On a mapping property of the Oseen operator with rotation

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The Oseen problem arises as the linearization of a steady-state Navier-Stokes flow past a translating body. If the body, in addition to the translational motion, is also rotating, the corresponding linearization of the equations of motion, written in a frame attached to the body, yields the Oseen system with extra terms in the momentum equation due to the rotation. These rotation terms are of lower order than the other terms in the equation. Consequently, a regularizing effect in terms of improved summability at infinity of solutions to the system is expected. In this paper, such a regularizing effect is identified by analyzing the mapping properties of the whole space Oseen operator with rotation. As an application, an asymptotic expansion of a steady-state, linearized Navier-Stokes flow past a rotating and translating body is established.

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1 Introduction

Consider a rigid body moving in a three-dimensional Navier-Stokes liquid. Assume the velocity of its center of mass and its angular velocity are both constant and directed along the same axis, which, without loss of generality, is taken to be the e_3-axis. Written

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in a frame attached the body, the linearized, steady-state equations of motion then read, in an appropriate non-dimensional form,

$$\begin{cases} -\Delta v - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\big) = f - \nabla p & \text{in } \Omega,\\ \operatorname{div} v = 0 & \text{in } \Omega,\\ v = \mathrm{e}_3 + \mathcal{T}\,\mathrm{e}_3\wedge x & \text{on } \partial\Omega,\\ \lim_{|x|\to\infty} v(x) = 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is an exterior domain, $v : \Omega \to \mathbb{R}^3$ the Eulerian velocity field of the liquid, $p : \Omega \to \mathbb{R}$ the corresponding pressure, $f : \Omega \to \mathbb{R}^3$ an external force, $\mathcal{R} > 0$ a dimensionless constant, and $\mathcal{T} > 0$ the magnitude of the dimensionless angular velocity.

The linear operator A on the left-hand side of $(1.1)_1$ can be written as

$$\mathrm{A}v = \mathrm{A}_{\mathrm{Oseen}}v - \mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\big), \tag{1.2}$$

where $\mathrm{A}_{\mathrm{Oseen}}$ is the classical Oseen operator

$$\mathrm{A}_{\mathrm{Oseen}}v := -\Delta v - \mathcal{R}\partial_3 v. \tag{1.3}$$

The terms $\mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\big)$ occur as a consequence of writing the Oseen system in a rotating frame of reference, namely the frame attached to the body. We shall refer to them as rotation terms. The operator A is referred to as the Oseen operator with rotation. Since $\mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\big)$ contains a zeroth order term, with respect to differentiability, one would expect a solution v of (1.1) to posses better summability at infinity in the case $\mathcal{T} > 0$ compared to the classical Oseen problem when $\mathcal{T} = 0$. In this paper, such improved summability of a solution v is identified and characterized by means of mapping properties of A in a setting of solenoidal vector-fields in $L_q(\mathbb{R}^3)^3$. It is already known, see [1] and [4], that A maps the Banach space, $q \in (1,2)$,

$$\begin{aligned} &\mathrm{X}_{q,\sigma} := \{v \in L_{1,loc}(\mathbb{R}^3)^3 \mid \operatorname{div} v = 0,\ \|v\|_{\mathrm{X}_{q,\sigma}} < \infty\},\\ &\|v\|_{\mathrm{X}_{q,\sigma}} := \|\nabla^2 v\|_q + \|\partial_3 v\|_q + \|\mathrm{e}_3\wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\|_q + \|\nabla v\|_{\frac{4q}{4-q}} + \|v\|_{\frac{2q}{2-q}}, \end{aligned} \tag{1.4}$$

homeomorphically onto the Banach space

$$L_{q,\sigma} := \{f \in L_q(\mathbb{R}^3)^3 \mid \operatorname{div} f = 0\} \tag{1.5}$$

of solenoidal L_q-fields, that is,

$$\mathrm{A} : \mathrm{X}_{q,\sigma} \to L_{q,\sigma} \tag{1.6}$$

is a homeomorphism. We shall compare this mapping property to the classical Oseen operator. Put, $q \in (1,2)$,

$$\begin{aligned} &\mathrm{X}^{\mathrm{Oseen}}_{q,\sigma} := \{v \in L_{1,loc}(\mathbb{R}^3)^3 \mid \operatorname{div} v = 0,\ \|v\|_{\mathrm{X}^{\mathrm{Oseen}}_{q,\sigma}} < \infty\},\\ &\|v\|_{\mathrm{X}^{\mathrm{Oseen}}_{q,\sigma}} := \|\nabla^2 v\|_q + \|\partial_3 v\|_q + \|\nabla v\|_{\frac{4q}{4-q}} + \|v\|_{\frac{2q}{2-q}}. \end{aligned}$$

