, g) + h(\Phi(k+1, g))$$

with

$$\begin{gathered}g(n) := \bar{g}(n + h(n)) + h(n), \\ h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1}, \\ K(n) := \ulcorner \sqrt{2(B(n) + 1)} \urcorner.\end{gathered}$$

Here $\Phi : \mathbb{N} \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is a rate of metastability (in the sense that we discussed above) corresponding to a given almost-orbit $u : [0, \infty) \to \overline{D(A)}$ of $\mathcal{F}$, i.e.

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g)\ \forall t \in [0, g(n)]\ (\|u(t+n) - S(t)u(n)\| \leq 2^{-k})$$

and $B(n) \in \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|u(n) - z\|^2$.

Proof. Let $u(\cdot) \in \overline{D(A)}$.

We consider the metastable (in the sense of Tao [22, 23]) version of an almost-orbit discussed in the form (+) as in Remark 3:

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g)\ \forall t \in [0, g(n)]\ (\varphi_t(n) \leq 2^{-k})\ (9)$$

where we have defined, for $n \in \mathbb{N}$,

$$\varphi_t(n) := \|u(t+n) - S(t)u(n)\|.$$

Notice that by the triangle inequality :

$$\|u(t+n) - z\| \leq \|u(t+n) - S(t)u(n)\| + \|w_n(t) - z\| = \varphi_t(n) + \|w_n(t) - z\|,$$

where $w_n(t)$ is as in Problem 3. Note that here $n \in \mathbb{N}$. We claim that

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Phi(k+1, g) + (B(\Phi(k+1, g)) + 2) \cdot 2^{\Theta_{K(\Phi(k+1,g))}(k+2)+1}$$
$$\forall x \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ (\|u(x) - z\| < 2^{-k})$$

where

$$\bar{n} := n + (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$
$$g(m) := \bar{g}(m + (B(m) + 2) \cdot 2^{\Theta_{K(m)}(k+2)+1})) + (B(m) + 2) \cdot 2^{\Theta_{K(m)}(k+2)+1}$$

so that

$$g(n) + n = \bar{g}(\bar{n}) + \bar{n},$$

where $B(m) : \mathbb{N} \to \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|u(m) - z\|^2$ and $K(m) := \ulcorner \sqrt{2(B(m) + 1} \urcorner$. Note that here $n \leq \Phi(k+1, g)$ is chosen for $k+1$ and g as defined above according to (9). By the monotonicity of $B(m)$ and $K(m)$ it follows that

$$\bar{n} \leq \Phi(k+1, g) + (B(\Phi(k+1, g)) + 2) \cdot 2^{\Theta_{K(\Phi(k+1,g))}(k+2)+1}.$$

To show the above claim, let $x = t + n \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]$ with

$$t \in [(B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1}, \bar{g}(\bar{n}) + (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1}] \subseteq [0, g(n)].$$

By (7) in the proof of Theorem 2 with t chosen in the above interval we have

$$\|w_n(t) - z\| < 2^{-k-1}$$

and, therefore,

$$\|u(x) - z\| = \|u(t+n) - z\| < \|u(t+n) - S(t)u(n)\| + 2^{-k-1} = \varphi_t(n) + 2^{-k-1}.$$

Thus, by our choice of n to be $n \leq \Phi(k+1, g)$ based on (9), we obtain :

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Phi(k+1, g) + (B(\Phi(k+1, g)) + 2) \cdot 2^{\Theta_{K(\Phi(k+1,g))}(k+2)+1}$$
$$\forall x \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ (\|u(x) - z\| < 2 \cdot 2^{-k-1} = 2^{-k}).$$

□

As previously mentioned, we will now show a result where the information on the almost-orbit is explicitly given and the rate of metastability $\Phi(k, g, ...)$- thus also the rate of metastability for the final result- can be computed via a result by the first author in [14].

The following is a corollary of Theorem 1 (Theorem 8 in [8]).

