Uniqueness of almost periodic-in-time solutions to Navier-Stokes equations in unbounded domains

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Abstract

We present a uniqueness theorem for almost periodic-in-time solutions to the Navier-Stokes equations in 3-dimensional unbounded domains. Thus far, uniqueness of almost periodic-in-time solutions to the Navier-Stokes equations in unbounded domain, roughly speaking, is known only for a small almost periodic-in-time solution in $BC(\mathbb{R}; L^3_w)$ within the class of solutions which have sufficiently small $L^\infty(L^3_w)$-norm. In this paper, we show that a small almost periodic-in-time solution in $BC(\mathbb{R}; L^3_w \cap L^{6,2})$ is unique within the class of all almost periodic-in-time solutions in $BC(\mathbb{R}; L^3_w \cap L^{6,2})$. The proof of the present uniqueness theorem is based on the method of dual equations.

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1 Introduction

In this paper, we consider a viscous incompressible fluid in 3-dimensional unbounded domains Ω. The motion of such a fluid is governed by the Navier-Stokes equations:

$$\text{(N-S)}\begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p = f, & t\in\mathbb{R}, \quad x\in\Omega,\\ \operatorname{div} u = 0, & t\in\mathbb{R}, \quad x\in\Omega,\\ u|_{\partial\Omega} = 0, & t\in\mathbb{R}, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t)\in\Omega\times\mathbb{R}$. Here f is a given external

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force. It is known that if f is almost periodic-in-time and small in some sense, then there exists a small almost periodic-in-time solution to (N-S). In this paper, we consider the uniqueness of almost periodic-in-time solutions to (N-S).

In case where the domain Ω is bounded, the problem of existence of time-periodic solutions was considered by several authors [31, 40, 14, 34, 29, 28, 37]; some of these authors even considered domains with boundaries moving periodically-in-time. Maremonti [24] was the first to prove the existence of time-periodic regular solutions to (N-S) in *unbounded* domains. He showed that if $\Omega = \mathbb{R}^3$ and if $f(t)$ is time periodic and small in some sense, then there exists a unique time-periodic solution u to (N-S) in the class

$$\{u \in C(\mathbb{R}; L^3_\sigma);\ \sup_t \|u(t)\|_3 < \gamma,\ \ \sup_t \|\nabla u(t)\|_2 < \infty\}, \tag{1.1}$$

where γ is a small number. The same problem in $\mathbb{R}^3_+$ is considered in [25]. Kozono-Nakao [17] showed that if $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$, $n \geq 3$, or $\Omega \subset \mathbb{R}^n$, $n \geq 4$, is an exterior domain, and if $f(t)$ is time periodic and small in some sense, then there exists a unique time-periodic solution u to (N-S) in the class $\{u \in C(\mathbb{R}; L^n_\sigma);\ \sup_t \|u(t)\|_r + \sup_t \|\nabla u(t)\|_q < \gamma\}$ ($2 < r < n$, $\frac{n}{2} < q < n$), where γ is a small number depending on Ω, r and q, see also [35]. Kubo [21] proved the same result as [17] in the case where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a perturbed half space or an aperture domain. While he assumed a null flux condition in case of an aperture domain, Crispo-Maremonti [4] proved existence of unique time-periodic solutions for given time-periodic fluxes.

With respect to 3-dimensional exterior domains, we mention the results given by Maremonti-Padula [26], Salvi [30], Yamazaki [39] and Galdi-Sohr [10]. Maremonti-Padula [26] showed that for any $\Omega \subset \mathbb{R}^3$, if $f(t)$ is time-periodic and can be expressed as $f = \nabla \cdot F$, where $f, F \in C(\mathbb{R}; L^2)$, then there exists at least one time-periodic weak solution u to (N-S) in the class $\nabla u \in L^2_{loc}(\mathbb{R}; L^2)$. Moreover, they showed under some symmetry assumptions on Ω and on f that there exists a unique time-periodic solution u to (N-S) in the class defined in (1.1). In the case where Ω is an exterior domain with a periodically moving boundary, Salvi [30] proved the existence of weak time-periodic solutions and of a strong periodic solution. In the case where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is an exterior domain, $\mathbb{R}^n$, or $\mathbb{R}^n_+$, Yamazaki [39] showed that if $f = \nabla \cdot F$, $F \in BUC(\mathbb{R}; L^{n/2,\infty})$ and $\sup_t \|F(t)\|_{L^{n/2,\infty}}$ is small, then there exists a unique mild solution u to (N-S) in the class

$$\{u \in C(\mathbb{R}; L^{n,\infty});\ \sup_t \|u(t)\|_{L^{n,\infty}} < \gamma\},$$

where $\gamma = \gamma(\Omega)$ is sufficiently small. In particular, he shows that if f is time-periodic or almost periodic-in-time, then the mild solution is time-periodic or almost periodic-in-time. In the case of a 3-dimensional exterior domain, Galdi-Sohr [10] proved the existence of a small periodic strong solution u in $C(\mathbb{R}; L^r(\Omega))$, $r > 3$, satisfying the condition that $\sup_{x,t}(1+|x|)|u(x,t)|$ is small, under the assumption that $f = \mathrm{div}\, F$ is periodic and small in some function spaces. Moreover, they proved the uniqueness of such solutions in the larger class of all periodic weak solutions v with $\nabla v \in L^2(0,T;L^2)$, satisfying the energy inequality $\int_0^T \|\nabla v\|_2^2\, d\tau \le -\int_0^T (F, \nabla v)\, d\tau$ and mild integrability conditions on the corresponding pressure; here T is a period of f.

Another type of uniqueness theorem for time-periodic L^3_w-solution was proved by the second author [36] without assuming the energy inequality. In the case of an exterior domain $\Omega \subset \mathbb{R}^3$, the whole space $\mathbb{R}^3$, the halfspace $\mathbb{R}^3_+$, a perturbed halfspace, or an aperture domain, it was shown in [36] that if u and v are time-periodic L^3_w-solutions in $L^2_{loc}(\mathbb{R}; L^{6,2})$ for the same force f, and if *one of them* is small, then $u = v$.

On the other hand, thus far, *uniqueness of almost periodic-in-time solutions* in unbounded domains is only known for a small almost periodic-in-time L^3_w-solution within the class of solutions which have sufficiently small $L^\infty(L^3_w)$-norm; i.e., if u and v are L^3_w-solutions for the same force f, and if *both of them* are small, then $u = v$, see [39]. In the present paper, we establish a new uniqueness theorem for almost periodic-in-time solutions. We show that if u and v are almost periodic-in-time solutions in

$$C(\mathbb{R}; L^3_w) \cap L^2_{loc}(\mathbb{R}; L^{6,2})$$

for the same force f, and if *one of them* is small, then $u = v$.

Our proof is based on an idea given by Lions-Masmoudi [23]. They proved the uniqueness of L^n-solutions to the initial-boundary value problem of (N-S) by using the backward initial-boundary value problem of *dual* equations. Of course, in the initial-boundary value problem of (N-S), the initial condition $u(0) = v(0)$ plays an important role in proving $u(t) = v(t)$ for $t > 0$. In our problem, however, we cannot assume $u(0) = v(0)$. To overcome this crucial difficulty, we construct a sequence of weak solutions of dual equations having a property similar to that of almost periodic functions.

2 Preliminaries and Results

Throughout this paper we impose the following assumption on the domain.

Assumption 1 *$\Omega \subset \mathbb{R}^3$ is an exterior domain, the half-space $\mathbb{R}^3_+$, the whole space $\mathbb{R}^3$, a perturbed half-space, or an aperture domain with $\partial\Omega \in C^\infty$.*

For the definitions of perturbed half-spaces and aperture domains, see Kubo-Shibata [22] and Farwig-Sohr [6, 7].

Before stating our results, we introduce some notation and function spaces. Let $C_{0,\sigma}^\infty(\Omega) = C_{0,\sigma}^\infty$ denote the set of all C^∞-real vector functions $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in Ω such that div $\phi = 0$. Similarly $C_{0,\sigma}^m$ is defined. Then L_σ^r is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r-norm $\|\cdot\|_r$. The symbol $(\cdot, \cdot)$ denotes the L^2- inner product and the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. Concerning Sobolev spaces we use the notations $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$, $k \in \mathbb{N}$, $1 \le p, q \le \infty$. Note that very often we will simply write L^r and $W^{k,p}$ instead of $L^r(\Omega)$ and $W^{k,p}(\Omega)$, respectively. Let $L^{p,q}(\Omega)$, $1 \le p, q \le \infty$, denote the Lorentz spaces and $\|\cdot\|_{p,q}$ denote the norm of $L^{p,q}(\Omega)$; for the definition and properties of $L^{p,q}(\Omega)$, see e.g. [1]. We note that $L^{p,\infty}$ is equivalent to the weak-L^p space (L_w^p) and $L^{p,p}$ is equivalent to L^p. Finally,

$$L_{uloc}^2(\mathbb{R}; L^{6,2}) = \{g \in L_{loc}^2(\mathbb{R}; L^{6,2}) \ ; \ \sup_t \|g\|_{L^2(t,t+1;L^{6,2})} < \infty\}$$

denotes the space of uniformly locally integrable L^2-function on $\mathbb{R}$ with values in $L^{6,2}(\Omega)$.

For a Banach space B, let B^* be the dual space of B. Let X be a Banach space of functions on Ω such that $L_\sigma^2 \cap X$ is dense in X; if $g \in L_\sigma^2 \cap X^*$ and ${}_{X^*}< g, \phi >_X = (g, \phi)$ for all $\phi \in L_\sigma^2 \cap X$, then we denote ${}_{X^*}< \cdot, \cdot >_X$ by $(\cdot, \cdot)$ for simplicity.

