Leading term at infinity of steady Navier-Stokes flow around a rotating obstacle

Reinhard Farwig
FB Mathematik, Technische Universität Darmstadt
64289 Darmstadt, Germany
and
Toshiaki Hishida*
Graduate School of Mathematics, Nagoya University
Nagoya 464-8602 Japan

Abstract

Consider a viscous incompressible flow around a body in $\mathbb{R}^3$ rotating with constant angular velocity ω. Using a coordinate system attached to the body, the problem is reduced to a modified Navier-Stokes system in a fixed exterior domain. This paper addresses the question of the asymptotic behavior of stationary solutions to the new system as $|x| \to \infty$. Under a suitable smallness assumption on the velocity field, u, and the net force on the boundary, N, we prove that the leading term of u is the so-called Landau solution U, a singular solution of the stationary Navier-Stokes system in $\mathbb{R}^3$ with external force $k\omega\delta_0$ and decaying as $1/|x|$; here $k \in \mathbb{R}$ is a suitable constant determined by N and δ_0 is the Dirac measure supported in the origin.

Key Words and Phrases. Navier-Stokes flow; rotating obstacle; leading term; asymptotic profile
2000 *Mathematics Subject Classification Numbers.* 35B40; 35Q30; 76D05

*Supported in part by Grant-in-Aid for Scientific Research, No. 19540170, from the Japan Society for the Promotion of Science

1 Introduction

1.1. Background

This paper is the continuation of our work [11] on studies of the asymptotic profile of viscous flow around a rotating obstacle. Let us consider the spatial behavior of steady solutions to the Navier-Stokes equation in three-dimensional exterior domains. We want to understand the decay structure at infinity of the flow in each case of the following three typical situations:

(1) the obstacle is at rest;
(2) the obstacle is translating with constant velocity;
(3) the obstacle is rotating with constant angular velocity.

For the last two cases we consider the steady flow in the coordinate system attached to the moving obstacle. The translating case (2) is relatively well known since a series of celebrated papers, for instance [15], by Finn in 1960's: The leading profile of the Navier-Stokes flow is the Oseen fundamental solution which exhibits a paraboloidal wake property behind the obstacle ([8], [14], [16]) in the sense that the remaining term decays faster. In this case the dominance of the linear part is due to the better decay structure outside the wake region, see also [7], [30] besides the literature above.

However, that is not the case when the obstacle is at rest, because the nonlinear part is balanced with the linear one. In fact, Deuring and Galdi [6] proved that the leading profile of the Navier-Stokes flow, which decays like $1/|x|$, is no longer the Stokes fundamental solution. Nazarov and Pileckas [29] gave a partial answer to the asymptotic representation of the Navier-Stokes flow, but the concrete profile of the leading term was not specified. On the other hand, we know that the coefficient of the leading term should be related to the total net force exerted on the boundary of the obstacle from the fluid, see (2.5); this was suggested by [25], [3], [26]. Since nonlinear effect must be involved in the leading term, it is reasonable to expect that a self-similar solution to the Navier-Stokes equation (2.11) may provide the leading profile; here, self-similarity is equivalent to (-1)-homogeneity, see (2.12), due to the nonlinear structure.

Recently, Šverák [32] gained the crucial insight that every self-similar Navier-Stokes flow which is smooth in $\mathbb{R}^3 \setminus \{0\}$ must have its own axis of symmetry and be a member of the family of exact solutions constructed by Landau [27] (who did it in order to describe jets from a thin pipe). We call a member of this family a *Landau solution*; see section 2 for details. The Landau solution is parametrized by a vectorial parameter, which stands for the direction of the axis of symmetry of the solution. Korolev and Šverák [24] proved that if the Navier-Stokes flow in the case (1) decays like $1/|x|$ and is small, then the Landau solution actually provides its leading term, where the vectorial parameter is given by the net force $\widetilde{N}$, see (2.5), of the given Navier-Stokes flow. We remark that the Landau solution is the leading term of the point singularity $1/|x|$ at 0 as well as for the decay at infinity for the Navier-Stokes flow as long as it is small enough; very recently, this has been shown by Miura and Tsai [28].

In this paper our attention centers on the rotating case (3) and we prove that the leading term at infinity of small Navier-Stokes flow, which decays like $1/|x|$, is still the Landau solution; however, this time the vectorial parameter is parallel to the axis of rotation of the obstacle. Thus the axis of rotation is the preferred direction in the sense that the flow is largely concentrated along that axis. Such anisotropy has been observed at the level of linear analysis by the present authors [11], in which the asymptotic expansion of the steady Stokes flow at infinity has been deduced. Roughly speaking, some knowledge of the linearized problem from [11] together with the result in [32] mentioned above yields our main result, see section 2.

When both (2) and (3) are taken into account, in particular, the obstacle is translating along the axis of rotation, it is proved in [20] that there is still a wake region; in this case, very probably, the leading term of the Navier-Stokes flow comes from the linear part which differs from the purely rotating case (3) discussed in the present paper.

1.2. Problem

In the last decade remarkable progress in the analysis of Navier-Stokes flow around a rotating obstacle has been made; however, many questions like that one addressed in this paper still remain open. The steady motion in the coordinate system attached to the rotating obstacle is governed by, see [2], [17], [21],

$$\begin{array}{rcll} -\Delta u-(\omega\times x)\cdot\nabla u+\omega\times u+\nabla p+u\cdot\nabla u &=& 0 & (x\in D), \\ \operatorname{div} u &=& 0 & (x\in D), \\ u &=& \omega\times x & (x\in\partial D), \\ u &\to& 0 & (|x|\to\infty), \end{array} \tag{1.1}$$

where an incompressible viscous fluid occupies an exterior domain $D\subset\mathbb{R}^3$ with smooth boundary ∂D; here (u,p) denotes the velocity and pressure of the fluid, $\omega=ae_3=(0,0,a)^T\in\mathbb{R}^3\setminus\{0\}$ is the constant angular velocity of the obstacle $\mathbb{R}^3\setminus D$ which consists of a finite number of rigid bodies, and $\times$ stands for the usual exterior product of three-dimensional vectors so that $\omega\times x=a(-x_2,x_1,0)^T$ is the velocity of the obstacle at the point x. Thus the boundary condition $(1.1)_3$ is the no-slip one. Concerning hyperbolic aspects of the equation $(1.1)_1$, we refer to [13] and [21], although that is not the point in this paper. Even if $|\omega|$ is large, problem (1.1) has at least one solution with finite Dirichlet integral ([2], [17], [31]), however, we have less information about the qualitative behavior of this solution for $|x| \to \infty$.

