Error analysis of finite element and finite volume methods for some viscoelastic fluids

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Abstract

We present the error analysis of a particular Oldroyd-B type model with the limiting Weissenberg number going to infinity. Assuming a suitable regularity of the exact solution we study the error estimates of a standard finite element method and of a combined finite element/finite volume method. Our theoretical result shows first order convergence of the finite element method and the error of the order $\mathcal{O}(h^{3/4})$ for the finite element/finite volume method. These error estimates are compared and confirmed by the numerical experiments.

Keywords: error analysis, Oldroyd-B type models, viscoelastic fluids, finite element method, finite volume method, finite difference method, high Weissenberg number, multiplicative trace inequality

1 Introduction

Many biological, industrial or geological fluids can no longer be described by a linear relation between the stress and the deformation tensor. These complex fluids fall into a class of the so-called non-Newtonian fluids. Recently, an increasing number of mathematicians have become interested in mathematical modeling and numerical simulation of complex fluids and there exists already a rich body of literature on mathematical analysis and problem-suited numerical methods. Mathematical models consist of the conservation laws describing the conservation of mass (div-free condition for the velocity vector) and momentum. The stress tensor is typically written as a sum of viscous stress tensor depending linearly on the deformation tensor and the extra stress due to the polymer contribution. In macroscopic models the latter is given by a complex constitutive

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equation to capture the corresponding viscoelastic properties. In order to describe the evolution of viscoelastic stress tensor various approaches can be used. Let us mention, for example, differential, integral models or macro-micro models based on the kinetic formulation of the probability distribution function. Well-known differential models are the Oldroyd-B or the Johnson-Segelman models, where an additional transport equation is added to describe the time evolution of the polymer stress tensor. In what follows we will concentrate on the differential viscoelastic models.

In the mathematical literature we can find already various results dealing with the questions of well-posedness of the viscoelastic flows and in particular with the Oldroyd-B model. Let us mention classical results on strong solutions published by Fernández-Cara, Guillén and Ortega [15] and of Guillopé and Saut [17], see also [19] for further related results on the existence of strong solutions in exterior domains obtained by Hieber, Naito and Shibata.

Recently, the global existence result for fully two- and three-dimensional flow has been obtained by Lions and Masmoudi [25] for the case of the co-rotational Oldroyd model. In [24] the local existence of solutions and global existence of small solutions of some rate type fluids have been shown, global existence of weak solutions for small data can be found, e.g., in [8]. In the recent work [1] Barrett and Boyaval studied the so-called diffusive Oldroyd-B model both from numerical as well as analytical point of view. For two space dimensions they were able to prove the global existence of weak solutions. The diffusive Oldroyd-B model has also been studied by Constantin and Kliegl in [7] and the global regularity in two space dimensions has been proven.

Concerning numerical simulation of the Oldroyd-B type flows a major obstacle is the high Weissenberg number problem. The so-called numerical blow-up is a widely known phenomenon in numerical simulations of high Weissenberg number viscoelastic flows. It is anticipated that the blow-up has various reasons: influence of domain singularities, missing analytical results on the well-posedness of global weak solutions and numerical instabilities. The latter is a purely numerical phenomenon that arises due to the inadequacy of polynomial interpolations to approximate spatial exponential profiles, which is the case of the elastic stress tensor. In [13] a new promising approach using the log-conformation tensor has been proposed, see also further related works [5], [2], [10] and the references therein.

The Oldroyd-B equations consist of the conservation laws of momentum and mass and an additional transport equation describing time evolution of the elastic tensor $\mathbb{T}^e$

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \nu \Delta \boldsymbol{u} + \nabla \pi = \operatorname{div}(\mathbb{T}^e) \tag{1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \tag{1b}$$

$$\partial_t \mathbb{T}^e + (\boldsymbol{u} \cdot \nabla)\mathbb{T}^e - (\nabla \boldsymbol{u})\mathbb{T}^e - \mathbb{T}^e(\nabla \boldsymbol{u})^\top = \frac{1}{We}\left(\tilde{\nu}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top) - \mathbb{T}^e\right). \tag{1c}$$

Here $\boldsymbol{u}$ and π denote the velocity vector and pressure, $\mathbb{T}^e$ stays for the elastic tensor, while the total stress tensor reads $-\pi\mathbb{I} + \nu(\nabla \boldsymbol{u} + \nabla(\boldsymbol{u})^\top) + \mathbb{T}^e$. We have denoted by a constant $\nu > 0$ the Newtonian part of the viscosity, $\tilde{\nu} > 0$ is the rest non-Newtonian viscosity part and We is the reference number expressing a ratio of the relaxation time to a typical flow time scale. Denoting λ the relaxation time of the non-Newtonian fluid and U, L the characteristic velocity and length, respectively, we have $We = \frac{\lambda U}{L}$.

In this paper we study the viscoelastic model with $\mathbb{T}^e = \mathbb{F}\mathbb{F}^\top$, where $\mathbb{F}$ is the deformation gradient of the material; $\mathbb{F} := \partial X/\partial x$ using the Langrangian and Eulerian coordinates X and x, respectively. Due to the incompressibility condition we have $\det \mathbb{F} = 1$. The model considered in this paper can be achieved as a limiting case of the Oldroyd-B model (1) when the Weissenberg number We is set to infinity. Consequently, we obtain from (1c) the following transport equation for the viscoelastic stress

$$\partial_t \mathbb{T}^e + (\boldsymbol{u} \cdot \nabla)\mathbb{T}^e - (\nabla \boldsymbol{u})\mathbb{T}^e - \mathbb{T}^e(\nabla \boldsymbol{u})^\top = \mathbf{0}.$$

Now writing the transport equation for the tensor $\mathbb{F}$ we obtain altogether the following system

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \nu \Delta \boldsymbol{u} + \nabla \pi = \operatorname{div}(\mathbb{F}\mathbb{F}^\top) \tag{2a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \tag{2b}$$

$$\partial_t \mathbb{F} + (\boldsymbol{u} \cdot \nabla)\mathbb{F} = (\nabla \boldsymbol{u})\mathbb{F}. \tag{2c}$$

The system (2a)–(2c) describes the unsteady motion of a viscoelastic fluid in a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary in the time interval $(0, T)$ and is complemented with the no-slip boundary condition

$$\boldsymbol{u}|_{\partial\Omega} = \mathbf{0} \tag{2d}$$

and the initial conditions

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0, \quad \mathbb{F}(0, \cdot) = \mathbb{F}_0. \tag{2e}$$

Note that there is no need to prescribe the boundary condition for $\mathbb{F}$ since (2c) is a first-order transport equation and $\boldsymbol{u} = 0$ on the boundary.

We should mention that the question of global in time existence of weak solutions of (2) is still open. Local existence of strong solutions has been proven in [21], [24] as well as in [6]. As far as we know this model has not yet been studied from the numerical point of view and our paper is the first contribution in this direction. More interestingly, we would like to point out that we do not need any particular stabilization techniques for high Weissenberg number problems and obtain stable and convergent results using some suitable numerical approximations that are typically used in computational fluid dynamics.

The main aim of this paper is to present error analysis of two finite element-type approximations of the problem (2). In particular, we combine the lowest order Taylor-Hood finite element discretization of the flow part (piecewise quadratic velocity and piecewise linear pressure) with either piecewise linear finite elements or finite volumes for the deformation gradient. The paper is organized as follows. In Section 2 we introduce space discretizations of (2). The main result on convergence rates is stated in Section 3. Sections 4 and 5 are devoted to the proof of the main result. In the second part of the proof we use a variant of the multiplicative trace inequality, which is proved in Section 6. Finally, we present results of a numerical test in Section 7.

2 Approximation

Before passing to the finite dimensional approximation, let us introduce the basic notation. Throughout the paper we shall use the space $\mathcal{C}^k(B)$ of continuously k-times differentiable functions in an compact set B, the Lebesgue space $L^p(\Omega)$, its subspace $L^p_0(\Omega)$ of functions with zero integral mean, the Sobolev space $W^{k,p}(\Omega)$ and the Sobolev space $W^{1,p}_0(\Omega)$ of functions with vanishing trace. Lebesgue and Sobolev spaces with values in some Banach space X will be denoted by $L^p(\Omega; X)$, $W^{k,p}(\Omega; X)$, respectively; $p \in [1, \infty)$. The norm in $L^p(\Omega)$ and $W^{k,p}(\Omega)$ will be denoted $\|\cdot\|_p$, $\|\cdot\|_{k,p}$, respectively. We shall employ the continuous embeddings

$$\begin{aligned} &W^{1,p}(\Omega) \hookrightarrow L^{\frac{2p}{2-p}}(\Omega), \quad p \in [1,2) \\ &W^{1,2}(\Omega) \hookrightarrow L^q(\Omega), \quad q \in [1,\infty) \\ &W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}), \quad p > 2 \\ &W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega) \end{aligned} \tag{3}$$

and the Sobolev inequality

$$\forall v \in W^{1,2}_0(\Omega): \ \|v\|_4^2 \le C\|v\|_2\|\nabla v\|_2. \tag{4}$$

Here and in what follows, C will stand for a generic positive constant whose value may change from line to line. If $fg \in L^1(\Omega)$ then we shall write (f, g) in place of $\int_\Omega fg$. The following variant of Young's inequality holds: For every $p \in (1, \infty)$ there exists $C = C(p) > 0$ such that for any $\alpha > 0$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$:

$$(f, g) \le \alpha \int_\Omega |f|^p + C\alpha^{-\frac{q}{p}} \int_\Omega |g|^q, \tag{5}$$

where $q = p/(p-1)$.