In this paper, we denote by C various constants. In particular, $C = C(*, \cdots, *)$ denotes a constant depending only on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition: $L^r(\Omega) = L_\sigma^r \oplus G_r$ $(1 < r < \infty)$, where $G_r = \{\nabla p \in L^r; p \in L_{loc}^r(\overline{\Omega})\}$, see Fujiwara-Morimoto [9], Miyakawa [27], Simader-Sohr [32], Borchers-Miyakawa [2], and Farwig-Sohr [6, 8]; P_r denotes the projection operator from L^r onto L_σ^r along G_r. The Stokes operator A_r on L_σ^r is defined by $A_r = -P_r\Delta$ with domain $D(A_r) = W^{2,r} \cap W_0^{1,r} \cap L_\sigma^r$. It is known that

$$(L_\sigma^r)^* \text{ (the dual space of } L_\sigma^r) = L_\sigma^{r'}, \quad A_r^* \text{ (the adjoint operator of } A_r) = A_{r'},$$

where $1/r + 1/r' = 1$. It is shown by Giga [11], Giga-Sohr [12], Borchers-Miyakawa [2] and Farwig-Sohr [6, 8] that $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}; t \ge 0\}$ of class C_0 in L_σ^r. Moreover, it is found that

$$\|u\|_{W^{2,r}} \le C\|(1 + A_r)u\|_r \quad \text{for all } u \in D(A_r) \tag{2.1}$$

with a constant $C = C(r, n, \Omega)$; see e.g. [13, Lemma 2.8].

In this paper, $\dot{W}^{1,r}_{0,\sigma}$ denotes the closure of $D(A_r)$ with respect to the norm $\|\phi\|_{\dot{W}^{1,r}} = \|\nabla\phi\|_r$, where $\nabla\phi = (\partial\phi^i/\partial x_j)_{i,j=1,\cdots,n}$. Its dual space $(\dot{W}^{1,2}_{0,\sigma})^*$ is equipped with the norm $\|\phi\|_{(\dot{W}^{1,2}_{0,\sigma})^*} = \sup\left\{\frac{|(\phi,\theta)|}{\|\nabla\theta\|_2}\ ;\ \theta \in \dot{W}^{1,2}_{0,\sigma}\right\}$.

Since $P_r u = P_q u$ for all $u \in L^r \cap L^q$ $(1 < r, q < \infty)$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u, P_q u$ as Pu for $u \in L^r \cap L^q$ and $A_r u, A_q u$ as Au for $u \in D(A_r) \cap D(A_q)$, respectively. Finally $L^{q,\infty}_\sigma$ denotes the space $PL^{q,\infty}(\Omega)$.

Following Kozono-Ogawa [18], we define mild $L^{3,\infty}$-solutions to (N-S).

Definition 1. *Let $f \in L^1_{loc}(\mathbb{R}; D(A_p)^* + D(A_q)^*)$ for some $1 < p, q < \infty$. A function $v \in C(\mathbb{R}; L^{3,\infty}_\sigma)$ is called a mild $L^{3,\infty}$-solution to (N-S) on $\mathbb{R}$ if v satisfies*

$$(2.2)\quad (v(t),\psi) = (e^{-(t-s)A}v(s),\psi) + \int_s^t \Big((v\cdot\nabla e^{-(t-\tau)A}\psi, v)(\tau) + < f(\tau), e^{-(t-\tau)A}\psi > \Big)\, d\tau$$

for all $\psi \in C^\infty_{0,\sigma}$ and all $t, s \in \mathbb{R}$ with $t > s$.

Next, we introduce the definition of almost periodic functions, cf. [5].

Definition 2. *Let B be a Banach space and $f \in C(\mathbb{R}; B)$. Then f is called an almost periodic functions in B if for all $\epsilon > 0$ there exists $L = L(\epsilon) > 0$ with the following property: For all $a \in \mathbb{R}$, there exists $T \in [a, a+L]$ such that*

$$\sup_{t\in\mathbb{R}} \|f(t+T) - f(t)\|_B \le \epsilon.$$

Now our main result reads as follows:

Theorem 1. *Let Ω satisfy Assumption 1. Then, there exists an absolute constant $\delta > 0$ such that if u and v are almost periodic-in-time mild $L^{3,\infty}$-solutions to (N-S) for the same external force f, if*

$$(2.3)\qquad u, v \in L^2_{uloc}(\mathbb{R}; L^{6,2}(\Omega)),$$

and

$$(2.4)\qquad \sup_t \|u\|_{3,\infty} < \delta,$$

then $u = v$.

Remark 1. We emphasize that the constant δ in (2.4) is independent of Ω. This constant will be determined in Section 3. We also note that our results are applicable to stationary solutions in L^3_w.

Remark 2. When $\Omega \subset \mathbb{R}^3$, $n \geq 3$, is an exterior domain with $\partial\Omega \in C^\infty$, Yamazaki [39] constructed small mild solutions in $BC(\mathbb{R}; L^{n,\infty}_\sigma)$ for small external forces f in some function space. Moreover, he showed that these solutions are almost periodic in $L^{n,\infty}(\Omega)$ if f is almost periodic in some space. If $n = 3$ and if f is sufficiently small and, e.g., $f \in BC(\mathbb{R}; L^{3,\infty})$, then standard arguments easily prove that Yamazaki's small solution belongs to $L^\infty(\mathbb{R}; L^{6,2})$. Indeed, there exists a number $\epsilon_0(\Omega) > 0$ such that for $a \in L^{3,\infty}$ with $\|a\|_{3,\infty} \leq \epsilon_0$ there exists a small local-in-time solution $v \in BC([0,T]; L^{3,\infty}_\sigma)$ with $t^{1/3}v \in L^\infty(0,T; L^9)$ and $v(0) = a$ by the same way as in [39]. Here the local existence time T can be chosen by $T = C/\sup_t \|f\|_{3,\infty}$ and we also have $\sup_{0\leq t\leq T} t^{1/3}\|v\|_9 < C(\epsilon_0)$. Hence, by using the uniqueness of small mild solutions in $BC([0,T]; L^{3,\infty})$ to the initial boundary value problem of (N-S) with initial data $v(0) = u(t)$, we see that Yamazaki's solution with $\sup_t \|u\|_{3,\infty} \leq \epsilon_0$ satisfies $\sup_{t+T/2\leq s\leq t+T} \|u(s)\|_{L^9} < C(\epsilon_0, \sup_t \|f\|_{3,\infty})$ for all $t \in \mathbb{R}$, which implies $u \in L^\infty(\mathbb{R}; L^9)$ and consequently $u \in L^\infty(\mathbb{R}; L^{6,2})$.

Remark 3. Kozono-Nakao [17] constructed time-periodic solutions $v \in BC(\mathbb{R}; L^n_\sigma)$ for small time-periodic external forces when $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$, $n \geq 3$, or $\Omega \subset \mathbb{R}^n$, $n \geq 4$, is an exterior domain [17]. In case where Ω is a perturbed half space or an aperture domain, time-periodic solutions in $BC(\mathbb{R}; L^n_\sigma)$ were constructed by Kubo [21]. It is straightforward to see that their existence theorems of solutions in $BC(\mathbb{R}; L^n_\sigma)$ hold for small external forces f without time-periodicity. In the same way as in [39], we observe that if the external force f is small and almost periodic in some Banach space, the small solutions $v \in BC(\mathbb{R}; L^n_\sigma)$ given in [17, 21] are almost periodic in L^n_σ. Moreover, in case $n = 3$, if $u \in C(\mathbb{R}; L^3_\sigma)$ is almost periodic in L^3 and if $f \in BC(\mathbb{R}; L^3))$, then $u \in L^2_{uloc}(\mathbb{R}; L^{6,2}(\Omega))$, i.e., condition (2.3) in Theorem 1 is automatically satisfied, see Appendix.

Before coming to the main lemma of the proof, Lemma 2.3 below, let us recall several properties of almost periodic functions and of the Stokes semigroup. It is straightforward to see that Definition 1 on almost periodic functions is equivalent to the following one:

Proposition 2.1. *$f \in C(\mathbb{R}; B)$ is almost periodic in B if and only if for all $\epsilon > 0$ there exists $l = l(\epsilon) > 0$ with the following property: For all $k \in \mathbb{Z}$, there exists $T_{\epsilon k} \in [-(k+1)l, -kl]$ such that*

$$\sup_{t\in\mathbb{R}} \|f(t + T_{\epsilon k}) - f(t)\|_B \leq \epsilon.$$

Proposition 2.2. *(i) If a Banach space B_1 is continuously embedded in a Banach space B_2 and if f is almost periodic in B_1, then f is almost periodic in B_2. Moreover, $\|f\|_{B_1}^2$ is almost periodic in $\mathbb{R}$.*

(ii) If f is almost periodic in a Banach space B, then $f \in BC(\mathbb{R}; B)$.

The proof of (i) is easy. For the proof of (ii), see [5, Theorem 6.1].

Proposition 2.3. *Assume that u, v are almost periodic in $L^{3,\infty}$ and F is almost periodic in $L^{6/5}(\Omega)$.*

(i) For all $\epsilon > 0$, there exists $l = l(\epsilon, u, v, F) > 0$ with the following property: For all $k \in \mathbb{Z}$, there exists $T_{\epsilon k} = T_{\epsilon k}(\epsilon, k, u, v, F) \in [-(k+1)l, -kl]$ such that

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \|u(t + T_{\epsilon k}) - u(t)\|_{3,\infty} \le \epsilon, \\ &\sup_{t\in\mathbb{R}} \|v(t + T_{\epsilon k}) - v(t)\|_{3,\infty} \le \epsilon, \\ &\sup_{t\in\mathbb{R}} \|F(t + T_{\epsilon k}) - F(t)\|_{6/5} \le \epsilon. \end{aligned} \tag{2.5}$$

(ii) $w := u - v$ is almost periodic in $L^{3,\infty}$.

For the proof, see [5, Theorems 6.9 and 6.7].

Lemma 2.1 ([15],[38],[12],[13],[2],[3],[39],[22],[20])**.** *For all $t > 0$ and $a \in L^p_\sigma$, the following inequalities are satisfied:*

$$\|e^{-tA}a\|_{q,1} \le Ct^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_{p,\infty} \quad \text{for } 1 < p < q < \infty, \tag{2.6}$$

$$\|\nabla e^{-tA}a\|_q \le Ct^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_p \quad \text{for } 1 < p \le q \le 3, \tag{2.7}$$

where $C = C(p, q)$.