In [18] Galdi constructed a unique solution which satisfies

$$|u(x)| \le \frac{C}{|x|}, \qquad |\nabla u(x)| + |p(x)| \le \frac{C}{|x|^2} \tag{1.2}$$

for large $|x|$ provided that $|\omega|$ is small enough. The present authors [10] generalized his result in the sense that there is a unique solution of class

$$u \in L_{3,\infty}(D), \qquad (\nabla u, p) \in L_{3/2,\infty}(D) \tag{1.3}$$

for the problem (1.1) with the external force $f = \text{div } F$ when both $|\omega|$ and $\|F\|_{L_{3/2,\infty}(D)}$ are small, where $L_{q,\infty}(D)$ denotes the the weak-L_q space. We note that the class (1.3) is consistent with the pointwise estimate (1.2). The stability of those steady flows was established by [19] and [23] within the framework of L_2 and L_q spaces, respectively.

The decay rate $1/|x|$ of the steady flow above is the same as that of the usual Navier-Stokes flow in which the obstacle is at rest. Thus a natural question arises: Can we catch the effect of rotation on the profile of the flow? Since this effect must be observed even for the Stokes equation, as a first step, we have started with the linear analysis [11] telling us what kind of effect on the profile the rotation of the obstacle causes. In fact, a heuristic observation in section 2 suggests a reasonable candidate of the leading term of the Navier-Stokes flow, and this paper gives an affirmative answer so far as small solutions in $L_{3,\infty}(D)$ are concerned. The key of the proof is the asymptotic expansion at infinity of the fundamental solution (3.20), which has been derived in [11] and is crucial to explain why the flow is concentrated along the axis of rotation, see Lemma 3.3. But we don't have enough information about pointwise estimates of the fundamental solution for $|x|$, $|y| \to \infty$ (unlike [24]); thus, we make use of (a variant of) $L_{q,\infty}$-estimates of solutions for the linearized problem developed in [10].

The next section provides our main theorem after introducing the Landau solution as well as some heuristic arguments. The last section is devoted to the proof.

2 Result

2.1. Heuristic observation

Let (v, q) be the Stokes flow, that is, the solution to (1.1) in which the nonlinear term is neglected. We set

$$N = \int_{\partial D} \nu \cdot T(v, q)\, dS$$

which stands for the net force on the boundary ∂D; here ν is the exterior unit normal to ∂D and $T(v, q)$ denotes the Cauchy stress tensor, that is,

$$T(v, q) = \nabla v + (\nabla v)^T - q\mathbb{I}, \qquad \mathbb{I} = (\delta_{ij})_{1 \le i,j \le 3}. \tag{2.1}$$

According to [11], the leading term of the Stokes flow $v(x)$ is given by

$$V(x) = \left(\frac{\omega}{|\omega|} \cdot N \right) E_{St}(x) \frac{\omega}{|\omega|}, \qquad \frac{\omega}{|\omega|} = e_3, \tag{2.2}$$

where $E_{St}(x)$ is the usual Stokes fundamental solution:

$$E_{St}(x) = \frac{1}{8\pi} \left(\frac{1}{|x|}\mathbb{I} + \frac{x \otimes x}{|x|^3} \right), \qquad x \otimes x = (x_i x_j)_{1 \le i,j \le 3}.$$

It is remarkable that the rate of decay of $v(x)$ can be controlled only by the third component of the net force in the sense that $v(x) = O(1/|x|^2)$ as $|x| \to \infty$ if and only if $e_3 \cdot N = 0$. Concerning the second term ($\sim 1/|x|^2$) of $v(x)$ we refer to [11]. The leading term $V(x)$ thus satisfies

$$-\Delta V + \nabla Q = (e_3 \cdot N) e_3\, \delta_0, \qquad \operatorname{div} V = 0$$

in $\mathcal{D}'(\mathbb{R}^3)$, where δ_0 denotes the Dirac measure at 0 and $Q(x) = \frac{(e_3 \cdot N) x_3}{4\pi |x|^3}$. But the pair (V, Q) enjoys

$$-\Delta V - (\omega \times x) \cdot \nabla V + \omega \times V + \nabla Q = (e_3 \cdot N) e_3\, \delta_0, \qquad \operatorname{div} V = 0 \tag{2.3}$$

as well, since

$$(e_3 \times x) \cdot \nabla V - e_3 \times V = 0. \tag{2.4}$$

Note that (2.4) holds for all vector fields which are symmetric about $\mathbb{R}e_3$ (x_3-axis). In fact, because such vector fields must be of the form

$$V = (W(r, x_3) \cos\theta, W(r, x_3) \sin\theta, V_3(r, x_3))^T$$

in cylindrical coordinates r, θ, x_3, we see that $(e_3 \times x) \cdot \nabla V = \partial_\theta V = e_3 \times V$. Furthermore, (2.4) holds in the sense div $[V \otimes (e_3 \times x) - (e_3 \times x) \otimes V] = 0$, cf. (2.6) below, in $\mathcal{D}'(\mathbb{R}^3)$ as long as $V(x) \sim 1/|x|$ near the origin.

Now, let (u, p) be the solution to the Navier-Stokes problem (1.1). Our goal is to find the leading term, which we denote by $U(x)$, of the velocity $u(x)$. In view of (2.2), it is reasonable to expect that the leading term U still keeps the properties

(i) symmetry about the axis of rotation (that is, $\mathbb{R}e_3$);
(ii) (-1)-homogeneity,

and that the quantity $e_3 \cdot \widetilde{N}$ controls the rate of decay of $u(x)$, where

$$\widetilde{N} = \int_{\partial D} \nu \cdot [T(u, p) - u \otimes u]\, dS, \qquad N = \int_{\partial D} \nu \cdot T(u, p)\, dS \tag{2.5}$$

and $T(u, p)$ is the stress tensor defined by (2.1). Note the relation

$$(\omega \times x) \cdot \nabla u - \omega \times u = \text{div}\, [u \otimes (\omega \times x) - (\omega \times x) \otimes u], \tag{2.6}$$

but

$$\int_{\partial D} \nu \cdot [T(u, p) + u \otimes (\omega \times x) - (\omega \times x) \otimes u - u \otimes u]\, dS = \widetilde{N} \tag{2.7}$$

owing to the no-slip condition $u|_{\partial D} = \omega \times x$. As in (2.3), one can also expect that the leading term U, together with some scalar field P, solves

$$-\Delta U - (\omega \times x) \cdot \nabla U + \omega \times U + \nabla P + U \cdot \nabla U = (e_3 \cdot \widetilde{N})\, e_3\, \delta_0, \quad \text{div}\, U = 0 \tag{2.8}$$

in $\mathcal{D}'(\mathbb{R}^3)$. This is equivalent to

$$-\Delta U + \nabla P + U \cdot \nabla U = (e_3 \cdot N)\, e_3\, \delta_0, \qquad \text{div}\, U = 0 \tag{2.9}$$

because U satisfies (2.4) under the property (i) above and because

$$e_3 \cdot N = e_3 \cdot \widetilde{N} \tag{2.10}$$

which follows from $u|_{\partial D} = \omega \times x$ together with $e_3 \cdot (\omega \times x) = 0$. Hence, by taking the property (ii) above into account, U is a self-similar solution to

$$-\Delta u + \nabla p + u \cdot \nabla u = 0, \qquad \text{div}\, u = 0 \qquad (x \in \mathbb{R}^3 \setminus \{0\}), \tag{2.11}$$

and, thanks to [32], U is a member of the family of the Landau solutions [27].

