Bounds on Kuhfittig's iteration schema in uniformly convex hyperbolic spaces

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Abstract: The purpose of this paper is to extract an explicit effective and uniform bound on the rate of asymptotic regularity of an iteration schema involving a finite family of nonexpansive mappings. The results presented in this paper contribute to the general project of proof mining as developed by the second author as well as generalize and improve various classical and corresponding quantitative results in the current literature. More precisely, we give a rate of asymptotic regularity of an iteration schema due to Kuhfittig for finitely many nonexpansive mappings in the context of uniformly convex hyperbolic spaces. The bound only depends on an upper bound on the distance between the starting point and some common fixed point, a lower bound $1/N \leq \lambda_n(1-\lambda_n)$, the error $\epsilon > 0$ and a modulus η of uniform convexity.

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1. Introduction and Preliminaries

This paper is a continuation of the case study in the general program of 'proof mining' – introduced by Kohlenbach in the 90's (see [17]) – which provides proof-theoretic tools to extract explicit and effective uniform bounds from ineffective proofs in functional analysis and in particular metric fixed point and ergodic theory. For various classes of proofs, the extraction of such uniform bounds is guaranteed by so-called 'logical metatheorems'. These metatheorems were developed in [16] for abstract bounded metric structures (including hyperbolic and $CAT(0)$ spaces) and bounded convex subsets of a normed space and generalized to unbounded structures in [7]. In [26] this was adapted to further structures such as $\mathbb{R}$-trees, Gromov's δ-hyperbolic spaces and uniformly convex W-hyperbolic spaces. In the context of these spaces, the metatheorems cover functions such as nonexpansive, Lipschitz, weakly quasi-nonexpansive or uniformly continuous maps among others. As an application of these metatheorems, strong uniform bounds have been extracted from numerous previously established convergence results in metric fixed point and ergodic theory, see, for example, [13, 14, 18, 19, 20, 21, 22, 23, 24, 27, 28, 31] and the references cited therein.

Fixed point theory for nonexpansive mappings is an active area of research in nonlinear functional analysis and it requires tools far beyond from metric fixed

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point theory. The problem of finding a common fixed point of a finite family of nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics. For example, finding a common fixed point of a finite family of nonexpansive mappings may be used to solve a convex minimization problem or a system of simultaneous equations. Hence, the analysis of iterative schemas for finite families of nonexpansive mappings is a problem of interest in such contexts.

In fixed point theory, various iterative schemas $\{x_n\}$ for computing fixed points of nonlinear mappings T have been studied. One of the most important notions in metric fixed point theory is the asymptotic regularity [5] of the iteration under consideration, i.e.

$$(*) \qquad d(x_n, T(x_n)) \overset{n\to\infty}{\to} 0.$$

This usually is an important first step in establishing (under suitable additional assumptions such as compactness conditions) the convergence of $\{x_n\}$ towards a fixed point. The concept of asymptotic regularity naturally leads to the issue of finding a rate of convergence in $(*)$ or even for the convergence of $\{x_n\}$ itself (in cases where strong convergence holds). Whereas the latter usually can be shown to be inherently noncomputable and to be effectively solvable only in the computationally weaker (though ineffectively equivalent) formulation of metastability (in the sense of Tao [33])

$$\forall \epsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n\, \forall i, j \in [n, n + g(n)]\, (d(x_i, x_j) < \epsilon),$$

rates of asymptotic regularity typically can be obtained. A series of papers with case studies in the general program of 'proof mining' has been carried out for the extraction of uniform bounds by examining the Krasnoselskii-Mann iteration of nonexpansive mappings. In [13], the second author analyzed a result due to Borwein-Reich-Shafrir [2] and obtained effective uniform bounds on the asymptotic behavior of the Krasnoselskii-Mann iteration which was extended in [20] to W-hyperbolic spaces and directionally nonexpansive mappings. Later on, in 2003, simpler bounds on asymptotic regularity of the Krasnoselskii-Mann iteration were computed in uniformly convex normed spaces [14] for more general iterations. It is worth mentioning that the bounds on the rate of asymptotic regularity established in [13] and [14] (see also, [1] and [12]) are related to single nonexpansive mappings. It is natural to analyze also proofs of convergence results for families of nonexpansive mappings for the extraction of bounds on asymptotic regularity. So far, no such bounds have been extracted in this context.

In 1981, Kuhfittig [25] established a fundamental theorem regarding the approximation of a common fixed point of a finite family of nonexpansive mappings in a strictly convex Banach space. More precisely, he proposed the following iteration schema:

Let C be a nonempty convex compact subset of a Banach space E and $\{T_i : 1 \leq i \leq k\}$ be a finite family of nonexpansive self-mappings with a nonempty set of common fixed points $F = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Let $U_0 = I$ be the identity mapping and $0 < \lambda < 1$, then using the mappings

$$\begin{aligned} U_1 &= (1-\lambda)I + \lambda T_1 U_0 \\ U_2 &= (1-\lambda)I + \lambda T_2 U_1 \\ &\vdots \\ U_k &= (1-\lambda)I + \lambda T_k U_{k-1}, \end{aligned}$$

one defines

$$x_0 \in C,\ x_{n+1} := (1-\lambda)x_n + \lambda T_k U_{k-1} x_n, \quad n \geq 0. \tag{1.1}$$

