Spectral Properties in L^q of an Oseen Operator Modelling Fluid Flow past a Rotating Body

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Abstract

We study the spectrum of a linear Oseen–type operator which arises from equations of motion of a viscous incompressible fluid in the exterior of a rotating compact body. Considering the operator in the function space $L^q_\sigma(\Omega), 1 < q < \infty$, we prove that the essential spectrum consists of an infinite set of overlapping parabolic regions in the left half–plane of the complex plane. The full spectrum coincides with the essential and continuous spectrum if $\Omega = \mathbb{R}^3$. Our approach is based on the Fourier transform in $\mathbb{R}^3$ and the transfer of the results to the exterior domain Ω.

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1 Introduction and main results

Suppose that $\mathcal{B}$ is a compact body in $\mathbb{R}^3$ which is rotating about the x_1–axis with a constant angular velocity $\omega > 0$. Denote by $\Omega(t)$ the exterior of $\mathcal{B}$ at time t and assume that $\Omega(t)$ is a domain with boundary of class $C^{1,1}$. Put $\boldsymbol{\omega} := \omega \boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is the unit vector oriented in the direction of the x_1–axis.

The flow of a viscous incompressible fluid in the exterior of the body $\mathcal{B}$ can be described by the Navier–Stokes equation in the space–time region $\{(\boldsymbol{x}, t) \in \mathbb{R}^3 \times \mathcal{I};\ t \in \mathcal{I},\ \boldsymbol{x} \in \Omega(t)\}$ where $\mathcal{I}$ is a time interval. The disadvantage of this description is the variability of the spatial domain $\Omega(t)$. This is why many authors use a time–dependent transformation of spatial coordinates which in fact also represents the rotation about the x_1–axis such that the body $\mathcal{B}$ is fixed and its exterior is just $\Omega := \Omega(0)$ in the new coordinate system. The system of equations after the transformation has the form

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} - (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} + \nabla p &= \boldsymbol{f} \qquad \text{in } \Omega \times \mathcal{I}, \\ \nabla \cdot \boldsymbol{u} &= 0 \qquad \text{in } \Omega \times \mathcal{I}. \end{aligned} \tag{1.1}$$

Among a series of results on qualitative properties of the system (1.1) and related linear problems, let us mention T. Hishida [17], [18], [19], G. P. Galdi [13], [14], R. Farwig, T. Hishida, D. Müller [6], R. Farwig [4], [5], M. Geissert, H. Heck, M. Hieber [15], R. Farwig, J. Neustupa [10], [11], R. Farwig, Š. Nečasová, J. Neustupa [9], R. Farwig, M. Krbec, Š. Nečasová [7], [8].

If $\boldsymbol{u}(\boldsymbol{x},t)$ tends to the constant velocity $\gamma\boldsymbol{e}_1$ for $|\boldsymbol{x}|\to\infty$, it is advantageous to write $\boldsymbol{u}=\boldsymbol{v}+\gamma\boldsymbol{e}_1$ and to deal with a new system for the unknown function $\boldsymbol{v}$:

$$\begin{aligned} \partial_t\boldsymbol{v}-\nu\Delta\boldsymbol{v}-(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}+\boldsymbol{\omega}\times\boldsymbol{v}+\gamma\partial_1\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\nabla p &= \boldsymbol{f} && \text{in } \Omega\times\mathcal{I}, \\ \nabla\cdot\boldsymbol{v} &= 0 && \text{in } \Omega\times\mathcal{I}. \end{aligned} \tag{1.2}$$

The analysis of this system is based on properties of the steady linear Oseen problem

$$\begin{aligned} \nu\Delta\boldsymbol{v}+(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}-\boldsymbol{\omega}\times\boldsymbol{v}-\gamma\partial_1\boldsymbol{v}-\nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{v} &= 0 && \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

This problem can be written in the form of one operator equation $A^{\omega}_{\gamma}\boldsymbol{v}=\boldsymbol{f}$, where A^{ω}_{γ} is the Oseen–type operator

$$A^{\omega}_{\gamma}\boldsymbol{v} \;=\; P_q\nu\Delta\boldsymbol{v}+P_q[(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}-\boldsymbol{\omega}\times\boldsymbol{v}-\gamma\partial_1\boldsymbol{v}] \tag{1.4}$$

with domain

$$D(A^{\omega}_{\gamma}) \;:=\; \bigl\{\boldsymbol{v}\in W^{2,q}(\Omega)^3\cap W^{1,q}_0(\Omega)^3\cap L^q_\sigma(\Omega);\ (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}\in L^q(\Omega)^3\bigr\} \tag{1.5}$$

in the function space $L^q_\sigma(\Omega)$, $1<q<\infty$; this function space as well as the Helmholtz projection P_q in $L^q(\Omega)^3$ will be defined below. We shall further treat the domain $D(A^{\omega}_{\gamma})$ as a Banach space with the norm

$$\|\boldsymbol{v}\|_{D(A^{\omega}_{\gamma})} \;:=\; \|\boldsymbol{v}\|_{2,q}+\|(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}\|_q. \tag{1.6}$$

The information on the spectrum of the linear operator A^{ω}_{γ} plays a fundamental role in studies of (1.3) and (1.2). The cases $q=2$ and $\gamma=0$ or $\gamma\neq 0$ were treated in our papers [10], [11], and the case $1<q<\infty$, $\gamma=0$ was studied in our paper [9] (together with Š. Nečasová). In this paper, **we consider the case $1<q<\infty$ and $\gamma\neq 0$.** We assume, without loss of generality, that $\gamma>0$.

We shall use the usual function spaces and norms:

- The norm in $L^q(\Omega)^3$ is denoted by $\|\,.\,\|_q=\|\,.\,\|_{q;\,\Omega}$.
- $W^{1,q}_0(\Omega)$ is the subspace of the Sobolev space $W^{1,q}(\Omega)$ consisting of functions vanishing on $\partial\Omega$ in the sense of traces.
- The norm in $W^{k,q}(\Omega)^3$, $k\in\mathbb{N}$, is denoted by $\|\,.\,\|_{k,q}=\|\,.\,\|_{k,q;\,\Omega}$.
- $C^\infty_{0,\sigma}(\Omega)$ denotes the linear space of all divergence–free vector fields from $C^\infty_0(\Omega)^3$.
- $L^q_\sigma(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^q(\Omega)^3$ and coincides with the space of all divergence–free (in the sense of distributions) vector fields $\boldsymbol{u}\in L^q(\Omega)^3$ such that $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$ in the sense of traces ([12], pp. 111–115); here $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$.
- P_q denotes the projection of $L^q(\Omega)^3$ onto $L^q_\sigma(\Omega)$, associated with the Helmholtz decomposition

$$L^q(\Omega)^3 \;=\; L^q_\sigma(\Omega)\oplus\bigl\{\nabla p\in L^q(\Omega)^3;\ p\in W^{1,q}_{loc}(\Omega)\bigr\}.$$

Let us recall definitions and basic properties from spectral theory of linear operators. Assume that X is a Banach space with the norm $\| \,.\, \|$, X^* is its dual and T is a closed linear operator in X. We denote by $D(T)$ the domain and by $R(T)$ the range of T and we assume that $D(T)$ is dense in X. This guarantees that the adjoint operator T^* exists.