Corollary 1. *(Corollary 9 in* [8]*) Let X be a real Banach space. Suppose that $A : D(A) \to 2^X$ is an m-ψ-strongly accretive operator on X. Suppose that Problem 2 has a strong solution for each $x_0 \in D(A)$. Then, for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of Problem 1 converges strongly to the zero z of A as $t \to \infty$.*

Note that although in [8] the above corollary is stated with the assumption of m-ψ-strong accretivity, ϕ-accretivity at zero and the range condition are sufficient conditions for the proof. (Note that clearly m-accretivity implies the range condition and that by Theorem 8 in [9] every m-ψ-strongly accretive operator is ϕ-accretive at zero).

The proof of the above corollary follows from Lemma 1(b) in [8] (shown as Proposition 7.1 in [21]), which states that under the conditions of the corollary the integral solution $u(\cdot)$ of Problem 1 is an almost-orbit of the nonexpansive semigroup

$$\mathcal{F} := \{S(t) \ : \ \overline{D(A)} \to \overline{D(A)} \ : \ t \geq 0\}$$

generated by $-A$ via the Crandall-Liggett formula, thus enabling the application of Theorem 1 (Theorem 8 in [8]) that gives immediately the result.

We will show the following theorem, which can be seen as a corollary of Theorem 3, in an analogy to Corollary 1 (Corollary 9 in [8]) being a corollary of Theorem 1 (Theorem 8 in [8]):

Theorem 4. *Let X be a real Banach space. Suppose that $A : D(A) \to 2^X$ is a uniformly accretive at zero operator on X with the range condition that has a modulus of accretivity Θ. Suppose that Problem 2 has a strong solution for each $x_0 \in D(A)$. Then, for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of Problem 1 converges strongly to the zero z of A as $t \to \infty$ and one has*

$$\forall k \in \mathbb{N} \ \forall \overline{g} : \mathbb{N} \to \mathbb{N} \ \exists \bar{n} \leq \Psi(k, \overline{g}, M, B, \Theta) \ \forall x \in [\bar{n}, \bar{n} + \overline{g}(\bar{n})] \ (\|u(x) - z\| < 2^{-k})$$

with rate of metastability

$$\Psi(k, \overline{g}, M, B, \Theta) = \tilde{g}^{(M \cdot 2^{k+1})}(0) + h(\tilde{g}^{(M \cdot 2^{k+1})}(0)),$$

where

$$\tilde{g}(n) := g(n) + n$$

with

$$g(n) := \overline{g}(n + h(n)) + h(n),$$
$$h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$
$$K(n) := \ulcorner \sqrt{2(B(n) + 1)} \urcorner.$$

Here $B(n) \in \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|u(n) - z\|^2$, $M \in \mathbb{N}$ is any upper bound on the integral $I := \int_0^\infty \|f(\xi)\| d\xi$, and in general the function iterations are defined recursively in the following way:

$$g^{(0)}(k) := k$$

$$g^{(i+1)}(k) := g(g^{(i)}(k)).$$

Proof. Let $s \geq 0$ be arbitrarily fixed. Set $u_s(t) = u(t + s)$, $f_s(t) = f(t + s)$, $v(t) = S(t)u(s)$ for $t \geq 0$. Then u_s is an integral solution of

$$(d/dt)u_s \in Au_s + f_s, u_s(0) = u(s).$$

and $(d/dt)v \in Av, v(0) = u(s)$ respectively. By a result in [1] (see (2.4) in p.124),

$$\|u_s(t) - v(t)\|^2 \leq 2 \int_0^t \|f_s(\xi)\| \|u_s(\xi) - v(\xi)\| d\xi.$$

Therefore (using Bihari's inequality)

$$\|u(t+s) - S(t)u(s)\| \leq \int_0^t \|f(s+\xi)\| d\xi.$$

(Note: the above argumentation was taken from the proof of Lemma 1(b) in [8] shown as Proposition 7.1 in [21].)

We now set

$$\varphi_t(s) := \|u(t+s) - S(t)u(s)\|$$

so we have

$$\varphi_t(s) \leq \int_0^t \|f(s+\xi)\| d\xi = \int_s^{s+t} \|f(\xi)\| d\xi.$$

Let $I := \int_0^\infty \|f(\xi)\| d\xi$ and let $M \in \mathbb{N}$ be any upper bound of I. We define a nonincreasing path on $[0, M]$ by

$$\bar{\varphi}(s) := M - \int_0^s \|f(\xi)\| d\xi$$

so that

$$\varphi_t(s) \leq \int_s^{s+t} \|f(\xi)\| d\xi = |\bar{\varphi}(s+t) - \bar{\varphi}(s)|.$$

We now claim that we can give a bound $\Phi(k, g, M)$ on the metastable version of the Cauchy property of this path, i.e.