If $k = 1$, then the iteration schema (1.1) reduces to the usual Krasnoselskii-Mann iteration of T_1 (for constant λ)

$$x_{n+1} = (1-\lambda)x_n + \lambda T_1 x_n,$$

which contains the Krasnoselskii iteration as a special case for $\lambda = \frac{1}{2}$. Under the assumption of the compactness of C, [25] established the strong convergence of $\{x_n\}$ in strictly convex Banach spaces as well as the weak convergence in uniformly convex Banach spaces that satisfy the Opial condition. In 2000, Rhoades [30] managed to eliminate the assumption of the Opial condition from the latter result.

Implicit in Kuhfittig's paper [25] is the following asymptotic regularity result:

Theorem 1.1. *Let E be a strictly convex Banach space, C a nonempty compact convex subset of E and $T_1, T_2, \ldots, T_k$ a family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let $x_0 \in C$, then for the sequence $\{x_n\}$ generated by (1.1), we have*

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \text{ for all } 1 \leq i \leq k.$$

Kuhfittig argues using the functions $S_i := T_i U_{i-1}$ for $1 \leq i \leq k$ which again are nonexpansive. Since $\{x_n\}$ is a Krasnoselskii-Mann iteration of S_k it follows by classical theorems due to Edelstein and Ishikawa that $\{x_n\}$ converges strongly to a fixed point p of S_k. He then shows

$$\forall q \in C \left(S_k(q) = q \implies \bigwedge_{i=1}^{k} (S_i q = q) \implies \bigwedge_{i=1}^{k} (T_i q = q) \right).$$

Taking than $q := p$ gives the strong convergence of $\{x_n\}$ to a common fixed point of $T_1, \ldots, T_k$ and so a fortiorily Theorem 1.1. That latter Theorem, however, already follows without any compactness assumption on C provided that E is uniformly convex.

Since the statement that $S_k(q) = q \Rightarrow T_i(q) = q$ can be written in the form

$$\forall q \in C\, \forall \epsilon > 0\, \exists \delta > 0\ \left(\|S_k q - q\| \leq \delta \implies \|T_i q - q\| < \epsilon \right),$$

general logical metatheorems for uniformly convex Banach spaces E (which do not hold for strictly convex Banach spaces though) from [16, 7, 17] guarantee the extractability of effective bounds $\Psi_i(D, \epsilon, N, \eta) > 0$ such that

$$\forall \epsilon > 0\ \left(\|S_k q - q\| \leq \Psi_i(D, \epsilon, N, \eta) \implies \|T_i q - q\| < \epsilon \right).$$

Here D is any upper bound on the distance $\|q - p\|$ for some $p \in F$ and $N \in \mathbb{N}$ is such that $\frac{1}{N} \leq \lambda(1-\lambda)$. Moreover, η is some modulus of uniform convexity for E.

Here one only has to observe that the definitions of the mappings U_i and the iteration schema (1.1) are purely universal so that one can apply e.g. Corollary 6.8.1 from [7] (which adopts without any problems to k maps $T_1, \ldots, T_k$ instead of one map T). Note that Ψ_i does not depend on q or p except for D.

Together with the fact $\{x_n\}$ is asymptotically regular w.r.t. S_k, (and that $\|x_n - p\| \leq \|x_0 - p\|$) this establishes already the asymptotic regularity w.r.t. each of $T_1, \ldots, T_k$.

Moreover, given any rate $\theta(D,\epsilon,N,\eta)$ of asymptotic regularity for S_k and for $D \geq \|x_0 - p\|$ (e.g. one can take the one from [14] (see also [1] and [12]), i.e.

$$\forall n \geq \theta(D,\epsilon,N,\eta)\ (\|x_n - S_k x_n\| \leq \epsilon),$$

one obtains that for $\Phi_i(D,\epsilon,N,\eta) := \theta(D,\Psi_i(D,\epsilon,N,\eta),N,\eta)$:

$$\forall n \geq \Phi_i(D,\epsilon,N,\eta)\ (\|x_n - T_i x_n\| \leq \epsilon).$$

Theorem 1.2. *Let E be a uniformly convex normed space, C be a nonempty convex subset of E and $T_1, T_2, \ldots, T_k$ a family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let $x_0 \in C$, then for the sequence $\{x_n\}$ generated by (1.1), we have*

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \text{ for all } 1 \leq i \leq k.$$

Moreover, for $N \in \mathbb{N}$ with $1/N \leq \lambda(1-\lambda)$ and $D \geq \|x_0 - p\|$ for some $p \in F$ one has an effective and highly uniform rate of convergence Φ_i :

$$\forall \epsilon > 0\, \forall n \geq \Phi_i(D,\epsilon,N,\eta)\ (\|x_n - T_i x_n\| \leq \epsilon) \text{ for all } 1 \leq i \leq k.$$

The aforementioned logical metatheorem even guarantees that for this quantitative asymptotic regularity result it is sufficient to assume that in the D-neighborhood of x_0 there are arbitrarily good common δ-approximate fixed points p_δ of $T_1, \ldots, T_k$, i.e.

$$\|p_\delta - T_i(p_\delta)\| < \delta, \text{ for all } 1 \leq i \leq k.$$

Since the logical metatheorem also applies to uniformly convex W-hyperbolic spaces (see [7, 17] for general W-hyperbolic spaces and [27] for the uniformly convex case) and the proof of Theorem 1.2 does so as well, everything said so far also extends to this more general setting which we, therefore, adopt in this paper.