Let Ω_h be a polynomial approximation of a computational domain Ω. Further, let $\{\mathcal{T}_h\}_{h>0}$ be a family of partitions of Ω_h into triangles and h be the length of the largest edge in $\mathcal{T}_h$; $\overline{\Omega}_h = \bigcup_{T\in\mathcal{T}_h} T$. To simplify our considerations we set that $\Omega = \Omega_h$. In the present work we do not include the theory of variational crimes that should be used when a smooth domain Ω is approximated by its polygonal approximation Ω_h, cf., e.g., [14]. For any $T \in \mathcal{T}_h$ and $k = 0, 1, 2, \ldots$ we denote by $\mathcal{P}^k(T)$ the space of k-th degree polynomials on T. We define the spaces

$$
\begin{aligned}
W_h &:= \{\boldsymbol{v} \in W_0^{1,2}(\Omega;\mathbb{R}^2);\ \forall T \in \mathcal{T}_h:\ \boldsymbol{v}_{|T} \in \mathcal{P}^2(T)^2\},\\
L_h &:= \{q \in L_0^2(\Omega) \cap \mathcal{C}(\overline{\Omega});\ \forall T \in \mathcal{T}_h:\ q_{|T} \in \mathcal{P}^1(T)\},\\
X_h &:= \{\mathbb{G} \in \mathcal{C}(\overline{\Omega};\mathbb{R}^{2\times 2});\ \forall T \in \mathcal{T}_h:\ \mathbb{G}_{|T} \in \mathcal{P}^1(T)^{2\times 2}\},\\
Z_h &:= \{\mathbb{G} \in L^2(\Omega;\mathbb{R}^{2\times 2});\ \forall T \in \mathcal{T}_h:\ \mathbb{G}_{|T} \in \mathcal{P}^0(T)\}.
\end{aligned}
$$

Let e be an interior edge shared by elements T_1 and T_2. Define the unit normal vectors $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ on e pointing exterior to T_1 and T_2, respectively. For a function φ, piecewise smooth on $\mathcal{T}_h$, with $\varphi^i = \varphi_{|T_i}$ we define the average $\{\varphi\}$ and the jump $[\varphi]$:

$$
\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\varphi] = \varphi^1 \boldsymbol{n}^1 + \varphi^2 \boldsymbol{n}^2 \quad \text{on } e \in \mathcal{E}_h^0,
$$

where $\mathcal{E}_h^0$ is the set of all interior edges. For a vector-valued function $\boldsymbol{\phi}$, piecewise smooth on $\mathcal{T}_h$, we define the average and the jump analogously

$$
\{\boldsymbol{\phi}\} = \frac{1}{2}(\boldsymbol{\phi}^1 + \boldsymbol{\phi}^2), \quad [\boldsymbol{\phi}] = \boldsymbol{\phi}^1 \cdot \boldsymbol{n}^1 + \boldsymbol{\phi}^2 \cdot \boldsymbol{n}^2 \quad \text{on } e \in \mathcal{E}_h^0.
$$

Let $\boldsymbol{\beta}$ be a vector-valued function, continuous across e. The upwind value of a quantity $\boldsymbol{\beta}\varphi$ is defined as follows:

$$
\{\boldsymbol{\beta}\varphi\}_u = \begin{cases} \boldsymbol{\beta}\varphi^1 & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}^1 > 0,\\ \boldsymbol{\beta}\varphi^2 & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}^1 < 0,\\ \boldsymbol{\beta}\{\varphi\} & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}^1 = 0. \end{cases}
$$

We introduce the following space semi-discretizations of (2).

A. Finite element approximation. Find $(\boldsymbol{u}_h, \pi_h, \mathbb{F}_h) \in \mathcal{C}^1([0,T]; W_h) \times \mathcal{C}([0,T]; L_h) \times \mathcal{C}^1([0,T]; X_h)$ such that

- for all $(\boldsymbol{v}_h, q_h, \mathbb{G}_h) \in W_h \times L_h \times X_h$ and $t \in (0,T)$ the following integral identities hold:

$$
\begin{aligned}
(\partial_t \boldsymbol{u}_h(t), \boldsymbol{v}_h) - (\boldsymbol{u}_h(t) \otimes \boldsymbol{u}_h(t), \nabla \boldsymbol{v}_h) - \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h(t))\boldsymbol{u}_h(t), \boldsymbol{v}_h)&\\
+ \nu(\nabla \boldsymbol{u}_h(t), \nabla \boldsymbol{v}_h) - (\pi_h(t), \operatorname{div} \boldsymbol{v}_h) + (\mathbb{F}_h(t)\mathbb{F}_h^\top(t), \nabla \boldsymbol{v}_h) &= 0, \quad \text{(6a)}\\
(q_h, \operatorname{div} \boldsymbol{u}_h(t)) &= 0, \quad \text{(6b)}
\end{aligned}
$$

$$
\begin{aligned}
(\partial_t \mathbb{F}_h(t), \mathbb{G}_h) - ((\boldsymbol{u}_h(t)\cdot\nabla)\mathbb{G}_h, \mathbb{F}_h(t)) - \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h(t))\mathbb{F}_h(t), \mathbb{G}_h)&\\
- ((\nabla \boldsymbol{u}_h(t))\mathbb{F}_h(t), \mathbb{G}_h) &= 0; \quad \text{(6c)}
\end{aligned}
$$

- $\boldsymbol{u}_h$ and $\mathbb{F}_h$ satisfy the initial conditions: for all $\boldsymbol{v}_h \in W_h$ and $\mathbb{G}_h \in X_h$

$$
(\boldsymbol{u}_h(0,\cdot), \boldsymbol{v}_h) = (\boldsymbol{u}_0, \boldsymbol{v}_h), \qquad (\mathbb{F}_h(0,\cdot), \mathbb{G}_h) = (\mathbb{F}_0, \mathbb{G}_h). \tag{7}
$$

B. Finite element-finite volume approximation. Find $(\boldsymbol{u}_h, \pi_h, \mathbb{F}_h) \in \mathcal{C}^1([0,T]; W_h) \times \mathcal{C}([0,T]; L_h) \times \mathcal{C}^1([0,T]; Z_h)$ such that

- for all $(\boldsymbol{v}_h, q_h, \mathbb{G}_h) \in W_h \times L_h \times Z_h$ and $t \in (0,T)$, the integral identities (6a), (6b) hold true and furthermore

$$(\partial_t \mathbb{F}_h(t), \mathbb{G}_h) + \sum_{e \in \mathcal{E}_h^0} (\{\boldsymbol{u}_h(t)\mathbb{F}_h(t)\}_u, [\mathbb{G}_h])_e - \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h(t))\mathbb{F}_h(t), \mathbb{G}_h) \\ - ((\nabla \boldsymbol{u}_h(t))\mathbb{F}_h(t), \mathbb{G}_h) = 0; \quad (8)$$

- $\boldsymbol{u}_h$ and $\mathbb{F}_h$ satisfy the initial conditions: for all $\boldsymbol{v} \in W_h$ and $\mathbb{G} \in Z_h$

$$(\boldsymbol{u}_h(0,\cdot), \boldsymbol{v}) = (\boldsymbol{u}_0, \boldsymbol{v}), \qquad (\mathbb{F}_h(0,\cdot), \mathbb{G}) = (\mathbb{F}_0, \mathbb{G}). \quad (9)$$

In what follows we shall assume that the family $\{\mathcal{T}_h\}$ is regular (see e.g. [3]). Consequently, the discrete spaces W_h, L_h, X_h and Z_h enjoy the following properties:

- (inf-sup condition) There exists a constant $C > 0$ independent of $h > 0$ such that for all $q_h \in L_h$

$$\sup_{\boldsymbol{v}_h \in W_h,\ \boldsymbol{v}_h \neq \boldsymbol{0}} \frac{(q_h, \operatorname{div} \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,2}} \geq C\|q_h\|_2; \quad (10)$$

- (interpolation into W_h) There exists an operator $\Pi_h^u : W_0^{1,2}(\Omega; \mathbb{R}^2) \to W_h$ such that
  - for all $\boldsymbol{v} \in W_0^{1,2}(\Omega; \mathbb{R}^2) \cap W^{k,q}(\Omega; \mathbb{R}^2)$, $1 \leq q \leq \infty$, $k \in \{1,2,3\}$, $r \in \{0, \dots, k\}$:
  $$\|\Pi_h^u \boldsymbol{v} - \boldsymbol{v}\|_{r,q} \leq C h^{k-r} \|\boldsymbol{v}\|_{k,q}, \quad (11)$$
  where $C > 0$ is independent of $h > 0$;
  - for all $\boldsymbol{v} \in W_0^{1,2}(\Omega; \mathbb{R}^2)$ and $q_h \in L_h$:
  $$(q_h, \operatorname{div} \Pi_h^u \boldsymbol{v}) = (q_h, \operatorname{div} \boldsymbol{v}); \quad (12)$$

- (interpolation into L_h) There exists an operator $\Pi_h^\pi : L_0^2(\Omega) \to L_h$ and a constant $C > 0$ independent of $h > 0$, such that
$$\|\Pi_h^\pi s - s\|_{r,q} \leq C h^{2-r} \|s\|_{2,q} \quad (13)$$
for all $s \in L_0^2(\Omega) \cap W^{2,q}(\Omega)$, $1 \leq q \leq \infty$, $r \in \{0,1\}$;

- (interpolation into X_h) There exists an operator $\Pi_h^F : L^2(\Omega; \mathbb{R}^{2\times2}) \to X_h$ and a constant $C > 0$ independent of $h > 0$, such that
$$\|\Pi_h^F \mathbb{G} - \mathbb{G}\|_{r,q} \leq C h^{2-r} \|\mathbb{G}\|_{2,q}, \quad (14)$$
for all $\mathbb{G} \in W^{2,q}(\Omega; \mathbb{R}^{2\times2})$, $1 \leq q \leq \infty$, $r \in \{0,1\}$.

- (interpolation into Z_h) There exists an operator $\Pi_h^0 : L^2(\Omega;\mathbb{R}^{2\times 2}) \to Z_h$ and a constant $C > 0$ independent of $h > 0$, such that

$$\|\Pi_h^0 \mathbb{G} - \mathbb{G}\|_q \le Ch\|\mathbb{G}\|_{1,q}, \tag{15}$$

for all $\mathbb{G} \in W^{1,q}(\Omega;\mathbb{R}^{2\times 2})$, $1 \le q \le \infty$.

The inequality (10) is the Babuška-Brezzi condition for the Taylor-Hood finite element (see e.g. [4]), (11)-(15) are standard properties of interpolation operators [3].

In the error analysis of the finite element-finite volume scheme we will need the following variant of *multiplicative trace inequality.*

Lemma 1. *Let $F \in W^{2,2}(\Omega)$. Then there exists a constant $c > 0$ independent of h such that*

$$\sum_{e\in\mathcal{E}_h^0} \|F - \Pi_h^0 F\|_{4,e} \le ch^{3/4}\|F\|_{2,2}. \tag{16}$$

The proof of this statement is postponed to Section 6.

For a smooth domain Ω and initial data of the form

$$\mathbb{F}_0 = \nabla \times \mathbf{\Phi}_0, \quad \nabla\mathbf{\Phi}_0 \in W^{k,2}(\Omega),\ \boldsymbol{u}_0 \in W^{k,2}(\Omega),\ k \ge 2, \tag{17}$$

it has been proved [24, Theorem 2.2] that there exists a positive time T, which depends only on $\|\nabla\mathbf{\Phi}_0\|_{2,2}$ and $\|\boldsymbol{u}_0\|_{2,2}$, such that the system (2) possesses a unique solution in the time interval $[0,T]$ with

$$\begin{aligned} &\partial_t^j \nabla^\alpha \boldsymbol{u} \in L^\infty(0,T;W^{k-2j-|\alpha|,2}(\Omega)) \cap L^2(0,T;W^{k-2j-|\alpha|+1,2}(\Omega)),\\ &\partial_t^j \nabla^\alpha \mathbb{F} \in L^\infty(0,T;W^{k-2j-|\alpha|,2}(\Omega)), \end{aligned} \tag{18}$$

for all j and α satisfying $2j + |\alpha| \le k$.

For the approximate problems we have the following result.

Lemma 2. *Let $T > 0$, $\boldsymbol{u}_0 \in L^2(\Omega;\mathbb{R}^2)$ and $\mathbb{F}_0 \in L^2(\Omega;\mathbb{R}^{2\times 2})$. Then there exists $h_0 > 0$ such that for every $h \in (0,h_0)$ problem A (or problem B, respectively) has a unique global in time solution $(\boldsymbol{u}_h, \pi_h, \mathbb{F}_h)$, which satisfies*

$$\|\boldsymbol{u}_h(\tau,\cdot)\|_2^2 + \|\mathbb{F}_h(\tau,\cdot)\|_2^2 + 2\nu\int_0^\tau \|\nabla\boldsymbol{u}_h\|_2^2 = \|\boldsymbol{u}_0\|_2^2 + \|\mathbb{F}_0\|_2^2 \tag{19}$$

for all $\tau \in (0,T)$.