By well-known results, see for example [2, Chapter VII],

$$\mathrm{A}_{\mathrm{Oseen}}:\mathrm{X}^{\mathrm{Oseen}}_{q,\sigma}\to L_{q,\sigma}$$

is a homeomorphism. Comparing $\mathrm{X}_{q,\sigma}$ to $\mathrm{X}^{\mathrm{Oseen}}_{q,\sigma}$, we see that the summability of the fields in the two spaces is the same, namely $v\in L_{2q/2-q}(\mathbb{R}^3)^3$. Apparently, the lower order rotation terms $\mathrm{e}_3\wedge x\cdot\nabla v-\mathrm{e}_3\wedge v$ in (1.2) do not have a regularizing effect resulting in better summability at infinity of the fields in $\mathrm{X}_{q,\sigma}$ over those in $\mathrm{X}^{\mathrm{Oseen}}_{q,\sigma}$. The reason for the apparent lack of such a regularizing effect is that kernel of the operator $\mathrm{T}^{rot}v:=\mathrm{e}_3\wedge x\cdot\nabla v-\mathrm{e}_3\wedge v$, considered as an operator from $\mathrm{X}_{q,\sigma}$ into $L_{q,\sigma}$, is non-trivial. To identify this kernel, we introduce the rotation matrix

$$Q(t):=\begin{pmatrix}\cos(\mathcal{T}t) & -\sin(\mathcal{T}t) & 0\\ \sin(\mathcal{T}t) & \cos(\mathcal{T}t) & 0\\ 0 & 0 & 1\end{pmatrix},\quad t\in\mathbb{R},$$

and observe that

$$\partial_t\Big[Q(t)v\big(Q(t)^\top x\big)\Big]=-\mathcal{T}Q(t)\big[\,\mathrm{e}_3\wedge y\cdot\nabla v(y)-\mathrm{e}_3\wedge v(y)\big]_{|y=Q(t)^\top x}. \tag{1.7}$$

From (1.7) we deduce that $\mathrm{T}^{rot}v=0$ if and only if v is invariant with respect to conjugation by $Q(t)$ for all $t\in\mathbb{R}$. Consequently, $\ker\mathrm{T}^{rot}$ coincides with the space

$$\mathrm{X}^{sym}_{q,\sigma}:=\{v\in\mathrm{X}_{q,\sigma}\mid\forall t\in\mathbb{R}:\ Q(t)v\big(Q(t)^\top\cdot\big)=v(\cdot)\}. \tag{1.8}$$

The main purpose of this paper is to factor out $\mathrm{X}^{sym}_{q,\sigma}$ in $\mathrm{X}_{q,\sigma}$, and analyze the operator A in the resulting complement space. In particular, it is the goal to identify the regularizing effect of the rotation terms in A in terms of improved summability at infinity of the fields in this complement space. As the first step, a decomposition of A into a direct sum $\mathrm{A}=\mathrm{A}_{\mathrm{Oseen}}\oplus\mathrm{A}_\perp$ of two homeomorphisms $\mathrm{A}_{\mathrm{Oseen}}:\mathrm{X}^{sym}_{q,\sigma}\to L^{sym}_{q,\sigma}$ and $\mathrm{A}_\perp:\mathrm{X}^{\perp}_{q,\sigma}\to L^{\perp}_{q,\sigma}$ is established, where

$$L^{sym}_{q,\sigma}:=\{f\in L_{q,\sigma}\mid\forall t\in\mathbb{R}:\ Q(t)f\big(Q(t)^\top\cdot\big)=f(\cdot)\}, \tag{1.9}$$