- $\mathrm{nul}\,(T)$ and $\mathrm{def}\,(T)$ denote the nullity and the deficiency of the operator T, respectively. If $R(T)$ is closed then $\mathrm{nul}\,(T) = \mathrm{def}\,(T^*)$ and $\mathrm{def}\,(T) = \mathrm{nul}\,(T^*)$ (see e.g. T. Kato [20, p. 234]).
- $\mathrm{nul}'(T)$ and $\mathrm{def}'(T) := \mathrm{nul}'(T^*)$ denote the approximate nullity and the approximate deficiency of T, respectively. We recall that $\mathrm{nul}'(T)$ is the maximum integer m ($m = +\infty$ being permitted) with the property that to each $\epsilon > 0$ there exists an m–dimensional linear manifold M_ϵ in $D(T)$ such that $\|T\boldsymbol{v}\| < \epsilon$ for all $\boldsymbol{v} \in M_\epsilon$, $\|\boldsymbol{v}\| = 1$. Note that $\mathrm{nul}\,(T) \leq \mathrm{nul}'(T)$ and $\mathrm{def}\,(T) \leq \mathrm{def}'(T)$ and the equalities hold if $R(T)$ is closed. On the other hand, if $R(T)$ is not closed then $\mathrm{nul}'(T) = \mathrm{def}'(T) = \infty$. The identity $\mathrm{nul}'(T) = \infty$ is equivalent to the existence of a non–compact sequence $\{\boldsymbol{u}_n\}$ on the unit sphere in X such that $T\boldsymbol{u}_n \to \boldsymbol{0}$ for $n \to \infty$, see [20, p. 233].
- We say that T is a Fredholm operator if its range $R(T)$ is closed in X and both the numbers $\mathrm{nul}\,(T)$ and $\mathrm{def}\,(T)$ are finite.
- The operator T is called semi–Fredholm if the range $R(T)$ is closed in X and at least one of the numbers $\mathrm{nul}\,(T)$ and $\mathrm{def}\,(T)$ is finite. Consequently, T is semi–Fredholm if and only if at least one of the numbers $\mathrm{nul}'(T)$ and $\mathrm{def}'(T)$ is finite.
- We denote by $\rho(T)$ the resolvent set of T, by $\sigma(T) = \mathbb{C} \setminus \rho(T)$ the spectrum of T and by $\sigma_{\mathrm{ess}}(T)$ the essential spectrum of T. Recall that $\rho(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $R(T - \lambda I) = X$ and the operator $T - \lambda I$ has a bounded inverse in X. Thus, $\mathrm{nul}\,(T - \lambda I) = \mathrm{nul}'(T - \lambda I) = \mathrm{def}\,(T - \lambda I) = \mathrm{def}'(T - \lambda I) = 0$ for $\lambda \in \rho(T)$. Moreover, $\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C};\ T - \lambda I \text{ is not semi}-\text{Fredholm}\}$. Both $\sigma(T)$ and $\sigma_{\mathrm{ess}}(T)$ are closed subsets of $\mathbb{C}$ and $\sigma_{\mathrm{ess}}(T) \subset \sigma(T)$.
- Let us also mention that $\sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \cup \sigma_{\mathrm{r}}(T)$ where the sets $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{r}}(T)$ are called the point spectrum, the continuous spectrum and the residual spectrum of T, respectively. They are mutually disjoint and they are defined in this way:
  - $\sigma_{\mathrm{p}}(T) := \big\{\lambda \in \mathbb{C};\ \mathrm{nul}\,(T - \lambda I) > 0\big\}$,
  - $\sigma_{\mathrm{c}}(T)$ is the set of $\lambda \in \mathbb{C}$ such that $\mathrm{nul}\,(T - \lambda I) = 0$, $R(T - \lambda I)$ is dense in X, but $R(T - \lambda I) \neq X$. In this case, $R(T - \lambda I)$ is not closed in X, which implies that $\mathrm{def}\,(T - \lambda I) = \mathrm{def}'(T - \lambda I) = \mathrm{nul}'(T - \lambda I) = \infty$.
  - $\sigma_{\mathrm{r}}(T)$ is the set of $\lambda \in \mathbb{C}$ such that $\mathrm{nul}\,(T - \lambda I) = 0$ and the range $R(T - \lambda I)$ is not dense in X.

  Obviously, $\sigma_{\mathrm{c}}(T) \subset \sigma_{\mathrm{ess}}(T)$. There are no generally valid relations between $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{r}}(T)$ on one hand and $\sigma_{\mathrm{ess}}(T)$ on the other hand. However, any point on the boundary of $\sigma(T)$ belongs to $\sigma_{\mathrm{ess}}(T)$ unless it is an isolated point of $\sigma(T)$, see [20, p. 244].
- The so-called approximate point spectrum $\sigma_{\mathrm{ap}}(T)$ of T consists of all points $\lambda \in \mathbb{C}$ such that there exists a sequence $\{\boldsymbol{u}_n\}$ in $D(T)$ such that $\|\boldsymbol{u}_n\| = 1$ and $(T - \lambda I)\boldsymbol{u}_n \to \boldsymbol{0}$ as $n \to \infty$. Obviously, $\mathrm{nul}'(T - \lambda I) > 0$ for $\lambda \in \sigma_{\mathrm{ap}}(T)$, which implies that $\lambda \in \sigma(T)$, and $\sigma_{\mathrm{p}}(T) \subset \sigma_{\mathrm{ap}}(T)$. Finally, if $\lambda \in \sigma_{\mathrm{c}}(T)$ then $\mathrm{nul}'(T - \lambda I) = \infty$, which also implies that there exists a sequence $\{\boldsymbol{u}_n\}$ with the properties required in the definition of $\sigma_{\mathrm{ap}}(T)$.

Hence $\lambda \in \sigma_{\rm ap}(T)$. We have thus shown that $\sigma_{\rm p}(T) \cup \sigma_{\rm c}(T) \subset \sigma_{\rm ap}(T) \subset \sigma(T)$ and $\sigma_{\rm ap}(T) \cup \sigma_{\rm r}(T) = \sigma(T)$. We note that the approximate spectrum has been introduced for the sake of completeness, but will not be used in this paper.

The main theorems of this paper concern the concrete operator A^ω_γ in the space $L^q_\sigma(\Omega)$:

Theorem 1.1. *Let* $1 < q < \infty$ *and* $\Omega = \mathbb{R}^3$. *Then*

(i) $\sigma(A^\omega_\gamma) = \sigma_{\rm c}(A^\omega_\gamma) = \sigma_{\rm ess}(A^\omega_\gamma) = \Lambda^\omega_\gamma$, *where*

$$\Lambda^\omega_\gamma \;:=\; \{\lambda = \alpha + {\rm i}\beta + {\rm i}k\omega \in \mathbb{C};\; \alpha, \beta \in \mathbb{R},\; k \in \mathbb{Z},\; \alpha \le -\nu\beta^2/\gamma^2\}.$$

(ii) If $q = 2$ *then* A^ω_γ *is a normal operator in* $L^q_\sigma(\mathbb{R}^3)$ *(* $= L^2_\sigma(\mathbb{R}^3)$*).*

Theorem 1.2. *Let* $1 < q < \infty$ *and* $\Omega \subset \mathbb{R}^3$ *be an exterior domain with boundary of class* $C^{1,1}$. *Then the spectrum of* A^ω_γ *lies in the left complex half plane* $\{\lambda \in \mathbb{C}; {\rm Re}\,\lambda \le 0\}$ *and consists of the essential spectrum* $\sigma_{\rm ess}(A^\omega_\gamma) = \Lambda^\omega_\gamma$ *and possibly a set* Γ *of isolated eigenvalues* $\lambda \in \mathbb{C} \setminus \Lambda^\omega_\gamma$ *with* ${\rm Re}\,\lambda < 0$ *and finite algebraic multiplicity, which can cluster only at points of* $\sigma_{\rm ess}(A^\omega_\gamma)$. *The set* Γ *of such isolated eigenvalues is independent of* $q \in (1, \infty)$.

The set Λ^ω_γ is a union of infinitely many equally shifted filled parabolas in the left half–plane of $\mathbb{C}$, see Fig. 1. Theorem 1.1 is proved in Section 3. The proof of Theorem 1.2 is given in Section 4.