For all $\phi \in \dot{W}^{1,2}_{0,\sigma}$ it holds that

$$\|\nabla e^{-tA}\phi\|_2 \le \|\nabla\phi\|_2\,, \quad t > 0, \tag{2.8}$$

and for $\phi \in L^2_\sigma$

$$2\int_0^\infty \|\nabla e^{-\tau A}\phi\|_2^2 \, d\tau = \|\phi\|_2^2. \tag{2.9}$$

For the proof of (2.8), (2.9) see e.g. [36, Proposition 2.1].

Lemma 2.2 ([19])**.** *Let $1 < p, q < \infty$ with $1/r := 1/p + 1/q < 1$. Then, for all $f \in L^{p,\infty}(\Omega)$ and $g \in L^{q,2}(\Omega)$, it holds that*

$$\|f \cdot g\|_{r,2} \le C\|f\|_{p,\infty}\|g\|_{q,2}, \tag{2.10}$$

where $C = C(p,q)$.

For $u \in \dot{W}_0^{1,2}(\Omega)$ it holds that

$$\|u\|_{6,2} \le C\|\nabla u\|_2, \tag{2.11}$$

where C is an absolute constant.

Finally, we come to the key lemma of the proof of uniqueness. If u and v are solutions to the Navier-Stokes equations, then $w := u - v$ satisfies

$$(U) \qquad \begin{cases} \partial_t w - \Delta w + w\cdot\nabla u + v\cdot\nabla w + \nabla p' = 0, \qquad t\in\mathbb{R},\ x\in\Omega, \\ \quad \mathrm{div}\ w = 0, \qquad t\in\mathbb{R},\ x\in\Omega, \\ \quad w|_{\partial\Omega} = 0. \end{cases}$$

Hence, if Ω is a bounded domain and if u, v belong to the Leray-Hopf class, under the hypotheses of Theorem 1, the usual energy method and the Poincaré inequality yield $\|w(t)\|_2^2 \le e^{-(t-s)}\|w(s)\|_2^2$ for $t > s$. Consequently, in the case of *bounded* domains, Theorem 1 is obvious. In the case where Ω is an *unbounded* domain, u and v do not belong to the energy class in general and the Poincaré inequality does not hold in general. Hence, since we cannot use the energy method, we will use the argument of Lions-Masmoudi [23].

We recall the dual equations of the above system (U).

$$(\mathrm{D}) \qquad \begin{cases} -\partial_t\psi - \Delta\psi - \sum_{i=1}^3 u^i\nabla\psi^i - v\cdot\nabla\psi + \nabla\pi &= F, \quad t<0,\ x\in\Omega, \\ \nabla\cdot\psi &= 0, \quad t<0,\ x\in\Omega, \\ \psi|_{\partial\Omega} &= 0. \end{cases}$$

In the following key lemma we construct a sequence of weak solutions of (D) having a property similar to that of almost periodic functions.

Lemma 2.3. *Let u and v be almost periodic in $L^{3,\infty}$ and $L^{3,\infty}_\sigma$, respectively. Assume that F is almost periodic-in-time in $L^{6/5}(\Omega)\cap L^2(\Omega)$ and $\sup_t \|u\|_{3,\infty} < \delta$. Then, for all $\epsilon \in (0,\delta]$, there exists a constant $l = l(\epsilon) > 1$ with the following property: For all $k = 1, 2, \cdots$, there exist $T_{\epsilon k} \in [-(k+1)l, -kl]$ and generalized weak solutions $\psi_{\epsilon k} \in L^2(3T_{\epsilon k}, 0; \dot{W}^{1,2}_{0,\sigma})$ of (D) in the sense*

(2.12)

$$\begin{aligned} &\int_{T_{\epsilon k}}^0 \big\{ -(g(t+T_{\epsilon k}), \psi_{\epsilon k}(t+T_{\epsilon k})) + (g(t), \psi_{\epsilon k}(t))\big\}\,dt \\ &= \int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t \Big\{ (\frac{d}{dt}g, \psi_{\epsilon k}) + (\nabla g, \nabla\psi_{\epsilon k}) - (g, \sum_{i=1}^3 u^i\nabla\psi^i_{\epsilon k}) - (g, v\cdot\nabla\psi_{\epsilon k}) - (g, F)\Big\}\,d\tau dt \end{aligned}$$

for all $g \in L^2_{loc}(\mathbb{R}; D(A_2) \cap (\dot W^{1,2}_{0,\sigma})^*)$ *with* $\frac{d}{dt} g \in L^2_{loc}(\mathbb{R}; L^2_\sigma \cap (\dot W^{1,2}_{0,\sigma})^*)$*. Moreover,*

$$\frac{1}{|T_{\epsilon k}|} \int_{3T_{\epsilon k}}^0 \|\nabla \psi_{\epsilon k}\|_2^2 \, d\tau \le C(1 + \sup_t \|F(t)\|_{6/5}^2), \tag{2.13}$$

$$\frac{1}{|T_{\epsilon k}|} \int_{2T_{\epsilon k}}^0 \|\nabla \psi_{\epsilon k}(t + T_{\epsilon k}) - \nabla \psi_{\epsilon k}(t)\|_2^2 \, d\tau \le C\epsilon^2 (\sup_t \|F(t)\|_{6/5}^2 + 1), \tag{2.14}$$

where C *is an absolute constant. Finally, (2.5) holds for those* $T_{\epsilon k}$ *and for* u, v, F*.*

We note that this lemma does not require the divergence-free condition on u. We shall prove Lemma 2.3 in the next section.

3 Proof of Lemma 2.3

Let $\epsilon \in (0, \delta]$, $k \in \mathbb{N}$ be fixed and $l = l(\epsilon, u, v, F)$ and $T_{\epsilon k} \in [-(k+1)l, -kl]$ be numbers as in Proposition 2.3 such that (2.5) holds. Without loss of generality, we may choose $l > 1$. By using the time-space mollifier $\rho_\lambda(t) * \rho_\lambda(|x|)*$ for u, v and F, it is straightforward to construct sequences $\{u_\lambda\}, \{v_\lambda\}, \{F_\lambda\}$ such that

$$\begin{aligned}
&v_\lambda \in BC^\infty(\mathbb{R}; W^{1,\infty} \cap L^{3,\infty}), \ \operatorname{div} v_\lambda = 0 \text{ in } \Omega,\\
&u_\lambda \in BC^\infty(\mathbb{R}; L^\infty \cap L^{3,\infty}), F_\lambda \in BC^\infty(\mathbb{R}; L^\infty \cap L^{6/5})\\
&u_\lambda, v_\lambda \to u, v \text{ in } L^4(3T_{\epsilon k}, 0; L^2 + L^4) \text{ and } F_\lambda \to F \text{ in } L^2(3T_{\epsilon k}, 0; L^{6/5}) \text{ as } \lambda \to 0+,\\
&\sup_t \|u_\lambda(t)\|_{3,\infty} \le \sup_t \|u(t)\|_{3,\infty} < \delta, \ \sup_t \|F_\lambda(t)\|_{6/5} \le \sup_t \|F(t)\|_{6/5}\\
&\sup_{t \in \mathbb{R}} \|u_\lambda(t + T_{\epsilon k}) - u_\lambda(t)\|_{3,\infty} \le \epsilon, \ \sup_{t \in \mathbb{R}} \|v_\lambda(t + T_{\epsilon k}) - v_\lambda(t)\|_{3,\infty} \le \epsilon\\
&\sup_{t \in \mathbb{R}} \|F_\lambda(t + T_{\epsilon k}) - F_\lambda(t)\|_{6/5} \le \epsilon\,.
\end{aligned} \tag{3.1}$$

Then, for any $\lambda > 0$ and $a \in L^2_\sigma(\Omega)$, the backward initial-boundary value problem

$$(\mathrm{D})_\lambda \qquad \begin{cases} -\partial_t \psi + (\lambda - \Delta)\psi - \displaystyle\sum_{i=1}^3 u_\lambda^i \nabla \psi^i - v_\lambda \cdot \nabla \psi + \nabla \pi &= F_\lambda, \quad t < 0, \ x \in \Omega,\\ \nabla \cdot \psi &= 0, \quad t < 0, \ x \in \Omega,\\ \psi|_{t=0} = a, \ \psi|_{\partial\Omega} &= 0 \end{cases}$$

has a unique solution $\psi_\lambda \in C((-\infty, 0]; L^2_\sigma) \cap C((-\infty, 0); D(A_2)) \cap C^1((-\infty, 0); L^2_\sigma)$ with $|t|^{1/2} \nabla \psi \in L^\infty_{loc}((-\infty, 0]; L^2)$. Indeed, by the usual iteration argument we observe that the integral equation

$$\psi_\lambda(t) = e^{tA} a - \int_t^0 e^{(t-\tau)A} P\big(\lambda \psi_\lambda - \sum_{i=1}^3 u_\lambda^i \nabla \psi_\lambda^i - v_\lambda \cdot \nabla \psi_\lambda - F_\lambda\big)(\tau) \, d\tau\,, \quad t < 0,$$

has a unique solution in $C([-T_*,0];L^2_\sigma)$ with $|t|^{1/2}\nabla\psi_\lambda \in L^\infty(-T_*,0;L^2)$, where $T_* = C\min(\frac{1}{\lambda},\frac{1}{\sup_t(\|u_\lambda\|_\infty+\|v_\lambda\|_\infty)^2})$ is independent of a. Hence ψ_λ can be extended to a solution on $(-\infty,0)$. Since $u_\lambda, v_\lambda \in C^\infty(\mathbb{R};L^\infty)$, by the above integral equation, for all $\alpha>0$ we have $\big(\lambda\psi_\lambda - \sum_{i=1}^n u^i_\lambda \nabla\psi^i_\lambda - v_\lambda\cdot\nabla\psi_\lambda - F_\lambda\big) \in C^\beta((-\infty,-\alpha);L^2)$ for some $\beta>0$. Consequently, ψ_λ satisfies $(\mathrm{D})_\lambda$ in the strong sense and $\psi_\lambda \in C((-\infty,0];L^2_\sigma)\cap C((-\infty,0);D(A_2))\cap C^1((-\infty,0);L^2_\sigma)$.