$\mathrm{X}^{\perp}_{q,\sigma}$ is the complement of $\mathrm{X}^{sym}_{q,\sigma}$ in $\mathrm{X}_{q,\sigma}$, and $L^{\perp}_{q,\sigma}$ the complement of $L^{sym}_{q,\sigma}$ in $L_{q,\sigma}$. It is then shown that $\mathrm{X}^{\perp}_{q,\sigma}\subset L_q(\mathbb{R}^3)^3$. This inclusion is a manifest of the aforementioned regularizing effect. More precisely, it shows for $f\in L^{\perp}_{q,\sigma}$ that the solution v to $\mathrm{A}v=f$ lies in $L_q(\mathbb{R}^3)^3$, and thus has better summability at infinity than the solution w to $\mathrm{A}_{\mathrm{Oseen}}w=f$, which at the outset is only in $L_{2q/2-q}(\mathbb{R}^3)^3$. Consequently, the decomposition shows that the Oseen operator with rotation can be written as the direct sum of the classical Oseen operator and an operator for which the regularizing effect of the rotation terms materializes.

As an application of the decomposition of A, an asymptotic expansion of a solution to (1.1) is presented in Section 3. The leading term in the expansion is expressed in terms of components of the Oseen fundamental solution, which is a direct consequence of the classical Oseen operator appearing in the direct sum of A.

2 Main Result

The main theorem of the paper reads:

Theorem 2.1. *Let $\mathcal{R} > 0, \mathcal{T} > 0$ and $1 < q < 2$. The Banach spaces $\mathrm{X}_{q,\sigma}$ and $L_{q,\sigma}$, defined in (1.4) and (1.5), can be decomposed in the direct sums*

$$\mathrm{X}_{q,\sigma} = \mathrm{X}_{q,\sigma}^{sym} \oplus \mathcal{W}_{q,\sigma}^{\perp}, \tag{2.1}$$

$$L_{q,\sigma} = L_{q,\sigma}^{sym} \oplus L_{q,\sigma}^{\perp}, \tag{2.2}$$

where $\mathrm{X}_{q,\sigma}^{sym}$ is given by (1.8), $L_{q,\sigma}^{sym}$ by (1.9), $L_{q,\sigma}^{\perp}$ a closed subspace of $L_{q,\sigma}$, and

$$\mathcal{W}_{q,\sigma}^{\perp} := \{v \in W_q^2(\mathbb{R}^3)^3 \mid \mathrm{e}_3 \wedge x \cdot \nabla v \in L_q(\mathbb{R}^3)^3,\ v \in L_{q,\sigma}^{\perp}\}. \tag{2.3}$$

The operator A, defined in (1.2), can be decomposed in the direct sum

$$\mathrm{A} = \mathrm{A}_{\mathrm{Oseen}} \oplus \mathrm{A}_{\perp}, \tag{2.4}$$

where $\mathrm{A}_{\mathrm{Oseen}}$ is defined by (1.3), and

$$\mathrm{A}_{\mathrm{Oseen}} : \mathrm{X}_{q,\sigma}^{sym} \to L_{q,\sigma}^{sym}, \tag{2.5}$$

$$\mathrm{A}_{\perp} : \mathcal{W}_{q,\sigma}^{\perp} \to L_{q,\sigma}^{\perp}, \tag{2.6}$$

are both homeomorphisms. Furthermore, the projection of $\mathrm{X}_{q,\sigma}$ onto $\mathrm{X}_{q,\sigma}^{sym}$ and of $L_{q,\sigma}$ onto $L_{q,\sigma}^{sym}$ is in both cases given by

$$\mathcal{P}h(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} Q(t)h\big(Q(t)^\top x\big)\,\mathrm{d}t, \tag{2.7}$$

where the integral is understood as the Bochner integral in $\mathrm{X}_{q,\sigma}$ and $L_{q,\sigma}$, respectively.