2. Uniformly Convex Hyperbolic Spaces

This section deals with the introduction and geometry of hyperbolic spaces. One can find different notions of hyperbolic space in the literature, see for example [8, 9, 16, 29]. We work in the setting of hyperbolic spaces as introduced by the second author [16], which are more restrictive than the space of hyperbolic type in [8] but more general than the hyperbolic spaces introduced in [29]. For a distinction, we denote it as $W-$hyperbolic space.

Recall that a W-hyperbolic space is a metric space (X,d) together with a map $W : X^2 \times [0,1] \to X$ satisfying:

$$\begin{aligned}
&(\mathrm{W1}) : d(u, W(x,y,\alpha)) \leq (1-\alpha)d(u,x) + \alpha d(u,y);\\
&(\mathrm{W2}) : d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha - \beta|\, d(x,y);\\
&(\mathrm{W3}) : W(x,y,\alpha) = W(y,x,(1-\alpha));\\
&(\mathrm{W4}) : d(W(x,z,\alpha), W(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w);
\end{aligned}$$

for all $x,y,z,w \in X$ and $\alpha, \beta \in [0,1]$. If the triplet (X,d,W) satisfies only (W1), then it coincides with the convex metric space introduced by Takahashi [32]. A subset C of a W-hyperbolic space X is convex if $W(x,y,\alpha) \in C$ for all $x,y \in C$ and $\alpha \in [0,1]$. Set $W(x,y,\alpha) := (1-\alpha)x \oplus \alpha y$, then one can define the mappings

$\{U_i\}_{i=1}^{k}$ as:

$$\begin{aligned} U_1 &= W(I, T_1U_0, \lambda) \\ U_2 &= W(I, T_2U_1, \lambda) \\ &\vdots \\ U_k &= W(I, T_kU_{k-1}, \lambda), \end{aligned}$$

and the corresponding iteration schema (1.1) as

$$x_0 \in C,\ x_{n+1} = W\left(x_n, T_kU_{k-1}x_n, \lambda\right),\ n \geq 0. \tag{2.1}$$

The class of W-hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert ball equipped with the hyperbolic metric [9] as well as Hadamard manifolds. $CAT(0)$ spaces in the sense of Gromov(see [4] for a detailed treatment) – as an important subclass of W-hyperbolic spaces – play the analogous role in the context of W-hyperbolic spaces as the Hilbert spaces do among all Banach spaces. In fact, a $CAT(0)$ space is a W-hyperbolic space which satisfies the (CN) inequality of Bruhat and Tits [6]. That is, for all $x, y_1, y_2 \in (X, d, W)$, if $d(y_0, y_1) = \frac{1}{2}d(y_1, y_2) = d(y_0, y_2)$, then the following inequality holds:

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{CN}$$

In this case, W is determined as the unique geodesic path in X connecting two points $x, y \in X$.

A W-hyperbolic space is uniformly convex [27] if for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that for all $u, x, y \in X$, we have

$$d\left(W(x, y, \frac{1}{2}), u\right) \leq (1-\delta)r$$

provided $d(x, u) \leq r, d(y, u) \leq r$ and $d(x, y) \geq \epsilon r$.

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is known as a modulus of uniform convexity of X . We call η monotone if it decreases with r (for a fixed ϵ), *i.e.*, $\forall \epsilon > 0, \forall r_2 \geq r_1 > 0\ (\eta\left(r_2, \epsilon\right) \leq \eta\left(r_1, \epsilon\right))$.

It turns out that $CAT(0)$ spaces are uniformly convex W-hyperbolic spaces [27] with modulus of uniform convexity $\eta(r, \epsilon) = \frac{\epsilon^2}{8}$. Thus, uniformly convex W-hyperbolic spaces are a natural generalization of both uniformly convex normed spaces and $CAT(0)$ spaces.

Lemma 2.1. *Let (X, d, W) be a uniformly convex W-hyperbolic space with monotone modulus of uniform convexity η. For $r > 0,\ \epsilon \in (0, 2],\ a, x, y \in X$ and $\lambda \in [0, 1]$, the inequalities*

$$d(x, a) \leq r,\ d(y, a) \leq r \text{ and } d(x, y) \geq \epsilon r$$

imply

$$d\left(W(x, y, \lambda), a\right) \leq \left(1 - 2\lambda\left(1-\lambda\right)\eta\left(r, \epsilon\right)\right) r.$$

3. Main Results

In this section, we establish a quantitative version of Theorem 1.2. We start with the following simple lemma.

