A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part I: Strong solutions

Giovanni P. Galdi*
Department of Mechanical Engineering and Materials Science
University of Pittsburgh, U.S.A
email:galdi@pitt.edu

Mads Kyed†
Fachbereich Mathematik
Technische Universität Darmstadt, Germany
email: kyed@mathematik.tu-darmstadt.de

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Consider a rigid body moving in a three-dimensional Navier-Stokes liquid with a prescribed velocity $\xi \in \mathbb{R}^3$ and a non-zero angular velocity $\omega \in \mathbb{R}^3 \setminus \{0\}$ that are constant when referred to a frame attached to the body. Linearizing the associated equations of motion, we obtain the Oseen ($\xi \neq 0$) or Stokes ($\xi = 0$) equations in a *rotating* frame of reference. We will consider the corresponding steady-state whole-space problem. Our main result in this first part concerns elliptic estimates of the solutions in terms of data in $L^q(\mathbb{R}^3)$. Such estimates have been established by R. Farwig in Tohoku Math. J., Vol. 58, 2006, for the Oseen case, and R. Farwig, T. Hishida, and D. Müller in Pac. J. Math, Vol. 215 (2), 2004, for the Stokes case. We introduce a new approach resulting in an elementary proof of these estimates. Moreover, our method yields more details on how the constants in the estimates depend on ξ and ω. In part II we will establish similar estimates in terms of data in the negative order homogeneous Sobolev space $D_0^{-1,q}(\mathbb{R}^3)$.

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1 Introduction

Consider a rigid body moving in a Navier-Stokes liquid that fills the whole three-dimensional space outside the body. We assume the body moves with a velocity $\xi \in \mathbb{R}^3$ and angular velocity $\omega \in \mathbb{R}^3 \setminus \{0\}$ that are constant when referred to a frame attached to the body. In this frame, we consider the linearized steady-state equations of motion. We assume that ξ and ω are both directed along the x_3-axis. Due to a simple transformation (see [8, Section 2]), this assumption can be made without loss of generality whenever $\xi \cdot \omega \neq 0$ or $\xi = 0$. After a suitable non-dimensionalization and reduction to a whole-space problem, the equations of motion then read

$$\begin{cases} -\Delta v + \nabla p - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $v : \mathbb{R}^3 \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \to \mathbb{R}$ denotes the Eulerian velocity and pressure, respectively. Moreover, $\mathcal{R} \geq 0$ and $\mathcal{T} > 0$ are dimensionless constants with $\mathcal{R} = 0$ if and only if $\xi = 0$. We refer the reader to [5] for the derivation of (1.1) and details on the physical background.

The above system is the classical steady-state whole-space Oseen ($\mathcal{R} > 0$) or Stokes ($\mathcal{R} = 0$) problem with the extra term $\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)$. This extra term stems from the rotating frame of reference and represents the main challenge of the problem. Due to the unbounded coefficient $\mathrm{e}_3 \wedge x$, the term can not be treated as a perturbation to the Oseen or Stokes operator.

We will prove elliptic L^q-estimates for solutions (v, p) to (1.1) in terms of the data f. Our main result in the Stokes case ($\mathcal{R} = 0$) reads:

Theorem 1.1. *Let $1 < q < \infty$, $\mathcal{R} = 0$, and $\mathcal{T} > 0$. For any $f \in L^q(\mathbb{R}^3)^3$ there exists a solution $(v, p) \in D^{2,q}(\mathbb{R}^3)^3 \times D^{1,q}(\mathbb{R}^3)$ to (1.1) that satisfies*

$$\|\nabla^2 v\|_q + \|\nabla p\|_q + \|\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)\|_q \leq C_1 \|f\|_q. \tag{1.2}$$

If $1 < q < 3$, then

$$\|\nabla v\|_{\frac{3q}{3-q}} \leq C_2 \|f\|_q. \tag{1.3}$$

If $1 < q < \frac{3}{2}$, then

$$\|v\|_{\frac{3q}{3-2q}} \leq C_3 \|f\|_q. \tag{1.4}$$

The constants C_1, C_2, C_3 are independent of $\mathcal{T}$. If $(\tilde{v}, \tilde{p}) \in D^{2,r}(\mathbb{R}^3)^3 \times D^{1,r}(\mathbb{R}^3)$, $1 < r < \infty$, is another solution to (1.1), then

$$\tilde{v} = v + \alpha\, \mathrm{e}_3 + \beta\, \mathrm{e}_3 \wedge x + \sigma(x_1, x_2, -2x_3), \quad \tilde{p} = p + \gamma \tag{1.5}$$

for some $\alpha, \beta, \sigma, \gamma \in \mathbb{R}$.