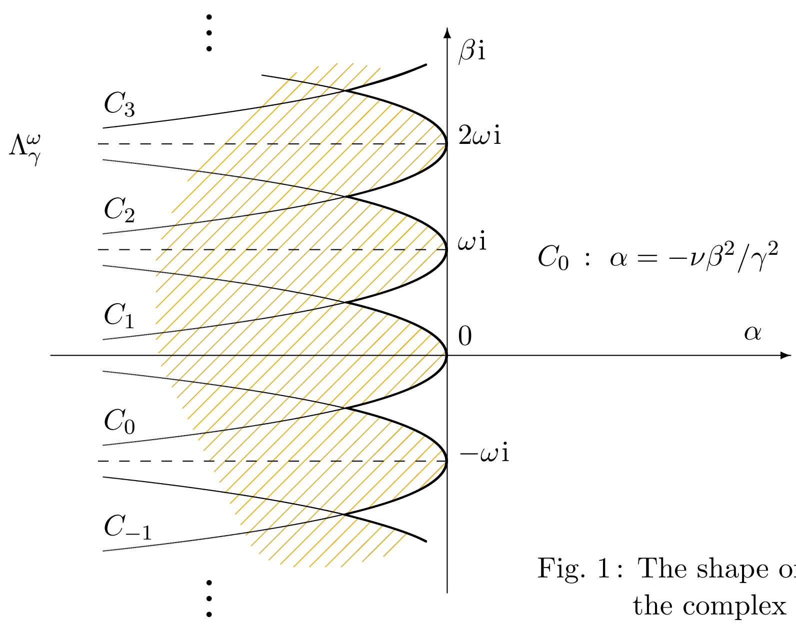

Fig. 1: The shape of set Λ^ω_γ in the complex domain $\mathbb{C}$

The question whether the identities of Theorem 1.1 (i) also hold in the case when Ω is an exterior domain in $\mathbb{R}^3$ is open. The reason consists in the application of the Fourier transform, which is a useful tool in $\mathbb{R}^3$ but cannot be used in a general exterior domain Ω.

2 Preliminary results

The domain $D(A^\omega_\gamma)$ of the operator A^ω_γ is dense in $L^q_\sigma(\Omega)$, because $C^\infty_{0,\sigma}(\Omega) \subset D(A^\omega_\gamma)$ and $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^q_\sigma(\Omega)$). Hence the adjoint operator $(A^\omega_\gamma)^*$ exists as a linear operator in $L^{q'}_\sigma(\Omega)$, where $q' = q/(q-1)$. The next lemma brings more information on both the operators A^ω_γ and $(A^\omega_\gamma)^*$.

Lemma 2.1. *The operator A^ω_γ is closed in $L^q_\sigma(\Omega)$ and generates a C_0–semigroup in $L^q_\sigma(\Omega)$. Its adjoint operator is*

$$(A^\omega_\gamma)^* \boldsymbol{v} \;=\; P_{q'} \nu \Delta \boldsymbol{v} + P_{q'}[-(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} + \gamma \partial_1 \boldsymbol{v}] \tag{2.1}$$

with domain

$$D((A^\omega_\gamma)^*) \;=\; \big\{ \boldsymbol{v} \in W^{2,q'}(\Omega)^3 \cap W^{1,q'}_0(\Omega)^3 \cap L^{q'}_\sigma(\Omega);\; (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} \in L^{q'}(\Omega)^3 \big\}. \tag{2.2}$$

It is a closed operator in $L^{q'}_\sigma(\Omega)$ and generates a C_0–semigroup in $L^{q'}_\sigma(\Omega)$.

Proof. The fact that A^ω_γ is a generator of a C_0–semigroup in $L^q_\sigma(\Omega)$ follows from [23, Theorem 1.1]. It also implies that A^ω_γ is a closed operator in $L^q_\sigma(\Omega)$ and that $R(A^\omega_\gamma - \zeta I) = L^q_\sigma(\Omega)$ for all $\zeta > 0$ sufficiently large. Let us denote by T^ω_γ the operator on the right hand side of (2.1) with the domain given by (2.2):

$$D(T^\omega_\gamma) = \big\{ \boldsymbol{v} \in W^{2,q'}(\Omega)^3 \cap W^{1,q'}_0(\Omega)^3 \cap L^{q'}_\sigma(\Omega);\; (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} \in L^{q'}(\Omega)^3 \big\}.$$

By analogy with A^ω_γ, the operator T^ω_γ is closed in $L^{q'}_\sigma(\Omega)$ and $R(T^\omega_\gamma - \zeta I) = L^{q'}_\sigma(\Omega)$ if $\zeta > 0$ is sufficiently large. It is easy to verify that the operators A^ω_γ and T^ω_γ are adjoint to each other in the sense of T. Kato [20, p. 167]; hence $T^\omega_\gamma \subset (A^\omega_\gamma)^*$. In order to show that $T^\omega_\gamma = (A^\omega_\gamma)^*$, we need to verify that T^ω_γ is the maximal operator adjoint to A^ω_γ. Suppose that $\boldsymbol{v} \in D((A^\omega_\gamma)^*)$ and put $\boldsymbol{f} = (\zeta I - (A^\omega_\gamma)^*)\boldsymbol{v}$. Since $\boldsymbol{f} \in R(T^\omega_\gamma - \zeta I)$, there exists $\boldsymbol{w} \in D(T^\omega_\gamma)$ such that $\boldsymbol{f} = (T^\omega_\gamma - \zeta I)\boldsymbol{w}$. Hence $((A^\omega_\gamma)^* - \zeta I)\boldsymbol{v} = (T^\omega_\gamma - \zeta I)\boldsymbol{w}$. Multiplying both sides of this identity by $\boldsymbol{u} \in D(A^\omega_\gamma)$ and integrating in Ω, we arrive at

$$\int_\Omega \boldsymbol{v} \cdot (A^\omega_\gamma - \zeta I)\boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \;=\; \int_\Omega \boldsymbol{w} \cdot (A^\omega_\gamma - \zeta I)\boldsymbol{u} \, \mathrm{d}\boldsymbol{x}.$$

Since this holds for all $\boldsymbol{u} \in D(A^\omega_\gamma)$, we get $\boldsymbol{v} = \boldsymbol{w} \in D(T^\omega_\gamma)$; thus $(A^\omega_\gamma)^* = T^\omega_\gamma$. As for A^ω_γ we conclude that $(A^\omega_\gamma)^*$ generates a C_0–semigroup in $L^{q'}_\sigma(\Omega)$ and is closed. □

Lemma 2.2. *There exist constants $c_1 > 0$ and $c_2 > 0$ such that if $\boldsymbol{v} \in D(A^\omega_\gamma)$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$ satisfy the equation $A^\omega_\gamma \boldsymbol{v} = \boldsymbol{f}$ then*

$$\|\boldsymbol{v}\|_{2,q} + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}\|_q \;\leq\; c_1 \|\boldsymbol{f}\|_q + c_2 \|\boldsymbol{v}\|_q. \tag{2.3}$$

Proof. If $\Omega = \mathbb{R}^3$ then (2.3) follows from [4, Theorem 1.1] and an interpolation argument. Now consider an exterior domain $\Omega \subset \mathbb{R}^3$ of class $C^{1,1}$. Let $\boldsymbol{v} \in D(A^\omega_\gamma)$ and $\boldsymbol{f} \in L^q_\sigma(\Omega)$ satisfy the equation $A^\omega_\gamma \boldsymbol{v} = \boldsymbol{f}$. Then there exists a pressure function p such that

$$\nabla p \;=\; \nu \Delta \boldsymbol{v} + (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} - \boldsymbol{\omega} \times \boldsymbol{v} - \gamma \partial_1 \boldsymbol{v} - \boldsymbol{f} \;\in L^q(\Omega)^3$$