Proof. As one easily verifies, for $h \in L_{q,\sigma}$ the function $t \to Q(t)h\big(Q(t)^\top x\big)$ is continuous as a mapping from $\mathbb{R}$ into $L_q(\mathbb{R}^3)^3$, and $\operatorname{div}\big[Q(t)h\big(Q(t)^\top x\big)\big] = 0$. Thus, $t \to Q(t)h\big(Q(t)^\top x\big)$ belongs to the space $C(\mathbb{R}; L_{q,\sigma})$. The integral in (2.7) is therefore well-defined in the sense of a Bochner integral in $L_{q,\sigma}$. It follows that $\mathcal{P}$ is a bounded linear operator $\mathcal{P} : L_{q,\sigma} \to L_{q,\sigma}$. Moreover, a direct calculation shows $\mathcal{P}^2 = \mathcal{P}$. Consequently, $\mathcal{P}$ is a continuous projection on the Banach space $L_{q,\sigma}$. Similarly, one may verify for $h \in \mathrm{X}_{q,\sigma}$, using the observation

$$\begin{aligned}\mathrm{e}_3 \wedge x \cdot \nabla\big[Q(t)h\big(Q(t)^\top x\big)\big] - \mathrm{e}_3 \wedge \big[Q(t)h\big(Q(t)^\top x\big)\big]&\\ = Q(t)\big[\,\mathrm{e}_3 \wedge y \cdot \nabla h(y) - \mathrm{e}_3 \wedge h(y)\big]_{|y=Q(t)^\top x}&,\end{aligned} \tag{2.8}$$

that $t \to Q(t)h\big(Q(t)^\top x\big)$ belongs to the space $C(\mathbb{R}; \mathrm{X}_{q,\sigma})$. It follows that $\mathcal{P}$ is also a continuous projection on the Banach space $\mathrm{X}_{q,\sigma}$. Putting $\mathcal{P}_\perp := \mathrm{Id} - \mathcal{P}$, we can now conclude

$$\mathrm{X}_{q,\sigma} = \mathcal{P}\mathrm{X}_{q,\sigma} \oplus \mathcal{P}_\perp \mathrm{X}_{q,\sigma}, \tag{2.9}$$

$$L_{q,\sigma} = \mathcal{P}L_{q,\sigma} \oplus \mathcal{P}_\perp L_{q,\sigma}. \tag{2.10}$$

Another direct calculation, utilizing again (2.8), shows that the diagram

$$\begin{array}{ccc} \mathrm{X}_{q,\sigma} & \xrightarrow{\mathrm{A}} & L_{q,\sigma} \\ {\scriptstyle\mathcal{P}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{P}} \\ \mathrm{X}_{q,\sigma} & \xrightarrow{\mathrm{A}} & L_{q,\sigma} \end{array}$$

commutes. Since $\mathrm{A} : \mathrm{X}_{q,\sigma} \to L_{q,\sigma}$ is a homeomorphism, see [1] or [4], it further follows that

$$\mathrm{A} = \mathrm{A}_{|\mathcal{P}\mathrm{X}_{q,\sigma}} \oplus \mathrm{A}_{|\mathcal{P}_\perp \mathrm{X}_{q,\sigma}}, \tag{2.11}$$

with $\mathrm{A}_{|\mathcal{P}\mathrm{X}_{q,\sigma}} : \mathcal{P}\mathrm{X}_{q,\sigma} \to \mathcal{P}L_{q,\sigma}$ and $\mathrm{A}_{|\mathcal{P}_\perp \mathrm{X}_{q,\sigma}} : \mathcal{P}_\perp \mathrm{X}_{q,\sigma} \to \mathcal{P}_\perp L_{q,\sigma}$ both homeomorphisms. Clearly, $\mathcal{P}\mathrm{X}_{q,\sigma} = \mathrm{X}_{q,\sigma}^{sym}$ and $\mathcal{P}L_{q,\sigma} = L_{q,\sigma}^{sym}$. Also, since for $v \in \mathrm{X}_{q,\sigma}^{sym}$

$$\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = -\partial_t \Big[Q(t) v\big(Q(t)^\top x\big)\Big]_{|t=0} = 0,$$