Our main result in the Oseen case ($\mathcal{R} > 0$) is the following theorem:

Theorem 1.2. *Let $1 < q < \infty$, $\mathcal{R}_0 > 0$, $0 < \mathcal{R} < \mathcal{R}_0$, and $\mathcal{T} > 0$. For any $f \in L^q(\mathbb{R}^3)^3$ there exists a solution $(v,p) \in D^{2,q}(\mathbb{R}^3)^3 \times D^{1,q}(\mathbb{R}^3)$ to (1.1) that satisfies*

$$\|\nabla^2 v\|_q + \|\nabla p\|_q \leq C_4 \|f\|_q, \tag{1.6}$$

with C_4 independent of $\mathcal{R}_0$, $\mathcal{R}$, and $\mathcal{T}$. Moreover,

$$\|\mathcal{R}\partial_3 v\|_q + \|\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)\|_q \leq C_5 \Big(1 + \mathcal{T}^{-2}\Big)\|f\|_q. \tag{1.7}$$

If $1 < q < 4$, then

$$\|\nabla v\|_{\frac{4q}{4-q}} \leq C_6 \Big(\mathcal{R}^{-\frac{1}{4}} + \mathcal{T}^{-\frac{1}{2}}\Big)\|f\|_q, \tag{1.8}$$

If $1 < q < 2$, then

$$\|v\|_{\frac{2q}{2-q}} \leq C_7 \Big(\mathcal{R}^{-\frac{1}{2}} + \mathcal{T}^{-1}\Big)\|f\|_q. \tag{1.9}$$

The constants C_5, C_6, C_7 depend only on $\mathcal{R}_0$. If $(\tilde{v},\tilde{p}) \in D^{2,r}(\mathbb{R}^3)^3 \times D^{1,r}(\mathbb{R}^3)$, $1 < r < \infty$, is another solution to (1.1), then

$$\tilde{v} = v + \alpha\,\mathrm{e}_3 + \beta\,\mathrm{e}_3 \wedge x, \quad \tilde{p} = p + \gamma \tag{1.10}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Remark 1.3. We note that (1.3) and (1.4) follow by Sobolev embedding from (1.2). Similarly, (1.3) and (1.4) also follow from (1.6). Thus, the solution in Theorem 1.2 (Oseen case) also satisfies (1.3) and (1.4). Comparing the estimates (1.8) and (1.9) to (1.3) and (1.4), we observe that the former are better estimates in terms of summability of the left-hand side. In fact, the improved summability of v and ∇v in (1.8) and (1.9) relative to (1.3) and (1.4) is exactly the same as obtained in the case $\mathcal{T} = 0$, *i.e*, the classical Oseen and Stokes system without the rotation terms.

The statements of Theorem 1.2 and Theorem 1.1 have already been established in [2] and [3], respectively, albeit with less information on how the constants in (1.8) and (1.9) depend on $\mathcal{R}$ and $\mathcal{T}$. Unlike the classical Oseen and Stokes problems, due to the term $\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)$ the estimates do not follow from a standard application of well-known Fourier multiplier theorems. Therefore, in [2] and [3] the estimates are established by a very technical and non-trivial application of the Littlewood-Payley decomposition. The purpose of this paper is to give an elementary proof using a different approach. More specifically, we employ an idea going back to [6] of transforming (1.1) into a time-dependent Stokes problem.

2 Preliminaries

In this section we recall basic notation and prove some simple lemmas. The experienced reader may skip this section and proceed directly to the proof of the main theorems in section 3.

By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue space with norm $\|\cdot\|_q$. By $D^{m,q}(\mathbb{R}^3)$ we denote the homogeneous Sobolev space with semi-norm $|\cdot|_{m,q}$, *i.e*,

$$|v|_{m,q} := \Big(\sum_{|\alpha|=m}\int_{\mathbb{R}^3} |\partial^\alpha v(x)|^q \,\mathrm{d}x\Big)^{\frac{1}{q}},\; D^{m,q} := \{v\in L^1_{loc}(\mathbb{R}^3) \mid |v|_{m,q}<\infty\}.$$

Unless otherwise indicated, differential operators act in the spatial variable only, *i.e.*, $\operatorname{div} u(x,t) := \operatorname{div}_x u(x,t)$, $\Delta u(x,t) := \Delta_x u(x,t)$ etc. We use $\mathrm{B}_R := \{x\in\mathbb{R}^3 \mid |x|<R\}$ to denote balls in $\mathbb{R}^3$. Finally, we note that constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear.

For notational purpose, we put

$$Lv := -\Delta v - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\big), \tag{2.1}$$
$$L^*v := -\Delta v + \mathcal{R}\partial_3 v + \mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\big). \tag{2.2}$$

Note that L^* is (formally) the adjoint of L.

The existence of a weak solution to (1.1) can be shown by standard methods. In fact, in the case $q=2$ the following is known:

Lemma 2.1. *Let $\mathcal{R}\geq 0$ and $\mathcal{T}>0$. Let $f\in C_0^\infty(\mathbb{R}^3)^3$. There exists a solution*

$$\begin{aligned} &v\in D^{2,2}(\mathbb{R}^3)^3\cap D^{1,2}(\mathbb{R}^3)^3\cap L^6(\mathbb{R}^3)^3\cap C^\infty(\mathbb{R}^3)^3,\\ &p\in D^{1,2}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)\cap C^\infty(\mathbb{R}^3)\end{aligned} \tag{2.3}$$

to problem (1.1) that satisfies

$$\|\nabla^2 v\|_2 + \|\nabla p\|_2 \leq C_8\|f\|_2, \tag{2.4}$$

with C_8 independent of $\mathcal{R}$ and $\mathcal{T}$. Moreover,

$$(v,p)\in\cap_{m=1}^\infty D^{m,2}(\mathbb{R}^3)^3\times D^{m,2}(\mathbb{R}^3). \tag{2.5}$$

