On the computational content of convergence proofs via Banach limits

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Abstract

This paper addresses new developments in the ongoing proof mining program, i.e. the use of tools from proof theory to extract effective quantitative information from prima facie ineffective proofs in analysis. Very recently, the current authors developed a method to extract rates of metastability (in the sense of Tao) from convergence proofs in nonlinear analysis that are based on Banach limits and so (for all what is known) rely on the axiom of choice. In this paper we apply this method to a proof due to Shioji and Takahashi on the convergence of Halpern iterations in spaces X with a uniformly Gâteaux differentiable norm. We design a logical metatheorem guaranteeing the extractability of highly uniform rates of metastability under the stronger condition of the uniform smoothness of X. Combined with our method of eliminating Banach limits this yields a full quantitative analysis of the proof by Shioji and Takahashi. We also give a sufficient condition for the computability of the rate of convergence of Halpern iterations.

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1 Introduction

The topic of computable analysis started with Alan Turing's seminal paper [1] in which he defined the notion of a computable real number as one which has a computable binary (or decimal) expansion. While being the right notion for single real numbers, it does not provide the proper notion of computable functions on reals as even the function $x, y \mapsto x+y$ would not be computable. This was corrected by Turing in [2] by giving up the uniqueness of the representation allowing (following Brouwer) overlapping intervals which is equivalent to the nowadays used definition in terms of computable Cauchy sequences of rational numbers with computable Cauchy rate (see [3] for much more detailed information). As this definition highlights already, the issue of effective rates of convergence and other effective bounds plays an important role in computable analysis. Often, however, the use of ineffective reasoning in analysis (both via the use of classical logic as well as by introducing noncomputable objects) provides an obstacle for obtaining effective information. This is particularly ubiquitous in the area of abstract nonlinear analysis.

Starting in [4] and continued in [5, 6, 7], general logical metatheorems have been developed which guarantee the extractability of highly uniform effective (and actually subrecursive) bounds

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(whose complexity reflects that of the proof principles used) from large classes of (prima facie highly ineffective proofs) in nonlinear functional analysis ('proof mining'). Here by 'uniform' we refer to the fact that the bounds are independent from metrically majorizable input data (without any compactness condition). For this to hold it is crucial that no separability conditions on the underlying structures must be imposed as the uniform version of separability is total boundedness and so (modulo completeness) compactness. Since structures such as complete metric or Banach spaces usually are represented (in proof theory as well as in reverse mathematics and computable analysis) as completions of countable dense substructures this requires a novel treatment of these structures. The approach is to 'hardwire' an abstract such space X as a kind of atom to the formal system (a suitable system $\mathcal{T}^\omega$ of arithmetic or analysis formulated in the language of functionals of all finite types over $\mathbb{N}$) by adding a new base type X and all finite types over the two base types $\mathbb{N}, X$ (of course also several spaces $X_1, \ldots, X_n$ can be treated simultaneously, see [7]). Then appropriate constants and axioms for the respective structure treated have to be added. The crucial conditions are:

- that the constants added can be majorized by functionals definable already in $\mathcal{T}^\omega$ or that they have a type $\mathbb{N}^n \to \mathbb{N}$ in which case they can be essentially majorized by themselves;
- that the new axioms have a monotone functional interpretation [7] by the same functionals which suffice for the interpretation of $\mathcal{T}^\omega$ and the majorization of the new constants.

The second point is automatically satisfied if the additional axioms are all purely universal which will be the case in this paper.

Structures treated so far are:

- bounded metric, hyperbolic and CAT(0)-spaces, (real) normed, uniformly convex and inner product spaces also with abstract bounded convex subsets $C \subseteq X$ in the normed case [4];
- unbounded metric, hyperbolic and CAT(0)-spaces and (real) normed spaces also with unbounded convex subsets [5];
- Gromov δ-hyperbolic spaces, $\mathbb{R}$-trees and uniformly convex hyperbolic spaces [6];
- complete metric and normed spaces [7].

Some obvious classes of spaces are missing in this list as they do not have the right uniformity built-in to have a monotone functional interpretation. As mentioned already this is the case for separable spaces as the monotone functional interpretation of separability upgrades the latter to the total boundedness of metrically bounded subsets and hence (in the presence of completeness and closedness) to compactness. Compact metric spaces, however, are much easier to treat via their representation as continuous image of the Cantor space (which is explicitly given in our formal framework). Another bad-behaved class are the strictly convex Banach spaces which (under monotone functional interpretation) get upgraded to uniformly convex Banach spaces.

Since our systems are based on full classical logic the theorems to be proven essentially have to be of the form of $\forall\exists$-sentences in order to allow for the existence of effective bounds.

Many theorems in nonlinear analysis are of the form that certain iterations (x_n) built up using some operator $T : X \to X$ and a starting point $x \in X$ (maybe further involving some sequence of scalars (λ_n), typically in $[0, 1]$) are strongly convergent. Since the Cauchy property is $\forall\exists\forall$ rather than $\forall\exists$ one has to express this in the (ineffectively equivalent) Herbrand normal form from logic

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \in [n, n + g(n)]\ (d(x_i, x_j) < 2^{-k})$$

which - in the context of Cauchy statements - has been called metastability by Tao [8, 9]. [1]
The aforementioned metatheorems now guarantee the extractability of effective bounds $\Phi(k, g)$ on '$\exists n$' (in logic referred to as the Kreisel 'no-counterexample interpretation', see [12, 13]) which are

[1] There are, however, important situations where even ineffective proofs guarantee effective rates of convergence, namely e.g. in the presence of uniqueness conditions [10, 7, 11] or in cases of monotone convergence [7].

highly uniform in the sense that they – in addition to k, g – only depend on norm bounds for x and majorants $T^* : \mathbb{N} \to \mathbb{N}$ for T

$$n \geq \|y\| \to T^*(n) \geq \|Ty\|, \quad n \in \mathbb{N},\, y \in X,$$

as well as certain moduli which make some assumptions on (λ_n) explicit. For large classes of maps T (e.g. for nonexpansive ones which are the only ones needed in this paper) the computation of T^* is trivial. This also is the case, when $T : C \to C$ where C is a bounded (convex) subset (as will be the case throughout this paper). Thus the bound is (essentially) independent from x and T as well as the underlying space X (except for data such as a modulus of uniform convexity etc.).