The usual energy calculation and Lemma 2.2 yield

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\|\psi_\lambda\|_2^2 + \lambda\|\psi_\lambda\|_2^2 + \|\nabla\psi_\lambda\|_2^2 &\le M\|u_\lambda\|_{3,\infty}\|\nabla\psi_\lambda\|_2^2 + M\|F_\lambda\|_{6/5}\|\nabla\psi_\lambda\|_2 \\ &\le (M\delta+\frac{1}{2})\|\nabla\psi_\lambda\|_2^2 + \frac{M^2}{2}\|F_\lambda\|_{6/5}^2,\end{aligned} \tag{3.2}$$

where M is an absolute constant. Let $\delta \le \frac{1}{8M}$. Then

$$-\frac{1}{2}\frac{d}{dt}\|\psi_\lambda\|_2^2 + \lambda\|\psi_\lambda\|_2^2 + \frac{1}{4}\|\nabla\psi_\lambda\|_2^2 \le C\|F\|_{6/5}^2. \tag{3.3}$$

Hence

$$\|\psi_\lambda(t)\|_2^2 \le e^{2\lambda t}\|\psi_\lambda(0)\|_2^2 + C\int_t^0 \|F(\tau)\|_{6/5}^2\,d\tau \qquad \text{for } t<0.$$

Let S be the map from $L^2_\sigma(\Omega)$ to $C((-\infty,0];L^2_\sigma(\Omega))$ defined by

$$S(t,a) = \psi_\lambda(t), \quad t\le 0,$$

where ψ_λ is the unique solution to $(\mathrm{D})_\lambda$ with $\psi_\lambda(0)=a$. Then,

$$\|S(T_{\epsilon k},a)\|_2^2 \le e^{2\lambda T_{\epsilon k}}\|a\|_2^2 + C\int_{T_{\epsilon k}}^0 \|F(\tau)\|_{6/5}^2\,d\tau.$$

In the same way as above, we easily have

$$\|S(T_{\epsilon k},a) - S(T_{\epsilon k},b)\|_2^2 \le e^{2\lambda T_{\epsilon k}}\|a-b\|_2^2.$$

Since $T_{\epsilon k}<0$, the above estimate implies that $S(T_{\epsilon k},\cdot): L^2_\sigma \to L^2_\sigma$ is a contraction operator. Hence, there exists $a_{\epsilon,k}\in L^2_\sigma$ such that $S(T_{\epsilon k},a_{\epsilon,k}) = a_{\epsilon,k}$.

Let $\phi_\lambda(t) := S(t,a_{\epsilon,k})$. Since $\phi_\lambda(T_{\epsilon k}) = \phi_\lambda(0) = a_{\epsilon,k}$, by inequality (3.3) we have

$$\int_{T_{\epsilon k}}^0 \|\nabla\phi_\lambda\|_2^2\,d\tau \le C\int_{T_{\epsilon k}}^0 \|F\|_{6/5}^2\,d\tau. \tag{3.4}$$

For any function g on $\Omega \times (-\infty, 0)$ let $dg(t) := g(t+T_{\epsilon k}) - g(t)$. Then $d\phi_\lambda$ satisfies

$$\begin{cases} -\partial_t d\phi_\lambda + (\lambda - \Delta) d\phi_\lambda - \sum_{i=1}^{3} (du_\lambda(t))^i \nabla \phi_\lambda^i (t+T_{\epsilon k}) & \\ -\sum_{i=1}^{3} u_\lambda^i \nabla d\phi_\lambda^i - (dv_\lambda(t)) \cdot \nabla \phi_\lambda(t+T_{\epsilon k}) - v_\lambda \cdot \nabla d\phi_\lambda + \nabla d\pi = dF_\lambda, \quad t<0, \\ \nabla \cdot d\phi_\lambda = 0, \ t<0, \quad d\phi_\lambda|_{t=0} = 0, \quad d\phi_\lambda|_{\partial\Omega} = 0. \end{cases}$$

Since by (3.1) $\sup_t \|u_\lambda\|_{3,\infty} < \delta$ and $\sup_t \|du_\lambda\|_{3,\infty} + \sup_t \|dv_\lambda\|_{3,\infty} < 2\epsilon$, the usual energy estimate yields

$$\begin{aligned} & -\frac{1}{2}\frac{d}{dt}\|d\phi_\lambda\|_2^2 + \lambda \|d\phi_\lambda\|_2^2 + \|\nabla d\phi_\lambda\|_2^2 \\ & \le 2M\epsilon \|\nabla d\phi_\lambda\|_2 \|\nabla \phi_\lambda(\cdot + T_{\epsilon k})\|_2 + M\delta \|\nabla d\phi_\lambda\|_2^2 + M\|dF_\lambda\|_{6/5}\|\nabla d\phi_\lambda\|_2 \\ & \le 8M^2\epsilon^2 \|\nabla \phi_\lambda(\cdot + T_{\epsilon k})\|_2^2 + (M\delta + \frac{1}{4})\|\nabla d\phi_\lambda\|_2^2 + 2M^2 \|dF_\lambda\|_{6/5}^2. \end{aligned} \tag{3.5}$$

Then, since $\sup_t \|dF_\lambda\|_{6/5}^2 \le \epsilon^2$ and since $d\phi_\lambda(0) = 0$, for $t < 0$ we have

$$\|d\phi_\lambda(t)\|_2^2 + \int_t^0 \|\nabla d\phi_\lambda\|_2^2 \, d\tau \le 16M^2\epsilon^2 \int_t^0 \|\nabla \phi_\lambda(\tau + T_{\epsilon k})\|_2^2 \, d\tau + C\epsilon^2(-t). \tag{3.6}$$

Since

$$\|\nabla \phi_\lambda(\tau + T_{\epsilon k})\|_2^2 = \|\nabla \phi_\lambda(\tau) + \nabla d\phi_\lambda(\tau)\|_2^2 \le 2\|\nabla \phi_\lambda(\tau)\|_2^2 + 2\|\nabla d\phi_\lambda(\tau)\|_2^2 \tag{3.7}$$

and since $32M^2\epsilon^2 \le 32M^2\delta^2 \le \frac{1}{2}$, by (3.6) we have

$$\|d\phi_\lambda(t)\|_2^2 + \frac{1}{2}\int_t^0 \|\nabla d\phi_\lambda\|_2^2 \, d\tau \le 32M^2\epsilon^2 \int_t^0 \|\nabla \phi_\lambda(\tau)\|_2^2 \, d\tau + C\epsilon^2(-t) \tag{3.8}$$

Hence from (3.4) and (3.8) we obtain

$$\int_{T_{\epsilon k}}^0 \|\nabla d\phi_\lambda\|_2^2 \, d\tau \le C\epsilon^2 \int_{T_{\epsilon k}}^0 \|F\|_{6/5}^2 \, d\tau + C\epsilon^2 |T_{\epsilon k}|. \tag{3.9}$$

The above estimate, (3.4) and (3.7) yields

$$\int_{2T_{\epsilon k}}^{T_{\epsilon k}} \|\nabla \phi_\lambda(t)\|_2^2 \, d\tau = \int_{T_{\epsilon k}}^0 \|\nabla \phi_\lambda(t + T_{\epsilon k})\|_2^2 \, d\tau \le C \int_{T_{\epsilon k}}^0 \|F\|_{6/5}^2 \, d\tau + C\epsilon^2 |T_{\epsilon k}|$$

and consequently

$$\int_{2T_{\epsilon k}}^0 \|\nabla \phi_\lambda(t)\|_2^2 \, d\tau \le C \int_{T_{\epsilon k}}^0 \|F\|_{6/5}^2 \, d\tau + C\epsilon^2 |T_{\epsilon k}|. \tag{3.10}$$

Repeating the previous step once again, we finally obtain

$$\int_{3T_{\epsilon k}}^{0} \|\nabla\phi_\lambda\|_2^2\, d\tau \le C\left(\int_{T_{\epsilon k}}^{0} \|F\|_{6/5}^2\, d\tau + \epsilon^2|T_{\epsilon k}|\right). \tag{3.11}$$

Hence, there exist a null sequence $\{\lambda_j\}$ and a function $\psi_{\epsilon k} \in L^2(3T_{\epsilon k}, 0; \dot{W}^{1,2}_{0,\sigma})$ such that

$$\phi_{\lambda_j} \to \psi_{\epsilon k} \text{ weakly in } L^2(3T_{\epsilon k}, 0; L^6_\sigma) \text{ and weakly in } L^2(3T_{\epsilon k}, 0; \dot{W}^{1,2}_{0,\sigma}) \tag{3.12}$$

Then (3.9), with $T_{\epsilon k}$ replaced by $2T_{\epsilon k}$ on the left side, and (3.11) yield (2.14) and (2.13), respectively.