2.2. The Landau solution

Let $b \in \mathbb{R}^3 \setminus \{0\}$ be a prescribed vector, that we call the Landau parameter. Then, among nontrivial smooth solutions of (2.11), Landau [27] (see also [5], [24], [32], [33]) found an exact solution which satisfies:

- axially symmetry about $\mathbb{R}b$;
- the homogeneity

$$U(x) = \frac{1}{|x|}\, U\left(\frac{x}{|x|}\right), \qquad P(x) = \frac{1}{|x|^2}\, P\left(\frac{x}{|x|}\right); \tag{2.12}$$

- $-\Delta U + \nabla P + U \cdot \nabla U = b\delta_0$ in $\mathcal{D}'(\mathbb{R}^3)$.

We set $x = |x|\sigma,\ \sigma = (\sigma_1, \sigma_2, \sigma_3)^T \in \mathbb{S}^2$ (unit sphere). When b is parallel to e_3, the Landau solution is of the form

$$\begin{aligned} U(x) &= \frac{2}{|x|}\left[\frac{c\sigma_3 - 1}{(c-\sigma_3)^2}\,\sigma + \frac{1}{c-\sigma_3}\, e_3\right], \\ P(x) &= \frac{4\,(c\sigma_3 - 1)}{|x|^2 (c-\sigma_3)^2} \end{aligned} \tag{2.13}$$

with parameter $c \in (-\infty, -1) \cup (1, \infty)$, and it satisfies

$$-\Delta U + \nabla P + U \cdot \nabla U = k e_3\, \delta_0, \qquad \operatorname{div} U = 0$$

in $\mathcal{D}'(\mathbb{R}^3)$, where k is given by

$$k = k(c) = \frac{8\pi c}{3(c^2-1)} \left(2 + 6c^2 - 3c(c^2-1)\log\frac{c+1}{c-1}\right). \tag{2.14}$$

This calculation was done by Cannone and Karch [5, Proposition 2.1]. Note that the function $k(\cdot)$ is monotonically decreasing on both $(-\infty, -1)$ and $(1, \infty)$, and fulfills

$$k(c) \to 0 \quad (|c| \to \infty); \quad k(c) \to -\infty \quad (c \to -1); \quad k(c) \to \infty \quad (c \to 1).$$

Hence, for every $\widetilde{k} \in \mathbb{R} \setminus \{0\}$, there is a unique $c \in (-\infty, -1) \cup (1, \infty)$ such that $k(c) = \widetilde{k}$. When $\widetilde{k} = 0$ (the Landau parameter $b = 0$), we may understand $(U, P) = (0, 0)$ as the solution (2.13) with $|c| \to \infty$.

2.3. Main theorem

Let $1 < q < \infty$. By $L_q(D)$ we denote the usual Lebesgue space. The weak-L_q space $L_{q,\infty}(D)$ is obtained as one of the Lorentz spaces via real interpolation of Lebesgue spaces. For a measurable function f, we know that $f \in L_{q,\infty}(D)$ if and only if

$$\sup_{t>0}\, t\,|\{x \in D;\ |f(x)| > t\}|^{1/q} < \infty.$$

This quantity is equivalent to the norm $\|f\|_{L_{q,\infty}(D)}$ (cf. [3, Section 2]), from which we easily see that $1/|x|^{n/q} \in L_{q,\infty}(D)$, where $n(=3)$ is the space dimension, and that $L_{q,\infty}(D)$ is strictly larger than $L_q(D)$. Some related function spaces over the whole space $\mathbb{R}^3$ will be introduced in subsection 3.2. We often use the same symbols for denoting the spaces of scalar and vector functions if there is no confusion.

Given a smooth solution (u,p) of the Navier-Stokes problem (1.1), we take N and $\widetilde{N}$ as in (2.5), and recall (2.10). Suppose $u \in L_{3,\infty}(D)$ rather than the pointwise estimate $|u(x)| \leq C/|x|$. We now single out a special solution as the leading term of $u(x)$. Let (U,P) be the Landau solution with the Landau parameter

$$b = (e_3 \cdot N)e_3 = (e_3 \cdot \widetilde{N})e_3; \tag{2.15}$$

that is, (U,P) is given by (2.13) with c determined by $k(c) = e_3 \cdot N$ and $k(\cdot)$ as in (2.14); it is the trivial solution in case $e_3 \cdot N = 0$. Since U is symmetric about $\mathbb{R}e_3$, as already observed, (U,P) solves not only (2.9), but also (2.8) in $\mathcal{D}'(\mathbb{R}^3)$.

Our main theorem reads as follows.

Theorem 2.1 *Let $\omega = ae_3$ with $a \in \mathbb{R} \setminus \{0\}$. For each $q_0 \in (3/2,3)$ there exists a constant $\eta = \eta(q_0) > 0$ such that if (u,p) is a smooth solution to* (1.1) *in the class $u \in L_{3,\infty}(D)$, $p \in L_{3/2,\infty}(D)$ and satisfies*

$$\|u\|_{L_{3,\infty}(D)} + |e_3 \cdot N| \leq \eta, \tag{2.16}$$

then, for every $q \in (q_0,3)$, we have

$$u - U|_D \in L_q(D), \qquad \|u - U\|_{L_q(D)} \leq C(|a|^{-3/q+1} + 1) \tag{2.17}$$

with some $C = C(q) > 0$, where U is the Landau solution for (2.15) *as explained above.*

From (2.17) we find that the remainder $u-U$ possesses better summabilty suggesting the pointwise decay $1/|x|^2$ at infinity; in this sense, the Landau solution U is the leading term of the small solution u. The singular behavior for $a \to 0$ of the remainder is also observed because the leading term of the usual Navier-Stokes flow found by [24] is different, namely the Landau solution for $b = \widetilde{N}$.