In [14], the first author extracted such a 'rate of metastability' $\Phi(k, g)$ for Halpern iterations

$$x_{n+1} := \frac{1}{n+2}x + \left(1 - \frac{1}{n+2}\right)Tx_n, \quad x_0 := x \in C,$$

for nonexpansive selfmappings $T : C \to C$ of a convex subset of a Hilbert space from a proof due to Wittmann [15] of the strong convergence of (x_n) (provided that C is bounded or – weaker – T has a fixed point). Wittmann's result has received considerable attention as it is an important nonlinear generalization of the well-known von Neumann mean ergodic theorem (see [16, 17] for effective metastable versions) as (x_n) coincides with the Cesàro mean for linear T.

Recently [18], the current authors extracted an explicit rate of metastability from a proof due to [19] of a generalization of Wittmann's theorem to CAT(0)-spaces. This result constitutes a significant extension of the hitherto context of proof mining as Saejung's proof makes use of Banach limits whose existence (for all what is known) requires some substantial use of the axiom of choice. Nevertheless, we developed a method to convert such proofs into more elementary proofs which no longer rely on Banach limits and can be analyzed by the existing logical machinery.

In this paper we extract a rate of metastability for another generalization (due to [20]) of Wittmann's theorem, namely to Banach spaces with a uniformly Gâteaux differentiable norm. The significance of this is twofold:

- As the proof again uses Banach limits we further substantiate our claim that the machinery developed in [18] to eliminate arguments based on Banach limits is indeed a general method. In fact, we can literally re-use most of the technical lemmas from [18] showing the striking modularity of this proof-theoretic approach based on (monotone) functional interpretation.

- The proof is based on the existence of a uniformly continuous (in a suitable sense) so-called duality mapping J which also plays an important role in numerous other proofs in nonlinear analysis. In the next section we indicate how this structure can be nicely incorporated into the framework of the logical metatheorems referred to above.

In our paper, all Banach spaces are real Banach spaces.

2 A logical metatheorem for real Banach spaces with a norm-to-norm uniformly continuous duality selection map

In this paper we study a proof that uses a smoothness property of Banach spaces, namely that the norm is uniformly Gâteaux differentiable. It turns out that this notion is not uniform enough to have a monotone functional interpretation or - rather - that the latter requires that the space even has a uniformly Fréchet differentiable norm, i.e. it is uniformly smooth.

The uniform smoothness of a space X can be universally axiomatized once a constant τ_X : $\mathbb{N} \to \mathbb{N}$ representing (a suitable notion of) a modulus of uniform smoothness is given. Then the corresponding metatheorem will guarantee the extractability of an effective uniform bound that (in addition to its usual input data) will only depend on δ_X. In the concrete application given in this paper it indeed will be the uniform smoothness (rather than uniform Gâteaux differentiability of the norm) which we need for this. This is via the norm-to-norm uniform continuity on bounded sets of the normalized duality map J of X which holds in uniformly smooth spaces whereas uniform Gâteaux differentiability only implies the norm-to-weak* uniform continuity of J.

Definition 2.1. *Let X be a Banach space and X^* its dual space. Then the mapping*

$$J : X \to 2^{X^*}, \quad Jx := \{y \in X^* : \langle x, y\rangle = \|x\|^2 = \|y\|^2\}$$

is called the (normalized) duality mapping of X. Here $\langle x, y\rangle$ denotes $y(x)$.

By the Hahn-Banach theorem it follows that Jx is always nonempty. If X is smooth (i.e. has a Gâteaux differentiable norm), then Jx is always single-valued and also the converse holds (see e.g. Theorems 4.3.1 and 4.3.2 in [21]). This single valued mapping is norm-to-norm uniformly continuous on bounded subsets provided that X is uniformly smooth and a modulus of uniform continuity can be obtained from a modulus of uniform smoothness for X (see Proposition 2.5 below). In our application, we only need a function $J : X \to X^*$ which selects in a uniformly continuous way a point from the duality set and will not insist on that the latter is single-valued.

Let us define a *space with a uniformly continuous duality selection map* (X, J) to be a real Banach space X together with a mapping $J : X \to X^*$ satisfying

(i) $\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2$ for all $x \in X$, and

(ii) J is norm-to-norm uniformly continuous on any bounded subsets of X.

Obviously, it suffices to require that J is norm-to-norm uniformly continuous on any open ball $B_d(0)$ (resp closed ball $\overline{B}_d(0)$), $d > 0$. By a *modulus* for the space with a uniformly continuous duality selection map (X, J) we shall understand a mapping $\omega : (0, \infty) \times (0, \infty) \to (0, \infty)$ such that for all $d > 0$, $\omega(d, \cdot)$ is a modulus of uniform continuity for the restriction of J to $\overline{B}_d(0)$, that is for all $\varepsilon > 0$ and $x, y \in \overline{B}_d(0)$,

$$\|x - y\| \leq \omega(d, \varepsilon) \text{ implies } \|Jx - Jy\| \leq \varepsilon. \tag{1}$$

Using standard continuity arguments one can easily see that the existence of a modulus ω satisfying (1) is equivalent to the existence of $\omega : \mathbb{Q}^+_* \times \mathbb{N} \to \mathbb{N}$ such that for any $d \in \mathbb{Q}^+_*, k \in \mathbb{N}$, and $x, y \in B_d(0)$,

$$\|x - y\| < 2^{-\omega(d,k)} \text{ implies } \|Jx - Jy\| \leq 2^{-k}. \tag{2}$$

Rather than having to formalize the proof of the existence of J and its continuity property we directly add constants J_X, ω_X and axioms (J_X) and (J_X, ω_X) to the formal framework expressing that for $x \in X$, $J_X x$ represents a linear operator $X \to \mathbb{R}$ with $\|J_X x\| \leq \|x\|$ and $J_X xx = \|x\|^2$, which - taken together - yields $\|J_X x\| = \|x\|$, i. e. $J_X xx = \|x\|^2 = \|J_X x\|^2$ and that J_X is norm-to-norm uniformly continuous on any bounded ball $B_d(0)$ with modulus of uniform continuity $\omega_X(d, \cdot)$. Instead of using the operator norm and stating $\|J_X x - J_X y\| \leq 2^{-k}$ we express things equivalently in the language of X as $\forall z \in X(|J_X xz - J_X yz| \leq 2^{-k} \cdot \|z\|)$.