All assertions above are also true for the adjoint problem

$$\begin{cases} L^*v+\nabla p = f & \text{in } \mathbb{R}^3,\\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3.\end{cases} \tag{2.6}$$

Proof. See [5, Lemma 4.14] and [8, Theorem 2] for the existence of a solution in the class (2.3) satisfying (2.4). A direct calculation shows that Δ commutes with the operator on the left-hand side of $(1.1)_1$, *i.e.*, with L. Combining this fact with the uniqueness result [7, Lemma 4.1], we may simply apply Δ to $(1.1)_1$ and iterate the argument to obtain (2.5). The adjoint problem (2.6) is dealt with in exactly the same manner. □

In section 3 we shall prove the main theorems for data $f \in C_0^\infty(\mathbb{R}^3)^3$. By a standard density argument, one can then extend the statements to all $f \in L^q(\mathbb{R}^3)^3$, as detailed in the following lemma:

Lemma 2.2. *If the statements in Theorem 1.1 and Theorem 1.2 are true for all $f \in C_0^\infty(\mathbb{R}^3)^3$, then they are true also for all $f \in L^q(\mathbb{R}^3)^3$.*

Proof. We only prove the part concerning Theorem 1.2. The assertion concerning Theorem 1.1 follows analogously. Assume therefore that the statements in Theorem 1.2 are true for all $f \in C_0^\infty(\mathbb{R}^3)^3$. Let $f \in L^q(\mathbb{R}^3)^3$ and choose $\{f_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^3)^3$ with $\lim_{n\to\infty} f_n = f$ in $L^q(\mathbb{R}^3)^3$. Let (v_n, p_n) be the solution to (1.1) with f_n as the right-hand side. Then choose $\alpha_n, \beta_n, \kappa_n \in \mathbb{R}^3$ and $\iota_n \in \mathbb{R}$ such that

$$0 = \int_{\mathrm{B}_1} \partial_1 v_n - \alpha_n \,\mathrm{d}x = \int_{\mathrm{B}_1} \partial_2 v_n - \beta_n \,\mathrm{d}x, \tag{2.7}$$

$$0 = \int_{\mathrm{B}_1} v_n - (\kappa_n + \alpha_n x_1 + \beta_n x_2) \,\mathrm{d}x, \tag{2.8}$$

and $0 = \int_{\mathrm{B}_1} p_n - \iota_n \,\mathrm{d}x$. Put $r_n := \kappa_n + \alpha_n x_1 + \beta_n x_2$. From (1.6)–(1.7) we see, using Poincaré's inequality, that $\{(v_n - r_n, p_n - \iota_n)\}_{n=1}^\infty$ is a Cauchy sequence in the Banach-space

$$\begin{aligned} &X_m := \{(v,p) \in L^1_{loc}(\mathbb{R}^3)^3 \times L^1_{loc}(\mathbb{R}^3) \mid \|(v,p)\|_{X_m} < \infty\},\\ &\|(v,p)\|_{X_m} := \|\nabla^2 v\|_q + \|\nabla p\|_q + \mathcal{R}\|\partial_3 v\|_q + \|v\|_{L^q(\mathrm{B}_m)} + \|p\|_{L^q(\mathrm{B}_m)} \end{aligned}$$

for all $m \in \mathbb{N}$. Consequently, there is an element $(v,p) \in \cap_{m\in\mathbb{N}} X_m$ with the property that $\lim_{n\to\infty}(v_n - r_n, p_n - \iota_n) = (v,p)$ in X_m for all $m \in \mathbb{N}$. Recall now (2.1). It follows that $\lim_{n\to\infty} L(v_n - r_n) + \nabla(p_n - \iota_n) = Lv + \nabla p$ in $\mathcal{D}'(\mathbb{R}^3)^3$. By construction, we have $\lim_{n\to\infty} Lv_n + \nabla p_n = \lim_{n\to\infty} f_n = f$ in $L^q(\mathbb{R}^3)^3$. We thus deduce that $\lim_{n\to\infty} Lr_n = f - (Lv + \nabla p)$ in $\mathcal{D}'(\mathbb{R}^3)^3$. Consequently, $f - (Lv + \nabla p)$ must be equal to Lr for some first order polynomial r. It follows that $(v + r, p) \in D^{2,q}(\mathbb{R}^3)^3 \times D^{1,q}(\mathbb{R}^3)$ solves (1.1). Moreover, since (v_n, p_n) satisfies (1.6) and (1.7), so does $(v + r, p)$. This proves the first part of the Theorem 1.2 with respect to f. If $1 < q < 4$ we repeat the argument above with the modification that we ignore (2.7) (put $\alpha_n = \beta_n = 0$), and add the term $\|\nabla v\|_{\frac{4q}{4-q}}$ to the X_m-norm. We then obtain a solution to (1.1) that also also satisfies (1.8). If $1 < q < 2$ we ignore both (2.7) and (2.8) (put $\alpha_n = \beta_n = \kappa_n = 0$), and add the term $\|v\|_{\frac{2q}{2-q}}$ to the X_m-norm. We then obtain a solution that also also satisfies (1.9). $\square$