we have $\mathrm{A}_{|\mathcal{P}\mathrm{X}_{q,\sigma}} = \mathrm{A}_{|\mathrm{X}_{q,\sigma}^{sym}} = \mathrm{A}_{\mathrm{Oseen}|\mathrm{X}_{q,\sigma}^{sym}}$. Consequently, (2.9), (2.10), and (2.11) establish the theorem once we show $\mathcal{P}_\perp \mathrm{X}_{q,\sigma} = \mathcal{W}_{q,\sigma}^\perp$. The inclusion $\mathcal{W}_{q,\sigma}^\perp \subset \mathcal{P}_\perp \mathrm{X}_{q,\sigma}$ is easy to verify. To prove the reverse inclusion, let $v \in \mathcal{P}_\perp \mathrm{X}_{q,\sigma}$ and assume, to begin with, that $v \in C^\infty(\mathbb{R}^3)^3$. Put

$$u(x,t) := Q(t) v\big(Q(t)^\top x\big), \quad F(x,t) := \partial_t u(x,t).$$

Both u and F are smooth and $\frac{2\pi}{\mathcal{T}}$-periodic in t. We can thus express these fields in terms of their Fourier series with respect to t:

$$u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)\,\mathrm{e}^{i\mathcal{T}kt}, \quad F(x,t) = \sum_{k\in\mathbb{Z}} F_k(x)\,\mathrm{e}^{i\mathcal{T}kt}, \tag{2.12}$$

with

$$u_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} u(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t, \quad F_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} F(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t.$$

Now employ Minkowski's integral inequality, recall $q \in (1,2)$, to obtain

$$\begin{aligned} \|v\|_q^q &= \left(\frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \left[\int_{\mathbb{R}^3} |u(x,t)|^q\,\mathrm{d}x\right]^{\frac{1}{q-1}} \mathrm{d}t\right)^{q-1} \\ &\leq \int_{\mathbb{R}^3} \left(\frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} |u(x,t)|^{\frac{q}{q-1}}\,\mathrm{d}t\right)^{q-1} \mathrm{d}x. \end{aligned}$$

By Hausdorff-Young's inequality for Fourier series, it then follows that

$$\|v\|_q^q \leq \int_{\mathbb{R}^3} \sum_{k\in\mathbb{Z}} |u_k(x)|^q \,\mathrm{d}x.$$

Observe that $u_0 = \mathcal{P}v = 0$ and $F_k = ku_k$. Consequently,

$$\begin{aligned} \|v\|_q^q \leq \sum_{k\neq 0} \|u_k\|_q^q \leq \sum_{k\neq 0} |k|^{-q} \|F_k\|_q^q \leq c_1 \|F\|_q^q = c_1 \|\partial_t u\|_q^q \\ = c_1 \big\|\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)\big\|_q^q. \end{aligned}$$

The above inequality implies

$$\|v\|_q \leq c_2 \|v\|_{\mathrm{X}_{q,\sigma}}. \tag{2.13}$$

We have established (2.13) for $v \in \mathcal{P}_\perp \mathrm{X}_{q,\sigma} \cap C^\infty(\mathbb{R}^3)^3$. Since $C^\infty_{0,\sigma}(\mathbb{R}^3)^3$ is dense in $L_{q,\sigma}$, and $\mathrm{A} : \mathrm{X}_{q,\sigma} \to L_{q,\sigma}$ is a homeomorphism, the fact that $\mathrm{A}^{-1}(\psi) \in C^\infty(\mathbb{R}^3)^3$ for $\psi \in C^\infty_{0,\sigma}(\mathbb{R}^3)^3$, which follows by standard regularity theory for elliptic systems, implies that $\mathrm{X}_{q,\sigma} \cap C^\infty(\mathbb{R}^3)^3$ is dense in $\mathrm{X}_{q,\sigma}$. Consequently, $\mathcal{P}_\perp \mathrm{X}_{q,\sigma} \cap C^\infty(\mathbb{R}^3)^3$ is dense in $\mathcal{P}_\perp \mathrm{X}_{q,\sigma}$, and (2.13) therefore extends to all $v \in \mathcal{P}_\perp \mathrm{X}_{q,\sigma}$. From (2.13), the inclusion $\mathcal{P}_\perp \mathrm{X}_{q,\sigma} \subset \mathcal{W}^\perp_{q,\sigma}$ follows. □