We don't really use the exact form (2.13) except for the observation that the smallness of $|e_3 \cdot N|$ implies that of $\|U\|_{L_{3,\infty}(\mathbb{R}^3)}$, see (3.13) below. However, equation (2.8) rather than (2.13) is essentially needed in the proof.

Remark 2.1 *We cannot conclude that P, associated with the Landau solution U above, is the leading term of the pressure p. The reason is as follows. Consider the linear whole space problem* (3.6) *with $f \in C_0^\infty(\mathbb{R}^3)$. Under the condition $e_3 \cdot \int_{\mathbb{R}^3} f = 0$, we obtain the better decay $u(x) = O(1/|x|^2)$, see Lemma 3.4; however, the same condition does not imply any better decay of the pressure such as $p(x) = O(1/|x|^3)$.*

3 Proof

3.1. Reduction to the whole space problem

For the proof of Theorem 2.1 it does not seem to be easy to treat directly the exterior problem (1.1). So, as in [24], we reduce (1.1) to the problem in the whole space $\mathbb{R}^3$. Then the information on the boundary ∂D will go to the external force of the equation, see (3.2) below. We use the following lemma ([1], [4], [16]) on the equation of continuity in order to recover solenoidality.

Lemma 3.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then there is a linear operator $\mathbb{B} : C_0^\infty(\Omega) \to C_0^\infty(\Omega)^n$ such that for $1 < q < \infty$ and $j \in \mathbb{N}_0$*

$$\|\nabla^{j+1}\mathbb{B}f\|_{L_q(\Omega)} \leq C\|\nabla^j f\|_{L_q(\Omega)}$$

with $C = C(\Omega, q, j) > 0$ independent of $f \in C_0^\infty(\Omega)$; moreover,

$$\mathrm{div}\ (\mathbb{B}f) = f$$

for all $f \in C_0^\infty(\Omega)$ with $\displaystyle\int_\Omega f(x)\,dx = 0$.

By B_ρ we denote the open ball centered at the origin with radius $\rho > 0$. We take $R > 0$ so that $\mathbb{R}^3 \setminus D \subset B_{R-2}$. Given a smooth solution (u, p) to (1.1), we set

$$\widetilde{u} = (1-\phi)u + \mathbb{B}[u \cdot \nabla\phi], \qquad \widetilde{p} = (1-\phi)p, \tag{3.1}$$

where $\phi \in C_0^\infty(B_R)$ is a given function such that $\phi(x) = 1$ in B_{R-1}, and $\mathbb{B}$ is the operator defined by Lemma 3.1 for the domain $A_R = B_R \setminus \overline{B_{R-2}}$. Note that $u \cdot \nabla\phi \in C_0^\infty(A_R)$ and

$$\int_{A_R} u \cdot \nabla\phi\,dx = \frac{1}{R-2}\int_{|x|=R-2}(-x)\cdot u\,dS = \int_{\partial D}\nu \cdot u\,dS = 0$$

on account of $u|_{\partial D} = \omega \times x$ together with the identity

$$\int_{\partial D}\nu \cdot (\omega \times x)\,dS = -\int_{\mathbb{R}^3\setminus D}\mathrm{div}\ (\omega \times x)\,dx = 0.$$

The pair $(\widetilde{u}, \widetilde{p})$ obeys

$$-\Delta \widetilde{u} - (\omega \times x) \cdot \nabla \widetilde{u} + \omega \times \widetilde{u} + \nabla \widetilde{p} + \widetilde{u} \cdot \nabla \widetilde{u} = g, \quad \text{div } \widetilde{u} = 0 \quad (x \in \mathbb{R}^3)$$

with some $g \in C_0^\infty(A_R)$ which satisfies

$$\int_{\mathbb{R}^3} g(x)\, dx = \widetilde{N}, \tag{3.2}$$

where $\widetilde{N}$ is the net force (2.5). We note that (3.2) follows only from the structure of the equation; in other words, we don't need any exact form of g. In fact, by (1.1) and (2.7) we have

$$\begin{aligned} &\int_{A_R} g(x)\, dx \\ &= -\int_{A_R} \text{div } [T(\widetilde{u}, \widetilde{p}) + \widetilde{u} \otimes (\omega \times x) - (\omega \times x) \otimes \widetilde{u} - \widetilde{u} \otimes \widetilde{u}]\, dx \\ &= -\int_{|x|=R} \frac{x}{R} \cdot [T(u, p) + u \otimes (\omega \times x) - (\omega \times x) \otimes u - u \otimes u]\, dS \\ &= \int_{\partial D} \nu \cdot [T(u, p) + u \otimes (\omega \times x) - (\omega \times x) \otimes u - u \otimes u]\, dS = \widetilde{N} \end{aligned}$$

which shows (3.2).

Let (U, P) be the Landau solution for $b = (e_3 \cdot N)e_3$. To regularize (U, P) around $x = 0$, one may follow the same manner as in (3.1):

$$\widetilde{U} = (1 - \phi)U + \mathbb{B}[U \cdot \nabla \phi], \qquad \widetilde{P} = (1 - \phi)P. \tag{3.3}$$

We observe, using (2.12),

$$\int_{A_R} U \cdot \nabla \phi\, dx = \frac{1}{R-2} \int_{|x|=R-2} (-x) \cdot U\, dS = \frac{1}{\varepsilon} \int_{|x|=\varepsilon} (-x) \cdot U\, dS = O(\varepsilon)$$

as $\varepsilon \to 0$ to obtain $\int_{A_R} U \cdot \nabla \phi\, dx = 0$. Then the pair $(\widetilde{U}, \widetilde{P})$ enjoys

$$-\Delta \widetilde{U} - (\omega \times x) \cdot \nabla \widetilde{U} + \omega \times \widetilde{U} + \nabla \widetilde{P} + \widetilde{U} \cdot \nabla \widetilde{U} = h, \quad \text{div } \widetilde{U} = 0 \quad (x \in \mathbb{R}^3)$$

with some $h \in C_0^\infty(A_R)$; one can also find

$$\int_{\mathbb{R}^3} h(x)\, dx = (e_3 \cdot \widetilde{N})e_3. \tag{3.4}$$

In fact, using a test function $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfying $\psi(x) = 1\,(|x| \le R)$ and $\psi(x) = 0\,(|x| \ge R+1)$, we see from (2.8) that

$$\begin{aligned}
&\int_{A_R} h(x)\,dx \\
&= -\int_{|x|=R} \frac{x}{R} \cdot [T(U,P) + U \otimes (\omega \times x) - (\omega \times x) \otimes U - U \otimes U]\,dS \\
&= \int_{\mathbb{R}^3} [T(U,P) + U \otimes (\omega \times x) - (\omega \times x) \otimes U - U \otimes U] \cdot \nabla\psi(x)\,dx \\
&= \langle (e_3 \cdot \widetilde{N}) e_3\, \delta_0,\, \psi \rangle
\end{aligned}$$

which yields (3.4). Then we set $(v, \varpi) = (\widetilde{u} - \widetilde{U}, \widetilde{p} - \widetilde{P})$, and get that in $\mathbb{R}^3$

$$\begin{aligned}
-\Delta v - (\omega \times x) \cdot \nabla v + \omega \times v + \nabla \varpi + v \cdot \nabla \widetilde{u} + \widetilde{U} \cdot \nabla v = g - h,& \\
\operatorname{div} v = 0.&
\end{aligned} \tag{3.5}$$