Proof. The semidiscrete systems (6), (7) or (6a), (6b), (8), (9) yield the system of ordinary differential equations, whose solution can be proven applying the standard arguments for the ordinary differential systems and showing the apriori estimates (19) following [24]. □

3 Main result

Now let us state the main result on the error rates.

Theorem 1. *Let the family* $\{\mathcal{T}_h\}_{h>0}$ *be regular, the initial data* $(\boldsymbol{u}_0, \nabla \times \boldsymbol{\Phi}_0)$ *satisfy* $\boldsymbol{u}_0, \nabla\boldsymbol{\Phi}_0 \in W^{2,2}(\Omega)$ *and* $[0,T]$ *be the maximal time interval in which the strong solution* $(\boldsymbol{u}, \pi, \mathbb{F})$ *to* (2) *satisfying (18) with* $k = 2$ *exists. Further, let* $(\boldsymbol{u}_h, \pi_h, \mathbb{F}_h)$ *be the solution to the problem A, i.e.* (6)–(7)*, satisfying (19). Then there exist constants* $h_0 > 0$ *and* $C > 0$ *such that for all* $h \in (0, h_0)$ *it holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(0,T;L^2(\Omega))} + \|\mathbb{F} - \mathbb{F}_h\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch. \tag{20}$$

Similarly, there exist constants $h_0 > 0$ *and* $C > 0$ *such that for all* $h \in (0, h_0)$ *it holds*

$$\|\boldsymbol{u} - \bar{\boldsymbol{u}}_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\boldsymbol{u} - \bar{\boldsymbol{u}}_h)\|_{L^2(0,T;L^2(\Omega))} + \|\mathbb{F} - \bar{\mathbb{F}}_h\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch^{3/4}, \tag{21}$$

where $(\bar{\boldsymbol{u}}_h, \bar{\pi}_h, \bar{\mathbb{F}}_h)$ *is the solution to the problem B, i.e.* (6a)*,* (6b)*,* (8) *and* (9)*, that satisfy (19).*

4 Error estimates for finite element approximation

The aim of this section is to prove the first part of Theorem 1. To this end we first realize that the following regularity property of the exact solution holds.

Regularity of the solution to (2)**.** In accordance with (17)–(18), the assumptions of Theorem 1 imply that the exact solution $(\boldsymbol{u}, \pi, \mathbb{F})$ satisfies

$$\begin{aligned}
\partial_t \boldsymbol{u} &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),\\
\boldsymbol{u} &\in L^\infty(0,T;W^{2,2}(\Omega)) \cap L^2(0,T;W^{3,2}(\Omega)),\\
\partial_t \mathbb{F} &\in L^\infty(0,T;L^2(\Omega)),\\
\mathbb{F} &\in L^\infty(0,T;W^{2,2}(\Omega)).
\end{aligned} \tag{22}$$

Using (2b) and (2c), we furthermore obtain from (22) that

$$\begin{aligned}
\nabla\pi &= \operatorname{div}(\mathbb{F}\mathbb{F}^\top) - \partial_t \boldsymbol{u} - \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nu\Delta\boldsymbol{u} \in L^2(0,T;W^{1,2}(\Omega)),\\
\partial_t \mathbb{F} &= (\nabla\boldsymbol{u})\mathbb{F} - (\boldsymbol{u} \cdot \nabla)\mathbb{F} \in L^\infty(0,T;W^{1,2}(\Omega)).
\end{aligned} \tag{23}$$

Equations satisfied by the errors. Multiplying (2) with $\boldsymbol{v}_h \in W_h$, $q_h \in L_h$ and $\mathbb{G}_h \in X_h$, respectively, and integrating over Ω, we obtain for a.a. $t \in (0,T)$ the following identities

$$
\begin{aligned}
(\partial_t \boldsymbol{u}(t), \boldsymbol{v}_h) - (\boldsymbol{u}(t) \otimes \boldsymbol{u}(t), \nabla \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{u}(t), \nabla \boldsymbol{v}_h) \\
- (\pi(t), \operatorname{div} \boldsymbol{v}_h) + (\mathbb{F}(t)\mathbb{F}^\top(t), \nabla \boldsymbol{v}_h) = 0, \quad (24a)
\end{aligned}
$$

$$
(q_h, \operatorname{div} \boldsymbol{u}(t)) = 0, \quad (24b)
$$

$$
(\partial_t \mathbb{F}(t), \mathbb{G}_h) - ((\boldsymbol{u}(t) \cdot \nabla)\mathbb{G}_h, \mathbb{F}(t)) - ((\nabla \boldsymbol{u}(t))\mathbb{F}(t), \mathbb{G}_h) = 0. \quad (24c)
$$

Let us denote $e_u := \boldsymbol{u} - \boldsymbol{u}_h$, $e_\pi := \pi - \pi_h$ and $e_F := \mathbb{F} - \mathbb{F}_h$. Taking the difference of (24) and (6) we obtain

$$
\begin{aligned}
\int_0^T \Big[(\partial_t e_u, \boldsymbol{v}_h) - (e_u \otimes \boldsymbol{u} + \boldsymbol{u}_h \otimes e_u, \nabla \boldsymbol{v}_h) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h) \\
+ \nu(\nabla e_u, \nabla \boldsymbol{v}_h) - (e_\pi, \operatorname{div} \boldsymbol{v}_h) + (\mathbb{F}\mathbb{F}^\top - \mathbb{F}_h\mathbb{F}_h^\top, \nabla \boldsymbol{v}_h)\Big] = 0, \quad (25a)
\end{aligned}
$$

$$
\int_0^T (q_h, \operatorname{div} e_u) = 0, \quad (25b)
$$

$$
\begin{aligned}
\int_0^T (\partial_t e_F, \mathbb{G}_h) - (u_i \mathbb{F} - u_{hi}\mathbb{F}_h, \partial_i \mathbb{G}_h) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \mathbb{G}_h) \\
- ((\nabla \boldsymbol{u})\mathbb{F} - (\nabla \boldsymbol{u}_h)\mathbb{F}_h, \mathbb{G}_h) = 0 \quad (25c)
\end{aligned}
$$

for any $(\boldsymbol{v}_h, q_h, \mathbb{G}_h) \in L^2(0,T;W_h) \times L^2(0,T;L_h) \times L^2(0,T;X_h)$.

Estimates of the errors. In order to derive estimates of e_u, e_π, e_F in suitable norms, we decompose

$$
e_u = (\boldsymbol{u} - \Pi_h^u \boldsymbol{u}) + (\Pi_h^u \boldsymbol{u} - \boldsymbol{u}_h) =: \eta_u + \delta_u.
$$

Similarly we introduce

$$
\eta_\pi := \pi - \Pi_h^\pi \pi, \quad \delta_\pi := \Pi_h^\pi \pi - \pi_h, \quad \eta_F := \mathbb{F} - \Pi_h^F \mathbb{F}, \quad \delta_F := \Pi_h^F \mathbb{F} - \mathbb{F}_h.
$$

The properties of the interpolation operators imply the following estimates

$$
\sup_{\tau \in (0,T)} \|\eta_u(\tau, \cdot)\|_2^2 \le Ch^4 \sup_{\tau \in (0,T)} \|\boldsymbol{u}(\tau, \cdot)\|_{2,2}^2,
$$

$$
\int_0^T \|\nabla \eta_u\|_2^2 \le Ch^4 \int_0^T \|\boldsymbol{u}\|_{3,2}^2,
$$

$$
\sup_{\tau \in (0,T)} \|\eta_F(\tau, \cdot)\|_2^2 \le Ch^4 \sup_{\tau \in (0,T)} \|\mathbb{F}(\tau, \cdot)\|_{2,2}^2.
$$

Hence, with the help of (22) we obtain

$$\sup_{\tau\in(0,T)}\left(\|e_u(\tau,\cdot)\|_2^2+\|e_F(\tau,\cdot)\|_2^2\right)+\nu\int_0^T\|\nabla e_u\|_2^2\le Ch^4$$
$$+\sup_{\tau\in(0,T)}\left(\|\delta_u(\tau,\cdot)\|_2^2+\|\delta_F(\tau,\cdot)\|_2^2\right)+\nu\int_0^T\|\nabla\delta_u\|_2^2. \quad (26)$$

The proof of (20) will be completed as soon as we estimate the δ-terms in the previous inequality by Ch^2. Due to the initial conditions (7) it holds that $\|\delta_u(0)\|_2=\|\delta_F(0)\|_2=0$. Hence, for any $\tau\in(0,T)$ we have

$$\begin{aligned}
&\frac{1}{2}\left(\|\delta_u(\tau,\cdot)\|_2^2+\|\delta_F(\tau,\cdot)\|_2^2\right)+\nu\int_0^\tau\|\nabla\delta_u\|_2^2\\
&=\frac{1}{2}\left(\|\delta_u(\tau,\cdot)\|_2^2-\|\delta_u(0,\cdot)\|_2^2+\|\delta_F(\tau,\cdot)\|_2^2-\|\delta_F(0,\cdot)\|_2^2\right)+\nu\int_0^\tau\|\nabla\delta_u\|_2^2\\
&\qquad=\int_0^\tau[(\partial_t\delta_u,\delta_u)+(\partial_t\delta_F,\delta_F)+\nu(\nabla\delta_u,\nabla\delta_u)]\\
&\qquad=\int_0^\tau[(\partial_t e_u,\delta_u)+(\partial_t e_F,\delta_F)+\nu(\nabla e_u,\nabla\delta_u)]\\
&\qquad-\int_0^\tau[(\partial_t\eta_u,\delta_u)+(\partial_t\eta_F,\delta_F)+\nu(\nabla\eta_u,\nabla\delta_u)]=J_1+J_2. \quad (27)
\end{aligned}$$

The last integral can be estimated using the Hölder and the Young inequality and (11)–(14)

$$\begin{aligned}
J_2&\le\left(\int_0^T\|\partial_t\eta_u\|_2^2\right)^{1/2}\left(\int_0^\tau\|\delta_u\|_2^2\right)^{1/2}+\left(\int_0^T\|\partial_t\eta_F\|_2^2\right)^{1/2}\left(\int_0^\tau\|\delta_F\|_2^2\right)^{1/2}\\
&\quad+\nu\left(\int_0^T\|\nabla\eta_u\|_2^2\right)^{1/2}\left(\int_0^\tau\|\nabla\delta_u\|_2^2\right)^{1/2}\\
&\le\frac{1}{2}\int_0^\tau\|\delta_u\|_2^2+\frac{1}{2}\int_0^T\|\partial_t\eta_u\|_2^2+\frac{1}{2}\int_0^\tau\|\delta_F\|_2^2+\frac{1}{2}\int_0^T\|\partial_t\eta_F\|_2^2\\
&\quad+\alpha\nu\int_0^\tau\|\nabla\delta_u\|_2^2+\frac{C\nu}{\alpha}\int_0^T\|\nabla\eta_u\|_2^2\\
&\le\frac{1}{2}\int_0^\tau\|\delta_u\|_2^2+Ch^2\int_0^T\|\partial_t\boldsymbol{u}\|_{1,2}^2+\frac{1}{2}\int_0^\tau\|\delta_F\|_2^2+Ch^2\int_0^T\|\partial_t\mathbb{F}\|_{1,2}^2\\
&\quad+\alpha\nu\int_0^\tau\|\nabla\delta_u\|_2^2+\frac{C\nu h^4}{\alpha}\int_0^T\|\boldsymbol{u}\|_{3,2}^2. \quad (28)
\end{aligned}$$