3 Applications

As an application of Theorem 2.1, an asymptotic expansion at infinity of a solution to (1.1) will be established, that is, of a linearized, steady-state Navier-Stokes flow past a rotating and translating body. An asymptotic expansion of a solution v is a decomposition $v(x) = \Gamma(x) + \mathfrak{R}(x)$, where Γ is an explicitly known function, referred to at the leading term, and $\mathfrak{R}$ some unspecified remainder decaying strictly faster than Γ as $|x| \to \infty$. Here, we shall establish an asymptotic expansion where the decay of the remainder is specified in an L_q-summability sense. For an expansion with a point-wise decay estimate, we refer the reader to [6].

We will identify the leading term as components of the Oseen fundamental solution given by

$$\begin{aligned} &\Gamma^{\mathcal{R}}_{\mathrm{O}} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^{3\times 3}, \quad \big[\Gamma^{\mathcal{R}}_{\mathrm{O}}(x)\big]_{ij} := (\delta_{ij}\Delta - \partial_i\partial_j)\Phi^{\mathcal{R}}(x), \\ &\Phi^{\mathcal{R}}(x) := \frac{1}{4\pi\mathcal{R}} \int_0^{\mathcal{R}(|x|+x_3)/2} \frac{1-\mathrm{e}^{-\tau}}{\tau} \,\mathrm{d}\tau, \end{aligned}$$

see also [2, Chapter VII.3].

The expansion is shown for a so-called Leray solution, which is a solution in $D^1_2(\Omega) \cap L_6(\Omega)$, where $D^1_2(\Omega)$ denotes the homogeneous Sobolev space of functions with bounded Dirichlet integral. The existence of a Leray solution to (1.1) is well-known, see for example [3].

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a C^2-smooth exterior domain, and $\mathcal{R}, \mathcal{T} > 0$. A solution*

$$(v,p) \in D^1_2(\Omega)^3 \cap L_6(\Omega)^3 \cap W^2_{2,loc}(\overline{\Omega})^3 \times W^1_{2,loc}(\overline{\Omega}) \tag{3.1}$$

satisfies

$$v = \Gamma^{\mathcal{R}}_{\mathrm{O}} \cdot \big(\mathcal{F} \cdot \mathrm{e}_3\big)\, \mathrm{e}_3 + \mathfrak{R}, \tag{3.2}$$

where

$$\forall q \in (4/3, \infty): \ \mathfrak{R} \in L_q(\Omega)^3, \tag{3.3}$$

and

$$\mathcal{F} := \int_{\partial\Omega} \mathrm{T}(v,p) \cdot n \,\mathrm{d}S, \tag{3.4}$$

with T *denoting the Cauchy stress tensor* $\mathrm{T}(v,p) := \nabla v + \nabla v^\top - pI$ *of the liquid.*

Remark 3.2. It is well-known that $\Gamma^{\mathcal{R}}_{\mathrm{O}}$ is *not* L_q-summable in a neighborhood of infinity for small q. More precisely, for $q \in [1,2]$ one can show, see for example [2, Chapter VII.3], $\Gamma^{\mathcal{R}}_{\mathrm{O}} \notin L_q\big(\mathbb{R}^3 \setminus \mathrm{B}_r(0)\big)$ for any $r > 0$. Thus, in the sense of summability the remainder term $\mathfrak{R}$ in the expansion (3.2) decays strictly faster as $|x| \to \infty$ than the leading term. Consequently, the decomposition (3.2) is a valid asymptotic expansion of v at spatial infinity.

Remark 3.3. Note that $\mathcal{F}$ equals the force exerted by the liquid on the body.