By $(\mathrm{D})_\lambda$, we have, for $g \in C([3T_{\epsilon k}, 0]; D(A_2))$ with $\frac{d}{dt}g \in C([3T_{\epsilon k}, 0]; L^2)$,

(3.13)

$$\int_{T_{\epsilon k}}^{0} \big\{ -(g(t+T_{\epsilon k}), \phi_{\lambda_j}(t+T_{\epsilon k})) + (g(t), \phi_{\lambda_j}(t)) \big\}\, dt = \int_{T_{\epsilon k}}^{0}\int_{t+T_{\epsilon k}}^{t} \lambda_j(g, \phi_{\lambda_j})\, d\tau dt$$
$$+ \int_{T_{\epsilon k}}^{0}\int_{t+T_{\epsilon k}}^{t} \Big\{ \big(\frac{d}{\partial t}g, \phi_{\lambda_j}\big) + (\nabla g, \nabla\phi_{\lambda_j}) - \big(g, \sum_i^3 u^i_{\lambda_j}\nabla\phi^i_{\lambda_j}\big) + v_{\lambda_j}\cdot\nabla\phi_{\lambda_j}\big) - (g, F_{\lambda_j}) \Big\}\, d\tau dt.$$

Let $g \in C([3T_{\epsilon k}, 0]; D(A_2) \cap (\dot{W}^{1,2}_{0,\sigma})^*)$ and $\frac{d}{dt}g \in C([3T_{\epsilon k}, 0]; L^2 \cap (\dot{W}^{1,2}_{0,\sigma})^*)$. Since

$$\int_{3T_{\epsilon k}}^{0} |(U\cdot\nabla\phi, W)|\, d\tau \le C\|U\|_{L^4(3T_{\epsilon k},0;L^2+L^4)}\|\nabla\phi\|_{L^2(3T_{\epsilon k},0;L^2)}\|W\|_{L^4(3T_{\epsilon k},0;L^\infty\cap L^4)},$$

letting $j \to \infty$, by (3.1) and (3.12) we have (2.12). Then, it is straightforward to see that (2.12) holds for $g \in L^2(3T_{\epsilon k}, 0; D(A_2) \cap (\dot{W}^{1,2}_{0,\sigma})^*)$ with $\frac{d}{dt}g \in L^2(3T_{\epsilon k}, 0; L^2 \cap (\dot{W}^{1,2}_{0,\sigma})^*)$. This proves Lemma 2.3. □

4 Proof of Theorem 1

The proof is based on the idea given by Lions-Masmoudi [23] whereby the uniqueness problem is reduced to the solvability of the dual equation. In order to prove Theorem 1, we establish the following two lemmata.

Lemma 4.1. *Let w be an almost periodic function in $L^{3,\infty}(\Omega)$. Assume that for any almost periodic function F in $L^2(\Omega) \cap L^{6/5}(\Omega)$ and any number $\epsilon > 0$ there exists a sequence $\{T_{\epsilon k}\}_{k=1}^\infty$ such that*

$$T_{\epsilon k} \to -\infty \quad \text{as} \quad k \to \infty, \tag{4.1}$$

$$\limsup_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2}\int_{T_{\epsilon k}}^{0}\int_{t+T_{\epsilon k}}^{t} (w(\tau), F(\tau))\, d\tau dt \le C\epsilon, \tag{4.2}$$

where C is independent of k and ϵ. Then $w \equiv 0$ in $\Omega \times \mathbb{R}$.

Proof of Lemma 4.1. With $F = w \cdot 1_{B(0,r)}$ for an arbitrary number $r > 0$, we have

$$\begin{aligned} C\epsilon &\geq \limsup_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2} \int_{T_{\epsilon k}}^{0} \int_{t+T_{\epsilon k}}^{t} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau dt \\ &\geq \limsup_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2} \int_{T_{\epsilon k}/2}^{0} \int_{T_{\epsilon k}}^{T_{\epsilon k}/2} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau dt \\ &= \limsup_{k\to\infty} \frac{1}{2|T_{\epsilon k}|} \int_{T_{\epsilon k}}^{T_{\epsilon k}/2} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau \, . \end{aligned}$$

Hence, since $T_{\epsilon k} \to -\infty$ as $k \to \infty$ and since $\epsilon \in (0, \delta]$ is arbitrary,

$$\liminf_{T\to-\infty} \frac{1}{2|T|} \int_{T}^{T/2} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau = 0. \tag{4.3}$$

Then, by a contradiction argument, we can prove that

$$\|w(t)\|_{L^2(B(0,r)\cap\Omega)} = 0 \ \text{ for all } r > 0, \ t \in \mathbb{R}, \text{ i.e., } w \equiv 0. \tag{4.4}$$

Indeed, assume that (4.4) is not true, then $\int_{t_0}^{t_0+1} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau > 2\alpha$ for some $\alpha > 0$, some $r > 0$ and some $t_0 \in \mathbb{R}$. Proposition 2.2 implies that $\|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)}$ is almost periodic. Hence, $G(t) := \int_t^{t+1} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau$ is almost periodic. Therefore, we find $L = L(\alpha) > 1$ such that for all $a \in \mathbb{R}$ there exists $T_a \in [a - L, a]$ satisfying $\sup_t \|G(t + T_a) - G(t)\|_{L^2(B(0,r)\cap\Omega)} \leq \alpha$ and consequently,

$$\int_{t_0+T_a}^{t_0+T_a+1} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} d\tau = G(t_0 + T_a) \geq G(t_0) - \alpha > \alpha.$$

Hence

$$\int_{t_0+a-L}^{t_0+a+1} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau > \alpha.$$

Letting $L^* = L + 1$, $a = -t_0 - 1 - kL^*$, we see that there exists $L^* > 0$ such that

$$\int_{-(k+1)L^*}^{-kL^*} \|w(\tau)\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau > \alpha \ \text{ for all } k \in \mathbb{Z}.$$

For $T < 0$ with $|T| >> 1$, choose $m \in \mathbb{N}$ such that $-(m + 1)L^* < T/2 \leq -mL^*$. Then

$$\frac{1}{|T|} \int_{T}^{T/2} \|w\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau \geq \frac{1}{2(m+1)L^*} \int_{-2mL^*}^{-(m+1)L^*} \|w\|^2_{L^2(B(0,r)\cap\Omega)} \, d\tau \geq \frac{(m-1)\alpha}{2(m+1)L^*} \, .$$

This contradicts (4.3) and hence we conclude that (4.4) is true. $\square$

Lemma 4.2. *Let Ω, u, v satisfy the hypotheses of Theorem 1, F be an arbitrary almost periodic function in $L^2(\Omega) \cap L^{6/5}(\Omega)$ and let $T_{\epsilon k} = T_{\epsilon k}(u, v, F) \in [-(k+1)l, -kl]$, $k \in \mathbb{N}$, be the negative numbers given in Lemma 2.3. Then $w := u - v$ satisfies (4.2) for all $\epsilon \in (0, \delta]$.*

Proof of Lemma 4.2. Looking at the system (U) in Section 2, for $t > 3T_{\epsilon k}$ let

$$
\begin{aligned}
&(4.5) \qquad & w(t) &= w_0(t) + w_1(t),\\
&(4.6) \qquad & w_0(t) &= e^{-(t-3T_{\epsilon k})A} w(3T_{\epsilon k})\\
&(4.7) \qquad & (w_1(t), \phi) &= \int_{3T_{\epsilon k}}^{t} \left\{ (w \cdot \nabla e^{-(t-\tau)A}\phi, u) + (v \cdot \nabla e^{-(t-\tau)A}\phi, w) \right\} d\tau \ \text{ for } \phi \in C^\infty_{0,\sigma}.
\end{aligned}
$$

We note that (4.7) holds for all $\phi \in L^2_\sigma$, and from (2.8) we conclude that $|(w_1(t), \phi)| \le C(u, v)(t - 3T_{\epsilon k})^{1/2} \|\nabla \phi\|_2$, since $w \otimes u, v \otimes w \in L^2(3T_{\epsilon k}, 0; L^2)$ by Lemma 2.2. Hence, $w_1 \in L^\infty(3T_{\epsilon k}, 0; (\dot{W}^{1,2}_{0,\sigma})^*)$. Let $t > T_{\epsilon k} (> 3T_{\epsilon k})$ and let us write, using the notation $⨍$ for integral means,

$$
\begin{aligned}
&⨍_{T_{\epsilon k}}^{0} ⨍_{t+T_{\epsilon k}}^{t} (w(\tau), F(\tau))\, d\tau dt\\
(4.8) \qquad &= ⨍_{T_{\epsilon k}}^{0} ⨍_{t+T_{\epsilon k}}^{t} (w_0(\tau), F(\tau))\, d\tau dt + ⨍_{T_{\epsilon k}}^{0} ⨍_{t+T_{\epsilon k}}^{t} (w_1(\tau), F(\tau))\, d\tau dt\\
&=: I_0 + I_1
\end{aligned}
$$

By Lemma 2.1, we have

$$
\begin{aligned}
\Big| \int_{t+T_{\epsilon k}}^{t} (w_0(\tau), F(\tau))\, d\tau \Big| \quad &\le \int_{t+T_{\epsilon k}}^{t} \|e^{-(\tau-3T_{\epsilon k})\Delta} w(3T_{\epsilon k})\|_6 \|F(\tau)\|_{6/5}\, d\tau\\
&\le C|T_{\epsilon k}|^{3/4} \sup_\tau \|w(\tau)\|_{3,\infty} \sup_\tau \|F(\tau)\|_{6/5}.
\end{aligned}
$$

Hence

$$
(4.9) \qquad |I_0| = \Big| ⨍_{T_{\epsilon k}}^{0} ⨍_{t+T_{\epsilon k}}^{t} (w_0(\tau), F(\tau))\, d\tau dt \Big| \le C|T_{\epsilon k}|^{-1/4} \sup_\tau \|w(\tau)\|_{3,\infty} \sup_\tau \|F(\tau)\|_{6/5}
$$

converges to 0 as $k \to \infty$ since $T_{\epsilon k} \to -\infty$.