Here $\alpha\in(0,1)$ is a number to be specified later. The regularity of the solution (22), (23) implies that

$$\boxed{J_2\le\frac{1}{2}\int_0^\tau\|\delta_u\|_2^2+\frac{1}{2}\int_0^\tau\|\delta_F\|_2^2+\alpha\nu\int_0^\tau\|\nabla\delta_u\|_2^2+Ch^2+\frac{C\nu h^4}{\alpha}.} \quad (29)$$

The first integral on the r.h.s. of (27) can be substituted from (25) as follows

$$J_1 = \int_0^\tau \big[(e_u \otimes \boldsymbol{u} + \boldsymbol{u}_h \otimes e_u, \nabla\delta_u) + \frac{1}{2}((\operatorname{div}\boldsymbol{u}_h)\boldsymbol{u}_h, \delta_u) + (e_\pi, \operatorname{div}\delta_u) - (\mathbb{F}\mathbb{F}^\top - \mathbb{F}_h\mathbb{F}_h^\top, \nabla\delta_u)$$
$$+ (u_i\mathbb{F} - u_{hi}\mathbb{F}_h, \partial_i\delta_F) + \frac{1}{2}((\operatorname{div}\boldsymbol{u}_h)\mathbb{F}_h, \delta_F) + ((\nabla\boldsymbol{u})\mathbb{F} - (\nabla\boldsymbol{u}_h)\mathbb{F}_h, \delta_F)\big]$$
$$=: \int_0^\tau \sum_{j=1}^{7} T_j. \quad (30)$$

We shall estimate the resulting terms $T_1, \ldots, T_7$ subsequently.

Term T_1 can be decomposed as follows

$$T_1 = (\eta_u \otimes \boldsymbol{u}, \nabla\delta_u) + \underbrace{(\delta_u \otimes \boldsymbol{u}, \nabla\delta_u)}_{=0} + (\boldsymbol{u}_h \otimes \eta_u, \nabla\delta_u) + (\boldsymbol{u}_h \otimes \delta_u, \nabla\delta_u),$$

where the second term vanishes since $\operatorname{div}\boldsymbol{u} = 0$. The remaining terms can be estimated with help of the Hölder inequality, (4), (5) and the properties of the interpolation operators:

$$\int_0^\tau |(\eta_u \otimes \boldsymbol{u}, \nabla\delta_u)| \le \int_0^\tau \|\eta_u\|_4 \|\boldsymbol{u}\|_4 \|\nabla\delta_u\|_2$$
$$\le C\int_0^\tau \|\eta_u\|_2^{1/2}\|\nabla\eta_u\|_2^{1/2}\|\boldsymbol{u}\|_2^{1/2}\|\nabla\boldsymbol{u}\|_2^{1/2}\|\nabla\delta_u\|_2$$
$$\le \alpha\nu\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha\nu}\left(\sup_{(0,T)}\|\eta_u\|_2\right)\left(\sup_{(0,T)}\|\boldsymbol{u}\|_2\right)\left(\int_0^T \|\nabla\boldsymbol{u}\|_2^2\right)^{1/2}\left(\int_0^T \|\nabla\eta_u\|_2^2\right)^{1/2}$$
$$\le \alpha\nu\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{Ch^4}{\alpha\nu}\left(\sup_{(0,T)}\|\boldsymbol{u}\|_{2,2}\right)^2\left(\int_0^T \|\boldsymbol{u}\|_{3,2}^2\right).$$

Similarly we get

$$\int_0^\tau |(\boldsymbol{u}_h \otimes \eta_u, \nabla\delta_u)| \le \alpha\nu\int_0^\tau \|\nabla\delta_u\|_2^2$$
$$+ \frac{Ch^4}{\alpha\nu}\left(\sup_{(0,T)}\|\boldsymbol{u}_h\|_2\right)\left(\sup_{(0,T)}\|\boldsymbol{u}\|_{2,2}\right)\left(\int_0^T \|\boldsymbol{u}_h\|_{1,2}^2\right)^{1/2}\left(\int_0^T \|\boldsymbol{u}\|_{3,2}^2\right)^{1/2}.$$

Finally, using the same inequalities we obtain

$$\int_0^\tau |(\boldsymbol{u}_h \otimes \delta_u, \nabla\delta_u)| \le \int_0^\tau \|\boldsymbol{u}_h\|_2^{1/2}\|\nabla\boldsymbol{u}_h\|_2^{1/2}\|\delta_u\|_2^{1/2}\|\nabla\delta_u\|_2^{3/2}$$
$$\le \alpha\nu\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{(\alpha\nu)^3}\left(\sup_{(0,T)}\|\boldsymbol{u}_h\|_2\right)^2\int_0^\tau \|\boldsymbol{u}_h\|_{1,2}^2\|\delta_u\|_2^2.$$

Due to (22), (23) and (19), the previous estimates altogether yield

$$\boxed{\int_0^\tau T_1 \leq 3\alpha\nu \int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{(\alpha\nu)^3}\int_0^\tau \|\boldsymbol{u}_h\|_{1,2}^2\|\delta_u\|_2^2 + \frac{Ch^4}{\alpha\nu}.} \tag{31}$$

Term T_2. We can write:

$$2T_2 = -((\operatorname{div}\eta_u)\boldsymbol{u}_h, \delta_u) - ((\operatorname{div}\delta_u)\boldsymbol{u}_h, \delta_u),$$

where, similarly as in the previous paragraph,

$$\begin{aligned}\int_0^\tau |((\operatorname{div}\eta_u)\boldsymbol{u}_h, \delta_u)| &\leq \int_0^\tau \|\nabla\eta_u\|_2\|\boldsymbol{u}_h\|_2^{1/2}\|\nabla\boldsymbol{u}_h\|_2^{1/2}\|\delta_u\|_2^{1/2}\|\nabla\delta_u\|_2^{1/2}\\ &\leq \alpha\nu\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha\nu}\left(\sup_{(0,T)}\|\boldsymbol{u}_h\|_2^2\right)\int_0^\tau \|\nabla\boldsymbol{u}_h\|_2^2\|\delta_u\|_2^2 + \int_0^T \|\nabla\eta_u\|_2^2,\end{aligned}$$

$$\begin{aligned}\int_0^\tau |((\operatorname{div}\delta_u)\boldsymbol{u}_h, \delta_u)| &\leq \int_0^\tau \|\nabla\delta_u\|_2^{3/2}\|\boldsymbol{u}_h\|_2^{1/2}\|\nabla\boldsymbol{u}_h\|_2^{1/2}\|\delta_u\|_2^{1/2}\\ &\leq \alpha\nu\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{(\alpha\nu)^3}\left(\sup_{(0,T)}\|\boldsymbol{u}_h\|_2^2\right)\int_0^\tau \|\nabla\boldsymbol{u}_h\|_2^2\|\delta_u\|_2^2.\end{aligned}$$

Hence

$$\boxed{\int_0^\tau T_2 \leq 2\alpha\nu\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{(\alpha\nu)^3}\int_0^\tau \|\nabla\boldsymbol{u}_h\|_2^2\|\delta_u\|_2^2 + Ch^4} \tag{32}$$

Term T_3. Thanks to (25b) and (12), we can write

$$\begin{aligned}T_3 &= (\eta_\pi, \operatorname{div}\delta_u) + \underbrace{(\delta_\pi, \operatorname{div} e_u)}_{=0} - \underbrace{(\delta_\pi, \operatorname{div}\eta_u)}_{=0}\\ &\leq \|\eta_\pi\|_2\|\nabla\delta_u\|_2 \leq \frac{Ch^2}{\alpha\nu}\|\pi\|_{1,2}^2 + \alpha\nu\|\nabla\delta_u\|_2^2.\end{aligned}$$

Thus

$$\boxed{\int_0^\tau T_3 \leq \alpha\nu\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{Ch^2}{\alpha\nu}.} \tag{33}$$

Terms T_4 and T_7. For all $\boldsymbol{v} \in W^{1,2}(\Omega;\mathbb{R}^2)$, $\mathbb{G},\mathbb{H} \in L^2(\Omega;\mathbb{R}^{2\times2})$ it holds: $((\nabla\boldsymbol{v})\mathbb{G},\mathbb{H}) = (\mathbb{H}\mathbb{G}^\top, \nabla\boldsymbol{v})$. Hence, we can write:

$$\begin{aligned}T_4 + T_7 &= -(\mathbb{F}\mathbb{F}^\top - \mathbb{F}_h\mathbb{F}_h^\top, \nabla\delta_u) + (\delta_F\mathbb{F}^\top, \nabla\boldsymbol{u}) - (\delta_F\mathbb{F}_h^\top, \nabla\boldsymbol{u}_h)\\ &= -(\eta_F\mathbb{F}^\top, \nabla\delta_u) - (\Pi_h^F\mathbb{F}\eta_F^\top, \nabla\delta_u) + (\delta_F\eta_F^\top, \nabla\boldsymbol{u}) + (\delta_F\Pi_h^F\mathbb{F}^\top, \nabla\eta_u)\\ &\quad + (\delta_F\delta_F^\top, \nabla\Pi_h^u\boldsymbol{u}) - (\Pi_h^F\mathbb{F}\delta_F^\top, \nabla\delta_u).\end{aligned}$$