In formulating (J_X) and (J_X, ω_X) we rely on the formal framework from [7] and the representation of real numbers, $\leq_{\mathbb{R}}$ etc. in terms of number theoretic functions. J_X then is an object of type $X \to X \to 1$ (where 1 denotes the type $\mathbb{N} \to \mathbb{N}$, that is the type of objects used to represent real numbers) and ω_X has type $\mathbb{N}^2 \to \mathbb{N}$:

$$\begin{aligned} (J_X) \quad &:\equiv \quad \forall x^X, y^X \big(J_X xx =_{\mathbb{R}} \|x\|_X^2 \wedge |J_X xy|_{\mathbb{R}} \leq_{\mathbb{R}} \|x\|_X \cdot_{\mathbb{R}} \|y\|_X \wedge \\ &\qquad \wedge \forall \alpha^1, \beta^1, u^X, v^X (J_X x(\alpha \cdot_X u +_X \beta \cdot_X v) =_{\mathbb{R}} \alpha \cdot_{\mathbb{R}} J_X xu +_{\mathbb{R}} \beta \cdot_{\mathbb{R}} J_X xv)\big) \\ (J_X, \omega_X) \quad &:\equiv \quad \forall x^X, y^X, z^X, k^{\mathbb{N}}, d^{\mathbb{N}} \big(\|x\|_X, \|y\|_X <_{\mathbb{R}} (d)_{\mathbb{R}} \wedge \|x - y\|_X <_{\mathbb{R}} 2^{-\omega_X(k,d)} \\ &\qquad \to |J_X xz -_{\mathbb{R}} J_X yz|_{\mathbb{R}} \leq_{\mathbb{R}} 2^{-k} \cdot_{\mathbb{R}} \|z\|_X\big) \end{aligned}$$

It is easy to see that (J_X) and (J_X, ω_X) (but not (J_X) alone) imply the extensionality of J_X

$$x =_X x' \wedge y =_X y' \to J_X xy =_{\mathbb{R}} J_X x'y',$$

so that we can safely use J_X in the usual set-theoretic way (whereas with (J_X) extensionality only holds for provably equal arguments; see [7]).

Let $\mathcal{T}^\omega$ be e.g. the finite type system for classical analysis WE-PA$^\omega$+DC$^\omega$+QF-AC and $\mathcal{T}^\omega[X,\|\cdot\|,J_X,\omega_X,C,\mathcal{C}]$ its extension by an abstract real normed space with the constants J_X,ω_X together with their above axioms, an abstract nonempty convex subset $C\subseteq X$ and a completeness axiom stating the completeness of X / closedness of C (see [7] for details). Then the logical metatheorems for Banach spaces from [7] hold if the extracted bound is allowed to depend on ω_X. We only formulate here a special instance of these theorems sufficient for our main application:

Theorem 2.2. *Let $A_\exists(k^{\mathbb{N}},g^{\mathbb{N}\to\mathbb{N}},x^X,T^{X\to X},n^{\mathbb{N}})$ be a purely existential formula containing only k,g,x,T,n free. Then the following rule holds: From a proof in $\mathcal{T}^\omega[X,\|\cdot\|,J_X,\omega_X,C,\mathcal{C}]$ of*

$$\forall k\in\mathbb{N}\,\forall g:\mathbb{N}\to\mathbb{N}\,\forall x\in C\,\forall T:C\to C\ \big(T \textit{ nonexpansive } \to \exists n\in\mathbb{N}\,A_\exists\big)$$

one can extract a computable $\Phi:\mathbb{N}\times\mathbb{N}\times\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}^2}\to\mathbb{N}$ bound such that

$$\forall k\in\mathbb{N}\,\forall g:\mathbb{N}\to\mathbb{N}\,\forall x\in C\,\forall T:C\to C\ \big(T \textit{ nonexpansive } \to \exists n\leq\Phi(k,b,g,\omega)\ A_\exists\big)$$

holds in any (real) Banach space X with a duality selection map J (used to interpret J_X) that has ω as modulus of uniform norm-to-norm continuity (used to interpret ω_X) and any closed b-bounded convex subset $C\subseteq X$. If C is not bounded, then one has to choose b such that $b\geq\|x\|,\|x-Tx\|$.

Note that Φ does not depend on x,T and depends on X,C only via ω resp. b.

Proof. The proofs in [4, Theorem 3.30] for bounded C and in [5, Corollary 6.6] for unbounded C easily extend to our situation (see also [5, Remark 4.13]) as both (J_X) and (J_X,ω_X) are purely universal (here we use that $=_{\mathbb{R}}$, $\leq_{\mathbb{R}}$ are purely universal while $<_{\mathbb{R}}$ is purely existential, see [7]), ω_X is trivially majorized by $\omega^*_X(n,m):=\max\{\omega_X(i,j): i\leq n, j\leq m\}$, and J_X is majorized by $J_X(n,m):=M(n\cdot m):=\lambda k\in\mathbb{N}.j(n\cdot m\cdot 2^{k+2},2^{k+1}-1)$ (see [4, Definition 2.9]) with the Cantor pairing function j and using the obvious extension of $\circ$ from [4, Definition 2.9] from $\mathbb{R}_+$ to $\mathbb{R}$) since $n\geq\|x\|\wedge m\geq\|y\|\to n\cdot m\geq\|x\|\cdot\|y\|\geq|Jxy|$, where J_X is interpreted via $(Jxy)_\circ\in\mathbb{N}^{\mathbb{N}}$. The completion axiom is incorporated as in [7, pp. 433-434] and the closedness of C can also easily be expressed in a purely universal way similarly to $(\mathcal{C})$. $\square$

Remark 2.3. *The extraction of the bound proceeds by monotone functional interpretation (see [7]) from the proof and its complexity faithfully reflects the complexity of the principles used in the proof. In our case in this paper this will yield a Φ of very restricted complexity.*