We next observe that it will be enough to prove the statements of the main theorems for either $1 < q \leq 2$ or $2 \leq q < \infty$. More specifically, we have the following lemma:

Lemma 2.3. *Assume for any $\varphi \in C_0^\infty(\mathbb{R}^3)^3$ the solution (ψ, η) from Lemma 2.1 to the adjoint problem*

$$\begin{cases} L^*\psi + \nabla\eta = \varphi & \text{in } \mathbb{R}^3, \\ \operatorname{div}\psi = 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{2.9}$$

satisfies the estimates in Theorem 1.1 and Theorem 1.2 for all $1 < q \leq 2$*. Then the solution* (v, p) *to* (1.1) *from Lemma 2.1 satisfies the estimates in Theorem 1.1 and Theorem 1.2 for all* $2 \leq q < \infty$*. Conversely, if for all* $\varphi \in C_0^\infty(\mathbb{R}^3)^3$ *the solution* (ψ, η) *from Lemma 2.1 to* (2.9) *satisfies the estimates in Theorem 1.1 and Theorem 1.2 for all* $2 \leq q < \infty$*, then the estimates are also true for* (v, p) *for all* $1 < q \leq 2$*.*

Proof. We will concentrate on just one case. The other cases follow by similar arguments. Assume for any $\varphi \in C_0^\infty(\mathbb{R}^3)^3$ that the solution (ψ, η) from Lemma 2.1 to the adjoint problem (2.9) satisfies (1.2), *i.e.*,

$$\|\nabla^2 \psi\|_q + \|\nabla \eta\|_q \leq c_1 \|\varphi\|_q, \tag{2.10}$$

for all $q \in [2, \infty)$. We will now show that the solution (v, p) from Lemma 2.1 to (1.1) satisfies (1.2) for the Hölder conjugate $q' = \frac{q}{q-1}$ of any $q \in [2, \infty)$. Exploiting that Δ commutes with L, and the summability properties of (v, p) and (ψ, η), we compute

$$\int_{\mathbb{R}^3} \Delta v \cdot \varphi \,\mathrm{d}x = \lim_{R \to \infty} \int_{\mathrm{B}_R} \Delta v \cdot L^* \psi \,\mathrm{d}x = \lim_{R \to \infty} \int_{\mathrm{B}_R} \Delta L v \cdot \psi \,\mathrm{d}x = \int_{\mathbb{R}^3} f \cdot \Delta \psi \,\mathrm{d}x,$$

where, when performing the partial integration, we use that

$$\begin{aligned} \int_{\mathrm{B}_R} \Delta v \cdot \big(\mathrm{e}_3 \wedge x \cdot \nabla \psi \big) \,\mathrm{d}x &= \int_{\partial \mathrm{B}_R} (\Delta v \cdot \psi)(\mathrm{e}_3 \wedge x) \cdot n \,\mathrm{d}S - \int_{\mathrm{B}_R} \big(\mathrm{e}_3 \wedge x \cdot \nabla \Delta v \big) \cdot \psi \,\mathrm{d}x \\ &= 0 - \int_{\mathrm{B}_R} \big(\mathrm{e}_3 \wedge x \cdot \nabla \Delta v \big) \cdot \psi \,\mathrm{d}x, \end{aligned}$$

which is due to $n = x/|x|$ on $\partial \mathrm{B}_R$. Using (2.10), we then obtain

$$\Big| \int_{\mathbb{R}^3} \Delta v \cdot \varphi \,\mathrm{d}x \Big| \leq \|f\|_{q'} \|\Delta \psi\|_q \leq \|f\|_{q'} \|\varphi\|_q.$$

Since φ is arbitrary, it follows that $\|\Delta v\|_{q'} \leq c_1 \|f\|_{q'}$, and thus, by standard theory, $\|\nabla^2 v\|_{q'} \leq c_2 \|f\|_{q'}$. The estimate $\|\nabla p\|_{q'} \leq c_3 \|f\|_{q'}$ follows simply from the fact that $-\Delta p = \operatorname{div} f$. Since $q \in [2, \infty)$ was arbitrary, we have thereby established (1.2) for all $q' \in (1, 2]$. □

3 Proof of Main Theorems

Before now proving the main theorems, we briefly sketch the main idea behind their proofs. The main challenge is to establish (1.2), (1.6), and (1.7) for a solution (v, p). By introducing the rotation matrix

$$Q(t) := \begin{pmatrix} \cos(\mathcal{T} t) & -\sin(\mathcal{T} t) & 0 \\ \sin(\mathcal{T} t) & \cos(\mathcal{T} t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and putting $u(x,t) := Q(t)v\big(Q(t)^\top x\big)$, $\pi(x,t) := p\big(Q(t)^\top x\big)$, we obtain a solution (u,π) to the time-dependent Oseen ($\mathcal{R} > 0$) or Stokes ($\mathcal{R} = 0$) problem in the whole space. In order to prove (1.2) and (1.6), we split this solution into a solution to a Cauchy problem with zero initial value, and a solution to a Cauchy problem with zero forcing term. We then prove the desired estimates by a simple analysis of these two systems. The main idea behind our proof of (1.7) is to exploit that the transformation above in fact yields functions u and π which are $\frac{2\pi}{T}$-periodic in t. We can thus expand (u,π) in a Fourier-series with respect to t. We will analyze the L^q-norm of v in terms of the Fourier coefficients in this series. As we shall see, each one of these coefficients solves an Oseen resolvent-like system. This information enables us to estimate their L^q-norms directly.