Using the regularity $\boldsymbol{u} \in L^2(0,T;W^{3,2}(\Omega;\mathbb{R}^2))$, $\mathbb{F} \in L^\infty(0,T;W^{2,2}(\Omega;\mathbb{R}^{2\times 2}))$ and the embeddings (3), we estimate the resulting terms as follows:

$$\int_0^\tau |(\eta_F \mathbb{F}^\top, \nabla \delta_u)| \le \int_0^\tau \|\mathbb{F}\|_\infty \|\eta_F\|_2 \|\nabla \delta_u\|_2$$
$$\le \alpha\nu \int_0^\tau \|\nabla \delta_u\|_2^2 + \frac{Ch^4}{\alpha\nu} \left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2} \right)^4,$$

$$\int_0^\tau |(\Pi_h^F \mathbb{F} \eta_F^\top, \nabla \delta_u)| \le \int_0^\tau \|\Pi_h^F \mathbb{F}\|_\infty \|\eta_F\|_2 \|\nabla \delta_u\|_2$$
$$\le \alpha\nu \int_0^\tau \|\nabla \delta_u\|_2^2 + \frac{Ch^4}{\alpha\nu} \left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2} \right)^4,$$

$$\int_0^\tau |(\delta_F \eta_F^\top, \nabla \boldsymbol{u})| \le \int_0^\tau \|\delta_F\|_2 \|\eta_F\|_3 \|\nabla \boldsymbol{u}\|_6 \le \int_0^\tau \|\delta_F\|_2^2 \|\boldsymbol{u}\|_{2,2}^2 + Ch^2 \int_0^T \|\mathbb{F}\|_{2,2}^2,$$

$$\int_0^\tau |(\delta_F \Pi_h^F \mathbb{F}^\top, \nabla \eta_u)| \le \int_0^\tau \|\delta_F\|_2 \|\Pi_h^F \mathbb{F}\|_\infty \|\nabla \eta_u\|_2$$
$$\le \left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2} \right)^2 \int_0^\tau \|\delta_F\|_2^2 + Ch^4 \int_0^T \|\boldsymbol{u}\|_{3,2}^2,$$

$$\int_0^\tau |(\delta_F \delta_F^\top, \nabla \Pi_h^u \boldsymbol{u})| \le C \int_0^\tau \|\delta_F\|_2^2 \|\boldsymbol{u}\|_{3,2},$$

$$\int_0^\tau |(\Pi_h^F \mathbb{F} \delta_F^\top, \nabla \delta_u)| \le \int_0^\tau \|\Pi_h^F \mathbb{F}\|_\infty \|\delta_F\|_2 \|\nabla \delta_u\|_2$$
$$\le \alpha\nu \int_0^\tau \|\nabla \delta_u\|_2^2 + \frac{C}{\alpha\nu} \left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2} \right)^2 \int_0^\tau \|\delta_F\|_2^2.$$

In summary we have

$$\boxed{\int_0^\tau T_4 + T_7 \le 3\alpha\nu \int_0^\tau \|\nabla \delta_u\|_2^2 + C \int_0^\tau \left(\frac{1}{\alpha\nu} + \|\boldsymbol{u}\|_{3,2} \right) \|\delta_F\|_2^2 + Ch^2 + \frac{Ch^4}{\alpha\nu}.} \tag{34}$$

Before proceeding to the remaining two terms we recall some basic identities related to the advective term $(\boldsymbol{u} \cdot \nabla)\mathbb{F}$.

Lemma 3. *Let $\boldsymbol{v} \in W^{1,2}(\Omega;\mathbb{R}^2)$ be such that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. Then*

$$\forall \mathbb{G} \in W^{1,2}(\Omega;\mathbb{R}^{2\times 2}) : \ (v_i \mathbb{G}, \partial_i \mathbb{G}) = \frac{1}{2}((\operatorname{div} \boldsymbol{v})\mathbb{G}, \mathbb{G}); \tag{35}$$

$$\forall \mathbb{G}, \mathbb{H} \in W^{1,2}(\Omega;\mathbb{R}^{2\times 2}) : \ (v_i \mathbb{G}, \partial_i \mathbb{H}) = -((\operatorname{div} \boldsymbol{v})\mathbb{G}, \mathbb{H}) - (v_i \mathbb{H}, \partial_i \mathbb{G}). \tag{36}$$

The proof is done by integrating by parts.

Terms T_5 and T_6. We rearrange these terms as follows

$$T_5 + T_6 = (u_i \eta_F, \partial_i \delta_F) + (e_{ui} \Pi_h^F \mathbb{F}, \partial_i \delta_F) + (u_{hi} \delta_F, \partial_i \delta_F) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \delta_F). \tag{37}$$

Using (36) we obtain

$$(u_i \eta_F, \partial_i \delta_F) = -(u_i \delta_F, \partial_i \eta_F), \tag{38}$$

$$(e_{ui} \Pi_h^F \mathbb{F}, \partial_i \delta_F) = ((\operatorname{div} \boldsymbol{u}_h)\Pi_h^F \mathbb{F}, \delta_F) - (e_{ui} \delta_F, \partial_i \Pi_h^F \mathbb{F}). \tag{39}$$

Due to (35), the last two terms in (37) can be rewritten as

$$\begin{aligned} (u_{hi} \delta_F, \partial_i \delta_F) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \delta_F) &= \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\delta_F, \delta_F) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \delta_F) \\ &= \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\Pi_h^F \mathbb{F}, \delta_F). \end{aligned} \tag{40}$$

Equations (37)–(40) and the fact that $\operatorname{div} \boldsymbol{u} = 0$ yield

$$T_5 + T_6 = -(u_i \delta_F, \partial_i \eta_F) - \frac{3}{2}((\operatorname{div} e_u)\Pi_h^F \mathbb{F}, \delta_F) - (e_{ui} \delta_F, \partial_i \Pi_h^F \mathbb{F}). \tag{41}$$

Decomposing $e_u = \eta_u + \delta_u$ and using similar arguments as in the previous paragraphs we can estimate terms on the r.h.s. of (41) as follows

$$\begin{aligned} -\int_0^\tau (u_i \delta_F, \partial_i \eta_F) &\leq \int_0^\tau \|\boldsymbol{u}\|_\infty \|\delta_F\|_2 \|\nabla \eta_F\|_2 \\ &\leq \frac{1}{2}\int_0^\tau \|\boldsymbol{u}\|_{2,2}^2 \|\delta_F\|_2^2 + \frac{1}{2}\int_0^T \|\nabla \eta_F\|_2^2 \\ &\leq \left(\sup_{(0,T)} \|\boldsymbol{u}\|_{2,2}\right)^2 \int_0^T \|\delta_F\|_2^2 + Ch^2 \left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2}\right)^2, \end{aligned} \tag{42}$$

$$\begin{aligned} -\frac{3}{2}\int_0^\tau ((\operatorname{div} e_u)\Pi_h^F \mathbb{F}, \delta_F) &\leq \frac{3}{2}\int_0^\tau \left(\|\nabla \eta_u\|_2 + \|\nabla \delta_u\|_2\right) \|\Pi_h^F \mathbb{F}\|_\infty \|\delta_F\|_2 \\ &\leq Ch^4 \int_0^T \|\boldsymbol{u}\|_{3,2}^2 + \alpha\nu \int_0^\tau \|\nabla \delta_u\|_2^2 + \frac{C}{\alpha\nu}\left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2}\right)^2 \int_0^\tau \|\delta_F\|_2^2, \end{aligned}$$

$$\begin{aligned} -\int_0^\tau (e_{ui} \delta_F, \partial_i \Pi_h^F \mathbb{F}) &\leq \int_0^\tau \left(\|\eta_u\|_4 + \|\delta_u\|_4\right) \|\delta_F\|_2 \|\nabla \Pi_h^F \mathbb{F}\|_4 \\ &\leq Ch^4 \int_0^T \|\boldsymbol{u}\|_{3,2}^2 + \alpha\nu \int_0^T \|\nabla \delta_u\|_2^2 + \frac{C}{\alpha\nu}\left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2}\right)^2 \int_0^T \|\delta_F\|_2^2. \end{aligned} \tag{43}$$

Altogether we have

$$\boxed{\int_0^\tau T_5 + T_6 \leq 2\alpha\nu \int_0^\tau \|\nabla \delta_u\|_2^2 + \frac{C}{\alpha\nu}\int_0^\tau \|\delta_F\|_2^2 + Ch^2 + Ch^4.} \tag{44}$$

Gronwall's inequality for the errors and the end of the proof. Collecting (29), (31), (32), (33), (34), (44) and inserting the result to (27), we obtain

$$\frac{1}{2}\left(\|\delta_u(\tau)\|_2^2 + \|\delta_F(\tau)\|_2^2\right) + (1 - C\alpha)\nu \int_0^\tau \|\nabla\delta_u\|_2^2$$
$$\leq C\int_0^\tau \left(1 + \frac{\|\boldsymbol{u}_h\|_{1,2}^2}{(\alpha\nu)^3}\right)\|\delta_u\|_2^2 + C\int_0^\tau \left(\frac{1}{\alpha\nu} + \|\boldsymbol{u}\|_{3,2}\right)\|\delta_F\|_2^2 + \frac{C}{\alpha\nu}\left(h^2 + h^4\right).$$

Choosing α sufficiently small and using the Gronwall inequality we obtain for $h \in (0, h_0)$

$$\sup_{(0,T)} \|\delta_u\|_2^2 + \sup_{(0,T)} \|\delta_F\|_2^2 + \int_0^T \|\delta_u\|_{1,2}^2 \leq Ch^2. \tag{45}$$

This in accordance with (26) completes the proof of Theorem 1.

Remark 1. *From the proof and the estimate* (19) *it can be deduced that the constant in* (45) *has the form* $C = c_1 \exp c_2$, *where*

$$c_i = c_i\left(\nu, \|\boldsymbol{u}_0\|_2^2 + \|\mathbb{F}_0\|_2^2, \int_0^T \|\boldsymbol{u}\|_{3,2}, \sup_{(0,T)} \|\boldsymbol{u}\|_{2,2}, \sup_{(0,T)} \|\mathbb{F}\|_{2,2}\right) > 0,\ i = 1, 2,$$

are independent of h.

Remark 2. *The question arises, whether the derived error estimate can be improved provided the exact solution is more regular. Essentially, the first order with respect to* h *arises due to* (42), *where* $\|\nabla\eta_F\|_2$ *can only be bounded by* $O(h)$ *due to the piecewise linear approximation of* $\mathbb{F}$. *In view of this, it seems that the result cannot be improved.*

Remark 3. *One can easily incorporate a forcing term* $\boldsymbol{f} \in L^2(0, T; L^2(\Omega; \mathbb{R}^2))$ *on the right hand side of the momentum equation. This would yield the same error estimates.*

5 Error estimates for finite element - finite volume approximation

The main aim of this section is to prove the second part of Theorem 1. The essential difference in the proof of error estimates for the solution of problem (6a), (6b), (8) and (9) is in the term $(\boldsymbol{u} \cdot \nabla)\mathbb{F}$. In particular, the term J_1 introduced in (27) is now substituted by the sum

$$J_1 = T_1 + T_2 + T_3 + T_4 + T_{FV} + T_7,$$

where T_i, $i \in \{1, 2, 3, 4, 7\}$ arise from (30) and

$$T_{FV} = \int_0^T \left(-((\boldsymbol{u} \cdot \nabla)\mathbb{F}, \delta_F) + \sum_{e\in\mathcal{E}_h^0} (\{\boldsymbol{u}_h\mathbb{F}_h\}_u, [\delta_F])_e - \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \delta_F)\right). \tag{46}$$

In spite of the weaker bound (15) for η_F, the estimates of T_i, $i \in \{1,2,3,4,7\}$ still yield terms of order at least h^2. It is therefore sufficient to estimate the error of T_{FV}.