The strong convergence result for Halpern iterations in [20] is proved under the **hypothesis** that the sequence (z_n) of the fixed points of the contractions $T_n(y):=\frac{1}{n+1}x+\left(1-\frac{1}{n+1}\right)Ty$ strongly converges (towards a fixed point of T) which is known in many cases such as for Hilbert spaces [22, 23], CAT(0)-spaces [24] and also uniformly smooth Banach spaces [25] for bounded, closed and convex C (which covers our context). Under this assumption (in fact already under the assumption of the plain Cauchy property of (z_n)), the proof of the strong convergence of the Halpern iteration (x_n) that results by our elimination of the use of Banach limits from the proof of [20] is basically constructive. Hence metatheorems for the constructive case [26] guarantee (and our proof displays this, see Theorem 3.4) a uniform effective procedure to transform a rate of convergence for (z_n) into one for (x_n). The problem, however, is that even in very simple cases (X being an effective Hilbert space and T a computable and linear nonexpansive map) there is no computable rate of the former as (see [16]) there is no for the latter. In fact, to show that that there is no effective operator which would effectively in a computable sequence of operators $(T(l))_l$ produce a rate of convergence for (z_n) is almost trivial and holds already for $X:=\mathbb{R}$ and $C:=[0,1]$ (similarly to [7, Theorem 18.4]). Only in certain cases, e.g. when, in particular, the norm $\|z\|$ of the limit $z:=\lim\limits_{n\to\infty}z_n$ is known, one gets computable rates of (z_n) (but not uniform ones and without any complexity information as the argument is based on unbounded search): see

Theorem 3.4.
What, however, **can** be obtained in many cases (not only computably but with low complexity) is a (fully uniform) **rate of metastability** for (z_n). Since the latter only ineffectively implies the convergence of (z_n) it is this feature which makes the proof of the convergence of (x_n) nonconstructive and forces us to also weaken the conclusion to the metastability of (x_n) (in logical terms this corresponds to applying a so-called negative translation prior to the actual functional interpretation).[2] So we actually use the above metatheorem in the form where we have as an additional input a (majorant of a) rate $K(\varepsilon, g)$ of metastability for (z_n) (or – equivalently – a selfmajorizing such rate) and extract a bound on the metastability of (x_n) that depends in addition to $\varepsilon, g, b, \omega$ also on K (in the case at hand it turns out that K not even needs to be majorizable as it gets applied only to a single function f^* that is defined in terms of g and the other input data). For the Hilbert case (as well as the CAT(0)-case, see [18]) a simple primitive recursive such K has been extracted in [14]. For the case of uniformly smooth Banach spaces [25] (i.e. the context of the present paper) this is to be left for future research.

2.1 Some examples

Let us recall that a Banach space X is

(i) *uniformly convex* if for all $\varepsilon \in (0,2]$ there exists $\delta \in (0,1]$ such that for all $x, y \in X$,

$$\|x\| \le 1, \quad \|y\| \le 1 \text{ and } \|x-y\| \ge \varepsilon \text{ imply } \left\|\frac{1}{2}(x+y)\right\| \le 1-\delta. \tag{3}$$

(ii) *uniformly smooth* whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$,

$$\|x\| = 1 \text{ and } \|y\| \le \delta \text{ imply } \|x+y\| + \|x-y\| < 2 + \varepsilon\|y\|. \tag{4}$$

A mapping $\eta : (0,2] \to (0,1]$ providing a $\delta := \eta(\varepsilon)$ satisfying (3) for given $\varepsilon \in (0,2]$ will be called a *modulus of uniform convexity*. Similarly, a function $\tau : (0,\infty) \to (0,\infty)$ providing such a $\delta := \tau(\varepsilon)$ satisfying (4) is said to be a *modulus of uniform smoothness*.

Remark 2.4. *The property of X being a uniformly smooth Banach space with a modulus $\tau_X : \mathbb{N} \to \mathbb{N}$ (formulated with 2^{-k} instead of ε/δ) can be axiomatized by a universal axiom over our framework (so that the logical metatheorems guarantee effective bounds depending additionally only on τ_X) as follows (using again that $\le_{\mathbb{R}}$ is universal while $<_{\mathbb{R}}$ is existential):*

$$\forall x^X, y^X \forall k \in \mathbb{N}\big(\|x\|_X >_{\mathbb{R}} 1 \wedge \|y\|_X <_{\mathbb{R}} 2^{-\tau_X(k)} \to \|\tilde{x} +_X y\| +_{\mathbb{R}} \|\tilde{x} -_X y\| \le_{\mathbb{R}} 2 +_{\mathbb{R}} 2^{-k} \cdot_{\mathbb{R}} \|y\|_X\big),$$

where $\tilde{x} := \dfrac{1}{\max_{\mathbb{R}}\{1, \|x\|_X\}} \cdot_X x$. Note that for x with $\|x\| > 1$ one has $\tilde{x} \in S_1$. Conversely, for $x \in S_1$ and $x' := 2 \cdot x$ one has $\|x'\| = 2 > 1$ and $\widetilde{x'} =_X x$. So in the axiom above we indeed quantify over all vectors $x \in S_1$.

Proposition 2.5. *(i) If X is uniformly smooth with modulus τ, then X^* is uniformly convex with modulus $\eta(\varepsilon) = \dfrac{\varepsilon}{4} \cdot \tau\left(\dfrac{\varepsilon}{2}\right)$.*

(ii) If X^* is uniformly convex with modulus η, then X is a space with a uniformly continuous duality selection map with modulus $\omega(d,\varepsilon) = \dfrac{\varepsilon}{3} \cdot \eta\left(\dfrac{\varepsilon}{d}\right)$ for all $\varepsilon \in (0,2]$ and $d \ge 1$. For $d < 1$ one can trivially define $\omega(d,\varepsilon) = \omega(1,\varepsilon)$ for all $\varepsilon > 0$, while for $\varepsilon > 2$, one defines $\omega(d,\varepsilon) = \omega(d,2)$ for all $d > 0$.

[2] That this can improve things even when applied to constructive proofs is discussed in [7, p. 171].