For simplicity, we assume that in the following all pressure functions have a vanishing mean on $\Omega_R = \Omega \cap B_R(\mathbf{0})$, i.e., $\int_{\Omega_R} p \, d\boldsymbol{x} = 0$, where $R > 0$ is chosen such that $\mathbb{R}^3 \setminus B_{R-1}(\mathbf{0}) \subset \Omega$. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ denote a cut-off function with values in $[0,1]$ such that

$$\eta(\boldsymbol{x}) = \begin{cases} 0 & \text{for } |\boldsymbol{x}| \le R-1, \\ 1 & \text{for } |\boldsymbol{x}| \ge R. \end{cases}$$

Then $(\eta \boldsymbol{v}, \eta p)$ can be considered as a solution of the whole space problem

$$\nu \Delta \boldsymbol{u} + (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u} - \gamma \partial_1 \boldsymbol{u} - \nabla \tilde{p} \;=\; \boldsymbol{f}_1, \quad \nabla \cdot \boldsymbol{u} \;=\; g_1 \quad \text{in } \mathbb{R}^3,$$

where

$$\boldsymbol{f}_1 = \eta \boldsymbol{f} + 2\nu \nabla \eta \cdot \nabla \boldsymbol{v} + \nu \boldsymbol{v} \Delta \eta + \big((\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \eta \big) \boldsymbol{v} - \gamma (\partial_1 \eta) \boldsymbol{v} - p \nabla \eta, \qquad g_1 = \boldsymbol{v} \cdot \nabla \eta.$$

Note that $\boldsymbol{f}_1$ coincides with $\eta \boldsymbol{f}$ up to perturbation terms with support in $\operatorname{supp}(\nabla \eta) \subset \overline{\Omega}_R$. Similarly, $\operatorname{supp} g_1 \subset \overline{\Omega}_R$. By [4, Theorem 1.1] there exists a solution $(\boldsymbol{u}, \tilde{p})$ satisfying the estimate

$$\|\nabla^2 \boldsymbol{u}\|_q + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u}\|_q + \|\nabla \tilde{p}\|_q \;\le\; c \big(\|\boldsymbol{f}\|_q + \|\boldsymbol{v}\|_{1,q;\,\Omega_R} + \|p\|_{q;\,\Omega_R} \big).$$

Moreover, by the uniqueness assertion in [4, Theorem 1.1], $(\eta \boldsymbol{v}, \eta p)$ satisfies the same estimate so that we get the inequality

$$\begin{aligned} &\|\nabla^2 (\eta \boldsymbol{v})\|_q + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla (\eta \boldsymbol{v})\|_q + \|\nabla (\eta p)\|_q \\ &\quad \le \; c \big(\|\boldsymbol{f}\|_q + \|\boldsymbol{v}\|_q + \|\boldsymbol{v}\|_{1,q;\,\Omega_R} + \|p\|_{q;\,\Omega_R} \big) . \end{aligned} \tag{2.4}$$

Next we consider $((1-\eta)\boldsymbol{v}, (1-\eta)p)$ as a solution of the Stokes problem

$$\nu \Delta \boldsymbol{u} - \nabla \tilde{p} = \boldsymbol{f}_2, \quad \operatorname{div} \boldsymbol{u} = g_2 \quad \text{in } \Omega_R, \qquad \boldsymbol{u} = \mathbf{0} \quad \text{on } \partial \Omega_R \,,$$

where $g_2 = -\boldsymbol{v} \cdot \nabla \eta$ and

$$\boldsymbol{f}_2 \;=\; (1-\eta) \big[\boldsymbol{f} - (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} + \gamma \, \partial_1 \boldsymbol{v} \big] - 2\nu \, \nabla \eta \cdot \nabla \boldsymbol{v} - \nu \, \boldsymbol{v} \, \Delta \eta + p \, \nabla \eta.$$

Hence classical *a priori* estimates for the Stokes system in the bounded domain Ω_R and the precise form of $\boldsymbol{f}_2$ imply that

$$\|\nabla^2 ((1-\eta)\boldsymbol{v})\|_{q;\,\Omega_R} + \|\nabla (1-\eta) p\|_{q;\,\Omega_R} \le \; c \big(\|\boldsymbol{f}\|_q + \|\boldsymbol{v}\|_{1,q;\,\Omega_R} + \|p\|_{q;\,\Omega_R} \big) . \tag{2.5}$$

Summing (2.4) and (2.5) and using an interpolation estimate for $\nabla \boldsymbol{v}$ we obtain the estimate

$$\|\nabla^2 \boldsymbol{v}\|_q + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}\|_q + \|\nabla p\|_q \le \; c \big(\|\boldsymbol{f}\|_q + \|\boldsymbol{v}\|_q + \|p\|_{q;\,\Omega_R} \big) \tag{2.6}$$

with a constant $c > 0$ independent of $\boldsymbol{f}$, $\boldsymbol{v}$, p.

Now we will prove (2.3) by contradiction, using (2.6). Suppose that for every $n \in \mathbb{N}$ there exist $\boldsymbol{v}_n \in D(A_\gamma^\omega)$ and p_n with $\nabla p_n \in L^q(\Omega)^3$ such that

$$1 = \|\nabla^2 \boldsymbol{v}_n\|_q + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}_n\|_q + \|\nabla p_n\|_q \ge \; n \big(\|\boldsymbol{f}_n\|_q + \|\boldsymbol{v}_n\|_q \big) \tag{2.7}$$

where $\boldsymbol{f}_n = A^\omega_\gamma \boldsymbol{v}_n$. Then we find a subsequence of $(\boldsymbol{v}_n, p_n)$, again denoted by $(\boldsymbol{v}_n, p_n)$, and $\boldsymbol{v} \in D(A^\omega_\gamma)$ and p with $\nabla p \in L^q(\Omega)^3$ such that in the weak sense

$$\nabla^2 \boldsymbol{v}_n \rightharpoonup \nabla^2 \boldsymbol{v}, \quad (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}_n \rightharpoonup (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}, \quad \nabla p_n \rightharpoonup \nabla p \quad \text{in } L^q(\Omega) \tag{2.8}$$

and in the strong sense

$$\boldsymbol{f}_n \to \boldsymbol{0}, \quad \boldsymbol{v}_n \to \boldsymbol{0} \quad \text{in } L^q(\Omega), \quad p_n \to p \quad \text{in } L^q(\Omega_R), \quad \int_{\Omega_R} p \, \mathrm{d}\boldsymbol{x} = 0 \tag{2.9}$$

as $n \to \infty$, and

$$\nu \Delta \boldsymbol{v} + (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} - \boldsymbol{\omega} \times \boldsymbol{v} - \gamma \partial_1 \boldsymbol{v} - \nabla p = 0, \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \partial\Omega.$$

By (2.8), (2.9) we conclude that $\boldsymbol{v} = 0$ and $\nabla p = \boldsymbol{0}$ in Ω. Since the limit function p was chosen with $\int_{\Omega_R} p \, \mathrm{d}\boldsymbol{x} = 0$, we have $p = 0$ in Ω. Using now (2.6) with $\boldsymbol{v}_n$, p_n, $\boldsymbol{f}_n$ instead of $\boldsymbol{v}$, p, $\boldsymbol{f}$, we observe that the left hand side equals one, while the right hand side tends to zero as $n \to \infty$. This is the contradiction to the assumption that (2.3) was wrong. $\square$

Let us further denote by $D_0(A^\omega_\gamma)$ the subspace of $D(A^\omega_\gamma)$ which contains only functions that have a compact support in $\overline{\Omega}$.

Lemma 2.3. *$D_0(A^\omega_\gamma)$ is a core of operator A^ω_γ, i.e., the graph of the restriction of the operator A^ω_γ to $D_0(A^\omega_\gamma)$ is dense in the graph of A^ω_γ in the norm of $L^q_\sigma(\Omega) \times L^q_\sigma(\Omega)$.*

Proof. Let $\boldsymbol{v} \in D(A^\omega_\gamma)$ and $\boldsymbol{f} = A^\omega_\gamma \boldsymbol{v}$. We will show that $[\boldsymbol{v}, A^\omega_\gamma \boldsymbol{v}]$ can be approximated by a sequence of elements $[\boldsymbol{v}^n, A^\omega_\gamma \boldsymbol{v}^n]$ where $\boldsymbol{v}^n \in D_0(A^\omega_\gamma)$, $n \in \mathbb{N}$.