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g, M)\ \forall s, t \in [n, n+g(n)]\ (|\bar{\varphi}(s) - \bar{\varphi}(t)| < 2^{-k})\ (*)$$

which by the monotonicity of $\bar{\varphi}(\cdot)$ can be equivalently written as

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g, M)\ (|\bar{\varphi}(n + g(n)) - \bar{\varphi}(n)| < 2^{-k})\ (**)$$

which is the no-counterexample interpretation of

$$\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \forall m \in \mathbb{N}\ (|\bar{\varphi}(n+m) - \bar{\varphi}(n)| < 2^{-k}).$$

$(*)$ (and hence $(**)$) yields

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g, M)\ \forall t \in [0, g(n)]\ (|\bar{\varphi}(n+t) - \bar{\varphi}(n))| < 2^{-k})$$

so that in turn

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g, M)\ \forall t \in [0, g(n)]\ (\varphi_t(n) < 2^{-k}).$$

To show the claim that we can give a bound $\Phi(k, g, M)$ for $(**)$, we follow the proof of 2.26 and 2.27 in [14] (see also Remark 2.29 there). For $g : \mathbb{N} \to \mathbb{N}$ define $\tilde{g} : \mathbb{N} \to \mathbb{N}$ by

$$\tilde{g}(n) := n + g(n)$$

and notice that

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists i \leq 2^k \cdot M\ (\bar{\varphi}(\tilde{g}^{(i)}(0)) - \bar{\varphi}(\tilde{g}^{(i+1)}(0)) < 2^{-k}),$$

where in general the function iterations are defined recursively in the following way:

$$g^{(0)}(k) := k$$

$$g^{(i+1)}(k) := g(g^{(i)}(k)),$$

for assuming on the contrary that for some $k \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$

$$\forall i \leq 2^k \cdot M(\bar{\varphi}(\tilde{g}^{(i)}(0)) - \bar{\varphi}(\tilde{g}^{(i+1)}(0) \geq 2^{-k})$$

by $\tilde{g}^{(0)}(0) = 0$ we obtain

$$\bar{\varphi}(0) - \bar{\varphi}(\tilde{g}^{(2^k \cdot M+1)}(0)) \geq (2^k \cdot M + 1) \cdot 2^{-k} > M$$

which is a contradiction. Now, because $\bar{\varphi}(\cdot)$ is nonincreasing, we obtain

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists i \leq 2^k \cdot M\ (|\bar{\varphi}(\tilde{g}^{(i)}(0)) - \bar{\varphi}(\tilde{g}^{(i)}(0) + g(\tilde{g}^{(i)}(0))| < 2^{-k}).$$

Hence we may take

$$\Phi(k, g, M) := \tilde{g}^{(M \cdot 2^k)}(0)(= \max\{\tilde{g}^{(i)}(0)\ :\ i \leq M \cdot 2^k\}).$$

From this point on the same pattern as in the proof of Theorem 3 is followed.

Assume that $u(\cdot) \in \overline{D(A)}$.

Recall that

$$K(n) := \ulcorner \sqrt{2B(n) + 1} \urcorner$$

where $B(n) \in \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|w_n(0) - z\|^2$ and that

$$w_s'(t) + A(w_s(t)) \ni 0, w_s(0) =: u(s).$$

We claim that we obtain

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Psi(k, \bar{g}, M, \Theta, B)\ \forall x \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ (\|u(x) - z\| < 2^{-k})$$

with a rate of metastability

$$\Psi(k, \overline{g}, M, B, \Theta) = \Phi(k+1, g, M) + h(\Phi(k+1, g, M)) = \tilde{g}^{(M \cdot 2^{k+1})}(0) + h(\tilde{g}^{(M \cdot 2^{k+1})}(0))$$

where

$$h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$

$M \in \mathbb{N}$ is any upper bound on the integral $I := \int_0^\infty \|f(\xi)\| d\xi$,

$$\tilde{g}(n) := g(n) + n,$$

and we have defined

$$g(n) := \overline{g}(n + h(n)) + h(n).$$

To prove the above claim, let

$$x \in [n + h(n), n + h(n) + \overline{g}(n + h(n))],$$

where $x := n + t$ for some $t \in [h(n), h(n) + \overline{g}(n + h(n))]$.