In order to estimate I_1, we consider an approximation of w_1. Let

$$
Y = L^2(3T_{\epsilon k}, 0; L^{6,2}(\Omega))
$$

and let

$$
(4.10) \qquad w_{1,\alpha}(t) := - \int_{3T_{\epsilon k}}^{t} e^{-(t-\tau)A} P(w \cdot \nabla u_\alpha + v \cdot \nabla w_\alpha)\, d\tau,
$$

where $\{u_\alpha\}, \{w_\alpha\} \subset BC(\mathbb{R}\times\Omega)\cap Y$ are approximation sequences such that

$$
\begin{aligned}
&u_\alpha \to u \ \text{ in } Y,\ w_\alpha \to w \ \text{ in } Y \ \text{ as } \alpha\to 0+, \quad \nabla u_\alpha, \nabla w_\alpha \in Y,\\
&\sup_t \|u_\alpha(t)\|_{3,\infty} \le C \sup_t \|u(t)\|_{3,\infty}, \quad \|u_\alpha\|_Y \le C\|u\|_Y, \quad \|w_\alpha\|_Y \le C\|w\|_Y,
\end{aligned}
\tag{4.11}
$$

where the constant C is independent of α. These sequences u_α, w_α are constructed by using the operator $(\rho_\alpha(t)*)e^{-\alpha A}$. Then,

$$
\left\{
\begin{aligned}
\frac{d}{dt}w_{1,\alpha} + Aw_{1,\alpha} &= -P(w\cdot\nabla u_\alpha + v\cdot\nabla w_\alpha) \quad \text{in } \ L^2(3T_{\epsilon k},0;L^2_\sigma),\\
w_{1,\alpha}(3T_{\epsilon k}) &= 0,
\end{aligned}
\right.
\tag{4.12}
$$

and the well-known L^2-maximal regularity (see e.g. [33, Chap. IV, Theorem 1.6.3]) yields

$$
\begin{aligned}
&w_{1,\alpha}\in L^2(3T_{\epsilon k},0;D(A_2)),\ \frac{d}{dt}w_{1,\alpha}\in L^2(3T_{\epsilon k},0;L^2_\sigma),\\
&\int_{3T_{\epsilon k}}^0 \Big\{\|\frac{d}{dt}w_{1,\alpha}\|_2^2 + \|Aw_{1,\alpha}\|_2^2\Big\}\,d\tau \le C\int_{3T_{\epsilon k}}^0 \|P(w\cdot\nabla u_\alpha + v\cdot\nabla w_\alpha)\|_2^2\,d\tau < \infty.
\end{aligned}
\tag{4.13}
$$

For all $\phi\in C_0(\Omega)$ and $3T_{\epsilon k}<t<0$, it holds that

$$
\begin{aligned}
&|(w_1(t)-w_{1,\alpha}(t),\phi)| = |(w_1(t)-w_{1,\alpha}(t),P\phi)|\\
&= \Big|\int_{3T_{\epsilon k}}^t (w\cdot\nabla e^{-(t-\tau)A}P\phi, u-u_\alpha) + (v\cdot\nabla e^{-(t-\tau)A}P\phi, w-w_\alpha)\,d\tau\Big|\\
&\le C\int_{3T_{\epsilon k}}^t (\|w\|_{3,\infty}\|u-u_\alpha\|_{6,2} + \|v\|_{3,\infty}\|w-w_\alpha\|_{6,2})\|\nabla e^{-(t-\tau)A}P\phi\|_2\,d\tau\\
&\le C\|\phi\|_2((\sup_t\|w\|_{3,\infty})\|u-u_\alpha\|_Y + (\sup_t\|v\|_{3,\infty})\|w-w_\alpha\|_Y),
\end{aligned}
$$

since by (2.9) $\int_{-\infty}^t \|\nabla e^{-(t-\tau)A}P\phi\|_2^2\,d\tau \le \|P\phi\|_2^2$. Thus we have

$$
\sup_{3T_{\epsilon k}\le t\le 0}\|w_1 - w_{1,\alpha}\|_2 \to 0 \ \text{ as } \alpha\to 0+\,. \tag{4.14}
$$

We shall substitute $w_{1,\alpha}$ as test function g in equation (2.12). To this end, we need to show that

$$
w_{1,\alpha}\in L^2(3T_{\epsilon k},0;(\dot W^{1,2}_{0,\sigma})^*),\quad \frac{d}{dt}w_{1,\alpha}\in L^2(3T_{\epsilon k},0;L^2_\sigma\cap(\dot W^{1,2}_{0,\sigma})^*)\,. \tag{4.15}
$$

Since for $\phi\in D(A_2)$

$$
\begin{aligned}
|(P(w\cdot\nabla u_\alpha + v\cdot\nabla w_\alpha)(t),\phi)| \ &\le\ |(w\cdot\nabla\phi,u_\alpha)| + |(v\cdot\nabla\phi,w_\alpha)|\\
&\le C\big(\|w(t)\|_{6,2}\sup_t\|u_\alpha\|_{3,\infty} + \|w_\alpha(t)\|_{6,2}\sup_t\|v\|_{3,\infty}\big)\|\nabla\phi\|_2\,,
\end{aligned}
\tag{4.16}
$$

we observe that (4.12) and an energy estimate yield

$$\sup_{3T_{\epsilon k}<t<0} \|w_{\alpha,1}(t)\|_2^2 + 2\int_{3T_{\epsilon k}}^0 \|\nabla w_{1,\alpha}\|_2^2\, d\tau$$
$$\leq C\|w\|_Y^2 \big(\sup_t \|u\|_{3,\infty}^2 + \sup_t \|v\|_{3,\infty}^2\big) + \int_{3T_{\epsilon k}}^0 \|\nabla w_{1,\alpha}\|_2^2\, d\tau,$$

which implies

$$\int_{3T_{\epsilon k}}^0 \|\nabla w_{1,\alpha}\|_2^2\, d\tau \leq C\|w\|_Y^2 \big(\sup_t \|u\|_{3,\infty}^2 + \sup_t \|v\|_{3,\infty}^2\big). \tag{4.17}$$

Since $|(Aw_{1,\alpha}, \phi)| \leq \|\nabla w_{1,\alpha}\|_2 \|\nabla \phi\|_2$ for $\phi \in D(A_2)$, from (4.16) we obtain

$$\frac{d}{dt} w_{1,\alpha} = -Aw_{1,\alpha} - P(w\cdot\nabla u_\alpha + v\cdot\nabla w_\alpha) \in L^2(3T_{\epsilon k}, 0; (\dot W_{0,\sigma}^{1,2})^*). \tag{4.18}$$

Since $w_{1,\alpha}(3T_{\epsilon k}) = 0$, by (4.18) we see that $w_{1,\alpha} \in C([3T_{\epsilon k}, 0]; (\dot W_{0,\sigma}^{1,2})^*)$. Therefore, from (4.13), (4.18) we obtain the assertion (4.15). Moreover, we easily conclude that even

$$\begin{aligned}
&\{w_{1,\alpha}\}_{\alpha>0} \text{ is bounded in } L^2(3T_{\epsilon k}, 0; \dot W_{0,\sigma}^{1,2}) (\subset Y),\\
&\{\tfrac{d}{dt} w_{1,\alpha}\}_{\alpha>0} \text{ is bounded in } L^2(3T_{\epsilon k}, 0; (\dot W_{0,\sigma}^{1,2})^*),\\
&\{w_{1,\alpha}\}_{\alpha>0} \text{ is bounded in } BC([3T_{\epsilon k}, 0]; (\dot W_{0,\sigma}^{1,2})^*).
\end{aligned} \tag{4.19}$$

Then (4.14) yields that

$$\begin{aligned}
&w_{1,\alpha} \to w_1 \ \text{ weakly in } L^2(3T_{\epsilon k}, 0; \dot W_{0,\sigma}^{1,2}) \text{ and weakly in } Y,\\
&w_{1,\alpha}(t) \to w_1(t) \text{ weakly} - * \ \text{ in } (\dot W_{0,\sigma}^{1,2})^* \text{ for all } t \in [3T_{\epsilon k}, 0].
\end{aligned} \tag{4.20}$$

Moreover, from (4.19) and (4.20), we obtain as in Lemma 2.3 for $U \in L^\infty(3T_{\epsilon k}, 0; L_\sigma^{3,\infty})$ and $\psi_{\epsilon k} \in L^2(3T_{\epsilon k}, 0; \dot W_{0,\sigma}^{1,2})$

$$\int_{t+T_{\epsilon k}}^t \big((w_{1,\alpha} - w_1)\cdot\nabla\psi_{\epsilon k}(\tau), U(\tau)\big)\, d\tau \to 0 \ \text{ as } \alpha \to +0,$$
$$\Big|\int_{t+T_{\epsilon k}}^t \big((w_{1,\alpha} - w_1)\cdot\nabla\psi_{\epsilon k}(\tau), U(\tau)\big)\, d\tau\Big|$$
$$\leq \big(\|w_{1,\alpha}\|_Y + \|w_1\|_Y\big)\|\psi_{\epsilon k}\|_{L^2(3T_{\epsilon k},0;\dot W^{1,2})} \sup_\tau \|U\|_{3,\infty} < C < \infty$$

for all $t \in [T_{\epsilon k}, 0]$, where C is a constant independent of α. Hence Lebesgue's Dominated Convergence Theorem yields

$$\int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t \big((w_{1,\alpha} - w_1)\cdot\nabla\psi_{\epsilon k}(\tau), U(\tau)\big)\, d\tau dt \to 0 \ \text{ as } \alpha \to +0. \tag{4.21}$$

Substituting $w_{1,\alpha}$ into equation (2.12) for g, by (4.12) we have

$$\begin{aligned}
&\int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t (F, w_{1,\alpha})\, d\tau dt \\
=&\int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t \Big\{ (\frac{d}{dt} w_{1,\alpha} + A w_{1,\alpha}, \psi_{\epsilon k}) - (w_{1,\alpha}\cdot\nabla\psi_{\epsilon k}, u) - (v\cdot\nabla\psi_{\epsilon k}, w_{1,\alpha}) \Big\}\, d\tau dt \\
&\quad + \int_{T_{\epsilon k}}^0 \big(w_{1,\alpha}(t+T_{\epsilon k}), \psi_{\epsilon k}(t+T_{\epsilon k})) - (w_{1,\alpha}(t), \psi_{\epsilon k}(t)\big)\, dt \\
=&\int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t \Big\{ (w\cdot\nabla\psi_{\epsilon k}, u_\alpha - u) + ((w - w_{1,\alpha})\cdot\nabla\psi_{\epsilon k}, u) + (v\cdot\nabla\psi_{\epsilon k}, w_\alpha - w_{1,\alpha}) \Big\}\, d\tau dt \\
&\quad + \int_{T_{\epsilon k}}^0 \big(w_{1,\alpha}(t+T_{\epsilon k}), \psi_{\epsilon k}(t+T_{\epsilon k})) - (w_{1,\alpha}(t), \psi_{\epsilon k}(t)\big)\, dt\,.
\end{aligned}$$