Proof. (i) A classical result states that X is uniformly smooth if and only if X^* is uniformly convex and the following Lindenstrauss duality formula holds (see e.g. [27, Proposition 1.e.2]): for all $\delta > 0$

$$\rho_X(\delta) = \sup\left\{\frac{\delta\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 \le \varepsilon \le 2\right\}, \tag{5}$$

where

$$\rho_X(\delta) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| \le \delta\right\} \tag{6}$$

is the modulus of smoothness of X, while

$$\delta_{X^*}(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x^* + y^*)\right\| : x^*, y^* \in X^*, \|x^*\| \le 1, \|y^*\| \le 1, \|x^* - y^*\| \ge \varepsilon\right\}. \tag{7}$$

is the modulus of convexity of X^*. One can see easily that $\rho_X(\delta) \le \frac{\delta\varepsilon}{4} \le \frac{\varepsilon}{4} \cdot \tau\left(\frac{\varepsilon}{2}\right)$ for all $\delta \le \tau\left(\frac{\varepsilon}{2}\right)$. Apply now (5) with $\delta := \tau\left(\frac{\varepsilon}{2}\right)$ to get that $\delta_{X^*}(\varepsilon) \ge \frac{\varepsilon}{2}\tau\left(\frac{\varepsilon}{2}\right) - \rho_X\left(\frac{\varepsilon}{2}\right) \ge \eta(\varepsilon)$.
(ii) Let $\varepsilon \in (0, 2]$, $d \ge 1$ and $x, y \in B_d(0)$ with $\|x - y\| \le \omega(d, \varepsilon)$. W.l.o.g. we may assume that $\|y\| \ge \|x\|$ and also that $\|y\| > \varepsilon/2$ since otherwise $\|Jx - Jy\| \le \|Jx\| + \|Jy\| = \|x\| + \|y\| \le \varepsilon$.

$$\begin{aligned}
\|Jx + Jy\| &\ge \frac{1}{\|y\|}\langle y, Jx + Jy\rangle = \frac{1}{\|y\|}\big(\langle x, Jx\rangle + \langle y, Jy\rangle - \langle x - y, Jx\rangle\big) \\
&\ge \frac{1}{\|y\|}\left(\|x\|^2 + \|y\|^2 - \|x\| \cdot \|x - y\|\right) \\
&\ge \frac{1}{\|y\|}\left((\|y\| - \omega(d, \varepsilon))^2 + \|y\|^2 - \|x\| \cdot \|x - y\|\right) \\
&> 2\|y\| - 2\omega(d, \varepsilon) - \frac{\|x\|}{\|y\|}\|x - y\| \ge 2\|y\| - 3\omega(d, \varepsilon).
\end{aligned}$$

Hence $\left\|\frac{1}{2}\left(\frac{1}{\|y\|}Jx + \frac{1}{\|y\|}Jy\right)\right\| > 1 - \frac{3\omega(d,\varepsilon)}{2\|y\|} > 1 - \frac{3\omega(d,\varepsilon)}{\varepsilon} = 1 - \eta\left(\frac{\varepsilon}{d}\right)$. By the uniform convexity of X^* one gets that $\left\|\frac{1}{\|y\|}Jx - \frac{1}{\|y\|}Jy\right\| < \frac{\varepsilon}{d}$ and so $\|Jx - Jy\| < \frac{\varepsilon\|y\|}{d} \le \varepsilon$. $\square$

Remark 2.6. *If $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$, where $\varepsilon_1 \le \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \le \tilde{\eta}(\varepsilon_2)$, then ω can be improved to $\omega(d, \varepsilon) := \frac{2}{3} \cdot \varepsilon \cdot \tilde{\eta}\left(\frac{\varepsilon}{d}\right)$: observe that with this ω*

$$\frac{3\omega(d,\varepsilon)}{2\|y\|} = \frac{\varepsilon \cdot \tilde{\eta}(\varepsilon/d)}{\|y\|} \le \frac{\varepsilon \cdot \tilde{\eta}(\varepsilon/\|y\|)}{\|y\|} = \eta\left(\frac{\varepsilon}{\|y\|}\right)$$

and so as before $\left\|\frac{1}{\|y\|}Jx - \frac{1}{\|y\|}Jy\right\| < \frac{\varepsilon}{\|y\|}$, i.e. $\|Jx - Jy\| < \varepsilon$.

It is well known that the Banach spaces L_p with $1 < p < \infty$ are both uniformly convex and uniformly smooth. A modulus of uniform convexity $\eta_p(\varepsilon)$ is given by

$$\eta_p(\varepsilon) = \varepsilon \cdot \tilde{\eta}_p(\varepsilon) \text{ where } \tilde{\eta}_p(\varepsilon) = \begin{cases} \frac{(p-1)}{8} \cdot \varepsilon, & 1 < p < 2, \\ \frac{1}{p \cdot 2^p} \cdot \varepsilon^{p-1}, & 2 \le p < \infty. \end{cases} \quad \text{(see [27, p. 63])}.$$

Since L_p^* is isometrically isomorphic with $L_{p'}$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p, we get (using the remark above) that L_p is a space with a uniformly continuous duality selection map with modulus $\omega(d, \varepsilon) = \frac{2\varepsilon}{3} \cdot \tilde{\eta}_{p'}\left(\frac{\varepsilon}{d}\right)$ for all $\varepsilon \in (0, 2]$ and $d \ge 1$.

3 An application to Halpern iterations

Let X be a Banach space, $C \subseteq X$ a convex closed subset and $T : C \to C$ be a nonexpansive mapping. The so-called *Halpern iteration* is defined as follows:

$$x_0 := x, \quad x_{n+1} := \lambda_{n+1}u + (1-\lambda_{n+1})Tx_n, \tag{8}$$

where $(\lambda_n)_{n\geq 1}$ is a sequence in $[0,1]$ and $x, u \in C$. We refer to [18] for a discussion.

For $t \in (0,1)$ and $u \in C$, define $T_t^u : C \to C$ by $T_t^u(y) = tu + (1-t)Ty$. Since T_t^u is a contraction, we apply Banach's Contraction Mapping Principle to get a unique fixed point $z_t^u \in C$:

$$z_t^u = tu + (1-t)Tz_t^u. \tag{9}$$

The following extension of Wittmann's result was obtained by Shioji and Takahashi [20].

Theorem 3.1. *Let X be a Banach space whose norm is uniformly Gâteaux differentiable, $C \subseteq X$ be closed and convex and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that*

(i) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_{n+1}$ *diverges and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ *converges;*

(ii) (z_t^u) *converges strongly to a fixed point z of T as $t \downarrow 0$.*

Then the Halpern iteration converges strongly to z.

In this paper we obtain an effective version of Theorem 3.1 by applying proof mining techniques to Shioji and Takahashi's proof, which is highly ineffective. Firstly, with the methods developed in [18], we eliminate the use of Banach limits from the proof. Secondly, we extract an effective and highly uniform rate of metastability, which is guaranteed to exist by Theorem 2.2.

Theorem 3.2. *Let (X, J) be a space with a uniformly continuous duality selection map with modulus ω, $C \subseteq X$ be a bounded convex closed subset with diameter d_C, $T : C \to C$ a nonexpansive mapping and $x, u \in C$. Let $M \in \mathbb{Z}_+$ be such that $M \geq d_C$.*

Assume that $\lim_{n\to\infty} \lambda_n = 0$ with rate of convergence α, $\sum_{n=1}^{\infty} \lambda_{n+1}$ diverges with rate of divergence θ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ converges with β being a Cauchy modulus of $s_n := \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$.