Proof of Theorem 3.1. Choose $\rho > 0$ so large that $\mathbb{R}^3 \setminus \Omega \subset \mathrm{B}_\rho$, where $\mathrm{B}_\rho := \{x \in \mathbb{R}^3 \mid |x| < \rho\}$. Let $\psi_\rho \in C^\infty(\mathbb{R}^3)$ be a "cut-off" function with $\psi_\rho = 0$ in B_ρ and $\psi_\rho = 1$ in $\mathbb{R}^3 \setminus \mathrm{B}_{2\rho}$. Since (v,p) solves (1.1), standard regularity theory for elliptic systems implies that $(v,p) \in C^\infty(\Omega \setminus \mathrm{B}_\rho)^3 \times C^\infty(\Omega \setminus \mathrm{B}_\rho)$. We can therefore define

$$\begin{aligned} &w : \mathbb{R}^3 \to \mathbb{R}^3, \quad w(x) := \psi_\rho(x) v(x) - \mathfrak{B}[\nabla\psi_\rho \cdot v](x), \\ &\mathfrak{p} : \mathbb{R}^3 \to \mathbb{R}, \quad \mathfrak{p}(x) := \psi_\rho(x) p(x), \end{aligned} \tag{3.5}$$

where $\mathfrak{B}$ denotes the so-called "Bogovskiĭ operator", that is, an operator

$$\mathfrak{B} : C^\infty_0(\mathrm{B}_{2\rho}) \to C^\infty_0(\mathrm{B}_{2\rho})^3$$

with the property that $\operatorname{div} \mathfrak{B}(f) = f$ whenever $\int_{\mathrm{B}_{2\rho}} f(x)\,\mathrm{d}x = 0$. We refer to [2, Theorem III.3.2] for details on this operator. Observe that

$$\begin{aligned} \int_{\mathrm{B}_{2\rho}} \nabla\psi_\rho \cdot v \,\mathrm{d}x &= \int_{\partial\,\mathrm{B}_{2\rho}} v \cdot n \,\mathrm{d}x = \int_{\partial\Omega} v \cdot n \,\mathrm{d}S \\ &= \int_{\partial\Omega} (\mathrm{e}_3 + \mathcal{T}\,\mathrm{e}_3 \wedge x) \cdot n \,\mathrm{d}S \\ &= \int_{\mathbb{R}^3\setminus\overline{\Omega}} \operatorname{div}(\mathrm{e}_3 + \mathcal{T}\,\mathrm{e}_3 \wedge x)\,\mathrm{d}x = 0, \end{aligned} \tag{3.6}$$

whence $(w,\mathfrak{p})$ is a smooth Leray solution to the whole space problem

$$\begin{cases} -\Delta w - \mathcal{R}\partial_3 w - \mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla w - \mathrm{e}_3\wedge w\big) = g - \nabla\mathfrak{p} & \text{in } \mathbb{R}^3,\\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

with $g\in C_0^\infty(\mathbb{R}^3)^3$ and $\operatorname{supp} g\subset \mathrm{B}_{2\rho}$. We compute

$$\begin{aligned}\int_{\mathbb{R}^3} g\,\mathrm{d}x &= \int_{\mathrm{B}_{2\rho}} \operatorname{div}\big[-\mathrm{T}(w,\mathfrak{p}) - \mathcal{R}w\otimes\mathrm{e}_3 - \mathcal{T}w\otimes(\mathrm{e}_3\wedge x) + \mathcal{T}(\mathrm{e}_3\wedge x)\otimes w\big]\,\mathrm{d}x\\ &= \int_{\partial\mathrm{B}_{2\rho}} \big[-\mathrm{T}(v,p) - \mathcal{R}v\otimes\mathrm{e}_3 - \mathcal{T}v\otimes(\mathrm{e}_3\wedge x) + \mathcal{T}(\mathrm{e}_3\wedge x)\otimes v\big]\cdot n\,\mathrm{d}S\\ &= -\int_{\partial\Omega} \big[-\mathrm{T}(v,p) - \mathcal{R}v\otimes\mathrm{e}_3 - \mathcal{T}v\otimes(\mathrm{e}_3\wedge x) + \mathcal{T}(\mathrm{e}_3\wedge x)\otimes v\big]\cdot n\,\mathrm{d}S.\end{aligned}$$