Proof of Theorem 1.1. Let $f \in C_0^\infty(\mathbb{R}^3)^3$, $q \in (2,\infty)$, and (v,p) be the solution to (1.1) from Lemma 2.1. Let $T > 0$. For $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ put

$$u(x,t) := Q(t)v\big(Q(t)^\top x\, \mathrm{e}_3\big),\ \pi(x,t) := p\big(Q(t)^\top x\big),\ F(x,t) := Q(t)f\big(Q(t)^\top x\big).$$

Then

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi = F & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = v(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{3.1}$$

By well-known results (see for example [9, Chap. 4, Sec. 5, Theorem 6]) there exists a solution $u_1 \in L^q(\mathbb{R}^3 \times (0,T))^3$ to

$$\begin{cases} \partial_t u_1 - \Delta u_1 + \nabla\pi = F & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u_1 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \lim\limits_{t\to 0^+} \|u_1(\cdot,t)\|_q = 0 & \end{cases}$$

satisfying

$$\|\nabla^2 u_1\|_{L^q(\mathbb{R}^3\times(0,T))} \leq c_1 \|F\|_{L^q(\mathbb{R}^3\times(0,T))},$$

with c_1 independent of T. It is also well known that

$$u_2(x,t) := (4\pi t)^{-3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-|x-y|^2/4t}\, v(y)\, \mathrm{d}y \tag{3.2}$$

satisfies $u_2 \in L^6(\mathbb{R}^3 \times (0,T))^3$ and solves

$$\begin{cases} \partial_t u_2 - \Delta u_2 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u_2 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \lim\limits_{t\to 0^+} \|u_2(\cdot,t) - v(\cdot)\|_6 = 0. & \end{cases}$$

Taking second order derivatives on both sides in (3.2), and applying Young's inequality, we obtain

$$\|\nabla^2 u_2(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leq c_2\, t^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}\, \|\nabla^2 v\|_2,$$

with c_2 independent of T. We claim that $u = u_1 + u_2$ in $\mathbb{R}^3 \times (0,T)$. This follows from the fact that $u_1 + u_2$ satisfies (3.1) combined with a uniqueness argument (see for example [7, Lemma 3.6]). We can now estimate

$$\begin{aligned}
(T-1)\|\nabla^2 v\|_q^q &= \int_1^T \int_{\mathbb{R}^3} |\nabla^2 u(x,t)|^q \,\mathrm{d}x\mathrm{d}t \\
&\leq c_3 \Big(\|\nabla^2 u_1\|^q_{L^q(\mathbb{R}^3\times(0,T))} + \int_1^T \|\nabla^2 u_2(\cdot,t)\|_q^q \,\mathrm{d}t \Big) \\
&\leq c_4 \Big(\|F\|^q_{L^q(\mathbb{R}^3\times(0,T))} + \int_1^T \big(t^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}\|\nabla^2 v\|_2\big)^q \,\mathrm{d}t \Big) \\
&\leq c_5 \big(T\|f\|_q^q + (T^{-\frac{3q}{2}(\frac{1}{2}-\frac{1}{q})+1} - 1)\|\nabla^2 v\|_2^q \big),
\end{aligned}$$

with c_5 independent of T, and also of $\mathcal{R}$ and $\mathcal{T}$. Dividing both sides by $T-1$ and subsequently letting $T \to \infty$ (note that $q > 2$ by assumption), we conclude that $\|\nabla^2 v\|_q \leq c_5 \|f\|_q$. Finally, we deduce directly from (1.1), by applying div on both sides in $(1.1)_1$, that $-\Delta p = \operatorname{div} f$. From this equation it follows that also $\|\nabla p\|_q \leq c_6 \|f\|_q$, with c_6 independent of $\mathcal{R}$ and $\mathcal{T}$. Hence, we have established (1.2) in the case $q \in (2,\infty)$ and $f \in C_0^\infty(\mathbb{R}^3)^3$.

We obtain directly from Lemma 2.1 that (1.2) also holds when $q = 2$. The estimates (1.3) and (1.4) are direct consequences of (1.2) and Sobolev's embedding theorem [4, Lemma II.2.2]. Consequently, we have established all the estimates in the theorem in the case $q \in [2,\infty)$ and $f \in C_0^\infty(\mathbb{R}^3)^3$. Analogously, we can show the same for the solution from Lemma 2.1 to the adjoint problem (2.6). Thus, by Lemma 2.3, we may then conclude (1.2)–(1.4) for all $q \in (1,\infty)$ and $f \in C_0^\infty(\mathbb{R}^3)^3$. Finally, by the density argument in Lemma 2.2, we obtain (1.2)–(1.4) in the general case $q \in (1,\infty)$ and $f \in L^q(\mathbb{R}^3)^3$.