Lemma 3.1. *Let C be a nonempty convex subset of a W-hyperbolic space (X, d, W)*

and $\{T_i\}_{i=1}^k$ *be a finite family of nonexpansive self-mappings. Then* $\{S_i\}_{i=1}^k$ *are nonexpansive and*

$$d\left(x_{n+1}, S_k x_{n+1}\right) \leq d\left(x_n, S_k x_n\right),$$

where $S_i := T_i U_{i-1}$ for $1 \leq i \leq k$.

Proof. We first show that $S_i := T_i U_{i-1}$ is nonexpansive for $1 \leq i \leq k$. Since $\{T_i\}_{i=1}^k$ are nonexpansive, therefore for any $x, y \in C$, we have

$$\begin{aligned}
d\left(S_i x, S_i y\right) &= d\left(T_i U_{i-1} x, T_i U_{i-1} y\right) \\
&\leq d\left(U_{i-1} x, U_{i-1} y\right) \\
&= d\left(W\left(x, T_{i-1} U_{i-2} x, \lambda\right), W\left(y, T_{i-1} U_{i-2} y, \lambda\right)\right) \\
&\leq (1-\lambda) d(x, y) + \lambda d\left(T_{i-1} U_{i-2} x, T_{i-1} U_{i-2} y\right) \ \text{(by W4)} \\
&\ \ \vdots \\
&\leq (1-\lambda^{k-1}) d(x, y) + \lambda^{k-1} d\left(T_1 x, T_1 y\right) \\
&\leq (1-\lambda^{k-1}) d(x, y) + \lambda^{k-1} d(x, y) \\
&= d(x, y).
\end{aligned}$$

Hence $S_i := T_i U_{i-1}$ and consequently U_i are nonexpansive for $1 \leq i \leq k$. Moreover, the last claim in the lemma (i.e. $d\left(x_{n+1}, S_k x_{n+1}\right) \leq d\left(x_n, S_k x_n\right)$) follows from the nonexpansivity of S_k and (2.1), see, for example [20, Proposition 3.4]. This completes the proof. ■

Theorem 3.2. *Let C be a nonempty convex subset of a uniformly convex W-hyperbolic space with monotone modulus of uniform convexity η and let $\{T_i\}_{i=1}^k$ be a finite family of nonexpansive self-mappings with $F = \cap_{i=1}^k F(T_i) \neq \emptyset$. Let $p \in F$ such that $d(x_0, p) \leq D > 0$ for $x_0 := x \in C$. Then for the sequence $\{x_n\}$ generated by (2.1), we have*

$$\forall \epsilon \in (0,2]\, \forall n \ \geq \Phi_i\left(D, \epsilon, \lambda, \eta\right) \ \left(d\left(x_n, T_i x_n\right) \leq \epsilon\right) \ \textit{for } 1 \leq i \leq k$$

where

$$\Phi_i := \theta\left(\widehat{\eta}^{(k-i+\min(1,k-1))}\left(\frac{\epsilon}{2}\right)\right);$$

with

$$\theta(\epsilon) := \left\lceil \frac{D}{\widehat{\eta}(\epsilon)} \right\rceil;$$

$$\widehat{\eta}(\epsilon) := \lambda(1-\lambda)\eta\left(D, \frac{\epsilon}{D+1}\right)\epsilon.$$

In fact, instead of $\lambda(1-\lambda)$ one can use $\frac{1}{N} \leq \lambda(1-\lambda)$ for $N > 0$, to have a bound $\widetilde{\Phi}_i(D, \epsilon, N, \eta)$ which no longer depends on λ.

Proof. The first part of this proof is essentially a special case of a result due to Leustean, see [27](and for the normed case due to Kohlenbach [14]) which we include for completeness. Let $p \in F$ be such that $d(x_0, p) \leq D$ where $D > 0$. Utilizing the nonexpansiveness of $\{S_i\}_{i=1}^k$, we have (using that $p \in F(S_k)$)

$$\begin{aligned}
d\left(x_{n+1}, p\right) &= d\left(W\left(x_n, S_k x_n, \lambda\right), p\right) \\
&\leq (1-\lambda) d\left(x_n, p\right) + \lambda d\left(S_k x_n, S_k p\right) \\
&\leq d\left(x_n, p\right). \qquad (3.1)
\end{aligned}$$

Hence, (3.1) implies that the sequence $\{d(x_n,p)\}_{n=1}^{\infty}$ is nonincreasing and bounded. Let $\theta(\epsilon) := \left\lceil \frac{D}{\widehat{\eta}(\epsilon)} \right\rceil - 1$. For $n \leq \theta(\epsilon)$, we can assume towards contradiction that

$$d(x_n, S_k x_n) > \epsilon \geq d(x_n,p)\left(\frac{\epsilon}{\alpha}\right), \tag{3.2}$$

where $\alpha \geq d(x_n,p)$.
Since $d(S_k x_n, p) = d(S_k x_n, S_k p) \leq d(x_n,p)$, we have that

$$\begin{aligned} \epsilon &< d(x_n, S_k x_n) \leq d(x_n,p) + d(p, S_k x_n) \\ &\leq 2d(x_n,p) \end{aligned}$$

so that

$$\frac{\epsilon}{d(x_0,p)} \leq \frac{\epsilon}{d(x_n,p)} \in (0,2].$$

Therefore, it follows from (3.2) and Lemma 2.1 that

$$\begin{aligned} d(x_{n+1},p) &= d(W(x_n, S_k x_n, \lambda), p) \\ &\leq \left(1 - 2\lambda(1-\lambda)\eta\left(d(x_n,p), \frac{\epsilon}{\alpha}\right)\right) d(x_n,p) \\ &= d(x_n,p) - 2\lambda(1-\lambda)\eta\left(d(x_n,p), \frac{\epsilon}{\alpha}\right) d(x_n,p). \end{aligned} \tag{3.3}$$