By (7) of Theorem 2 we have

$$\forall k \in \mathbb{N}\ \forall n \in \mathbb{N}\ \forall t \geq (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1}\ (\|w_n(t) - z\| < 2^{-k})$$

Therefore, as we assumed that $t \geq h(n)$, the condition

$$\|w_n(t) - z\| < 2^{-k-1}$$

is satisfied.

Now, by the triangle inequality, we obtain

$$\begin{aligned} \|u(n + t) - z\| &\leq \|u(n + t) - S(t)u(n)\| + \|S(t)u(n) - z\| \\ &= \|u(n + t) - S(t)u(n)\| + \|w_n(t) - z\| \\ &= \varphi_t(n) + \|w_n(t) - z\| \end{aligned}$$

$$
\begin{aligned}
&< \varphi_t(n) + 2^{-k-1} \\
&\le \int_n^{n+t} \|f(\xi)\| d\xi + 2^{-k-1} \\
&\le \int_n^{n+h(n)+\bar{g}(n+h(n))} \|f(\xi)\| d\xi + 2^{-k-1} \\
&= |\bar{\varphi}(n+g(n)) - \bar{\varphi}(n)| + 2^{-k-1}.
\end{aligned}
$$

By the (metastable) Cauchy property above we have

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \le \Phi(k+1, g, M)\ (\varphi_{g(n)}(n) \le |\bar{\varphi}(n+g(n)) - \bar{\varphi}(n)| < 2^{-k-1})$$

where we recall that $g(n) = h(n) + \bar{g}(n + h(n))$ and $\Phi(k, g, M)$ is as before. Hence we obtain

$$
\begin{aligned}
&\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists n \le \tilde{g}^{(M\cdot 2^{k+1})}(0) \\
&\qquad \forall t \in [h(n), h(n) + \bar{g}(n+h(n))]\ (\|u(n+t) - z\| < 2 \cdot 2^{-k-1} = 2^{-k}).
\end{aligned}
$$

Thus, for n chosen as above (taking $\bar{n} := n + h(n)$ and using the monotonicity of $h(\cdot)$) we get

$$
\begin{aligned}
&\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \le \tilde{g}^{(M\cdot 2^{k+1})}(0) + h(\tilde{g}^{(M\cdot 2^{k+1})}(0)) \\
&\qquad \forall x \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ (\|u(x) - z\| < 2^{-k}).
\end{aligned}
$$

□

Corollary 2. *(quantitative form of Corollary (10) in [8]) Let X be a real Banach space with the Radon-Nikodym property. Let $A : D(A) \to 2^X$ be an m-accretive and uniformly accretive at zero operator with $0 \in Az$ and modulus of accretivity Θ. Then for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of Problem 1 converges strongly to z as $t \to \infty$ with a rate of metastability as in Theorem 4.*

Proof. As shown in [2] Chapter 7, because X has the Radon-Nikodym property, the integral solution of Problem 2 is a strong solution. Because A is m-accretive, A satisfies the range condition. Thus the result follows directly by Theorem 4. □

Note that it is well-known that every reflexive Banach space has the Radon-Nikodym property. (Also note that in [1](Theorem 2.2, page 131) it is shown that assuming that X is a reflexive Banach space, the integral solution of Problem 2 is a strong solution).