Let $t_k := \dfrac{1}{k+1}$, $k \geq 1$ and assume that $(z_{t_k}^u)$ is Cauchy with rate of metastability K, i.e.

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\ \exists K_1 \leq K(\varepsilon, g)\ \forall i,j \in [K_1, K_1 + g(K_1)]\ \left(\|z_{t_i}^u - z_{t_j}^u\| \leq \varepsilon\right). \tag{10}$$

Then the Halpern iteration (x_n) is Cauchy and for all $\varepsilon \in (0,2)$ and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leq \Sigma(\varepsilon, \omega, g, M, K, \theta, \alpha, \beta)\ \forall m,n \in [N, N+g(N)]\ (\|x_n - x_m\| \leq \varepsilon), \tag{11}$$

where

$$\Sigma(\varepsilon, g, \omega, M, \theta, \alpha, \beta, K) := \theta^+\left(\Gamma - 1 + \left\lceil \ln\left(\frac{12M}{\varepsilon^2}\right)\right\rceil\right) + 1, \quad \textit{with}$$

$$\Gamma = \max\left\{\chi_k^*(\varepsilon^2/12) \mid \left\lceil \frac{1}{\varepsilon_0} \right\rceil \leq k \leq K(\varepsilon_0, f^*) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil \right\}, \theta^+(n) = \max\{\theta(i) \mid i \leq n\},$$

$$\delta = \frac{\varepsilon^2}{24M(4+M)}, \quad \varepsilon_0 = \min\{\delta, \omega(M,\delta)\}, \quad \chi_k^*(\varepsilon, \omega) = \chi_k\left(\frac{\varepsilon}{2}, \omega\right),$$

$$\chi_k(\varepsilon, \omega) = \tilde{\Phi}\left(\omega\left(M, \frac{\varepsilon_k}{M}\right)\right) + \tilde{P}_k(\varepsilon), \varepsilon_k = \frac{\varepsilon}{\tilde{P}_k(\varepsilon) + 1},$$

$$\tilde{P}_k(\varepsilon) = \left\lceil \frac{12M^2(k+1)}{\varepsilon} \cdot \Phi\left(\frac{\varepsilon}{12M(k+1)}\right)\right\rceil, f(k) = \max\left\{\left\lceil \frac{2M^2}{\Delta_k^*(\varepsilon^2/4, g)}\right\rceil, k\right\} - k,$$

$$f^*(k) = f\left(k + \left\lceil\frac{1}{\varepsilon_0}\right\rceil\right) + \left\lceil\frac{1}{\varepsilon_0}\right\rceil, \qquad g_{\varepsilon,k}(n) = n + g\left(n + \chi_k^*\left(\frac{\varepsilon}{3}, \omega\right)\right),$$

$$\tilde{\Phi}(\varepsilon) = \theta\left(\beta\left(\frac{\varepsilon}{4M}\right) + 1 + \left\lceil \ln\left(\frac{2M}{\varepsilon}\right)\right\rceil\right) + 1, \qquad \Phi(\varepsilon) = \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}\right), \alpha\left(\frac{\varepsilon}{4M}\right)\right\},$$

$$\Theta_k(\varepsilon) = \theta\left(\chi_k^*\left(\frac{\varepsilon}{3}, \omega\right) - 1 + \left\lceil \ln\left(\frac{3M}{\varepsilon}\right)\right\rceil\right) + 1, \quad \Delta_k^*(\varepsilon, g) = \frac{\varepsilon}{3g_{\varepsilon,k}\left(\Theta_k(\varepsilon) - \chi_k^*\left(\frac{\varepsilon}{3}, \omega\right)\right)}.$$

Remark 3.3. *For the most important case $\lambda_n := 1/(n+1)$ the moduli θ, α, β are all easily computable. In fact, one can avoid the use of the exponential θ by using $\lim\limits_{n\to\infty} \prod_{n=1}^{\infty}(1-\lambda_{n+1}) = 0$ instead of the divergence of $\sum_{n=1}^{\infty} \lambda_{n+1}$ (see [18] for details on this).*

Theorem 3.4. *Let $\lambda_n := 1/(n+1), n \geq 1$, $t_k := 1/(k+1)$, $k \geq 1$ and denote $z_{t_k}^u$ by z_k^u.*

(i) If $K(\varepsilon)$ is a rate of convergence of (z_k^u), then the bound in Theorem 3.2 gives a rate of convergence of (x_n).

(ii) If X is an effective Hilbert space and T, u are computable, then (z_k^u) has a computable rate of convergence iff $\|z - u\|$ is computable, where $z := \lim\limits_{k\to\infty} z_k^u$.

Proof. (i) $K(\varepsilon/2)$ is a witness (not only a bound) of metastability for any function g (i.e. we can take $K_1 := K(\varepsilon/2)$ in (10)). Hence we can replace in the bound Σ from Theorem 3.2 $K(\varepsilon_0, f^*)$ by $K(\varepsilon_0/2)$ which makes the bound independent of g since g only enters via the definition of f^*. Also note that the maximum in the definition of Γ can be replaced by just taking $k := K(\varepsilon_0/2)$. Then (11) holds with $N := \Sigma$ for all g where now Σ is independent of g.

(ii) From [14, p. 2789] it follows that a rate of convergence for (z_k^u) is given by a rate of convergence of the nondecreasing and M-bounded sequence $(\|z_k^u - u\|^2)$ which is computable provided that the limit $\|z-u\|^2 = \lim\limits_{k\to\infty} \|z_k^u - u\|^2$ is. Conversely, if we have a computable rate of convergence for (z_k^u), then z and hence $\|z - u\|$ is computable. □

3.1 Technical lemmas

In the following, (X, J) is a space with a uniformly continuous duality selection map with modulus ω, $C \subseteq X$ is a bounded convex closed subset with diameter d_C and $T : C \to C$ is a nonexpansive mapping. We consider $M \in \mathbb{Z}_+$ with $M \geq d_C$. Thus, $M \geq \|x - y\|$ for all $x, y \in C$.

Lemma 3.5.