Assume $(\tilde{v},\tilde{p}) \in D^{2,r}(\mathbb{R}^3)^3 \times D^{1,r}(\mathbb{R}^3)$ is another solution to (1.1). Then $(w,\mathfrak{q}) := (v-\tilde{v}, p-\tilde{p})$ satisfies (1.1) with a homogeneous right-hand side. Applying div to $(1.1)_1$, we immediately find that $\Delta\mathfrak{q} = 0$, which, since $\nabla\mathfrak{q} \in L^q(\mathbb{R}^3)^3 + L^r(\mathbb{R}^3)^3$, implies $\mathfrak{q} = \gamma$ for some $\gamma \in \mathbb{R}$. Moreover, since Δ and L commute, we observe that $(\Delta w, \Delta\mathfrak{q})$ also satisfies (1.1) with a homogeneous right-hand side. Recalling that $\Delta w = L^q(\mathbb{R}^3)^3 + L^r(\mathbb{R}^3)^3$, we deduce from [7, Lemma 4.1] that $\Delta w = 0$ and hence $w = Ax + b$ for some $A \in \mathbb{R}^{3\times 3}$ and $b \in \mathbb{R}^3$. By $(1.1)_2$, $\operatorname{div}(Ax) = \operatorname{Tr} A = 0$. In addition, we find that

$$\partial_t \big[Q(t) w\big(Q(t)^\top x\big) \big] = Q(t)\big(\Delta w - \nabla\mathfrak{q}\big) = 0,$$

whence $Q(t)\big(AQ(t)^\top x + b\big)$ is t-independent. As a direct consequence hereof we see that $b = \alpha\, \mathrm{e}_3$ for some $\alpha \in \mathbb{R}$, and conclude that $Q(t)AQ(t)^\top x$ is t-independent. We may now exploit this t-independence by considering combinations of the values $t = 0, t = \frac{\pi}{2\mathcal{T}}, t = \frac{\pi}{\mathcal{T}}, t = \frac{3\pi}{2\mathcal{T}}$ and $x = \mathrm{e}_1, x = \mathrm{e}_2, x = \mathrm{e}_3$ in this expression. We then obtain $Ax = \beta\, \mathrm{e}_3 \wedge x + \sigma(x_1, x_2, -2x_3)$ for some $\beta, \sigma \in \mathbb{R}$. □

Proof of Theorem 1.2. Let $f \in C_0^\infty(\mathbb{R}^3)^3$ and (v,p) be the solution from Lemma 2.1. Let $T>0$. For $(x,t)\in\mathbb{R}^3\times\mathbb{R}$ put

$$\begin{aligned} \mathfrak{u}(x,t) &:= Q(t)v\big(Q(t)^\top x - \mathcal{R}t\,\mathrm{e}_3\big), \quad \mathfrak{p}(x,t) := p\big(Q(t)^\top x - \mathcal{R}t\,\mathrm{e}_3\big),\\ F(x,t) &:= Q(t)f\big(Q(t)^\top x - \mathcal{R}t\,\mathrm{e}_3\big). \end{aligned}$$

Then $(\mathfrak{u},\mathfrak{p})$ satisfies (3.1). Estimate (1.6) now follows by the same argument that was used to show (1.2) in the proof of Theorem 1.1.

Next we show that (v,p) satisfies (1.7). We consider first the case $q\in(1,2]$. For $(x,t)\in\mathbb{R}^3\times\mathbb{R}$ put

$$u(x,t) := Q(t)v\big(Q(t)^\top x\big),\ \pi(x,t) := p\big(Q(t)^\top x\big),\ F(x,t) := Q(t)f\big(Q(t)^\top x\big).$$

Note that u, π, and F are smooth and $\frac{2\pi}{\mathcal{T}}$-periodic in the t variable. We can therefore expand these fields in their Fourier-series. More precisely, we have

$$u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\ \pi(x,t) = \sum_{k\in\mathbb{Z}} \pi_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\ F(x,t) = \sum_{k\in\mathbb{Z}} F_k(x)\,\mathrm{e}^{i\mathcal{T}kt},$$

with

$$\begin{aligned} u_k(x) &:= \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} u(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t, \quad \pi_k(x) := \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} \pi(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t,\\ F_k(x) &:= \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} F(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t. \end{aligned}$$

As one may easily verify,

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi - \mathcal{R}\partial_3 u = F & \text{in } \mathbb{R}^3\times\mathbb{R},\\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}. \end{cases}$$

It follows that the Fourier coefficients each satisfies

$$\begin{cases} i\mathcal{T}k u_k - \Delta u_k + \nabla\pi_k - \mathcal{R}\partial_3 u_k = F_k & \text{in } \mathbb{R}^3,\\ \operatorname{div} u_k = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.3}$$