Proof. (Error estimate for term $(\boldsymbol{u}\cdot\nabla)\mathbb{F}$)
Let us first denote by F one component of the tensor $\mathbb{F}$, similarly by G_h one component of $\mathbb{G}_h$. Then we can write

$$((\boldsymbol{u}\cdot\nabla)F, G_h) = \sum_{T\in\mathcal{T}_h}\int_T \operatorname{div}(\boldsymbol{u}\mathrm{F})\mathrm{G}_h\,d\boldsymbol{x} = \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{F}\;[\mathrm{G}_h]\;dS$$
$$= \sum_{e\in\mathcal{E}_h^0}\int_e \{\boldsymbol{u}F\}[\mathrm{G}_h]\;dS.$$

For the approximate solution it holds

$$\int_e \{\boldsymbol{u}_h\mathrm{F}_h\}_u\;[\mathrm{G}_h]\;dS = \int_e \{\boldsymbol{u}_h\mathrm{F}_h\}[\mathrm{G}_h]\;dS + \int_e \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}[\mathrm{F}_h][\mathrm{G}_h]\;dS \quad \forall\;\mathrm{G}_h\in Z_h.$$

Thus, we can rewrite (46) as follows:

$$T_{FV} = \int_0^T \sum_{e\in\mathcal{E}_h^0}\int_e \{(\boldsymbol{u}_h-\boldsymbol{u})\mathrm{F}\}[\delta_F]\;dS\;dt + \int_0^T \sum_{e\in\mathcal{E}_h^0}\int_e \{\boldsymbol{u}_h(\mathrm{F}_h-\mathrm{F})\}[\delta_F]\;dS\;dt$$
$$+\int_0^T \sum_{e\in\mathcal{E}_h^0}\int_e \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}[\mathrm{F}_h-\mathrm{F}][\delta_F]\;dS\;dt - \frac{1}{2}\int_0^T\int_\Omega \operatorname{div}\boldsymbol{u}_h\mathrm{F}_h\delta_F = \sum_{i=1}^4 I_i.$$

In what follows, we will estimate these terms separately.

Term I_1.

$$
\begin{aligned}
I_1 &= \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \{(\boldsymbol{u}_h - \boldsymbol{u})\mathrm{F}\}[\delta_F]\ dS\ dt = \int_0^T \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\boldsymbol{u}_h - \boldsymbol{u})\cdot \boldsymbol{n}\ \mathrm{F}\ \delta_F\ dS\ dt \\
&= \int_0^T \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}\big((\boldsymbol{u}_h - \boldsymbol{u})\mathrm{F}\big)\delta_F\ d\boldsymbol{x}\ dt \\
&= -\int_0^T \int_\Omega \operatorname{div}\delta_u\ \mathrm{F}\delta_F\ d\boldsymbol{x}\ dt - \int_0^T \int_\Omega \operatorname{div}\eta_u\ \mathrm{F}\delta_F\ d\boldsymbol{x}\ dt \\
&- \int_0^T \int_\Omega (\delta_u\cdot\nabla)\mathrm{F}\ \delta_F\ d\boldsymbol{x}\ dt - \int_0^T \int_\Omega (\eta_u\cdot\nabla)\mathrm{F}\ \delta_F\ d\boldsymbol{x}\ dt \\
&\le \int_0^T c_1\|\nabla\delta_u\|_{2,\Omega}\|\mathrm{F}\|_{2,2,\Omega}\|\delta_F\|_{2,\Omega} + c_2\|\nabla\eta_u\|_{2,\Omega}\|\mathrm{F}\|_{2,2,\Omega}\|\delta_F\|_{2,\Omega} \\
&+ \int_0^T c_3\|\delta_u\|_{4,\Omega}\|\nabla\mathrm{F}\|_{4,\Omega}\|\delta_F\|_{2,\Omega} + c_4\|\eta_u\|_{4,\Omega}\|\nabla\mathrm{F}\|_{4,\Omega}\|\delta_F\|_{2,\Omega} \\
&\le \alpha\nu \int_0^T \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha\nu}\Big(\sup_{(0,T)}\|\mathrm{F}\|_{2,2}\Big)^2 \int_0^T \|\delta_F\|_2^2 \\
&+ C\int_0^T \|\delta_F\|_2^2 + Ch^4\Big(\sup_{(0,T)}\|\mathrm{F}\|_{2,2}\Big)^2 \int_0^T \|\boldsymbol{u}\|_{3,2}^2.
\end{aligned}
$$

Terms $I_2 + I_4$.

$$
\begin{aligned}
I_2 &= \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \{\boldsymbol{u}_h(\mathrm{F}_h - \mathrm{F})\}[\delta_F]\ dS\ dt \\
&= -\int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \{\boldsymbol{u}_h\delta_F\}[\delta_F]\ dS\ dt - \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \{\boldsymbol{u}_h\eta_F\}[\delta_F]\ dS\ dt = A_1 + A_2
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= -\int_0^T \int_\Omega \frac{1}{2}\operatorname{div}\boldsymbol{u}_h\delta_F^2\ d\boldsymbol{x}\ dt = \int_0^T \int_\Omega \frac{1}{2}\operatorname{div} e_u\Pi_h^0\mathrm{F}\delta_F\ d\boldsymbol{x}\ dt - I_4 \\
&= \int_0^T \int_\Omega \frac{1}{2}\operatorname{div}\eta_u\Pi_h^0\mathrm{F}\delta_F\ d\boldsymbol{x}\ dt + \int_0^T \int_\Omega \frac{1}{2}\operatorname{div}\delta_u\Pi_h^0\mathrm{F}\delta_F\ d\boldsymbol{x}\ dt - I_4 \\
&\le C\int_0^T \|\eta_u\|_{1,2}\|\delta_F\|_2 + C\int_0^T \|\nabla\delta_u\|_2\|\delta_F\|_2 - I_4 \\
&\le Ch^2 + \alpha\nu\int_0^T \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha\nu}\int_0^T \|\delta_F\|_2^2 - I_4.
\end{aligned}
$$

$$
\begin{aligned}
A_2 &\leq \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}\{|\eta_F|\}|[\delta_F]|\ dS\ dt \\
&\leq \int_0^T \sum_{e\in\mathcal{E}_h^0} \|c_e^{1/2}[\delta_F]\|_{2,e}\|c_e^{1/2}\|_{4,e}\|\{|\eta_F|\}\|_{4,e} \\
&\leq \alpha \int_0^T \sum_{e\in\mathcal{E}_h^0} \|c_e^{1/2}[\delta_F]\|_{2,e}^2 + \frac{C}{\alpha}h^{3/2}\left(\sup_{(0,T)}\|\mathbb{F}\|_{2,2}\right)^2 \int_0^T \|\nabla\boldsymbol{u}_h\|_2^2\ dt,
\end{aligned}
$$

where η_F was estimated using Lemma 1 and $c_e := \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}$. Further, we have

$$
\begin{aligned}
I_3 &= -\int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}[\eta_F][\delta_F]\ dS\ dt - \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}[\delta_F][\delta_F]\ dS\ dt \\
&= B_1 + B_2,
\end{aligned}
$$

where B_1 can be estimated in the same way as A_2 and

$$
B_2 = -\int_0^T \sum_{e\in\mathcal{E}_h^0} \|c_e^{1/2}[\delta_F]\|_{2,e}^2,
$$

hence

$$
I_3 \leq (\alpha-1)\int_0^T \sum_{e\in\mathcal{E}_h^0} \|c_e^{1/2}[\delta_F]\|_{2,e}^2 + \frac{C}{\alpha}h^{3/2}\left(\sup_{(0,T)}\|\mathbb{F}\|_{2,2}\right)^2 \int_0^T \|\nabla\boldsymbol{u}_h\|_2^2\ dt.
$$

Note that I_4 is already subtracted in A_1, thus we can finally conclude:

$$
\boxed{T_{FV} \leq C\alpha\nu\int_0^T \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha\nu}\int_0^T \|\delta_F\|_2^2 + \frac{C}{\alpha}h^{3/2} + Ch^2 + Ch^4}. \tag{47}
$$

Substituting (47) in (27) and taking into account that the rest terms T_1, T_2, T_3, T_4, T_5 and T_7 are at least of order h^2 conclude the proof of the second part of Theorem 1. □

6 Multiplicative trace inequality

As mentioned above we now proceed with the proof of Lemma 1. We first state the following auxiliary result.

Lemma 4. *There exists a constant $c > 0$ such that for any simplex T in $\mathbb{R}^d$, $d \in \{2,3\}$, with $h := \operatorname{diam} T$, and $v \in H^1(T)$, we have*

$$
\|v\|_{L^4(\partial T)}^4 \leq c\left[4\|v\|_{L^6(T)}^3|v|_{H^1(T)} + \frac{d}{h_T}\|v\|_{L^4(T)}^4\right]. \tag{48}
$$

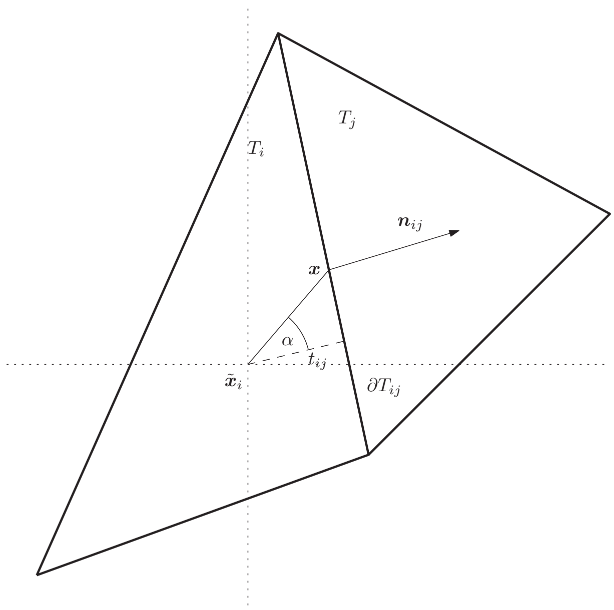

Figure 1: Example of a triangulation with a notation used in Lemma 4.

Proof. The proof of Lemma 4 is analogous to the proof of Lemma 3.1 in [11]. We prove Lemma 4 for the sake of consistency. Let us denote by $\tilde{\boldsymbol{x}}_i$ the center of the largest d-dimensional ball inscribed in $T = T_i$. W.l.o.g. put $\tilde{\boldsymbol{x}}_i$ to the origin of the coordinate system, see Figure 1. We start with the relation

$$\int_{\partial T} v^4 \; \boldsymbol{x} \cdot \boldsymbol{n} \; dS = \int_T \nabla \cdot (v^4 \; \boldsymbol{x}) \; d\boldsymbol{x} \qquad v \in H^1(T).$$

Let us denote $\boldsymbol{n}_{ij}$ the outer normal to T_i on $\partial T_{ij} = \partial T_i \cap \partial T_j$, where T_j is a neighbor of T_i, i.e.

$$j \in S(i) \equiv \{j; T_j \cap T_i \neq 0\}.$$

We have

$$\boldsymbol{x} \cdot \boldsymbol{n}_{ij} = |\boldsymbol{x}||\boldsymbol{n}_{ij}| \cos\alpha = |\boldsymbol{x}| \cos\alpha = t_{ij}, \quad \boldsymbol{x} \in \partial T_{ij}, \; j \in S(i),$$

where t_{ij} is the distance of $\tilde{\boldsymbol{x}}_i$ from ∂T_{ij}. Clearly,

$$t_{ij} \geq \rho_{T_i} \qquad j \in S(i),$$

where ρ_{T_i} is the radius of the inscribed ball.