Let $\eta \in C^\infty_0(\mathbb{R}^3)$ be radially symmetric, with values in $[0, 1]$, such that

$$\eta(\boldsymbol{x}) = \begin{cases} 1 & \text{for } |\boldsymbol{x}| \le 1, \\ 0 & \text{for } 2 \le |\boldsymbol{x}|. \end{cases}$$

Denote $K_1 := \{\boldsymbol{x} \in \mathbb{R}^3;\ 1 < |\boldsymbol{x}| < 2\}$ and, more generally, for $r > 0$, let $K_r := \{\boldsymbol{x} \in \mathbb{R}^3;\ r < |\boldsymbol{x}| < 2r\}$. Due to M. E. Bogovskij [2], there exists a bounded linear operator $\mathfrak{B} : L^q(K_1) \longrightarrow W^{1,q}_0(K_1)^3$ such that $\operatorname{div} \mathfrak{B} f = f$ for all $f \in L^q(K_1)$ satisfying $\int_{K_1} f \, \mathrm{d}\boldsymbol{x} = 0$. The operator $\mathfrak{B}$ is bounded from $W^{1,q}_0(K_1)$ to $W^{2,q}_0(K_1)^3$ as well.

Let $n \in \mathbb{N}$ be so large that $\mathbb{R}^3 \setminus B_n(\boldsymbol{0}) \subset \Omega$. Put $\boldsymbol{v}^n(\boldsymbol{x}) := \eta(\boldsymbol{x}/n)\, \boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{V}^n(\boldsymbol{x})$ with the correction term $\boldsymbol{V}^n(\boldsymbol{x})$ being equal to $\boldsymbol{U}^n(\boldsymbol{x}/n)$, where

$$\boldsymbol{U}^n(\boldsymbol{y}) = \begin{cases} \mathfrak{B}\big[\nabla\eta(\boldsymbol{y}) \cdot \boldsymbol{v}(n\boldsymbol{y})\big] & \text{for } \boldsymbol{y} \in K_1, \\ \boldsymbol{0} & \text{for } \boldsymbol{y} \in \mathbb{R}^3 \setminus K_1. \end{cases}$$

The function $\boldsymbol{v}^n$ is divergence–free, it coincides with $\boldsymbol{v}$ in $\Omega \cap B_n(\boldsymbol{0})$ and its support is a subset of the closure of $\Omega \cap B_{2n}(\boldsymbol{0})$. Due to the continuity of the operator $\mathfrak{B}$ from $L^q(K_1)$ to $W^{1,q}_0(K_1)^3$ and from $W^{1,q}_0(K_1)$ to $W^{2,q}_0(K_1)^3$, the function $\boldsymbol{U}^n$ satisfies the estimates

$$\|\nabla \boldsymbol{U}^n\|_{q;\,K_1} \le C \left\|\nabla\eta \cdot \boldsymbol{v}(n\,\cdot)\right\|_{q;\,K_1} \le C \left\|\boldsymbol{v}(n\,\cdot)\right\|_{q;K_1} = C\, n^{-3/q}\, \|\boldsymbol{v}\|_{q;\,K_n}\,,$$

$$
\begin{aligned}
\|\nabla^2 \boldsymbol{U}^n\|_{q;K_1} \;&\le\; C \left\|\nabla\big(\nabla\eta\cdot\boldsymbol{v}(n\,\cdot)\big)\right\|_{q;\,K_1} \le C\left\|\boldsymbol{v}(n\,\cdot)\right\|_{q;\,K_1} + C\left\|\nabla\boldsymbol{v}(n\,\cdot)\right\|_{q;\,K_1}\\
&=\; C\,n^{-3/q}\,\|\boldsymbol{v}\|_{q;\,K_n} \,+ C\,n^{1-3/q}\,\|\nabla\boldsymbol{v}\|_{q;\,K_n}\,.
\end{aligned}
$$

This means that

$$
\begin{aligned}
\|\nabla \boldsymbol{V}^n\|_{q;\,K_n} \;&=\; n^{-1+3/q}\,\|\nabla\boldsymbol{U}^n\|_{q;\,K_1} \;\le\; \frac{C}{n}\,\|\boldsymbol{v}\|_{q;\,K_n},\\
\|\nabla^2 \boldsymbol{V}^n\|_{q;\,K_n} \;&=\; n^{-2+3/q}\,\|\nabla^2\boldsymbol{U}^n\|_{q;\,K_1} \le \frac{C}{n^2}\,\|\boldsymbol{v}\|_{q;\,K_n} + \frac{C}{n}\,\|\nabla\boldsymbol{v}\|_{q;\,K_n}\,.
\end{aligned}
$$

Using also the fact that $\boldsymbol{V}^n(\boldsymbol{x}) = \boldsymbol{0}$ for $|\boldsymbol{x}| = n$, we derive that

$$
\|\boldsymbol{V}^n\|_{q;\,K_n} \;\le\; C n\,\|\nabla\boldsymbol{V}^n\|_{q;\,K_n} \;\le\; C\,\|\boldsymbol{v}\|_{q;\,K_n}.
$$

All generic constants C are independent of n. Thus, $\|\boldsymbol{V}^n\|_{2,q} \to 0$ as $n\to\infty$. The same result also holds on the $\|\,.\,\|_{2,q}$–norm of the difference $\eta(\boldsymbol{x}/n)\,\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{x})$. Consequently,

$$
\boldsymbol{v}^n \;\longrightarrow\; \boldsymbol{v} \qquad \text{in } W^{2,q}(\Omega)^3 \quad \text{for } n\to\infty. \tag{2.10}
$$

Furthermore, since $(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\eta(\boldsymbol{x}/n) = 0$, we have

$$
\begin{aligned}
\|A^\omega_\gamma\boldsymbol{v} - A^\omega_\gamma\boldsymbol{v}^n\|_{q;\,\Omega} \;&\le\; C\left(\|\boldsymbol{v}-\boldsymbol{v}_n\|_{2,q;\,\Omega} + \left\|(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v} - (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}^n\right\|_{q;\,\Omega}\right)\\
&\le\; C\left(\|\boldsymbol{v}-\boldsymbol{v}_n\|_{2,q;\,\Omega} + \left\|(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\big(1-\eta(\boldsymbol{x}/n)\big)\,\nabla\boldsymbol{v}\right\|_{q;\,\Omega} + \left\|(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{V}^n\right\|_{q;\,\Omega}\right)\\
&\le\; C\left(\|\boldsymbol{v}-\boldsymbol{v}_n\|_{2,q;\,\Omega} + \|(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}\|_{q;\,\Omega\smallsetminus B_n(0)} + \; n\,\|\nabla\boldsymbol{V}^n\|_{q;\,\Omega}\right)\\
&\longrightarrow\; 0 \quad \text{for} \quad n\to\infty.
\end{aligned} \tag{2.11}
$$

We can now observe from (2.10) and (2.11) that

$$
[\boldsymbol{v}^n, A^\omega_\gamma\boldsymbol{v}^n] \;\longrightarrow\; [\boldsymbol{v}, A^\omega_\gamma\boldsymbol{v}] \qquad \text{in } \; L^q_\sigma(\Omega)\times L^q_\sigma(\Omega)
$$

as $n\to\infty$. The proof is completed. □

3 The case $\Omega = \mathbb{R}^3$

If $\Omega = \mathbb{R}^3$, $1 < q < \infty$ and $\boldsymbol{v}\in D(A^\omega_\gamma)$, then the terms $\nu\Delta\boldsymbol{v}$, $(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v}$ and $\gamma\partial_1\boldsymbol{v}$ belong to $L^q_\sigma(\mathbb{R}^3)$. Hence the projection P_q in (1.4) can be omitted so that

$$
A^\omega_\gamma\boldsymbol{v} \;=\; \nu\Delta\boldsymbol{v} + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v} - \gamma\partial_1\boldsymbol{v}. \tag{3.1}
$$

By analogy, the adjoint operator – as an operator in $L^{q'}_\sigma(\mathbb{R}^3)$ – can be simplified to

$$
(A^\omega_\gamma)^*\boldsymbol{v} \;=\; \nu\Delta\boldsymbol{v} - (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} + \gamma\partial_1\boldsymbol{v}\,. \tag{3.2}
$$

The next lemma provides an information on solutions of the resolvent equation

$$
A^\omega_\gamma\boldsymbol{v} - \lambda\boldsymbol{v} \;=\; \boldsymbol{f} \tag{3.3}
$$

for $\boldsymbol{f}\in L^q_\sigma(\mathbb{R}^3)$. Recall the definition of the set Λ^ω_γ from Theorem 1.1.