Letting $\alpha \to 0+$, from (4.11), (4.20)-(4.21) we obtain

(4.22)

$$\begin{aligned}
&\int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t (F, w_1)\, d\tau dt \\
=&\int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t \Big\{ (w_0\cdot\nabla\psi_{\epsilon k}, u) + (v\cdot\nabla\psi_{\epsilon k}, w_0) \Big\}\, d\tau dt \\
&\qquad + \int_{T_{\epsilon k}}^0 (w_1\big(t+T_{\epsilon k}), \psi_{\epsilon k}(t+T_{\epsilon k})) - (w_1(t), \psi_{\epsilon k}(t)\big)\, dt \\
\le&\int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t \big|(w_0\cdot\nabla\psi_{\epsilon k}, u)\big|\, d\tau dt + \int_{T_{\epsilon k}}^0 \int_{t+T_{\epsilon k}}^t \big|(v\cdot\nabla\psi_{\epsilon k}, w_0)\big|\, d\tau dt \\
&+ \Big| \int_{T_{\epsilon k}}^0 (w_1(t+T_{\epsilon k}), \psi_{\epsilon k}(t+T_{\epsilon k}) - \psi_{\epsilon k}(t))\, dt \Big| + \Big| \int_{T_{\epsilon k}}^0 (w_1(t+T_{\epsilon k}) - w_1(t), \psi_{\epsilon k}(t))\, dt \Big| \\
=&J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Since by (2.6) $\|w_0(\tau)\|_{6,2} = \|e^{-(\tau-3T_{\epsilon k})A} w(3T_{\epsilon k})\|_{6,2} \le C(\tau - 3T_{\epsilon k})^{-1/4} \|w(3T_{\epsilon k})\|_{3,\infty}$, we get from (2.13) that

$$\begin{aligned}
J_1 \le& \int_{T_{\epsilon k}}^0 \int_{2T_{\epsilon k}}^0 \|w_0(\tau)\|_{6,2} \|\nabla\psi_{\epsilon k}(\tau)\|_2 \|u(\tau)\|_{3,\infty}\, d\tau dt \\
\le& C|T_{\epsilon k}| \sup_t \|u(t)\|_{3,\infty} \int_{2T_{\epsilon k}}^0 (\tau - 3T_{\epsilon k})^{-1/4} \|w(3T_{\epsilon k})\|_{3,\infty} \|\nabla\psi_{\epsilon k}(\tau)\|_2\, d\tau \\
\le& C|T_{\epsilon k}|^{7/4} \sup_t \|u\|_{3,\infty} \sup_t \|w\|_{3,\infty} \left(\frac{1}{|T_{\epsilon k}|} \int_{2T_{\epsilon k}}^0 \|\nabla\psi_{\epsilon k}(\tau)\|_2^2\, d\tau \right)^{1/2} \\
\le& C|T_{\epsilon k}|^{7/4} \sup_t \|u\|_{3,\infty} \sup_t \|w\|_{3,\infty} (1 + \sup_t \|F(t)\|_{6/5}^2)^{1/2}\,.
\end{aligned}$$

Hence, we have

$$\lim_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2} J_1 = 0. \tag{4.23}$$

Similarly, we have

$$\lim_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2} J_2 = 0. \tag{4.24}$$

By the definition (4.7) of w_1 and (2.8), we have

$$\begin{aligned}
J_3 \le& \Big| \int_{T_{\epsilon k}}^0 \int_{3T_{\epsilon k}}^{t+T_{\epsilon k}} (w(\tau)\cdot\nabla e^{-(t+T_{\epsilon k}-\tau)A}(\psi_{\epsilon k}(t+T_{\epsilon k}) - \psi_{\epsilon k}(t)), u(\tau))\, d\tau dt \Big| \\
&+ \Big| \int_{T_{\epsilon k}}^0 \int_{3T_{\epsilon k}}^{t+T_{\epsilon k}} (v(\tau)\cdot\nabla e^{-(t+T_{\epsilon k}-\tau)A}(\psi_{\epsilon k}(t+T_{\epsilon k}) - \psi_{\epsilon k}(t)), w(\tau))\, d\tau dt \Big| \\
\le& C\Big(\sup_t \|w\|_{3,\infty} \int_{3T_{\epsilon k}}^{T_{\epsilon k}} \|u\|_{6,2}\, d\tau + \sup_t \|v\|_{3,\infty} \int_{3T_{\epsilon k}}^{T_{\epsilon k}} \|w\|_{6,2}\, d\tau \Big) \\
&\times \int_{T_{\epsilon k}}^0 \|\nabla(\psi_{\epsilon k}(t+T_{\epsilon k}) - \psi_{\epsilon k}(t))\|_2\, dt
\end{aligned}$$

so that by (2.14)

$$\begin{aligned}
\frac{1}{|T_{\epsilon k}|^2} J_3 \le& C\big(\sup_t \|w\|_{3,\infty} \|u\|_{L^2_{uloc}(\mathbb{R};L^{6,2})} + \sup_t \|v\|_{3,\infty} \|w\|_{L^2_{uloc}(\mathbb{R};L^{6,2})} \big) \\
&\times \frac{1}{|T_{\epsilon k}|} \int_{T_{\epsilon k}}^0 \|\nabla(\psi_{\epsilon k}(t+T_{\epsilon k}) - \psi_{\epsilon k}(t))\|_2\, dt \\
\le& C\epsilon.
\end{aligned} \tag{4.25}$$

Similarly,

$$\begin{aligned}
J_4 =& \Big| \int_{T_{\epsilon k}}^0 (w_1(t+T_{\epsilon k}) - w_1(t), \psi_{\epsilon k}(t)) dt \Big| \\
\le& \Big| \int_{T_{\epsilon k}}^0 \Big\{ \int_{3T_{\epsilon k}}^{t+T_{\epsilon k}} (w(\tau)\cdot\nabla e^{-(t+T_{\epsilon k}-\tau)A}\psi_{\epsilon k}(t), u(\tau))\, d\tau \\
&\qquad - \int_{3T_{\epsilon k}}^{t} (w(\tau)\cdot\nabla e^{-(t-\tau)A}\psi_{\epsilon k}(t), u(\tau))\, d\tau \Big\}\, dt \Big| \\
&\qquad\qquad + \text{ a similar term with } (w,u) \text{ replaced by } (v,w) \\
=& \Big| \int_{T_{\epsilon k}}^0 \Big\{ \int_{2T_{\epsilon k}}^{t} (w(\tau+T_{\epsilon k})\cdot\nabla e^{-(t-\tau)A}\psi_{\epsilon k}(t), u(\tau+T_{\epsilon k}))\, d\tau \\
&- \Big(\int_{2T_{\epsilon k}}^{t} + \int_{3T_{\epsilon k}}^{2T_{\epsilon k}} \Big) (w(\tau)\cdot\nabla e^{-(t-\tau)A}\psi_{\epsilon k}(t), u(\tau))\, d\tau \Big\}\, dt \Big| \ + \text{ similar terms}.
\end{aligned}$$

Hence

$$
\begin{aligned}
J_4 \le & \int_{T_{\epsilon k}}^{0} \int_{2T_{\epsilon k}}^{t} \|w \otimes u(\tau + T_{\epsilon k}) - w \otimes u(\tau)\|_2 \|\nabla e^{-(t-\tau)A} \psi_{\epsilon k}(t)\|_2 \, d\tau dt \\
& + \Big| \int_{T_{\epsilon k}}^{0} \int_{3T_{\epsilon k}}^{2T_{\epsilon k}} (w(\tau) \cdot \nabla e^{-(t-\tau)A} \psi_{\epsilon k}(t), u(\tau)) \, d\tau dt \Big| \; + \; \text{similar terms} \\
=: & \, P_1 + P_2 + \; \text{similar terms.}
\end{aligned} \tag{4.26}
$$

Since, using (2.5),

$$
\|w \otimes u(\tau + T_{\epsilon k}) - w \otimes u(\tau)\|_2 \le C\epsilon (\|w(\tau + T_{\epsilon k})\|_{6,2} + \|u(\tau)\|_{6,2}) \;\; \text{for all } k \in \mathbb{N}, \tag{4.27}
$$

by (2.13) we have

$$
P_1 \le C\epsilon |T_{\epsilon k}| \int_{T_{\epsilon k}}^{0} \|\nabla \psi_{\epsilon k}(t)\|_2 dt \le C\epsilon |T_{\epsilon k}|^2. \tag{4.28}
$$

Next we estimate P_2.

$$
\begin{aligned}
P_2 \le & \int_{T_{\epsilon k}}^{0} \int_{-3(k+1)l}^{-2kl} \big|(w \otimes u(\tau), \nabla e^{-(t-\tau)A} \psi_{\epsilon k}(t))\big| \, d\tau dt \\
\le & \int_{T_{\epsilon k}}^{0} \sum_{j=2k}^{3k+2} \int_{-(j+1)l}^{-jl} \big|(w \otimes u(\tau), \nabla e^{-(t-\tau)A} \psi_{\epsilon k}(t))\big| \, d\tau dt \\
\le & \int_{T_{\epsilon k}}^{0} \sum_{j=2k}^{3k+2} \int_{-(j+1)l}^{-jl} \big|(w \otimes u(\tau) - w \otimes u(\tau - T_{\epsilon j}), \nabla e^{-(t-\tau)A} \psi_{\epsilon k}(t))\big| \, d\tau dt \\
& + \int_{T_{\epsilon k}}^{0} \sum_{j=2k}^{3k+2} \int_{-(j+1)l}^{-jl} \big|(w \otimes u(\tau - T_{\epsilon j}), \nabla e^{-(t-\tau)A} \psi_{\epsilon k}(t))\big| \, d\tau dt \\
=: & P_{2,1} + P_{2,2} \, .
\end{aligned} \tag{4.29}
$$