Now, we have the following estimate

$$\int_{\partial T_i} v^4 \, \boldsymbol{x} \cdot \boldsymbol{n} \, dS = \sum_{j \in S(i)} \int_{\partial T_i \cap \partial T_j} v^4 \, \boldsymbol{x} \cdot \boldsymbol{n}_{ij} \, dS = \sum_{j \in S(i)} t_{ij} \int_{\partial T_i \cap \partial T_j} v^4 \, dS$$
$$\geq \rho_{T_i} \sum_{j \in S(i)} \int_{\partial T_i \cap \partial T_j} v^4 \, dS = \rho_{T_i} \|v\|^4_{L^4(\partial T_i)}. \tag{49}$$

Further, it holds

$$\begin{aligned}
\int_{T_i} \nabla \cdot (v^4 \boldsymbol{x}) \, d\boldsymbol{x} &= \int_{T_i} v^4 \, \nabla \cdot \boldsymbol{x} + \boldsymbol{x} \cdot \nabla v^4 \, d\boldsymbol{x} = d \int_{T_i} v^4 \, d\boldsymbol{x} + 4 \int_{T_i} v^3 \, \boldsymbol{x} \cdot \nabla v \, d\boldsymbol{x} \\
&\leq d\|v\|^4_{L^4(T_i)} + 4 \int_{T_i} |v^3 \, \boldsymbol{x} \cdot \nabla v| \, d\boldsymbol{x} \\
&\leq d\|v\|^4_{L^4(T_i)} + 4 \sup_{\boldsymbol{x} \in T_i} |\boldsymbol{x}| \int_{T_i} |v|^3 . |\nabla v| \, d\boldsymbol{x} \\
&\leq d\|v\|^4_{L^4(T_i)} + 4 h_{T_i} \|v\|^3_{L^6(T_i)} |v|_{H^1(T_i)}. \qquad (50)
\end{aligned}$$

From (49) and (50) we have

$$\begin{aligned}
\rho_{T_i} \|v\|^4_{L^4(\partial T_i)} \leq \int_{\partial T_i} v^4 \, \boldsymbol{x} \cdot \boldsymbol{n} \, dS = \int_{T_i} \nabla \cdot (v^4 \, \boldsymbol{x}) \, d\boldsymbol{x} \\
\leq d\|v\|^4_{L^4(T_i)} + 4 h_{T_i} \|v\|^3_{L^6(T_i)} |v|_{H^1(T_i)},
\end{aligned}$$

which finally yields

$$\begin{aligned}
\|v\|^4_{L^4(\partial T_i)} &\leq \frac{d}{\rho_{T_i}} \|v\|^4_{L^4(T_i)} + 4 \frac{h_{T_i}}{\rho_{T_i}} \|v\|^3_{L^6(T_i)} |v|_{H^1(T_i)} \\
&\leq c \left(4 \|v\|^3_{L^6(T_i)} |v|_{H^1(T_i)} + \frac{d}{h_{T_i}} \|v\|^4_{L^4(\partial T_i)} \right).
\end{aligned}$$

□

Having proven the desired property (48) we can finally present the proof of Lemma 1.

Proof of Lemma 1. Set $v = F - \Pi^0_h F$, where $F \in W^{2,2}(\Omega)$. Then, due to Lemma 4 it holds

$$\begin{aligned}
\|F - \Pi^0_h F\|^4_{L^4(\partial T)} = \|\eta_F\|^4_{L^4(\partial T)} &\leq c_1 \left[4h^3 \|F\|^3_{W^{1,6}(T)} |F|_{H^1(T)} + \frac{2}{h} h^4 \|F\|^4_{W^{1,4}(T)} \right] \\
&\leq c_2 \left[h^3 \|F\|^4_{W^{2,2}(T)} + h^3 \|F\|^4_{W^{2,2}(T)} \right] \\
&\leq c_3 h^3 \|F\|^4_{W^{2,2}(T)}.
\end{aligned}$$

This implies

$$\sum_{e \in \mathcal{E}^0_h} \|F - \Pi^0_h F\|_{4,e} \leq c h^{3/4} \|F\|_{2,2}.$$

□

7 Numerical experiments

In order to demonstrate validity of our theoretical error estimates we perform experimental error analysis. Let us consider the flow in a rectangular domain $\Omega = (0,1)^2$ driven by the boundary condition

$$\boldsymbol{u} = \begin{cases} (4x(x-1), 0) & \text{if } y = 1, \\ (0,0) & \text{otherwise,} \end{cases}$$

with the initial conditions $\boldsymbol{u}_0 = (0,0)$, $\mathbb{F}_0 = \mathbb{I}$ and the viscosity $\nu = 1$. We have compared the following three methods:

a) finite element method (FEM) for velocity, pressure and viscoelastic stress,

b) FEM for velocity and pressure, dual finite volume method (FVM) for viscoelastic stress,

c) finite difference method (FDM) for velocity and pressure, FVM for viscoelastic stress.

The case a) is a standard finite element method based on the Taylor-Hood finite elements of the fluid part (piecewise quadratic velocity and piecewise linear pressure) combined with the piecewise linear approximation of the deformation gradient $\mathbb{F}$. In the case b) the deformation gradient was approximated by piecewise constants on dual elements, that arise by connecting the barycenters of primary elements with the edge midpoints. In the case c) we have combined the finite difference approximation of the fluid equations with the finite volume approximation of the deformation gradient $\mathbb{F}$. The latter method can be considered to be another variant of a combined finite element and finite volume method when a regular rectangular grid is used. To keep the paper self-consistent let us describe in what follows the combined finite difference/finite volume method in some more details: we first discretize the domain by dividing it into regular rectangular mesh cells and then apply the staggered finite difference approximation for the fluid equations. It means that the discretization nodes for velocities u_1 and u_2 are the midpoints of edges in x or $y-$direction, respectively. Nodes for pressure and deformation gradient are at the cell centers. We would like to point out that this finite difference approximation of the Navier-Stokes part is the well-known MAC (Marker and Cell) method frequently used, e.g., in engineering in order to approximate incompressible flows. As reported in [16] this method was introduced by the Los Alamos group: B.J. Daly, F.H. Harlow, J.P. Shannon and J.E. Welch in their 1965 report [9], but an idea of this discretization has already existed in the Russian paper by V.I. Lebedev in 1964, cf. [22].

Further, we treat the viscous term $\Delta\boldsymbol{u}$ and the viscoelastic term $\operatorname{div}(\mathbb{F}\mathbb{F}^\top)$ implicitly in time, whereas the convective part $\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})$ is approximated in an explicit manner. To enforce the incompressibility condition, we use the Chorin projection method. The evolution equation of the deformation gradient $\mathbb{F}$ is

	e_u		∇e_u		e_π		e_F	
h	$L^\infty(L^2)$	EOC	$L^2(L^2)$	EOC	$L^2(L^2)$	EOC	$L^\infty(L^2)$	EOC
1/8	2.08e-02	2.11	1.77e+02	1.00	4.19e+02	0.96	1.37e-01	0.73
1/16	4.80e-03	2.02	8.85e+01	1.00	2.14e+02	0.99	8.21e-02	0.89
1/32	1.18e-03	2.00	4.42e+01	0.99	1.07e+02	1.00	4.41e-02	0.94
1/64	2.95e-04	2.00	2.21e+01	0.99	5.36e+01	1.00	2.29e-02	0.95
1/128	7.37e-05		1.10e+01		2.68e+01		1.17e-02	

(a) FEM

	e_u		∇e_u		e_π		e_F	
h	$L^\infty(L^2)$	EOC	$L^2(L^2)$	EOC	$L^2(L^2)$	EOC	$L^\infty(L^2)$	EOC
1/8	2.07e-02	2.00	1.77e+02	1.00	4.73e+02	0.88	1.94e-01	0.73
1/16	5.16e-03	1.40	8.87e+01	0.98	2.55e+02	0.94	1.16e-01	0.88
1/32	1.94e-03	1.34	4.47e+01	0.98	1.33e+02	0.96	6.33e-02	0.90
1/64	7.67e-04	1.35	2.25e+01	0.98	6.83e+01	0.97	3.37e-02	0.91
1/128	3.00e-04		1.13e+01		3.47e+01		1.78e-02	

(b) FEM/dual FVM

	e_u		∇e_u		e_π		e_F	
h	$L^\infty(L^2)$	EOC	$L^2(L^2)$	EOC	$L^2(L^2)$	EOC	$L^\infty(L^2)$	EOC
1/8	4.22e-02	0.86	3.61e-01	0.86	2.94e-01	1.27	1.58e-01	0.88
1/16	2.32e-02	1.25	1.98e-01	1.04	1.22e-01	1.54	8.56e-02	1.07
1/32	9.73e-03	1.82	9.62e-02	1.22	4.22e-02	1.76	4.09e-02	1.23
1/64	2.77e-03	2.25	4.14e-02	1.59	1.25e-02	2.03	1.75e-02	1.59
1/128	5.81e-04		1.38e-02		3.06e-03		5.80e-03	

(c) FDM/FVM

Table 1: Error norms and experimental order of convergence for driven cavity problem.

approximated in the explicit way using the upwind finite volume scheme. More precisely, we have

$$\mathbb{F}_i^{k+1} = \mathbb{F}_i^k - \frac{\Delta t}{|T_i|} \sum_{j \in S(i)} H_{i\,j}(\mathbb{F}_i^k, \mathbb{F}_j^k) + (\nabla_h \boldsymbol{u}^k)_i \mathbb{F}_i^k,$$

where $|T_i|$ is the volume of an arbitrary (rectangular) mesh cell T_i, Δt is the time step, $\mathbb{F}_i^k, \mathbb{F}_i^{k+1}$ are the piecewise constant approximations of $\mathbb{F}$ on T_i at time t_k, t_{k+1}, respectively, and $(\nabla_h \boldsymbol{u}^k)_i$ is the piecewise constant approximation of $\nabla \boldsymbol{u}^k$ on T_i. Further, $H_{i\,j}(\mathbb{F}_i^k, \mathbb{F}_j^k)$ is the appropriate numerical flux function, that approximates the cell interface integral $\int_{T_i \cap T_j} \mathbb{F}^k \boldsymbol{u}^k \cdot \boldsymbol{n} dS$, where $\boldsymbol{n}$ is the outer normal, $j \in S(i)$, and $S(i)$ is the index set of all neighbours corresponding to the cell T_i. In our computations we have applied the upwind numerical flux for $H_{i\,j}$. In order to obtain stable finite difference-finite volume scheme, we have to subiterate between the finite difference approximation of the fluid part and the finite volume approximation of the deformation gradient. Finally, we have the combined scheme described in Algorithm 1.