In the case $k=0$, (3.3) reduces to the classical Oseen system. By well-known theory, see for example [4, Theorem VII.4.1], we thus have

$$\|\nabla^2 u_0\|_q + \mathcal{R}\|\partial_3 u_0\|_q \le c_1\|F_0\|_q \le c_2\|f\|_q, \tag{3.4}$$

with c_2 independent of $\mathcal{R}$ and $\mathcal{T}$. Consider now $k\neq 0$. By Minkowski's integral inequality, and the fact that (v,p) satisfies (1.6), we find that

$$\|\nabla^2 u_k\|_q \le \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} \bigg(\int_{\mathbb{R}^3} |\nabla^2 u(x,t)|^q\,\mathrm{d}x\bigg)^{1/q}\mathrm{d}t = \|\nabla^2 v\|_q \le \|f\|_q,$$

and similarly $\|\nabla\pi_k\|_q \leq \|f\|_q$. We can thus conclude from (3.3) that

$$\begin{aligned} |\mathcal{T}k|\|u_k\|_q &\leq \|\Delta u_k\|_q + \|\nabla\pi_k\|_q + \mathcal{R}\|\partial_3 u_k\|_q + \|F_k\|_q \\ &\leq c_3\|f\|_q + \mathcal{R}\|\partial_3 u_k\|_q, \end{aligned} \tag{3.5}$$

with c_3 independent of $\mathcal{R}$ and $\mathcal{T}$. By Nirenberg's inequality (see [10, p.125]),

$$\|\partial_3 u_k\|_q \leq c_4(\varepsilon\|u_k\|_q + \varepsilon^{-1}\|\nabla^2 u_k\|_q) \tag{3.6}$$

for all $\varepsilon > 0$. Choosing $\varepsilon = |\mathcal{T}k|/(2\mathcal{R}c_4)$ in (3.6) and applying the resulting estimate in (3.5), it follows that

$$\|u_k\|_q \leq c_5\frac{1}{|\mathcal{T}k|}\left(1 + \frac{\mathcal{R}^2}{|\mathcal{T}k|}\right)\|f\|_q, \tag{3.7}$$

with c_5 independent of $\mathcal{R}$ and $\mathcal{T}$. At this point we make the important observation that $v(x) = u(x,0) = \sum_{k\in\mathbb{Z}} u_k(x)$. We thus see that v can be written as u_0, which solves the classical Oseen problem and therefore satisfies the corresponding L^q-estimates, and a sum of functions u_k, each of which satisfies the even better L^q-estimate (3.7). To exploit this observation, we put

$$v_1 := v - u_0, \tag{3.8}$$

and define

$$U(x,t) := Q(t)v_1\big(Q(t)^\top x\big) = u(x,t) - u_0(x) = \sum_{k\neq 0} u_k(x)\,\mathrm{e}^{i\mathcal{T}kt}.$$

The first equality above follows from the fact that $Q(t)u_0\big(Q(t)^\top x\big) = u_0(x)$ for all $t \in \mathbb{R}$, which one easily verifies directly from the definition of u_0. Since

$$\|v_1\|_q^q = \left(\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} \left[\int_{\mathbb{R}^3} |U(x,t)|^q\,\mathrm{d}x\right]^{\frac{1}{q-1}}\mathrm{d}t\right)^{q-1},$$

an application of Minkowski's inequality yields (recall that $1 < q \leq 2$)

$$\|v_1\|_q^q \leq \int_{\mathbb{R}^3}\left(\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} |U(x,t)|^{\frac{q}{q-1}}\,\mathrm{d}t\right)^{q-1}\mathrm{d}x.$$

By the Hausdorff-Young inequality for Fourier series (see for example [1, Proposition 4.2.7]), we then obtain

$$\|v_1\|_q^q \leq \int_{\mathbb{R}^3}\sum_{k\neq 0}|u_k(x)|^q = \sum_{k\neq 0}\|u_k\|_q^q.$$

Using (3.7), it follows that

$$\|v_1\|_q^q \leq c_6\sum_{k\neq 0}\frac{1}{|\mathcal{T}k|^q}\left(1 + \frac{\mathcal{R}^2}{\mathcal{T}}\right)^q\|f\|_q^q \leq c_7\frac{1}{\mathcal{T}^q}\left(1 + \frac{\mathcal{R}^2}{\mathcal{T}}\right)^q\|f\|_q^q, \tag{3.9}$$

with c_7 independent of $\mathcal{R}$ and $\mathcal{T}$. From (3.4) and (1.6), we also deduce

$$\|\nabla^2 v_1\|_q \leq c_8 \|f\|_q, \tag{3.10}$$

with c_8 independent of $\mathcal{R}$ and $\mathcal{T}$. Combining now (3.4), (3.9), and (3.10), we finally obtain

$$\begin{aligned}\|\mathcal{R}\partial_3 v\|_q &\leq \mathcal{R}\|\partial_3 u_0\|_q + \mathcal{R}\|\partial_3 v_1\|_q \\ &\leq c_2\|f\|_q + \mathcal{R}c_9(\|v_1\|_q + \|\nabla^2 v_1\|_q) \leq c_{10}\Big(1 + \frac{1}{\mathcal{T}^2}\Big)\|f\|_q,\end{aligned}$$

where $c_{10} = c_{10}(\mathcal{R}_0)$, but otherwise independent of $\mathcal{R}$ and $\mathcal{T}$. We have thereby shown (1.7) in the case $q \in (1, 2]$ and $f \in C_0^\infty(\mathbb{R}^3)^3$. Analogously, we can prove the same for the solution from Lemma 2.1 to the adjoint problem (2.6). Thus, by the duality argument in Lemma 2.3, we conclude that (v, p) satisfies (1.7) for all $q \in (1, \infty)$.