By (2.13) and (4.27) we have, since $l > 1$ and $|T_{\epsilon k}| \ge kl$,

$$
\begin{aligned}
P_{2,1} \le & C\epsilon \int_{T_{\epsilon k}}^{0} \sum_{j=2k}^{3k+2} \int_{-(j+1)l}^{-jl} (\|w(\tau)\|_{6,2} + \|u(\tau - T_{\epsilon j})\|_{6,2}) \|\nabla \psi_{\epsilon k}(t)\|_2 \, d\tau dt \\
\le & C\epsilon \Big(\int_{T_{\epsilon k}}^{0} \|\nabla \psi_{\epsilon k}(t)\|_2 \, dt \Big) \, k \, l \, (\|w\|_{L^2_{uloc}(\mathbb{R}; L^{6,2})} + \|u\|_{L^2_{uloc}(\mathbb{R}; L^{6,2})}) \\
\le & C\epsilon |T_{\epsilon k}|^2 \, .
\end{aligned} \tag{4.30}
$$

Concerning $P_{2,2}$ we find, since $w \otimes u \in L^2(-l, l; L^2(\Omega))$, a sequence $\{g_n\}_{n=1}^{\infty}$ such that

$$
g_n \in C([-l, l]; C_0^1(\Omega)), \qquad \int_{-l}^{l} \|u \otimes w - g_n\|_2^2 \, d\tau \le 1/n^2 \, . \tag{4.31}
$$

Then,

$$
\begin{aligned}
P_{2,2} \le & \int_{T_{\epsilon k}}^{0} \sum_{j=2k}^{3k+2} \int_{-(j+1)l}^{-jl} \left|(w \otimes u(\tau - T_{\epsilon j}) - g_n(\tau - T_{\epsilon j}), \nabla e^{-(t-\tau)A} \psi_{\epsilon k}(t))\right| \, d\tau dt \\
& + \int_{T_{\epsilon k}}^{0} \sum_{j=2k}^{3k+2} \int_{-(j+1)l}^{-jl} \left|(g_n(\tau - T_{\epsilon j}), \nabla e^{-(t-\tau)A} \psi_{\epsilon k}(t))\right| \, d\tau dt \\
& =: K_1 + K_2.
\end{aligned} \tag{4.32}
$$

Since $-l \le \tau - T_{\epsilon j} \le l$ for $\tau \in [-(j+1)l, -jl]$,

$$
\int_{-(j+1)l}^{-jl} \|w \otimes u(\tau - T_{\epsilon j}) - g_n(\tau - T_{\epsilon j})\|_2 \, d\tau \le \int_{-l}^{l} \|w \otimes u(\tau) - g_n(\tau)\|_2 \, d\tau \le C \frac{l^{1/2}}{n}.
$$

Hence, by (2.13) we have

$$
K_1 \le C|T_{\epsilon k}| \sum_{j=2k}^{3k+2} \frac{l^{1/2}}{n} \le C|T_{\epsilon k}| \frac{(k+2)l^{1/2}}{n} \le \frac{C|T_{\epsilon k}|^2}{l^{1/2} n} \, . \tag{4.33}
$$

By integration by parts, we have

$$
\begin{aligned}
K_2 \le & \int_{T_{\epsilon k}}^{0} \sum_{j=2k}^{3k+2} \int_{-(j+1)l}^{-jl} \|\nabla g_n(\tau - T_{\epsilon j})\|_{12/11} \|e^{-(t-\tau)A} \psi_{\epsilon k}(t)\|_{12} \, d\tau dt \\
\le & C \sup_{-l \le s \le l} \|\nabla g_n(s)\|_{12/11} \int_{T_{\epsilon k}}^{0} \sum_{j=2k}^{3k+2} \int_{-(j+1)l}^{-jl} (t-\tau)^{-1/8} \|\psi_{\epsilon k}(t)\|_6 \, d\tau dt \\
\le & C \sup_{-l \le s \le l} \|\nabla g_n(s)\|_{12/11} \int_{T_{\epsilon k}}^{0} \int_{-(3k+3)l}^{-2kl} (t-\tau)^{-1/8} \|\psi_{\epsilon k}(t)\|_6 \, d\tau dt \\
\le & C \sup_{-l \le s \le l} \|\nabla g_n(s)\|_{12/11} (kl)^{7/8} \int_{T_{\epsilon k}}^{0} \|\psi_{\epsilon k}(t)\|_6 \, dt \\
\le & C \sup_{-l \le s \le l} \|\nabla g_n(s)\|_{12/11} |T_{\epsilon k}|^{15/8} \, .
\end{aligned} \tag{4.34}
$$

Therefore, from (4.33) and (4.34) we obtain $\limsup_{k\to\infty} P_{2,2}/|T_{\epsilon k}|^2 \le C/(l^{1/2}n)$ for all $n \in \mathbb{N}$, which yields

$$
\limsup_{k\to\infty} \frac{P_{2,2}}{|T_{\epsilon k}|^2} = 0. \tag{4.35}
$$

By combining (4.30) and (4.35) with (4.29) we find that $\limsup_{k\to\infty} \frac{P_2}{|T_{\epsilon k}|^2} \le C\epsilon$; by analogy, the "similar terms" in (4.26) can be estimated in the same way. Summarizing the last estimate and (4.28), (4.26) we get

$$
\limsup_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2} J_4 \le C\epsilon. \tag{4.36}
$$

Finally, from (4.22), (4.23), (4.24), (4.25) and (4.36), we obtain for I_1, see (4.8),

$$\limsup_{k\to\infty} |I_1| = \limsup_{k\to\infty} \left| \fint_{T_{\epsilon k}}^{0} \fint_{t+T_{\epsilon k}}^{t} (w_1(\tau), F(\tau))\, d\tau dt \right| \le C\epsilon.$$

This and (4.9) yields the assertion (4.2), which proves Lemma 4.2. □

Obviously, Lemmata 4.1 and 4.2 complete the proof of Theorem 1.

5 Appendix

In this appendix, we will show that if a mild solution $u \in C(\mathbb{R}; L^3_\sigma)$ on $\mathbb{R}$ is almost periodic in L^3_σ, then u always satisfies (2.3). It suffices to prove the following theorem.

Theorem 2. *Let Ω satisfy Assumption 1. If a mild solution $u \in C(\mathbb{R}; L^3_\sigma)$ on $\mathbb{R}$ to (N-S) is almost-periodic in L^3_σ and if $f \in L^\infty(\mathbb{R}; L^3)$, then u belongs to $L^\infty(\mathbb{R}; L^9_\sigma)$.*

We prove Theorem 2 by using an argument similar to that in [16, Theorems 2.3, 2.10]. First, we recall the local existence theorem for the initial-boundary value problem of (N-S).

Proposition 5.1. *Let Ω satisfy Assumption 1. Then, there exist numbers $\delta_0 = \delta_0(\Omega) > 0$, $C_0 = C_0(\Omega) > 0$ and $C_1 = C_1(\Omega)$ with the following property. If $v_0 \in L^3_\sigma$ and $g_0 \in L^3_\sigma \cap L^9_\sigma$ satisfy $\|v_0 - g_0\|_3 \le \delta_0$ and if $f \in L^\infty(\mathbb{R}_+; L^3)$, then there exists a mild L^3-solution v on $[0, T^*]$ to (N-S) with initial data $v(0) = v_0$ such that*

$$\sup_{0<t<T^*} t^{1/3} \|v(t)\|_9 < C_1,$$

where

$$T^* = C_0 \min\{\|g_0\|_9^{-3}, \|f\|^{-1}_{L^\infty(\mathbb{R}_+; L^3)}\}.$$

Proof of Theorem 2. Let $\delta_0 > 0$ be the number given in Proposition 5.1. Since u is almost periodic in L^3, there exists $L = L(\delta_0) > 0$ with the following property: For all $a \in \mathbb{R}$, we can find a number $T(a) \in [a, a+L]$ such that

$$\sup_t \|u(t) - u(t - T(a))\|_3 < \delta_0/2. \tag{5.1}$$

Since $u \in C([-L, 0]; L^3_\sigma)$ and $L^3_\sigma \cap L^9_\sigma$ is dense in L^3_σ, there exists a function $g \in L^\infty(-L, 0; L^3_\sigma \cap L^9_\sigma)$ such that

$$\sup_{-L\le t\le 0} \|u(t) - g(t)\|_3 < \delta_0/2. \tag{5.2}$$

Hence, by (5.1) and (5.2), for all $a \in \mathbb{R}$, it holds that

$$\|u(a) - g(a - T(a))\|_3 \le \|u(a) - u(a - T(a)\|_3 + \|u(a - T(a)) - g(a - T(a))\|_3 < \delta_0,$$

since $a - T(a) \in [-L, 0]$. Letting

$$\tilde{T}^* := C_0 \min\{\|g\|^{-3}_{L^\infty(-L,0;L^9)}, \|f\|^{-1}_{L^\infty(\mathbb{R};L^3)}\},$$

and using Proposition 5.1 with $(v_0, g_0) = (u(a), g(a - T(a)))$ we observe that for all $a \in \mathbb{R}$ there exists a mild L^3-solution v on $[0, \tilde{T}^*]$ to (N-S) with the initial data $v(0) = u(a)$ and the external force $f(\cdot + a)$ such that

$$\sup_{0<t<\tilde{T}^*} t^{1/3}\|v(t)\|_9 < C_1.$$

Then, by using the uniqueness theorem of mild solutions in $C([0,T); L^3_\sigma)$, see [23], we conclude that

$$\sup_{0<t<\tilde{T}^*} t^{1/3}\|u(t + a)\|_9 < C_1$$

for all $a \in \mathbb{R}$. Since g and hence also $\tilde{T}^*$ are independent of $a \in \mathbb{R}$, the assertion $u \in L^\infty(\mathbb{R}; L^9)$ is proved. □

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