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad x, y \in X. \tag{12}$$

Proof. Remark that $\|x+y\|^2 = \langle x, J(x+y)\rangle + \langle y, J(x+y)\rangle) \leq \|x\| \cdot \|x+y\| + \langle y, J(x+y)\rangle \leq \frac{1}{2}(\|x\|^2 + \|x+y\|^2) + \langle y, J(x+y)\rangle$. □

Given a sequence $(a_n)_{n\geq 1}$ of real numbers, we shall consider for all $m, p \geq 1$ the average $C_{m,p}(a_n) = \dfrac{1}{p}\sum\limits_{i=m}^{m+p-1} a_i$. As in [18], the use of Banach limits in Shioji-Takahashi's proof can be eliminated in favor of elementary lemmas on the finitary objects $C_{m,p}$.

Let $x, u \in C$, $t \in (0,1)$, (λ_n) be a sequence in $[0,1]$, (x_n) be the Halpern iteration defined by (8) and z_t^u given by (9). Define for all $n \geq 1$

$$\gamma_n^t := 2\langle u - z_t, J(x_n - z_t)\rangle - tM^2. \tag{13}$$

Let us recall that (x_n) is said to be *asymptotically regular* if $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. A rate of convergence of $(\|x_n - Tx_n\|)$ towards 0 will be called a rate of asymptotic regularity.

Proposition 3.6. *(i) For all $n \geq 1$, $\gamma_n^t \leq \frac{3M}{t}\|x_n - Tx_n\|$.*

(ii) If (x_n) is asymptotically regular with rate of asymptotic regularity φ, then for all $\varepsilon \in (0,2)$

$$\forall p \geq P(\varepsilon,t,M,\varphi)\, \forall m \geq 1 \ \left(C_{m,p}(\gamma_n^t) \leq \varepsilon\right), \tag{14}$$

where $P(\varepsilon,t,M,\varphi) = \left\lceil \frac{6M^2}{t\varepsilon}\,\varphi\left(\frac{t\varepsilon}{6M}\right)\right\rceil$.

(iii) Assume that (x_n) is asymptotically regular with rate of asymptotic regularity φ and that $\lim\limits_{n\to\infty}\|x_n - x_{n+1}\| = 0$ with rate of convergence $\tilde{\varphi}$. Then for all $\varepsilon \in (0,2)$

$$\forall n \geq \psi(\varepsilon,\omega,t,M,\varphi,\tilde{\varphi})\ (\gamma_n^t \leq \varepsilon), \tag{15}$$

where $\psi(\varepsilon,\omega,t,M,\varphi,\tilde{\varphi}) = \tilde{\varphi}(\omega(M,\varepsilon'/M)) + P(\varepsilon/2,t,M,\varphi)$, with P given by (ii) and $\varepsilon' = \frac{\varepsilon}{P(\varepsilon/2,t,M,\varphi)+1}$.

Proof. For simplicity, we shall denote z_t^u by z_t.
(i) Firstly, let us remark that $x_n - z_t = (1-t)(x_n - Tz_t) + t(x_n - u)$ so that, by (12), we get $\|x_n - z_t\|^2 \leq (1-t)^2\|x_n - Tz_t\|^2 + 2t\langle x_n - u, J(x_n - z_t)\rangle$. On the other hand,

$$\langle x_n - u, J(x_n - z_t)\rangle \quad = \quad \|x_n - z_t\|^2 - \langle u - z_t, J(x_n - z_t)\rangle\,.$$

It follows that

$$\begin{aligned}
\|x_n - z_t\|^2 &\leq (1-t)^2\|x_n - Tz_t\|^2 + 2t\|x_n - z_t\|^2 - 2t\langle u - z_t, J(x_n - z_t)\rangle \\
&\leq (1-t)^2\left(\|x_n - Tx_n\| + \|x_n - z_t\|\right)^2 + 2t\|x_n - z_t\|^2 - 2t\langle u - z_t, J(x_n - z_t)\rangle \\
&\leq 3M(1-t)^2\|x_n - Tx_n\| + (t^2+1)\|x_n - z_t\|^2 - 2t\langle u - z_t, J(x_n - z_t)\rangle\,,
\end{aligned}$$

so that $0 \leq \frac{3M(1-t)^2}{t}\|x_n - Tx_n\| + tM^2 - 2\langle u - z_t, J(x_n - z_t)\rangle$. We get finally that

$$\gamma_n^t \leq \frac{3M(1-t)^2}{t}\|x_n - Tx_n\| \leq \frac{3M}{t}\|x_n - Tx_n\|.$$

(ii) Let $\varepsilon \in (0,2)$ and denote $a_n := \frac{3M}{t}\|x_n - Tx_n\|$. By (i) we get that $C_{m,p}(\gamma_n^t) \leq C_{m,p}(a_n)$ for all $m \geq 1$, $p \geq 1$. Furthermore, $\lim\limits_{n\to\infty} a_n = 0$ with rate of convergence $\varphi\left(\frac{t\varepsilon}{3M}\right)$ and $L := \frac{3M^2}{t}$ is an upper bound for (a_n). Apply now [18, Lemma 8.5] for (a_n).
(iii) We have that $|\gamma_{n+1}^t - \gamma_n^t| = |\langle u - z_t, J(x_{n+1} - z_t)\rangle - \langle u - z_t, J(x_n - z_t)\rangle|$. Since $\|x_{n+1} - z_t\|$, $\|x_n - z_t\|$, $\|u - z_t\| \leq M$ for all n and $\lim\limits_{n\to\infty}\|(x_{n+1} - z_t) - (x_n - z_t)\| = \lim\limits_{n\to\infty}\|x_{n+1} - x_n\| = 0$ with rate of convergence $\tilde{\varphi}$, we obtain from the uniform continuity of J that $\lim\limits_{n\to\infty}|\gamma_{n+1}^t - \gamma_n^t| = 0$ with rate of convergence $\theta(\varepsilon,\omega,M,\tilde{\varphi}) = \tilde{\varphi}(\omega(M,\varepsilon/M))$. Apply now (ii) and [18, Lemma 8.4]. $\square$

Lemma 3.7.

$$\|x_{n+1} - z_t^u\|^2 \leq (1-\lambda_{n+1})\|x_n - z_t^u\|^2 + \lambda_{n+1}\gamma_{n+1}^t + 2tM^2, \quad u,x \in C, t \in (0,1), n \geq 0. \tag{16}$$