Algorithm 1

Given $\boldsymbol{u}^k, p^k, \mathbb{F}^k$, set $\boldsymbol{u}^{k,0} = \boldsymbol{u}^k, \mathbb{F}^{k,0} = \mathbb{F}^k, \pi^{k,0} = \pi^k$.
for $\ell = 0, 1, \cdots$ **do**
  solve the viscoelastic equation:
  $\mathbb{F}_i^{k,\ell+1} = \mathbb{F}_i^{k,0} - \frac{\Delta t}{|T_i|} \sum_{j \in S(i)} H_{i\,j}(\mathbb{F}_i^{k,\ell}, \mathbb{F}_j^{k,\ell}) + \Delta t (\nabla_h \boldsymbol{u}^{k,\ell})_i \mathbb{F}_i^{k,\ell}$
  solve the fluid part in two steps:
  $(\mathbb{I} - \Delta t \Delta_h) \boldsymbol{u}^* = \boldsymbol{u}^{k,0} + \Delta t \nabla_h \cdot (-\boldsymbol{u}^{k,l} \otimes \boldsymbol{u}^{k,l} + (\mathbb{F}\mathbb{F}^\top)^{k,\ell+1})$
  $\frac{1}{\Delta t}(\mathbf{u}^{k,\ell+1} - \mathbf{u}^*) = -\nabla_h \pi^{k,\ell+1}$
  if $(\|\mathbf{u}^{k,\ell+1} - \mathbf{u}^{k,\ell}\| \leq \xi \|\mathbf{u}^{k,\ell}\|$, $\|\pi^{k,\ell+1} - \pi^{k,\ell}\| \leq \xi \|\pi^{k,\ell}\|$, and $\|\mathbb{F}^{k,\ell+1} - \mathbb{F}^{k,\ell}\| \leq \xi \|\mathbb{F}^{k,\ell}\|$ for enough small ξ) **then**
    break
  end if
end for
Update solution: $\mathbf{u}^{k+1} = \mathbf{u}^{k,\ell+1}, \pi^{k+1} = \pi^{k,\ell+1}, \mathbb{F}^{k+1} = \mathbb{F}^{k,\ell+1}$.

In order to compute experimental error orders we use a series of regular triangular meshes consisting of 8 to 256 elements in each direction (methods a) and b)) as well as regular rectangular meshes with 4 to 128 elements (method c)). Calculations were run for the time interval $(0, 0.2)$ with a fixed timestep 0.001 that satisfies the CFL stability condition for the finest mesh. Our experimental error analysis, that is presented in Table 1, yields the results comparable with the theoretical results, cf. Theorem 1. Indeed, simulations using the finite element method a) confirm the first order error for $\nabla \boldsymbol{u}$ in $L^2(0, T; L^2(\Omega))$ and for $\mathbb{F}$ in $L^\infty(0, T; L^2(\Omega))$. Moreover, we also show that the pressure π is approximated in $L^2(0, T; L^2(\Omega))$ with the first order and the velocity $\boldsymbol{u}$ with the second order error in $L^\infty(0, T; L^2(\Omega))$. The same experimental orders of convergence are obtained by the finite element/dual finite volume method b).

Note however that the we have a slightly worse convergence for the velocity measured in $L^\infty(0,T;L^2(\Omega))$ than in the method a), though still superlinear. These experimental results also indicate that our theoretical error estimate (21) may be suboptimal for $\boldsymbol{u}$. Furthermore, the experimental error analysis of the combined finite difference/finite volume method shows the second order error for $\boldsymbol{u}$ in $L^\infty(0,T;L^2(\Omega))$ as well as for pressure π in $L^2(0,T;L^2(\Omega))$ and the superlinear convergence for $\nabla\boldsymbol{u}$ in $L^2(0,T;L^2(\Omega))$ and for $\mathbb{F}$ in $L^\infty(0,T;L^2(\Omega))$, see Table 1.

Remark 4. *History of the error analysis of the MAC scheme for the Navier-Stokes equations, or its linear version the Stokes equations, is very interesting. Although the method has been used successfully since 1965, its theoretical numerical analysis was not carried out until 1992 by Nicolaides and his collaborators [26, 28]. They reinterpreted the MAC scheme as a finite volume approximation of the velocity-vorticity equations on the dual (co-volume) meshes and proved the first order error estimates for pressure and velocity in the standard L^2 norms. Moreover, Nicolaides and Wu have reported in [27] that the experimental order of convergence using regular grids is about 2 for the velocity and pressure and about 1.5 for the gradients of velocity. We would like to point out that our numerical experiment, presented in Tab. 1 exactly confirms these results. There has been quite a number of papers where the convergence order of the MAC scheme was analyzed from the theoretical point of view. For example, in 1996 Girault and Lopez [16] showed that the finite difference equations of the MAC method can be derived by combining a velocity-vorticity mixed finite element method of degree one with an adequate quadrature formula. This paper confirms that the MAC method, applied to the steady-state incompressible Navier-Stokes equations, satisfies the error estimates of order one. Later, in 2008 Kanschat [20] showed that the MAC scheme is algebraically equivalent to the first order local discontinuous Galerkin method with a proper quadrature and also obtained first order convergence of the scheme. Further, in 2014 Herbin et al. [18] have shown the convergence of the MAC scheme for the Navier-Stokes equations on non-uniform grids. In [12] a related method, the so-called colocated finite volume scheme for the Stokes problem is studied and its first order convergence is proven theoretically. Here it is also reported that the second order convergence for velocity is obtained in numerical experiments.*

Nevertheless, until very recently the superconvergence of the MAC scheme has not been confirmed theoretically. In 2014 Li and Sun [23] succeeded to show the second order convergence of velocity and pressure measured in the L^2 norms even for irregular rectangular grids. The authors do not reinterpret the MAC method as the finite volume or finite element scheme, but work directly in the finite difference framework. Careful and elegant analysis of various sources of errors shows that although the local truncation errors are only first order, a suitable cancelation of local errors yields after summation the second order global errors for both velocity and pressure. Our numerical experiments presented in Table 1 for the combined finite difference/finite volume method confirm theoretical results of Li and Sun. As a consequence the velocity gradients $\nabla\boldsymbol{u}$ as well *as the deformation gradient $\mathbb{F}$ converge superlinearly with the order 1.5.*

In what follows we present graphs of the solution at the final time $T = 0.2$. In Figure 2 we plot the streamlines, pressure and velocity components. Figure 3 presents all components of the deformation gradient $\mathbb{F}$. Further, Figure 4 illustrates time evolution of the kinetic energy $\frac{1}{2}\boldsymbol{u}^2$ and the L^∞-norm of trace of the stress tensor $\mathbb{F}\mathbb{F}^\top$. Although the kinetic energy is bounded, we see clearly that L^∞-norm of the stress tensor is not bounded.

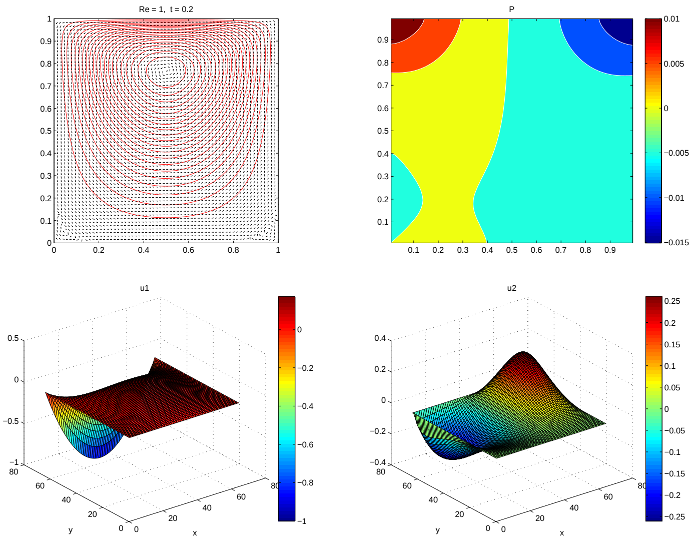

Figure 2: Graph of the solution at the final time $T = 0.2$: streamline (top left), pressure (top right) and velocity components u_1 and u_2 (bottom).

Conclusions

In this paper we have studied a particular variant of the Oldroyd-B visocelastic model having the limiting relaxation time going to infinity. Assuming global in time existence of enough regular weak solution we have studied the error estimates of a standard finite element method A and of a combined finite element/dual finite volume method B. Main theoretical result Theorem 1 shows the first order error estimates for a standard finite element discretization using the Taylor-Hood elements for the velocities and pressures and the linear elements for the deformation gradient $\mathbb{F}$. These results are also confirmed by a number of numerical experiments, a representative choice is presented in this paper. Furthermore, our theoretical results indicate the errors of order $\mathcal{O}(h^{3/4})$ for the combined finite element/dual finite volume method. Numerical experiments for the above two methods confirm our theoretical results. Moreover, using the finite difference/finite volume approximation on a rectangular grid, we get even the second order convergence of the velocity $\boldsymbol{u}$ and of pressure π, which is a consequence of the superconvergence of the special finite difference approximation of the Navier-Stokes equations. Using the recent theoretical results of Li and Sun [23] it would be interesting to study theoretically also the convergence of the combined finite difference/finite volume scheme for our viscoelastic model in future.

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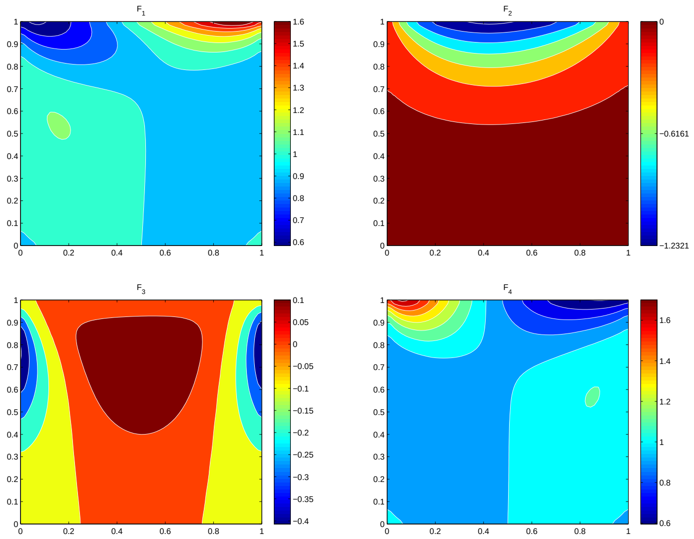

Figure 3: Graph of the solution at the final time $T = 0.2$; four components of the deformation gradient $\mathbb{F}_{ij}$, $i, j = 1, 2$, from the top left to the bottom right.

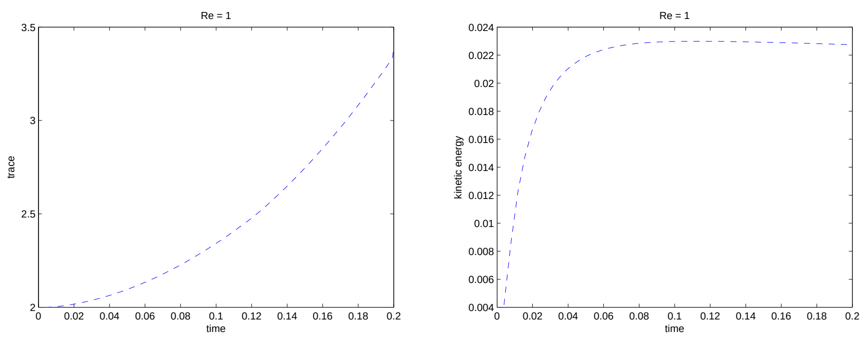

Figure 4: Time evolution of the maximum trace of the elastic stress tensor $\mathbb{T}^e$ (left) and of the kinetic energy (right).