Since 'η' is monotone and $d(x_n,p) \leq d(x_0,p) \leq D$, therefore the estimate (3.3) implies that($\alpha := D+1$)

$$\begin{aligned} d(x_{n+1},p) &\leq d(x_n,p) - \lambda(1-\lambda)\eta\left(D, \frac{\epsilon}{D+1}\right)\epsilon \\ &= d(x_n,p) - \widehat{\eta}(\epsilon), \end{aligned} \tag{3.4}$$

where $\widehat{\eta}(\epsilon) := \lambda(1-\lambda)\eta\left(D, \frac{\epsilon}{D+1}\right)\epsilon$.
If the estimate (3.4) is true for $n \leq \theta(\epsilon)$, then we have

$$\begin{aligned} d\left(x_{\theta(\epsilon)+1}, p\right) &\leq d(x_0,p) - (\theta(\epsilon)+1)\widehat{\eta}(\epsilon) \\ &< d(x_0,p) - D \leq 0, \end{aligned}$$

which is a contradiction. Hence there exists an $n \leq \theta(\epsilon)$ such that $d(x_n, S_k x_n) \leq \epsilon$. Moreover, from Lemma 3.1, we have the desired asymptotic regularity for the $k^{th}-$mapping 'S_k' with a witness $\theta(\epsilon)$, *i.e.*

$$\forall n \geq \theta(\epsilon) \ \ (d(x_n, S_k x_n) \leq \epsilon). \tag{3.5}$$

It remains to show the asymptotic regularity for the mappings T_i for $1 \leq i \leq k$ with a possible sequence of witnesses starting from Φ_i.
Let $q \in C_D := \{q \in C : d(q,p) \leq D\}$ and assume that $d(q, S_k q) \leq \widehat{\eta}(\epsilon)$. Next, we show that $d(q, S_{k-1} q) \leq \epsilon$, where $S_{k-1} = T_{k-1} U_{k-2}$. The claim trivially holds for $d(q,p) \leq \frac{\epsilon}{2}$. Since

$$\begin{aligned} d(q, S_{k-1} q) &\leq d(q,p) + d(S_{k-1} p, S_{k-1} q) \\ &\leq 2d(q,p) \leq \epsilon. \end{aligned}$$

So we may assume that $d(q,p) > \frac{\epsilon}{2}$ and

$$d(q, S_{k-1} q) > \epsilon \geq d(q,p)\left(\frac{\epsilon}{\beta}\right), \tag{3.6}$$

where $\beta \geq d(q,p)$. Note that $\frac{\epsilon}{\beta} \leq 2$. Define

$$z := U_{k-1}q = W(q, T_{k-1}U_{k-2}q, \lambda). \tag{3.7}$$

Since $p \in F = \cap_{i=1}^{k} F(U_i) = \cap_{i=1}^{k} F(S_i)$, therefore we have

$$d(S_{k-1}q, p) \leq d(q,p). \tag{3.8}$$

It follows from (3.6), (3.8) and Lemma 2.1, that

$$\begin{aligned} d(W(q, S_{k-1}q, \lambda), p) &\leq \left(1 - 2\lambda(1-\lambda)\eta\left(d(q,p), \frac{\epsilon}{\beta}\right)\right) d(q,p) \\ &= d(q,p) - 2\lambda(1-\lambda)\eta\left(d(q,p), \frac{\epsilon}{\beta}\right) d(q,p). \end{aligned} \tag{3.9}$$

The nonexpansivity of T_k implies that

$$d(T_k z, p) \leq d(z,p). \tag{3.10}$$

Since $T_k z = S_k q$, we have $d(T_k z, q) \leq \widehat{\eta}(\epsilon)$. Therefore utilizing (3.10), we have

$$\begin{aligned} d(q,p) &\leq d(q, T_k z) + d(T_k z, p) \\ &\leq d(z,p) + \widehat{\eta}(\epsilon) \\ &= d(W(q, S_{k-1}q, \lambda), p) + \widehat{\eta}(\epsilon), \end{aligned} \tag{3.11}$$

which contradicts (3.9) as

$$\begin{aligned} \widehat{\eta}(\epsilon) \quad &: \;= \lambda(1-\lambda)\eta\left(D, \frac{\epsilon}{D+1}\right)\epsilon \\ &< 2\lambda(1-\lambda)\eta\left(d(q,p), \frac{\epsilon}{D+1}\right) d(q,p), \ (\beta := D+1) \end{aligned}$$