4. Application

In [8] the following nonlinear boundary value problem is studied :

Problem 4.

$$
\begin{gathered}
u_t - div\ (|Du|^{p-2}Du) + \varphi(x, u) = f,\ \ on\ (0, \infty) \times \Omega, \\
-\frac{\partial u}{\partial \eta} \in \beta(u)\ \ on\ [0, \infty) \times \partial\Omega, \\
u(0, x) = u_0(x) \in L^q(\Omega)\ \ in\ \Omega,
\end{gathered}
$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $f \in L^1((0,\infty), L^q(\Omega))$, $1 \le p, q < \infty$, $\frac{\partial u}{\partial \eta} = \langle |Du|^{p-2}Du, \eta\rangle$, η the unit outward normal on $\partial\Omega$, Du the gradient of u, β a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(1) *for almost all* $x \in \Omega$, $r \to \varphi(x,r)$ *is continuous and nondecreasing,*
(2) *for every* $r \in \mathbb{R}$, $x \to \varphi(x,r)$ *is in* $L^1(\Omega)$,
(3) $\varphi(x,0) = 0$, $\varphi(x,r) \neq 0$ *whenever* $r \neq 0$ *and there exist* $\lambda > 0$, $\alpha \geq 2$ *such that* $\varphi(x,r)r \geq \lambda |r|^\alpha$.

In [8] it is shown that the above problem can be written in the form :

$$u'(t) + \mathcal{B}u(t) = f(t), \; 0 < t < \infty$$

$$u(0) = u_0$$

where for any $q \geq 1$, $u_0 \in L^q(\Omega)$, $f \in L^1((0,\infty), L^q(\Omega))$ and $\mathcal{B}$ is shown to be an m-ϕ-accretive at zero operator in $L^q(\Omega)$ with

$$\phi(x) := C_{\alpha,\Omega,\lambda} \|x\|_q^\alpha$$

for some constant $C_{\alpha,\Omega,\lambda}$ (which can be explicitly computed in α, λ and an upper bound on the measure of Ω) and zero $z = 0$. It is shown in [8] that $u(t)$ converges strongly in $L^q(\Omega)$ to $z = 0$ as $t \to \infty$ as it is shown that Problem 2 has a strong solution. Then, because $\mathcal{B}$ is, moreover, uniformly ϕ-accretive at zero (thus has a well-defined modulus of accretivity), the rate of metastability is given by Theorem 4. In particular, we have

$$\forall k \in \mathbb{N}\; \forall \bar{g} \in \mathbb{N} \to \mathbb{N}\; \exists \bar{n} \leq \Psi(k, \bar{g}, M, B, \alpha, C_{\alpha,\Omega,\lambda})\; \forall t \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\; (\|u(t)\| < 2^{-k})$$

with a rate of metastability

$$\Psi(k, \bar{g}, M, B, \alpha, C_{\alpha,\Omega,\lambda}) = \tilde{g}^{(M \cdot 2^{k+1})}(0) + h(\tilde{g}^{(M \cdot 2^{k+1})}(0))$$

where $h(n) := (B(n) + 2) \cdot 2^{\Theta(k+2)+1}$,

$$\tilde{g}(n) := g(n) + n,$$

$$g(n) := \bar{g}(n + h(n)) + h(n),$$

$$\bar{n} := n + h(n),$$

$B(n)$ is a nondecreasing upper bound on

$$\frac{1}{2}\|u(n)\|^2,$$

$M \in \mathbb{N}$ is any upper bound on the integral $I = \int_0^\infty \|f(t)\| dt$, and $\Theta(k)$ may be estimated (in terms of $C_{\alpha,\Omega,\lambda}$ and α) as follows:

For any $q \geq 1$ and assuming $\|x\|_q \geq 2^{-k}$, let $\Theta(k)$ be such that

$$\Theta(k) \geq \min n\{C_{\alpha,\Omega,\lambda}\|x\|_q^\alpha \geq C_{\alpha,\Omega,\lambda} \cdot (2^{-k})^\alpha \geq 2^{-n}\}.$$

We have

$$\log_2(C_{\alpha,\Omega,\lambda} \cdot (2^{-k})^\alpha) \geq \log_2 2^{-n}$$

therefore for n fulfilling

$$n \geq k \cdot \alpha - \log_2 C_{\alpha,\Omega,\lambda}$$

we may take

$$\Theta(k) \geq k \cdot \alpha - \log_2 C_{\alpha,\Omega,\lambda}.$$

Notice that because $g(r) := C_{\alpha,\Omega,\lambda} \cdot r^\alpha$ is nondecreasing, the modulus of accretivity Θ for $\mathcal{B}$ depends only on $k \in \mathbb{N}$ but not on K (see Remark 1).