3.2. Linear theory

The linear theory for the whole space problem

$$-\Delta u - (\omega \times x) \cdot \nabla u + \omega \times u + \nabla p = f, \qquad \operatorname{div} u = 0 \qquad (x \in \mathbb{R}^3) \tag{3.6}$$

has been developed by [12], [22] and [10] (see also [9] in which the Oseen term as well as rotation is taken into account) in Lebesgue and even Lorentz spaces. However, here we need - due to the presence of the Coriolis term $\omega \times u$ - a variant in order to ensure the summability of the velocity u itself as well as ∇u.

Let $1 < q < \infty$. We denote by $\|\cdot\|_q$ and by $\|\cdot\|_{q,\infty}$ the norms of $L_q(\mathbb{R}^3)$ and $L_{q,\infty}(\mathbb{R}^3)$, respectively, where the latter space is defined as in subsection 2.3. We also introduce the homogeneous Sobolev spaces over $\mathbb{R}^3$ adopted in [22], [10]. Set

$$\dot{W}_q^1(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla(\cdot)\|_q} = \{u \in L_{q,\mathrm{loc}}(\mathbb{R}^3);\ \nabla u \in L_q(\mathbb{R}^3)\}/\mathbb{R}.$$

Let $1 < q_0 < q < q_1 < \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. For $r = 1$ and $r = \infty$ we define

$$\dot{W}_{q,r}^1(\mathbb{R}^3) = \Big(\dot{W}_{q_0}^1(\mathbb{R}^3),\, \dot{W}_{q_1}^1(\mathbb{R}^3)\Big)_{\theta,r}$$

with norm $\|\nabla(\cdot)\|_{q,r}$, where $(\cdot,\cdot)_{\theta,r}$ denotes the real interpolation functor. Finally, the space $\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)$ is defined as the dual space of $\dot{W}_{q',1}^1(\mathbb{R}^3)$, where $1/q' + 1/q = 1$.

We begin with the following proposition, a special case of [10, Proposition 3.2] which follows essentially from the L_q-theory in [22]. When (u, p) satisfies (3.6) in the sense of distributions, we simply call (u, p) solution to (3.6).

Proposition 3.1 *Let* $1 < q < \infty$ *and suppose* $f \in \dot{W}^{-1}_{q,\infty}(\mathbb{R}^3)$. *Then problem* (3.6) *has a solution* $(u, p) \in \dot{W}^1_{q,\infty}(\mathbb{R}^3) \times L_{q,\infty}(\mathbb{R}^3)$ *subject to the estimate*

$$\|\nabla u\|_{q,\infty} + \|p\|_{q,\infty} + \|(\omega \times x) \cdot \nabla u - \omega \times u\|_{\dot{W}^{-1}_{q,\infty}(\mathbb{R}^3)} \le C\|f\|_{\dot{W}^{-1}_{q,\infty}(\mathbb{R}^3)},$$

where $C = C(q) > 0$ *is independent of* ω. *The solution is unique in the class above up to a constant multiple of* ω *for* u.

Let $1 < q < 3$ and let (u, p) be the solution obtained in Proposition 3.1. Then, by [3, Lemma 5.6], we have the embedding relation

$$\|u + \beta\|_{q_*,\infty} \le C\|\nabla u\|_{q,\infty}, \qquad \frac{1}{q_*} = \frac{1}{q} - \frac{1}{3}, \tag{3.7}$$

with some constant vector $\beta = \beta(u) \in \mathbb{R}^3$. But we don't know whether β is parallel to $\omega = ae_3$, i.e., $\omega \times \beta = 0$; thus, $u + \beta$ could fail to be a solution to (3.6) as long as the class of pressure is kept as $p \in L_{q,\infty}(\mathbb{R}^3)$.

To get around this difficulty, we introduce another class for the pressure function, namely, $L_{q,\infty}(\mathbb{R}^3) + Y$ where

$$Y = \{\alpha \cdot x;\ \alpha \in \mathbb{R}^3 \text{ with } \omega \cdot \alpha = 0\}. \tag{3.8}$$

Note that, given $\alpha \in \mathbb{R}^3$, there is a vector $\beta \in \mathbb{R}^3$ satisfying $\alpha = \omega \times \beta$ if and only if $\omega \cdot \alpha = 0$. For later use we prepare the following variant of Proposition 3.1.

Proposition 3.2 *Let* $1 < q < 3$, $\frac{1}{q_*} = \frac{1}{q} - \frac{1}{3}$, *and suppose* $f \in \dot{W}^{-1}_{q,\infty}(\mathbb{R}^3)$. *Then problem* (3.6) *has a solution of class*

$$u \in \dot{W}^1_{q,\infty}(\mathbb{R}^3) \cap L_{q_*,\infty}(\mathbb{R}^3), \qquad p \in L_{q,\infty}(\mathbb{R}^3) + Y \tag{3.9}$$

subject to the estimate

$$\|\nabla u\|_{q,\infty} + \|u\|_{q_*,\infty} \le C\|f\|_{\dot{W}^{-1}_{q,\infty}(\mathbb{R}^3)}, \tag{3.10}$$

where $C = C(q) > 0$ *is independent of* ω. *The solution is unique in the class*

$$u \in L_{q_*,\infty}(\mathbb{R}^3), \qquad p \in L_{q,\infty}(\mathbb{R}^3) + Y. \tag{3.11}$$

Proof. By $(u_0, p_0) \in \dot{W}^1_{q,\infty}(\mathbb{R}^3) \times L_{q,\infty}(\mathbb{R}^3)$ we denote the solution to (3.6) obtained in Proposition 3.1. Let $\beta = \beta(u_0) \in \mathbb{R}^3$ be a constant vector with $u_0 + \beta \in L_{q_*,\infty}(\mathbb{R}^3)$ and (3.7). Then the pair of functions

$$u = u_0 + \beta, \qquad p = p_0 - (\omega \times \beta) \cdot x$$

belongs to the class (3.9) with (3.10) and satisfies (3.6) in the sense of distributions.