We now show that (v, p) satisfies (1.8)–(1.9). Consider first $1 < q < 4$. By well known theory ([4, Theorem VII.4.1]),

$$\|\nabla u_0\|_{\frac{4q}{4-q}} \leq c_{11}\mathcal{R}^{1/4}\|F_0\|_q \leq c_{11}\mathcal{R}^{1/4}\|f\|_q. \tag{3.11}$$

By the Sobolev's embedding theorem (see for example [4, Lemma II.2.2]), (3.9), and (3.10), it follows that

$$\begin{aligned}\|\nabla v_1\|_{\frac{4q}{4-q}} &\leq c_{12}\|\nabla v_1\|_q^{\frac14}\|\nabla^2 v_1\|_q^{\frac34} \leq c_{13}(\|v_1\|_q + \|\nabla^2 v_1\|_q)^{\frac14}\|\nabla^2 v_1\|_q^{\frac34} \\ &\leq c_{14}\Big[1 + \frac{1}{|\mathcal{T}|}\Big(1 + \frac{\mathcal{R}^2}{|\mathcal{T}|}\Big)\Big]^{\frac14}\|f\|_q \leq c_{15}\Big(1 + \mathcal{T}^{-\frac12}\Big)\|f\|_q,\end{aligned} \tag{3.12}$$

with $c_{15} = c_{15}(\mathcal{R}_0)$. Combining (3.8), (3.11), and (3.12) gives us (1.8). Consider next $1 < q < 2$. It is well known ([4, Theorem VII.4.1]) that

$$\|u_0\|_{\frac{2q}{2-q}} \leq c_{16}\mathcal{R}^{1/2}\|F_0\|_q \leq c_{16}\mathcal{R}^{1/2}\|f\|_q. \tag{3.13}$$

Using again Sobolev's embedding theorem, (3.9), and (3.10), we find that

$$\begin{aligned}\|v_1\|_{\frac{2q}{2-q}} &\leq c_{17}\|v_1\|_{\frac{3q}{3-q}}^{\frac12}\|\nabla v_1\|_{\frac{3q}{3-q}}^{\frac12} \leq c_{18}(\|v_1\|_q + \|\nabla^2 v_1\|_q)^{\frac12}\|\nabla^2 v_1\|_q^{\frac12} \\ &\leq c_{19}\Big[1 + \frac{1}{|\mathcal{T}|}\Big(1 + \frac{\mathcal{R}^2}{|\mathcal{T}|}\Big)\Big]^{\frac12}\|f\|_q \leq c_{20}\Big(1 + \mathcal{T}^{-1}\Big)\|f\|_q,\end{aligned} \tag{3.14}$$

with $c_{20} = c_{20}(\mathcal{R}_0)$. Combining (3.8), (3.13), and (3.14) yields (1.9).

We have now shown (1.6)–(1.9) in the case $f \in C_0^\infty(\mathbb{R}^3)^3$. By the density argument in Lemma 2.2, we extend this conclusion to the general case of $f \in L^q(\mathbb{R}^3)^3$.

It remains to prove uniqueness. Assume $(\tilde{v}, \tilde{p}) \in D^{2,r}(\mathbb{R}^3)^3 \times D^{1,r}(\mathbb{R}^3)$ is another solution to (1.1). Then $(w, \mathfrak{q}) := (v - \tilde{v}, p - \tilde{p})$ satisfies (1.1) with a homogeneous right-hand side. By the same argument as in the proof of Theorem 1.1, this implies that $\mathfrak{q} = \gamma$

for some $\gamma \in \mathbb{R}$, $w = Ax + b$ for some $A \in \mathbb{R}^{3\times 3}$ and $b \in \mathbb{R}^3$, and $\operatorname{Tr} A = 0$. In addition, since

$$\partial_t \big[Q(t) w\big(Q(t)^\top x - \mathcal{R}t\, \mathrm{e}_3\big)\big] = Q(t)\big(\Delta w - \nabla \mathfrak{q}\big) = 0,$$

it follows that $Q(t)\big(AQ(t)^\top x - \mathcal{R}tA\, \mathrm{e}_3 + b\big)$ is t-independent. As a direct consequence hereof we see that $b = \alpha\, \mathrm{e}_3$ for some $\alpha \in \mathbb{R}$, and $A\, \mathrm{e}_3 = 0$. This then means that $Q(t)AQ(t)^\top x$ is t-independent, from which we finally conclude, by a similar argument as in the proof of Theorem 1.1, that $Ax = \beta\, \mathrm{e}_3 \wedge x$ for some $\beta \in \mathbb{R}$. □

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