Inserting the boundary values $(1.1)_3$ for v on $\partial\Omega$, an elementary calculation similar to (3.6) shows that all but the first term in the last integral above vanish. Thus

$$\int_{\mathbb{R}^3} g\,\mathrm{d}x = \int_{\partial\Omega}\mathrm{T}(v,p)\cdot n\,\mathrm{d}S = \mathcal{F}. \tag{3.7}$$

From [5, Theorem 4.4] we obtain the additional regularity

$$\forall q\in(1,2):\ w\in\mathrm{X}_{q,\sigma},\quad \mathfrak{p}\in L_{\frac{3q}{3-q}}(\mathbb{R}^3),\quad \nabla\mathfrak{p}\in L_q(\mathbb{R}^3)^3.$$

We thus have $\mathrm{A}w = g-\nabla\mathfrak{p}$ in $L_{q,\sigma}$ for all $q\in(1,2)$. From Theorem 2.1 it follows that

$$\mathrm{A}_{\mathrm{Oseen}}\mathcal{P}w = \mathcal{P}\big(g-\nabla\mathfrak{p}\big), \tag{3.8}$$

$$\forall q\in(1,2):\ \mathcal{P}_\perp w\in\mathcal{W}^\perp_{q,\sigma}, \tag{3.9}$$

where $\mathcal{P}_\perp := \mathrm{Id}-\mathcal{P}$. Since both g and $\mathfrak{p}$ are smooth,

$$\begin{aligned}\mathcal{P}\big(g-\nabla\mathfrak{p}\big) &= \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} Q(t)g\big(Q(t)^\top x\big)\,\mathrm{d}t - \nabla\Big[\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}\mathfrak{p}\big(Q(t)^\top x\big)\,\mathrm{d}t\Big]\\ &=: G(x) - \nabla Q(x),\end{aligned} \tag{3.10}$$

where the integrals are classical, evaluated pointwise in x, whence Q enjoys the same summability properties as $\mathfrak{p}$, and $G\in C_0^\infty(\mathbb{R}^3)^3$. Combining (3.8) and (3.10), we find

$$\begin{cases} -\Delta\mathcal{P}w - \mathcal{R}\partial_3\mathcal{P}w = G-\nabla Q & \text{in } \mathbb{R}^3,\\ \operatorname{div}\mathcal{P}w = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

By classical results for the Oseen system, see for example [2, Chapter VII.3], it then follows that

$$\mathcal{P}w(x) = \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot \Big(\int_{\mathbb{R}^3} G(y)\,\mathrm{d}y \Big) + O\big(\nabla\Gamma_{\mathrm{O}}^{\mathcal{R}}(x)\big) \quad \text{as} \quad |x| \to \infty. \tag{3.11}$$

Recall (3.7) and compute

$$\begin{aligned} \int_{\mathbb{R}^3} G(y)\,\mathrm{d}y &= \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} Q(t) \cdot \int_{\mathbb{R}^3} g\big(Q(t)^\top y\big)\,\mathrm{d}y\mathrm{d}t \\ &= \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} Q(t) \cdot \mathcal{F}\,\mathrm{d}t = \big(\mathcal{F} \cdot \mathrm{e}_3\,\big)\,\mathrm{e}_3\,. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$w = \mathcal{P}w + \mathcal{P}_\perp w = \Gamma_{\mathrm{O}}^{\mathcal{R}} \cdot \big(\mathcal{F} \cdot \mathrm{e}_3\,\big)\,\mathrm{e}_3 + O\big(\nabla\Gamma_{\mathrm{O}}^{\mathcal{R}}\big) + \mathcal{P}_\perp w. \tag{3.13}$$

It is well-known, see for example [2, Chapter VII.3], that

$$\forall q \in (4/3, \infty): \;\; \nabla\Gamma_{\mathrm{O}}^{\mathcal{R}} \in L_q(\mathbb{R}^3 \setminus \mathrm{B}_r)^{3\times3\times3} \text{ for any } r > 0. \tag{3.14}$$

Moreover, (3.9) implies $\mathcal{P}_\perp w \in L_q(\mathbb{R}^3)$ for all $q \in (1, \infty)$. Consequently, since $v(x) = w(x)$ for $|x| > 2\rho$, (3.13) implies the expansion (3.2) with a remainder satisfying (3.3). □

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