Acknowledgements:

Ulrich Kohlenbach has been supported by the German Science Foundation (DFG

Project KO 1737/5-2).

Angeliki Koutsoukou-Argyraki has been supported by the International Research Training Group 1529 'Mathematical Fluid Dynamics' funded by DFG and JSPS.

References

[1] Barbu, V. : *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publishing, Leyden, The Netherlands, 1976.

[2] Benilan, Ph., Grandall, M.G. and Pazy, A. : *Nonlinear Evolution Equations in Banach Spaces*, unpublished book

[3] Browder, F.E. : *Nonlinear accretive operators in Banach spaces*, Bull. Amer. Math. Soc. 73, 470-476, 1967

[4] Bruckner, A.M. : *Differentiation of Real Functions*, Springer-Verlag, New York, 1978

[5] Cioranescu, I. : *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic, Dordrecht, 1990.

[6] Crandall, M.G., Liggett, T.M. : *Generation of semigroups of nonlinear transformations on general Banach spaces*, Am. J. Math. **93**, 265-298, 1971.

[7] Deimling, K. : *Nonlinear Functional Analysis* , Springer, Berlin, 1985.

[8] García-Falset, J. : *The asymptotic behavior of the solutions of the Cauchy problem generated by ϕ-accretive operators*, J. Math. Anal. Appl. **310**, 594-608, 2005.

[9] García-Falset, J. and Morales, C.H : *Existence theorems for m-accretive operators in Banach Spaces*, J. Math. Anal. Appl. **309**, 453-461, 2005.

[10] Günzel, D. and Kohlenbach, U. : *Logical metatheorems for abstract spaces axiomatized in positive bounded logic*, Preprint 2014.

[11] Kato, T. : *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan 19, 508-520, 1967

[12] Komura, Y. : *Nonlinear semi-groups in Hilbert space*, J. Math. Soc. Japan 19, 493-507, 1967

[13] Kohlenbach, U.: *Effective moduli from ineffective uniqueness proofs. An unwinding of de La Vallée Poussin's proof for Chebycheff approximation*, Ann. Pure Appl. Logic **64**, 27-94, 1993.

[14] Kohlenbach, U. : *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*, Springer Monographs in Mathematics, 2008.

[15] Kohlenbach, U. : *On quantitative versions of theorems due to F.E. Browder and R. Wittmann*, Adv. Math. **226**, 2764-2795, 2011.

[16] Kohlenbach, U. and Leuştean, L. *Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces*, Journal of the European Mathematical Society **12** 71-92, 2010.

[17] Körnlein, D. and Kohlenbach, U. : *Effective rates of convergence for Lipschitzian pseudocontractive mappings in general Banach spaces*, Nonlinear Analysis **74** 5253-5267, 2011.

[18] Körnlein, D. and Kohlenbach, U. : *Rate of metastability for Bruck's iteration of pseudocontractive mappings in Hilbert space*, Numerical Functional Analysis and Optimization **35** 20-31, 2014.

[19] Kreisel, G. : *On the interpretation of non-finitist proofs, part I*, Journal of Symbolic Logic **16** 241-267, 1951.

[20] Kreisel, G. : *On the interpretation of non-finitist proofs, part II*, Journal of Symbolic Logic **17**(1) 43-58, 1952.

[21] Miyadera, I. and Kobayasi, K. : *On the asymptotic behaviour of almost-orbits of nonlinear contraction semigroups in Banach spaces*, Nonlinear Analysis, Theory, Methods & Applications, Vol. 6, No.4, pp. 349-365, 1982.

[22] Tao, T. : *Soft analysis, hard analysis, and the finite convergence principle*, Essay posted May 23, 2007, appeared in: 'T. Tao, Structure and Randomness: Pages from Year One of a Mathematical Blog. AMS, 298pp., 2008'.

[23] Tao, T. : *Norm convergence of multiple ergodic averages for commuting transformations*, Ergodic Theory Dynam. Systems **28**, 657-688, 2008.

E-mail address, Ulrich Kohlenbach: `kohlenbach@mathematik.tu-darmstadt.de`
E-mail address, Angeliki Koutsoukou-Argyraki: `koutsoukou@mathematik.tu-darmstadt.de`