Next we show uniqueness. Let (u, p) be a solution of class (3.11) to (3.6) with $f = 0$; note that consequently u and p are tempered distributions. Writing $p = p_0 + \alpha \cdot x$ with some $p_0 \in L_{q,\infty}(\mathbb{R}^3)$, we immediately see $p_0 = 0$ on account of $\Delta p_0 = \Delta p = 0$ since

$$\text{div } [(\omega \times x) \cdot \nabla u - \omega \times u] = (\omega \times x) \cdot \nabla \text{div } u = 0.$$

Set $\beta := \alpha \times \omega / |\omega|^2$. Then it follows from $\omega \cdot \alpha = 0$ that

$$\omega \times \beta = \frac{1}{|\omega|^2}[(\omega \cdot \omega)\alpha - (\omega \cdot \alpha)\omega] = \alpha = \nabla(\alpha \cdot x) = \nabla p.$$

As a consequence, we find

$$-\Delta(u + \beta) - (\omega \times x) \cdot \nabla(u + \beta) + \omega \times (u + \beta) = 0$$

in $\mathcal{S}'(\mathbb{R}^3)$. As shown in [12], [22] by use of the Fourier transform, we get

$$\text{supp } \widehat{(u + \beta)} \subset \{0\},$$

which implies that $u + \beta$ is a polynomial, and hence so is u. Since $u \in L_{q_*,\infty}(\mathbb{R}^3)$, we obtain $u = 0$. Further, $\alpha = \nabla p = 0$, and thus $p = \alpha \cdot x = 0$. This completes the proof. □

3.3. Proof of Theorem 2.1

We start with the lemma on the extended functions defined by (3.1) and (3.3).

Lemma 3.2 *We have*

$$\widetilde{u},\ \widetilde{U} \in L_{3,\infty}(\mathbb{R}^3), \qquad \widetilde{p},\ \widetilde{P} \in L_{3/2,\infty}(\mathbb{R}^3).$$

Furthermore,

$$\|\widetilde{u}\|_{3,\infty} \leq C\|u\|_{L_{3,\infty}(D)}, \qquad \|\widetilde{U}\|_{3,\infty} \leq C\|U\|_{3,\infty} \tag{3.12}$$

and

$$\|\widetilde{U}\|_{3,\infty} \to 0 \qquad \textit{when} \quad e_3 \cdot N \to 0. \tag{3.13}$$

Proof. One may regard $u \mapsto \widetilde{u}$ defined by (3.1) as the mapping from $L_q(D)$ to $L_q(\mathbb{R}^3)$ to see that it is bounded for every $q \in (1,\infty)$ by Lemma 3.1. Hence real interpolation implies that this mapping is continuous from $L_{q,\infty}(D)$ to $L_{q,\infty}(\mathbb{R}^3)$ for every $q \in (1,\infty)$. The class $u \in L_{3,\infty}(D)$ for the Navier-Stokes flow under consideration thus yields $\widetilde{u} \in L_{3,\infty}(\mathbb{R}^3)$; and, similarly, we get $\widetilde{U} \in L_{3,\infty}(\mathbb{R}^3)$ by considering $U \mapsto \widetilde{U}$ defined by (3.3). We have also obtained (3.12). The assertion for the pressure functions is obvious.

In view of (2.13), (2.14) with $k(c) = e_3 \cdot N$, we see that $\|U\|_{3,\infty} \to 0$ if and only if $|c| \to \infty$, i.e., if $e_3 \cdot N \to 0$. This combined with (3.12) yields (3.13). □

We may regard (3.5) as a linear problem for the unknown (v, ϖ). When (v, ϖ) satisfies (3.5) in the sense of distributions, it is simply called solution to (3.5). We have the following results on uniqueness (Proposition 3.3) and summability (Proposition 3.4) of solutions.

Proposition 3.3 *There is a constant* $\gamma > 0$ *such that the solution* (v, ϖ) *to* (3.5) *in the class*

$$v \in L_{3,\infty}(\mathbb{R}^3), \qquad \varpi \in L_{3/2,\infty}(\mathbb{R}^3) + Y \tag{3.14}$$

is unique provided

$$\|\widetilde{u}\|_{3,\infty} + \|\widetilde{U}\|_{3,\infty} \le \gamma, \tag{3.15}$$

where Y *is as in* (3.8).

Proof. Let (v, ϖ) be the solution to (3.5) in which the right-hand side $g - h$ is replaced by 0. Set

$$f := -(v \cdot \nabla \widetilde{u} + \widetilde{U} \cdot \nabla v) = -\operatorname{div}\,(\widetilde{u} \otimes v + v \otimes \widetilde{U}). \tag{3.16}$$

Then we find $f \in \dot{W}^{-1}_{3/2,\infty}(\mathbb{R}^3)$ with

$$\|f\|_{\dot{W}^{-1}_{3/2,\infty}(\mathbb{R}^3)} \le \|\widetilde{u} \otimes v + v \otimes \widetilde{U}\|_{3/2,\infty} \le C\left(\|\widetilde{u}\|_{3,\infty} + \|\widetilde{U}\|_{3,\infty}\right)\|v\|_{3,\infty}$$

by the weak Hölder inequality (cf. [3, Lemma 2.1]). By virtue of the class (3.14) one can regard (v, ϖ) as the solution to (3.6) with force (3.16) obtained in Proposition 3.2 (for $q = 3/2$) because of the uniqueness assertion of that proposition, see (3.11). Combining the estimate above with (3.10), we get

$$\|v\|_{3,\infty} \le C_3\left(\|\widetilde{u}\|_{3,\infty} + \|\widetilde{U}\|_{3,\infty}\right)\|v\|_{3,\infty}$$

which concludes $v = 0$ under the condition (3.15) with $\gamma = 1/(2C_3)$ (the constant C_3 is the same as in (3.25) below with $q_* = 3$). Thus $\nabla \varpi = 0$, yielding $\varpi = 0$ by (3.14). □

Proposition 3.4 *For each* $q_0 \in (3/2, 3)$ *there is a constant* $\widetilde{\gamma}(q_0) \in (0, \gamma]$ *(with* γ *as in* (3.15)*) such that problem* (3.5) *has a solution* (v, ϖ) *of class* (3.14) *and*

$$v \in L_{q_0,\infty}(\mathbb{R}^3) \tag{3.17}$$

subject to

$$\|v\|_q \le C(|a|^{-3/q+1} + 1) \qquad \text{for all } q \in (q_0, 3) \tag{3.18}$$

with some $C = C(q) > 0$ *provided*

$$\|\widetilde{u}\|_{3,\infty} + \|\widetilde{U}\|_{3,\infty} \le \widetilde{\gamma}(q_0). \tag{3.19}$$

We postpone the proof of Proposition 3.4 to that of Theorem 2.1.