Lemma 3.1. *Suppose that $\lambda \in \mathbb{C} \setminus \Lambda^\omega_\gamma$. There exists a constant $c_3 = c_3(\lambda, q) > 0$ such that if $\boldsymbol{f} \in L^q_\sigma(\mathbb{R}^3)$ and $\boldsymbol{v} \in D(A^\omega_\gamma)$ satisfy the resolvent equation (3.3) then*

$$\|\boldsymbol{v}\|_q \leq c_3 \, \|\boldsymbol{f}\|_q. \tag{3.4}$$

Proof. The linear space $D_0(A^\omega_\gamma)$ is a core of $A^\omega_\gamma - \lambda I$ due to Lemma 2.3. Thus, it is sufficient to prove (3.4) only for $\boldsymbol{v} \in D_0(A^\omega_\gamma)$.

Equation (3.3) can be written in the form

$$\nu\Delta\boldsymbol{v} + (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v} - \boldsymbol{\omega} \times \boldsymbol{v} - \gamma\partial_1\boldsymbol{v} - \lambda\boldsymbol{v} \;=\; \boldsymbol{f}. \tag{3.5}$$

Due to the geometry of the problem, it is reasonable to use the cylindrical coordinates (x_1, r, φ) in $\mathbb{R}^3$ with the axis being the x_1–axis; then $r^2 = x_2^2 + x_3^2$. The term $(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v}$ in (3.5), which equals $\omega(-x_3\,\partial_2\boldsymbol{v} + x_2\,\partial_3\boldsymbol{v})$, can be simplified to

$$(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v} \;=\; \omega\partial_\varphi\boldsymbol{v}. \tag{3.6}$$

We shall denote by $\mathcal{F}$ the Fourier transform, by $\mathcal{F}_{-1}$ its inverse, by $\widehat{\ }$ Fourier images of functions, by $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ their Cartesian Fourier variables, and we put $s = |\boldsymbol{\xi}|$. Applying $\mathcal{F}$ to (3.5), we obtain

$$(\lambda + \mathrm{i}\gamma\xi_1 + \nu s^2)\widehat{\boldsymbol{v}} - \omega\partial_\phi\widehat{\boldsymbol{v}} + \boldsymbol{\omega} \times \widehat{\boldsymbol{v}} \;=\; -\widehat{\boldsymbol{f}}. \tag{3.7}$$

Here $\partial_\phi\widehat{\boldsymbol{v}}$ denotes the angular derivative

$$\partial_\phi\widehat{\boldsymbol{v}} \;=\; (\boldsymbol{e}_1 \times \boldsymbol{\xi}) \cdot \nabla\widehat{\boldsymbol{v}} \;\equiv\; -\xi_3\,\frac{\partial\widehat{\boldsymbol{v}}}{\partial\xi_2} + \xi_2\,\frac{\partial\widehat{\boldsymbol{v}}}{\partial\xi_3}$$

when using cylindrical coordinates (ξ_1, ρ, ϕ) in the space of the Fourier variables. The equation $\operatorname{div}\boldsymbol{v} = 0$ (following from the fact that $\boldsymbol{v} \in D(A^\omega_\gamma)$) leads to the condition $\mathrm{i}\boldsymbol{\xi} \cdot \widehat{\boldsymbol{v}} = 0$. Now $\widehat{\boldsymbol{v}}$ can be considered to be a solution of the first order ordinary differential equation (3.7) with respect to the angular variable ϕ. Writing $\widehat{\boldsymbol{v}}$ in the form

$$\widehat{\boldsymbol{v}}(\rho, \phi, \xi_1) \;=\; O(\phi)\,\widehat{\boldsymbol{w}}(\rho, \phi, \xi_1)$$

where

$$O(t) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix},$$

one verifies that $\omega\partial_\phi\widehat{\boldsymbol{v}} \;=\; O(\phi)\,\omega\partial_\phi\widehat{\boldsymbol{w}} + \boldsymbol{\omega} \times [O(\phi)\widehat{\boldsymbol{w}}]$. Hence (3.7) is equivalent to the equation

$$-\omega\partial_\phi\widehat{\boldsymbol{w}} + (\lambda + \mathrm{i}\gamma\xi_1 + \nu s^2)\widehat{\boldsymbol{w}} \;=\; -O(\phi)^T\,\widehat{\boldsymbol{f}}\,, \tag{3.8}$$

or using the definition

$$a(\boldsymbol{\xi}) = \lambda + \mathrm{i}\gamma\xi_1 + \nu s^2, \quad s = |\boldsymbol{\xi}|\,,$$

to the inhomogeneous ordinary first order linear differential equation with respect to ϕ

$$\omega\partial_\phi\widehat{\boldsymbol{w}} - a(\boldsymbol{\xi})\widehat{\boldsymbol{w}} = O(\phi)^T\widehat{\boldsymbol{f}}\,.$$

Its solution $\widehat{\boldsymbol{w}}$ satisfies

$$\widehat{\boldsymbol{w}}(\xi_1,\rho,\phi+2\pi) = \mathrm{e}^{2\pi\, a(\boldsymbol{\xi})/\omega}\, \widehat{\boldsymbol{w}}(\xi_1,\rho,\phi) + \frac{1}{\omega}\int_0^{2\pi} \mathrm{e}^{(2\pi-t)a(\boldsymbol{\xi})/\omega}\, O(t+\phi)^T\, \widehat{\boldsymbol{f}}(\xi_1,\rho,t+\phi)\, \mathrm{d}t\,.$$

Since $\widehat{\boldsymbol{w}}$ is 2π–periodic in the variable ϕ, we have

$$\widehat{\boldsymbol{w}}(\xi_1,\rho,\phi) \;=\; \frac{1}{\omega}\int_0^{2\pi} \frac{\mathrm{e}^{(2\pi-t)a(\boldsymbol{\xi})/\omega}}{1-\mathrm{e}^{2\pi\, a(\boldsymbol{\xi})/\omega}}\, O(t+\phi)^T\, \widehat{\boldsymbol{f}}(\xi_1,\rho,t+\phi)\, \mathrm{d}t \tag{3.9}$$

and consequently

$$\widehat{\boldsymbol{v}}(\xi_1,\rho,\phi) \;=\; \frac{1}{\omega}\int_0^{2\pi} \frac{\mathrm{e}^{-ta(\boldsymbol{\xi})/\omega}}{\mathrm{e}^{-2\pi\, a(\boldsymbol{\xi})/\omega}-1}\, O(t)^T\, \widehat{\boldsymbol{f}}(\xi_1,\rho,t+\phi)\, \mathrm{d}t. \tag{3.10}$$

Returning to the Cartesian variables $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$ in (3.10), we obtain

$$\begin{aligned}\widehat{\boldsymbol{v}}(\boldsymbol{\xi}) \;&=\; \frac{1}{\omega}\int_0^{2\pi} \Psi(\lambda,\boldsymbol{\xi},t)\; O(t)^T\, \widehat{\boldsymbol{f}}\big(O(t)\boldsymbol{\xi}\big)\, \mathrm{d}t\\ &=\; \frac{1}{\omega}\int_0^{2\pi} \Psi(\lambda,\boldsymbol{\xi},t)\; O(t)^T\, \mathcal{F}[\boldsymbol{f}(O(t)\,\cdot\,)](\boldsymbol{\xi})\, \mathrm{d}t \end{aligned}\tag{3.11}$$

where

$$\Psi(\lambda,\boldsymbol{\xi},t) \;=\; \frac{\mathrm{e}^{-ta(\boldsymbol{\xi})/\omega}}{\mathrm{e}^{-2\pi\, a(\boldsymbol{\xi})/\omega}-1}.$$

In order to complete the proof, we shall need the next lemma.