Hence, we conclude that

$$\forall q \in C_D \, \forall \epsilon \in (0,2] \ \left(d(q, S_k q) \leq \widehat{\eta}(\epsilon) \Longrightarrow d(q, S_{k-1}q) \leq \epsilon\right).$$

The above argument holds as well for all S_i, where $2 \leq i \leq k$:

$$\forall q \in C_D \, \forall \epsilon \in (0,2] \ \left(d(q, S_i q) \leq \widehat{\eta}(\epsilon) \Longrightarrow d(q, S_{i-1}q) \leq \epsilon\right).$$

In total, applying this successively to $k, k-1, k-2, \ldots, i \geq 2$ instead of i and to $\widehat{\eta}^{(k-i)}\left(\frac{\epsilon}{2}\right) \leq \frac{\epsilon}{2}$ as ϵ, we get that

$$\forall q \in C_D \, \forall \epsilon \in (0,2] \ \left(d(q, S_k q) \leq \widehat{\eta}^{(k-i+1)}\left(\frac{\epsilon}{2}\right) \Longrightarrow d(q, S_i q) \leq \widehat{\eta}\left(\frac{\epsilon}{2}\right), d(q, S_{i-1}q) \leq \frac{\epsilon}{2}\right)$$

and so for $2 \leq i \leq k$

$$\forall q \in C_D \, \forall \epsilon \in (0,2] \ \left(d(q, S_k q) \leq \widehat{\eta}^{(k-i+1)}\left(\frac{\epsilon}{2}\right) \Longrightarrow d(q, S_i q), d(q, S_{i-1}q) \leq \frac{\epsilon}{2}\right).$$

Since each $T_1, \ldots, T_k$ is nonexpansive, this implies that for $2 \leq i \leq k$ and $q \in C_D$ with $d(q, S_k q) \leq \widehat{\eta}^{(k-i+1)}$:

$$\begin{aligned} d(q, T_i q) &\leq d(q, S_i q) + d(S_i q, T_i q) \\ &= d(q, S_i q) + d(T_i U_{i-1} q, T_i q) \\ &\leq d(q, S_i q) + d(U_{i-1} q, q) \\ &\leq d(q, S_i q) + \lambda d(S_{i-1} q, q) \leq \frac{\epsilon}{2} + \lambda \frac{\epsilon}{2} < \epsilon, \text{ for } i \geq 2, \end{aligned}$$

and (for $i = 1$)

$$d(q, T_1 q) \leq d(q, S_1 q) + d(U_0 q, q) \leq \frac{\epsilon}{2} + 0 < \epsilon.$$

So, in total we have shown that

$$\forall q \in C_D\, \forall \epsilon \in (0,2]\ \left(d\left(q, S_k q\right) \leq \widehat{\eta}^{(k-i+1)}\left(\frac{\epsilon}{2}\right) \Longrightarrow d\left(q, T_i q\right) \leq \epsilon\right). \tag{3.12}$$

Now, let $1 \leq i \leq k$ and $n \geq \Phi_i\left(D, \frac{\epsilon}{2}, \lambda, \eta\right)$. Then by (3.5), we have

$$d\left(x_n, S_k x_n\right) \leq \widehat{\eta}^{(k-i+1)}\left(\frac{\epsilon}{2}\right).$$

Moreover, since the only assumption on $q \in C_D$ used in the proof above was just this property $d(q,p) \leq D$ which also holds for x_n by $d\left(x_n, p\right) \leq d\left(x_0, p\right) \leq D$, we can use '$x_n$' as '$q$' in the proof above and obtain

$$d\left(x_n, T_i x_n\right) \leq \epsilon.$$

This completes the proof. ■

Corollary to the proof. Let $p \in F$. For $D > 0$, define $C_D := \{q \in C : d(q,p) \leq D\}$. Then the following result holds:

$$\forall q \in C_D\, \forall \epsilon \in (0,2]\ \left(d\left(q, S_k q\right) \leq \widehat{\eta}^{(k)}\left(\frac{\epsilon}{2}\right) \Longrightarrow \bigwedge_{i=1}^{k} d\left(q, T_i q\right) \leq \epsilon\right),$$

where $\widehat{\eta}$ is defined as above.

Remark 3.3. Note that if $\eta\left(r,\epsilon\right)$ can be written as $\eta\left(r,\epsilon\right) := \epsilon\tilde{\eta}\left(r,\epsilon\right)$ where $\tilde{\eta}$ increases with ϵ (for a fixed r), *i.e.*, $\forall r > 0, \forall \epsilon_2 \geq \epsilon_1 > 0\ \left(\tilde{\eta}\left(r,\epsilon_2\right) \geq \tilde{\eta}\left(r,\epsilon_1\right)\right)$, then we can replace η by $\tilde{\eta}$ in the bound $\Phi_i\left(D, \epsilon, \lambda, \tilde{\eta}\right)$ while computing $\widehat{\eta}\left(\epsilon\right)$ from the estimates (3.3) and (3.9), respectively. First, we elaborate this fact for the estimate (3.3) with $\alpha := d\left(x_n, p\right)$ as follows:

$$\begin{aligned}
\lambda(1-\lambda)\tilde{\eta}\left(D, \frac{\epsilon}{D+1}\right)\epsilon &< 2\lambda\left(1-\lambda\right)\tilde{\eta}\left(D, \frac{\epsilon}{d\left(x_n,p\right)}\right)\epsilon \\
&= 2\lambda\left(1-\lambda\right)\frac{\epsilon}{d\left(x_n,p\right)}\tilde{\eta}\left(D, \frac{\epsilon}{d\left(x_n,p\right)}\right)d\left(x_n,p\right) \\
&= 2\lambda\left(1-\lambda\right)\eta\left(D, \frac{\epsilon}{d\left(x_n,p\right)}\right)d\left(x_n,p\right) \\
&\leq 2\lambda\left(1-\lambda\right)\eta\left(d(x_n,p), \frac{\epsilon}{d\left(x_n,p\right)}\right)d\left(x_n,p\right).
\end{aligned}$$

Similarly, for the estimate (3.9) with $\beta := d\left(q,p\right)$, we have

$$\begin{aligned}
\lambda(1-\lambda)\tilde{\eta}\left(D, \frac{\epsilon}{D+1}\right)\epsilon &< 2\lambda(1-\lambda)\tilde{\eta}\left(D, \frac{\epsilon}{d\left(q,p\right)}\right)\epsilon \\
&= 2\lambda(1-\lambda)\frac{\epsilon}{d\left(q,p\right)}\tilde{\eta}\left(D, \frac{\epsilon}{d\left(q,p\right)}\right)d\left(q,p\right) \\
&= 2\lambda(1-\lambda)\eta\left(D, \frac{\epsilon}{d\left(q,p\right)}\right)d\left(q,p\right) \\
&\leq 2\lambda(1-\lambda)\eta\left(d(q,p), \frac{\epsilon}{d\left(q,p\right)}\right)d\left(q,p\right).
\end{aligned}$$

Examples. Note that Banach spaces L_p and l_p are uniformly convex [11] with modulus of uniform convexity

$$\eta\left(\epsilon\right) := \frac{1}{p}\left(\frac{\epsilon}{2}\right)^p \text{ for } p \geq 2.$$

Take $\eta(\epsilon) := \epsilon\tilde{\eta}(\epsilon)$, we get

$$\tilde{\eta}(\epsilon) := \frac{1}{p}\left(\frac{\epsilon^{p-1}}{2^p}\right),$$

which – uniform in ϵ and p (independent of r and λ) – reconcile with the Remark 2.4; as a consequence, we get rate of asymptotic regularity $\Phi(D,\epsilon,p)$ for L_p and l_p spaces. Moreover, for $X = \mathbb{R}$ with the Euclidean norm as well as for $CAT(0)$ spaces, we can choose $\tilde{\eta}(\epsilon) := \frac{1}{2}$ against $\eta(\epsilon) := \epsilon\left(\frac{1}{2}\right)$ and $\tilde{\eta}(\epsilon) := \frac{\epsilon}{8}$ against $\eta(\epsilon) := \epsilon\left(\frac{\epsilon}{8}\right)$, respectively, which gives rate of convergence Φ. For L_2 and $\mathbb{R}$, these rates are optimal even for constant $\lambda = \frac{1}{2}$.

If we define iteration schema (2.1) with a general λ_n as:

$$x_{n+1} = W\left(x_n, T_k U_{k-1} x_n, \lambda_n\right),\ n \geq 0, \tag{3.13}$$

provided that $\{U_i\}_{i=1}^{k-1}$ are computed with a constant $\lambda \in (0,1)$ (see [3] where this situation is considered). Then we have the following result.

Theorem 3.4. *Let C be a nonempty convex subset of a uniformly convex W-hyperbolic space with monotone modulus of uniform convexity η and let $\{T_i\}_{i=1}^{k}$ be a finite family of nonexpansive self-mappings with $F = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Let $\lambda \in (0,1)$ and $\{\lambda_n\}$* in $[0,1]$ *with $g : \mathbb{N} \to \mathbb{N}$ be such that for all $n \in \mathbb{N}$,*

$$\sum_{n=1}^{g(n)} \lambda_n(1-\lambda_n) \geq n.$$

Let $p \in F$ such that $d(x_0,p) \leq D > 0$ for $x_0 := x \in C$. Then for the sequence $\{x_n\}$ generated by (3.13), we have

$$\forall \epsilon \in (0,2]\, \forall n\ \geq \Phi_i(D,\epsilon,g,\lambda,\eta)\ \ (d(x_n,T_i x_n) \leq \epsilon)\ \ \textit{for } 1 \leq i \leq k,$$

where

$$\Phi_i := \theta\left(\widehat{\eta}^{(k-i+\min(1,k-1))}\left(\frac{\epsilon}{2}\right)\right);$$

with

$$\theta(\epsilon) := g\left(\left\lceil \frac{D+1}{\eta\left(D+1, \frac{\epsilon}{D+1}\right)\epsilon} \right\rceil\right);$$

$$\widehat{\eta}(\epsilon) := \lambda(1-\lambda)\eta\left(D+1, \frac{\epsilon}{D+1}\right)\epsilon.$$

In fact, instead of $\lambda(1-\lambda)$ one can use $\frac{1}{N} \leq \lambda(1-\lambda)$ for $N > 0$, to have a bound $\widetilde{\Phi}_i(D,\epsilon,g,N,\eta)$ which no longer depends on λ.