Proof of Theorem 2.1. By (3.12) together with (3.13), the condition (2.16) implies (3.19) when we take a suitable constant $\eta = \eta(q_0) > 0$. Then we see from Proposition 3.4 that $\widetilde{u} - \widetilde{U}$ is in the class (3.17) and also enjoys

$$\widetilde{u} - \widetilde{U} \in L_q(\mathbb{R}^3), \qquad \|\widetilde{u} - \widetilde{U}\|_q \le C(|a|^{-q/3+1} + 1)$$

for all $q \in (q_0, 3)$ because, by Proposition 3.3 and Lemma 3.2, the only solution of class (3.14) is $(\widetilde{u} - \widetilde{U}, \widetilde{p} - \widetilde{P})$. Since $u - U = \widetilde{u} - \widetilde{U}$ for $|x| \ge R$, we obtain (2.17). □

For the proof of Proposition 3.4, we introduce the fundamental solution for (3.6):

$$\begin{aligned} \Gamma(x, y) &= \int_0^\infty O(at)^T (G\mathbb{I} + H)(O(at)x - y, t)\, dt, \\ Q(x, y) &= \frac{x - y}{4\pi |x - y|^3} \end{aligned} \tag{3.20}$$

where

$$G(x, t) = (4\pi t)^{-3/2} e^{-|x|^2/(4t)}, \qquad H(x, t) = \int_t^\infty \nabla^2 G(x, s)\, ds,$$

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $G(x, t)\mathbb{I} + H(x, t)$ is the fundamental solution of the usual unsteady Stokes equation; see [11, Section 2] for the derivation of the fundamental solution. The following asymptotic expansion of $\Gamma(x, y)$ near infinity has been proved in [11, Section 4].

Lemma 3.3 *For $|y| \le R$ and $|x| \to \infty$ we have*

$$\Gamma(x,y) = \Phi(x) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^2}\right)$$

with

$$\Phi(x) = \frac{1}{8\pi|x|^3} \begin{pmatrix} 0 & 0 & x_1x_3 \\ 0 & 0 & x_2x_3 \\ 0 & 0 & |x|^2 + x_3^2 \end{pmatrix}.$$

Set

$$\begin{aligned} v_0(x) &= \int_{\mathbb{R}^3} \Gamma(x,y)(g-h)(y)\,dy, \\ \varpi_0(x) &= \int_{\mathbb{R}^3} Q(x,y)\cdot(g-h)(y)\,dy. \end{aligned} \tag{3.21}$$

Then (v_0, ϖ_0) satisfies (3.6) with $f = g - h$ (and is a representative of the solution obtained in [12]). Since $g - h \in C_0^\infty(A_R)$, we find

$$v_0 \in \dot{W}^1_{3/2,\infty}(\mathbb{R}^3) \cap L_{3,\infty}(\mathbb{R}^3), \qquad \varpi_0 \in L_{3/2,\infty}(\mathbb{R}^3), \tag{3.22}$$

see [10, Proposition 3.3]. The crucial point is the following lemma, which tells us the reason why good summability properties at infinity can be deduced in Proposition 3.4.

Lemma 3.4 *The function v_0 given by* (3.21) *satisfies*

$$\begin{aligned} &v_0(x) = O\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty, \\ &\|v_0\|_{q,\infty} \le C(|a|^{-3/q+1} + 1) \qquad \text{for all } q \in [3/2, 3] \end{aligned} \tag{3.23}$$

with some $C = C(q) > 0$.

Proof. Let $|x| \ge 2R$. Since $g - h \in C_0^\infty(A_R)$, it follows from Lemma 3.3 that

$$v_0(x) = \left(e_3 \cdot \int_{\mathbb{R}^3} (g-h)(y)\,dy\right) \frac{1}{8\pi}\left(\frac{e_3}{|x|} + \frac{x_3 x}{|x|^3}\right) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^2}\right).$$

By (3.2) and (3.4) we find

$$e_3 \cdot \int_{\mathbb{R}^3} (g-h)(y)\,dy = 0,$$

which proves (3.23) for $q = 3/2$. This combined with $\|v_0\|_{3,\infty} \le C$ completes the proof. $\square$

We note that $\varpi_0(x) = O(1/|x|^3)$ cannot be in general expected, see Remark 2.1.

Let $1 < q < 3$. We take $q_* \in (3/2, \infty)$ so that $1/q_* = 1/q - 1/3$. Given $v \in L_{q_*,\infty}(\mathbb{R}^3)$, we denote by Mv the velocity part of the unique solution to (3.6) with $f = -(v \cdot \nabla \widetilde{u} + \widetilde{U} \cdot \nabla v)$ obtained in Proposition 3.2. Then problem (3.5) is rewritten as

$$v = v_0 + Mv. \tag{3.24}$$

We have $f \in \dot{W}^{-1}_{q,\infty}(\mathbb{R}^3)$ with

$$\|f\|_{\dot{W}^{-1}_{q,\infty}(\mathbb{R}^3)} \le C \left(\|\widetilde{u}\|_{3,\infty} + \|\widetilde{U}\|_{3,\infty} \right) \|v\|_{q_*,\infty},$$

by which together with (3.10) there is a constant $C_{q_*} > 0$ independent of ω such that $Mv \in \dot{W}^1_{q,\infty}(\mathbb{R}^3) \cap L_{q_*,\infty}(\mathbb{R}^3)$ with

$$\|Mv\|_{q_*,\infty} \le C_{q_*} \left(\|\widetilde{u}\|_{3,\infty} + \|\widetilde{U}\|_{3,\infty} \right) \|v\|_{q_*,\infty} \tag{3.25}$$

for all $v \in L_{q_*,\infty}(\mathbb{R}^3)$.