Lemma 3.2. *If $\lambda\notin\Lambda_\gamma^\omega$ then there exists a positive constant c_4 depending only on γ, ω and on the position of λ in $\mathbb{C}\smallsetminus\Lambda_\gamma^\omega$ such that*

$$\forall\,\boldsymbol{\xi}\in\mathbb{R}^3:\quad \big|\mathrm{e}^{-2\pi\,(\lambda+\mathrm{i}\gamma\xi_1+\nu s^2)/\omega}-1\big| \;\geq\; c_4. \tag{3.12}$$

Proof. The modulus of $\mathrm{e}^{-2\pi\, a(\boldsymbol{\xi})/\omega}-1$ is bounded below by a positive constant depending only on λ, γ and ω (i.e. independent of $\boldsymbol{\xi}$) if the distance $d_\lambda(\boldsymbol{\xi},k)$ of the complex number $-2\pi\, a(\boldsymbol{\xi})/\omega$ from $2\pi\mathrm{i}k$ (for $k\in\mathbb{Z}$) is bounded below by another positive constant, independent of $\boldsymbol{\xi}$ and k. Let $\lambda=\alpha+\mathrm{i}\beta$ and $\rho^2=\xi_2^2+\xi_3^2$ so that

$$\begin{aligned} d_\lambda^2(\boldsymbol{\xi},k) \;&=\; \frac{4\pi^2}{\omega^2}\,\big|\alpha+\mathrm{i}\beta+\mathrm{i}\gamma\xi_1+\nu\xi_1^2+\nu\rho^2+\mathrm{i}k\omega\big|^2\\ &=\; \frac{4\pi^2}{\omega^2}\,\big(\alpha+\nu\xi_1^2+\nu\rho^2\big)^2+\frac{4\pi^2}{\omega^2}\,\big(\beta+\gamma\xi_1+k\omega\big)^2. \end{aligned}$$

Assume that $\min_{\xi\in\mathbb{R}^3} d_\lambda^2(\boldsymbol{\xi},k)=0$ for some fixed $k\in\mathbb{Z}$; the minimum exists because $\lim_{|\xi|\to\infty} d_\lambda^2(\boldsymbol{\xi},k)=\infty$. Then $\xi_1=-(\beta+k\omega)/\gamma$ and

$$\alpha \;=\; -\nu\rho^2-\frac{\nu}{\gamma^2}\,(\beta+k\omega)^2 \;\leq\; -\frac{\nu}{\gamma^2}\,(\beta+k\omega)^2,$$

but this inequality contradicts with $\lambda=\alpha+\mathrm{i}\beta\notin\Lambda_\gamma^\omega$. Hence

$$D_\lambda(k) \;:=\; \min_{\xi\in\mathbb{R}^3}\; d_\lambda(\boldsymbol{\xi},k) \;>\; 0$$

for every $k \in \mathbb{Z}$. Moreover, the sequence $\{D_\lambda(k)\}_{k\in\mathbb{Z}}$ does not converge to zero as $k \to +\infty$ or $k \to -\infty$, because this would imply the existence of $\boldsymbol{\xi}(k) \in \mathbb{R}^3$ such that $d_\lambda\big(\boldsymbol{\xi}(k), k\big) \to 0$ as $k \to +\infty$ or $k \to -\infty$. Consequently $\beta + \gamma\xi_1(k) + k\omega \to 0$, $|\xi_1(k)| \to \infty$ and $\alpha + \nu\xi_1(k)^2 + \nu\rho(k)^2 \to \infty$, which is in contradiction with $d_\lambda\big(\boldsymbol{\xi}(k), k\big) \to 0$. Thus Lemma 3.2 is proved. □

Continuation of the proof of Lemma 3.1. Lemma 3.2 implies that, for fixed $\lambda \in \mathbb{C} \setminus \Lambda^\omega_\gamma$, the modulus of $\Psi(\lambda, \boldsymbol{\xi}, t)$ is bounded uniformly with respect to $t \in (0, 2\pi)$ and $\boldsymbol{\xi} \in \mathbb{R}^3$. Further, if $i \in \{2; 3\}$ and $j \in \{1; 2; 3\}$, then

$$\frac{\partial\Psi}{\partial\xi_i} \;=\; \frac{2\nu\xi_i}{\omega}\,\frac{-t\,\mathrm{e}^{-ta(\boldsymbol{\xi})/\omega}\big(\mathrm{e}^{-2\pi a(\boldsymbol{\xi})/\omega} - 1\big) + 2\pi\,\mathrm{e}^{-(2\pi+t)a(\boldsymbol{\xi})/\omega}}{\big[\mathrm{e}^{-2\pi\, a(\boldsymbol{\xi})/\omega} - 1\big]^2}$$

and $|\xi_i|\,\big|\partial\Psi/\partial\xi_i\big|$ can be estimated as follows:

$$\begin{aligned} |\xi_i|\left|\frac{\partial\Psi}{\partial\xi_i}\right| \;&\le\; \left|\frac{2\nu s^2}{\omega}\,\frac{t\,\mathrm{e}^{-ta(\boldsymbol{\xi})/\omega} + (2\pi - t)\,\mathrm{e}^{-(2\pi+t)a(\boldsymbol{\xi})/\omega}}{\big[\mathrm{e}^{-2\pi\, a(\boldsymbol{\xi})/\omega} - 1\big]^2}\right| \\ &\le\; C(\lambda)\left[s^2\,\frac{t}{\omega}\,\mathrm{e}^{-\nu s^2\, t/\omega} + s^2\,\frac{2\pi + t}{\omega}\,\mathrm{e}^{-\nu s^2\,(2\pi+t)/\omega}\right]. \end{aligned}$$

We observe that the right hand side is less than or equal to a constant independent of $\boldsymbol{\xi}$ and t. We can similarly estimate $|\xi_j|\,\big|\partial\Psi/\partial\xi_1\big|$ and all other terms of the form

$$|\xi_1|^{\kappa_1}\,|\xi_2|^{\kappa_2}\,|\xi_3|^{\kappa_3}\left|\frac{\partial^{\kappa_1+\kappa_2+\kappa_3}\Psi}{\partial\xi_1^{\kappa_1}\,\partial\xi_2^{\kappa_2}\,\partial\xi_3^{\kappa_3}}\right|, \quad \kappa_1,\,\kappa_2,\,\kappa_3 \in \{0, 1\}\,.$$

Applying the inverse Fourier transform to (3.11), we arrive at the formula

$$\boldsymbol{v}(\boldsymbol{x}) \;:=\; \frac{1}{\omega}\int_0^{2\pi} O(t)^T\,\mathcal{F}_{-1}\Big[\Psi(\lambda, \boldsymbol{\xi}, t)\,\mathcal{F}[\boldsymbol{f}(O(t)\,\cdot\,)](\boldsymbol{\xi})\Big](\boldsymbol{x})\;\mathrm{d}t.$$