Proof. Note that for $i = k = 1$, the bound $\Phi_i := \theta(\epsilon)$ is independent of $\widehat{\eta}(\epsilon)$ and consequently the witness $\theta(\epsilon)$ is merely the rate of asymptotic regularity extracted in [27]. For $1 \leq i \leq k \geq 2$, the conclusion follows by utilizing that by [27] $\theta(\epsilon)$ is a rate of asymptotic regularity for $\{x_n\}$ w.r.t. S_k and (3.12). This completes the proof. ■

Remark 3.5 (1) It is observed that the bound Φ_i in Theorem 3.2 and Theorem 3.4 is independent of the choice of X, C, x_0 and T_i.
(2) The assumption on C to be bounded is actually weakened by setting a bound on the displacement of the starting point x_0 and some common fixed point p. Further,

the assumption of a common fixed point can be relaxed by approximate $\delta-$common fixed points p_δ in some $D-$neighborhood of x_0.

Let $D > 0$ be such that for each $\delta > 0$ there exists a $p_\delta \in C$ with

$$d(p_\delta, T_i p_\delta) < \delta \wedge d(p_\delta, x_0) \le D \text{ for } 1 \le i \le k.$$

Note that, inductively,

$$\begin{aligned} d(p_\delta, U_i p_\delta) \le d(p_\delta, S_i p_\delta) &\le d(p_\delta, T_i p_\delta) + d(T_i p_\delta, T_i U_{i-1} p_\delta) \\ &\le d(p_\delta, T_i p_\delta) + d(p_\delta, U_{i-1} p_\delta) \\ &< \delta + (i-1)\delta \\ &= i \cdot \delta. \end{aligned}$$

Hence, each $\frac{\delta}{k}-$common fixed point of $\{T_i\}_{i=1}^k$ is a $\delta-$common fixed point of $\{S_i\}_{i=1}^k$. Now, we provide a sketch of the proof of Theorem 3.2 with such a $\delta-$common fixed point. For the first part of the proof dealing with the asymptotic regularity w.r.t. S_k, the appropriate δ has been computed already in [27] in the hyperbolic space setting as well as in [14] in the normed space setting. For the reminder of the proof, we reason as follows: let $\delta \le \frac{1}{2}\widehat{\eta}(\epsilon) := \frac{1}{4}\lambda(1-\lambda)\eta\left(D+1, \frac{\epsilon}{D+1}\right)\epsilon \le \frac{1}{2}$. Then the following modified estimates will replace the original ones in the proof of Theorem 3.2:

$(3.9^{'})$

$$d(q,p) + \delta \ge d(W(q, S_{k-1}q, \lambda), p) + 2\lambda(1-\lambda)\eta\left(d(q,p)+\delta, \frac{\epsilon}{\beta}\right)(d(q,p)+\delta),$$

where $\beta := d(q,p) + \delta$, and

$$(3.11^{'}) \qquad d(q,p) + \delta \le d(W(q, S_{k-1}q, \lambda), p) + 2\delta + \widehat{\eta}(\epsilon).$$

Since $2(d(q,p)+\delta) > \epsilon$, therefore $(3.11^{'})$ contradicts $(3.9^{'})$ since we have that

$$\begin{aligned} \widehat{\eta}(\epsilon) + 2\delta &= \frac{1}{2}\lambda(1-\lambda)\eta\left(D+1, \frac{\epsilon}{D+1}\right)\epsilon + 2\delta \\ &\le \lambda(1-\lambda)\eta\left(D+1, \frac{\epsilon}{D+1}\right)\epsilon \\ &< 2\lambda(1-\lambda)\eta\left(d(q,p)+\delta, \frac{\epsilon}{D+1}\right)(d(q,p)+\delta). \end{aligned}$$

(3) It is worth mentioning that the corollary to the proof of Theorem 3.2 also allows one to compute common approximate fixed points of $T_1, \ldots, T_k$ in the following way: let $G_k(x) := (1-\alpha)x_0 + \alpha S_k x$, for all $x \in C$ with $0 < \alpha < 1$. Hence $G_k(x)$ has a unique fixed point by Banach Contraction Principle and, in particular, arbitrarily good approximate fixed points which – for bounded C – provide arbitrarily good approximate fixed points of S_k (for α close enough to 1). Then the corollary to the proof of Theorem 3.2 yields arbitrarily good common approximate fixed points for $T_1, \ldots, T_k$.
(4) Theorem 3.2 provides a quantitatively strengthened version of the asymptotic regularity hidden in the the proofs of [25] and [30].
(5) Theorem 3.4 generalizes [27, Theorem 14] for a finite family of nonexpansive mappings.

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