Proof. For simplicity, we shall denote z_t^u by z_t. One has $x_{n+1} - z_t = (1-\lambda_{n+1})(Tx_n - z_t) + \lambda_{n+1}(u - z_t)$ and $Tx_n - z_t = (Tx_n - Tz_t) + t(Tz_t - u)$. Applying twice (12) we get that

$$\begin{aligned}
\|x_{n+1} - z_t\|^2 &\leq (1-\lambda_{n+1})^2\|Tx_n - z_t\|^2 + 2\lambda_{n+1}\langle u - z_t, J(x_{n+1} - z_t)\rangle \\
&\leq (1-\lambda_{n+1})^2\big(\|x_n - z_t\|^2 + 2t\langle Tz_t - u, J(Tx_n - z_t)\rangle\big) + \lambda_{n+1}\gamma_{n+1}^t + \lambda_{n+1}tM^2 \\
&\leq (1-\lambda_{n+1})\big(\|x_n - z_t\|^2 + 2tM^2\big) + \lambda_{n+1}\gamma_{n+1}^t + \lambda_{n+1}tM^2 \\
&\leq (1-\lambda_{n+1})\|x_n - z_t\|^2 + \lambda_{n+1}\gamma_{n+1}^t + 2tM^2.
\end{aligned}$$

$\square$

3.2 Proof of Theorem 3.2

Let $\varepsilon \in (0,2)$ and $g : \mathbb{N} \to \mathbb{N}$ be fixed. By [18, Proposition 6.1], we have that (x_n) is asymptotically regular with rate of asymptotic regularity Φ and $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ with rate of convergence $\tilde{\Phi}$. To make the proof easier to read, we shall omit parameters $M, \Phi, \tilde{\Phi}, \theta, \alpha, \beta$ for all the functionals which appear in the following. For $t_k := 1/(k+1)$, let us denote $z^u_{t_k}$ simply by z^u_k and $\gamma^{t_k}_n$ by γ^k_n. By Proposition 3.6.(iii), we obtain that $\gamma^k_n \le \varepsilon$ for each $k \ge 0$ and $n \ge \chi_k(\varepsilon, \omega)$.

We apply (10) for ε_0 and f^* to get the existence of $K_1 \le K(\varepsilon_0, f^*)$ such that $\|z^u_k - z^u_l\| \le \varepsilon_0$ for all $k, l \in [K_1, K_1 + f^*(K_1)]$. Let $K_0 := K_1 + \lceil 1/\varepsilon_0 \rceil$. Then $\lceil 1/\varepsilon_0 \rceil \le K_0 \le K(\varepsilon_0, f^*) + \lceil 1/\varepsilon_0 \rceil$ and it is easy to see that $\|z^u_k - z^u_l\| \le \varepsilon_0$ for all $k, l \in [K_0, K_0 + f(K_0)]$.

Let $P := K_0 + f(K_0) = \max\left\{ \left\lceil \dfrac{2M^2}{\Delta^*_{K_0}(\varepsilon^2/4, g)} \right\rceil, K_0 \right\}$. Then for all $n \ge 1$,

$$\begin{aligned}
\gamma^P_n - \gamma^{K_0}_n &= \big(2\left\langle u - z^u_P, J(x_n - z^u_P)\right\rangle - 2\left\langle u - z^u_P, J(x_n - z^u_{K_0})\right\rangle\big) + \\
&\quad +2\left\langle z^u_{K_0} - z^u_P, J(x_n - z^u_{K_0})\right\rangle + \left(\frac{1}{K_0+1} - \frac{1}{P+1}\right)M^2 \\
&\le 2\delta\|u - z^u_P\| + 2\left\langle z^u_{K_0} - z^u_P, J(x_n - z^u_{K_0})\right\rangle + \left(\frac{1}{K_0+1} - \frac{1}{P+1}\right)M^2 \\
&\quad \text{as } \|(x_n - z^u_P) - (x_n - z^u_{K_0})\| = \|z^u_{K_0} - z^u_P\| \le \varepsilon_0 \le \omega(M,\delta), \text{ so we can apply (1)} \\
&\le 2\delta M + 2\|x_n - z^u_{K_0}\| \cdot \|z^u_{K_0} - z^u_P\| + \frac{1}{K_0+1}M^2 \\
&\le 2\delta M + 2M\varepsilon_0 + \varepsilon_0 M^2 \le \delta M(4+M) = \frac{\varepsilon^2}{24} \quad \text{as } K_0 \ge \left\lceil \frac{1}{\varepsilon_0} \right\rceil.
\end{aligned}$$

It follows that $\gamma^P_n \le \gamma^{K_0}_n + \varepsilon^2/24 \le \varepsilon^2/12$ for all $n \ge \chi_{K_0}\left(\varepsilon^2/24, \omega\right) = \chi^*_{K_0}\left(\varepsilon^2/12, \omega\right)$. Applying Lemma 3.7 with $t := 1/(P+1)$, we get that for all $n \ge 1$,

$$\begin{aligned}
\|x_{n+1} - z^u_P\|^2 &\le (1-\lambda_{n+1})\|x_n - z^u_P\|^2 + \lambda_{n+1}\gamma^P_{n+1} + \frac{2M^2}{P+1} \\
&\le (1-\lambda_{n+1})\|x_n - z^u_P\|^2 + \lambda_{n+1}\gamma^P_{n+1} + \Delta^*_{K_0}\left(\frac{\varepsilon^2}{4}, g\right)
\end{aligned}$$

since $P \ge \left\lceil \dfrac{2M^2}{\Delta^*_{K_0}(\varepsilon^2/4, g)} \right\rceil$, hence $\dfrac{1}{P+1} \le \dfrac{\Delta^*_{K_0}(\varepsilon^2/4, g)}{2M^2}$. It follows that we can apply [18, Lemma 5.2] with $\varepsilon := \dfrac{\varepsilon^2}{4}$ to conclude that $\|x_n - z^u_P\|^2 \le \varepsilon^2/4$ for all $n \in [N, N + g(N)]$, where $N := \Theta_{K_0}(\varepsilon^2/4) \le \theta^+(\Gamma - 1 + \lceil \ln(12M/\varepsilon^2) \rceil) + 1 = \Sigma(\varepsilon, g, \omega)$. We conclude that

$$\|x_n - x_m\| \le \|x_n - z^u_P\| + \|x_m - z^u_P\| \le \varepsilon \quad \text{for all } n, m \in [N, N + g(N)].$$

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