Proof of Proposition 3.4. Fix $q_0 \in (3/2, 3)$ arbitrarily. We employ (3.25) with $q_* = q_0$ and $q_* = 3$. From them combined with (3.23) we obtain a solution $v \in L_{q_0,\infty}(\mathbb{R}^3) \cap L_{3,\infty}(\mathbb{R}^3)$ of (3.24) subject to

$$\|v\|_{q_0,\infty} \le 2\|v_0\|_{q_0,\infty}, \qquad \|v\|_{3,\infty} \le 2\|v_0\|_{3,\infty}$$

under the condition (3.19) with $\widetilde{\gamma}(q_0) = \min\,\{1/(2C_{q_0}), \gamma\}$; here, recall $\gamma = 1/(2C_3)$ in Proposition 3.3. Those estimates together with (3.23) yield (3.18) by real interpolation. Since the pressure associated with Mv is of class (3.9) with $q = 3/2$ and since we know (3.22) for ϖ_0, the obtained solution (v, ϖ) is of class (3.14) as well as (3.17). We have completed the proof. $\square$

References

[1] M. E. Bogovskiĭ, Solution of the first boundary value problem for the equation of continuity of an incompressible medium, Soviet Math. Dokl. **20** (1979), 1094–1098.

[2] W. Borchers, Zur Stabilität und Faktorisierungsmethode für die Navier-Stokes Gleichungen inkompressibler viskoser Flüssigkeiten, Habilitationsschrift, Universität Paderborn, 1992.

[3] W. Borchers and T. Miyakawa, On stability of exterior stationary Navier-Stokes flows, Acta Math. **174** (1995), 311–382.

[4] W. Borchers and H. Sohr, On the equations rot $v = g$ and div $u = f$ with zero boundary conditions, Hokkaido Math. J. **19** (1990), 67–87.

[5] M. Cannone and G. Karch, Smooth or singular solutions to the Navier-Stokes system?, J. Differential Equations **197** (2004), 247–274.

[6] P. Deuring and G. P. Galdi, On the asymptotic behavior of physically reasonable solutions to the stationary Navier-Stokes system in three-dimensional exterior domains with zero velocity at infinity, J. Math. Fluid Mech. **2** (2000), 353–364.

[7] R. Farwig, The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces, Math. Z. **211** (1992), 409–447.

[8] R. Farwig, The stationary Navier-Stokes equations in a 3D-exterior domain, Recent Topics on Mathematical Theory of Viscous Incompressible Fluid, 53–115, Eds. H. Kozono and Y. Shibata, Lecture Notes in Num. Appl. Anal. **16**, Kinokuniya, Tokyo, 1998.

[9] R. Farwig, An L^q-analysis of viscous fluid flow past a rotating obstacle, Tohoku Math. J. **58** (2006), 129–147.

[10] R. Farwig and T. Hishida, Stationary Navier-Stokes flow around a rotating obstacle, Funkcial. Ekvac. **50** (2007), 371–403.

[11] R. Farwig and T. Hishida, Asymptotic profile of steady Stokes flow around a rotating obstacle, Preprint Nr. 2578, TU Darmstadt, 2009.

[12] R. Farwig, T. Hishida and D. Müller, L^q-theory of a singular winding integral operator arising from fluid dynamics, Pacific J. Math. **215** (2004), 297–312.

[13] R. Farwig and J. Neustupa, On the spectrum of a Stokes-type operator arising from flow around a rotating body, Manuscripta Math. **122** (2007), 419–437.

[14] R. Farwig, and H. Sohr, Weighted estimates for the Oseen equations and the Navier–Stokes equations in exterior domains, Theory of the Navier–Stokes Equations, 11–31, Eds. J.G. Heywood, K. Masuda, R. Rautmann and V.A. Solonnikov, World Scientific, Ser. Advances Math. Appl. Sciences **47**, Singapore 1998

[15] R. Finn, On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems, Arch. Rational Mech. Anal. **19** (1965), 363–406.

[16] G. P. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I: Linearized Steady Problems, Vol. II: Nonlinear Steady Problems, revised edition, Springer, New York, 1994.

[17] G. P. Galdi, On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications, Handbook of Mathematical Fluid Dynamics, Vol. I, 653–791, North-Holland, Amsterdam, 2002.

[18] G. P. Galdi, Steady flow of a Navier-Stokes fluid around a rotating obstacle, J. Elasticity **71** (2003), 1–31.

[19] G. P. Galdi and A. L. Silvestre, Strong solutions to the Navier-Stokes equations around a rotating obstacle, Arch. Rational Mech. Anal. **176** (2005), 331–350.

[20] G. P. Galdi and A. L. Silvestre, Further results on steady-state flow of a Navier-Stokes liquid around a rigid body. Existence of the wake, Kyoto Conference on the Navier-Stokes Equations and their Applications, RIMS Kôkyûroku Bessatsu **B1** (2007), 127–143.

[21] T. Hishida, An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle, Arch. Rational Mech. Anal. **150** (1999), 307–348.

[22] T. Hishida, L^q estimates of weak solutions to the stationary Stokes equations around a rotating body, J. Math. Soc. Japan **58** (2006), 743–767.

[23] T. Hishida and Y. Shibata, L_p-L_q estimate of the Stokes operator and Navier-Stokes flows in the exterior of a rotating obstacle, Arch. Rational Mech. Anal. **193** (2009), 339–421.

[24] A. Korolev and V. Šverák, On the large-distance asymptotics of steady state solutions of the Navier-Stokes equations in 3D exterior domains, arXiv:math/07110560, preprint (2007).

[25] H. Kozono and H. Sohr, On stationary Navier-Stokes equations in unbounded domains, Ricerche Mat. **42** (1993), 69–86.

[26] H. Kozono, H. Sohr and M. Yamazaki, Representation formula, net force and energy relation to the stationary Navier-Stokes equations in 3-dimensional exterior domains, Kyushu J. Math. **51** (1997), 239–260.

[27] L. D. Landau, A new exact solution of Navier-Stokes equations, Dokl. Acad. Sci. URSS **43** (1944), 286–288.

[28] H. Miura and T.-P. Tsai, Point singularities of 3D stationary Navier-Stokes flows, preprint.

[29] S. A. Nazarov and K. Pileckas, On steady Stokes and Navier-Stokes problems with zero velocity at infinity in a three-dimensional exterior domain, J. Math. Kyoto Univ. **40** (2000), 475–492.

[30] Y. Shibata, On an exterior initial boundary value problem for Navier-Stokes equations, Quart. Appl. Math. **57** (1999), 117–155.

[31] A. L. Silvestre, On the existence of steady flows of a Navier-Stokes liquid around a moving rigid body, Math. Methods Appl. Sci. **27** (2004), 1399–1409.

[32] V. Šverák, On Landau's solutions of the Navier-Stokes equations, arXiv: math/0604550, preprint (2006).

[33] G. Tian and Z. Xin, One-point singular solutions to the Navier-Stokes equations, Topological Methods Nonlinear Anal. **11** (1998), 135–145.