Using Lizorkin's multiplier theorem (see e.g. [12, p. 375]) and the estimates of Ψ and its derivatives discussed above, we derive the inequality

$$\Big\|\mathcal{F}_{-1}\Big[\Psi(\lambda, \boldsymbol{\xi}, t)\,\mathcal{F}[\boldsymbol{f}(O(t)\,\cdot\,)](\boldsymbol{\xi})\Big]\Big\|_q \;\le\; c_5\,\|\boldsymbol{f}\|_q\,, \quad t \in [0, 2\pi]\,, \tag{3.13}$$

where $c_5 = c_5(\lambda, q)$. Then (3.11) and (3.13) imply that there exists $c_3 > 0$, independent of $\boldsymbol{f}$ and $\boldsymbol{v}$, such that $\boldsymbol{v}$ satisfies the estimate (3.4). □

Lemma 3.3. *Suppose that $\lambda \in \mathbb{C} \setminus \Lambda^\omega_\gamma$. Then $\lambda \in \rho(A^\omega_\gamma)$.*

Proof. The estimates (2.3 and (3.4) in Lemmas 2.2 and 3.1 imply that the range of $A^\omega_\gamma - \lambda I$ is closed in $L^q_\sigma(\mathbb{R}^3)$ and that the operator is injective. By similar arguments the same result does hold for its adjoint, $(A^\omega_\gamma)^* - \overline{\lambda}I$, on the dual space $L^{q'}_\sigma(\mathbb{R}^3)$ of $L^q_\sigma(\mathbb{R}^3)$. Since $(A^\omega_\gamma)^* - \overline{\lambda}I$ is injective, we conclude that $R(A^\omega_\gamma - \lambda I) = L^q_\sigma(\mathbb{R}^3)$. This proves that $\lambda \in \rho(A^\omega_\gamma)$. □

Lemma 3.4. *Let $1 < q < \infty$. Then $\sigma_{\mathrm{p}}(A^\omega_\gamma) = \emptyset$.*

Proof. Let $\lambda = \alpha + \mathrm{i}\beta \in \Lambda^\omega_\gamma$, $\alpha \le 0$, $\beta \in \mathbb{R}$, and let $\boldsymbol{v} \in D(A^\omega_\gamma)$ satisfy the equation $A^\omega_\gamma \boldsymbol{v} - \lambda \boldsymbol{v} = \mathbf{0}$. Applying the Fourier transform we arrive at the identity

$$\omega \partial_\phi \widehat{\boldsymbol{v}} - (\lambda + \mathrm{i}\gamma\xi_1 + \nu|\boldsymbol{\xi}|^2)\,\widehat{\boldsymbol{v}} - \boldsymbol{\omega} \times \widehat{\boldsymbol{v}} \;=\; \mathbf{0}.$$

(We are using the same notation as in the proof of Lemma 3.1.) First we consider the simpler case when $1 < q \le 2$, in which $\widehat{\boldsymbol{v}}$ is a function from $L^{q'}(\mathbb{R}^3)^3$. Denoting $\widehat{\boldsymbol{w}}(\rho,\phi,\xi_1) = O(\phi)^T\,\widehat{\boldsymbol{v}}(\rho,\phi,\xi_1)$, we arrive, by analogy with (3.8), at the equation

$$\omega \partial_\phi \widehat{\boldsymbol{w}} - (\lambda + \mathrm{i}\gamma\xi_1 + \nu s^2)\widehat{\boldsymbol{w}} \;=\; \mathbf{0}, \quad s = |\boldsymbol{\xi}|\,. \tag{3.14}$$

Solving explicitly this ordinary differential equation, we obtain

$$\widehat{\boldsymbol{w}}(\rho,\phi+2\pi,\xi_1) \;=\; \widehat{\boldsymbol{w}}(\rho,\phi,\xi_1)\ \mathrm{e}^{2\pi\,(\lambda+\mathrm{i}\gamma\xi_1+\nu s^2)}.$$

Since $\widehat{\boldsymbol{w}}$ is 2π–periodic in the variable ϕ we get the impossible condition $\alpha + \nu s^2 = 0$ for a.a. $\boldsymbol{\xi} \in \mathbb{R}^3$ unless $\widehat{\boldsymbol{w}} = \mathbf{0}$. Hence $\widehat{\boldsymbol{v}} = \mathbf{0}$ a.e. in $\mathbb{R}^3$ and also $\boldsymbol{v} = \mathbf{0}$ in $L^q_\sigma(\mathbb{R}^3)$.

In the general case $q > 2$ let us again fix $\lambda \in \Lambda^\omega_\gamma$, $\lambda = \alpha + \mathrm{i}\beta + \mathrm{i}k\omega$ where $\alpha \le -\nu\beta^2/\gamma^2$, $k \in \mathbb{Z}$. Consider $\boldsymbol{v} \in D(A^\omega_\gamma)$ and $\widehat{\boldsymbol{w}}(\rho,\phi,\xi_1) = O(\phi)^T\,\widehat{\boldsymbol{v}}(\rho,\phi,\xi_1)$ as above. Since the coefficients of $O(\phi)^T$ are either constant or $\cos\phi = \xi_2/\rho$ or $\sin\phi = \xi_3/\rho$, the function $\boldsymbol{w}$ (the inverse Fourier transform of $\widehat{\boldsymbol{w}}$) is defined by an application of 2D–Riesz transforms to $\boldsymbol{v}$. Hence $\boldsymbol{w} \in L^q(\mathbb{R}^3)^3$.

First we determine the support of tempered distributions $\widehat{\boldsymbol{w}}$ solving (3.14).

Assertion 3.1. *Let $\widehat{\boldsymbol{w}} \in \mathcal{S}(\mathbb{R}^3)^3$ be a solution of (3.14). Then*

$$\operatorname{supp}\widehat{\boldsymbol{w}} \subset D := \{\boldsymbol{\xi} \in \mathbb{R}^3; \alpha + \nu s^2 = 0,\ \beta + \gamma\xi_1 \in \omega\mathbb{Z}\}.$$

Proof of Assertion 3.1. Given $\boldsymbol{\psi} \in C^\infty_0(\mathbb{R}^3 \setminus D)^3$ we solve the equation

$$-\omega\partial_\phi \boldsymbol{\Psi} - a(\boldsymbol{\xi})\boldsymbol{\Psi} \;=\; \boldsymbol{\psi} \tag{3.15}$$

where $a(\boldsymbol{\xi}) = \lambda + \mathrm{i}\gamma\xi_1 + \nu s^2$, $s = |\boldsymbol{\xi}|$. Obviously, (3.15) yields the solution

$$\boldsymbol{\Psi}(\phi) \;=\; \mathrm{e}^{-a(\boldsymbol{\xi})\phi/\omega}\Big(\boldsymbol{\Psi}_0 - \frac{1}{\omega}\int_0^\phi \mathrm{e}^{a(\boldsymbol{\xi})\phi'/\omega}\boldsymbol{\psi}(\phi')\,\mathrm{d}\phi'\Big)$$

when omitting the variables ρ, ξ_1 in $\boldsymbol{\Psi}$ and $\boldsymbol{\psi}$. Since $\boldsymbol{\Psi}$ is 2π-periodic in ϕ, the initial value $\boldsymbol{\Psi}_0$ must satisfy the condition

$$\big(\mathrm{e}^{-2\pi a(\boldsymbol{\xi})/\omega} - 1\big)\boldsymbol{\Psi}_0 \;=\; \frac{1}{\omega}\,\mathrm{e}^{-2\pi a(\boldsymbol{\xi})/\omega}\int_0^{2\pi} \mathrm{e}^{a(\boldsymbol{\xi})\phi'/\omega}\boldsymbol{\psi}(\phi')\,\mathrm{d}\phi'.$$

This equation is uniquely solvable for the unknown $\boldsymbol{\Psi}_0$ if

$$2\pi a(\boldsymbol{\xi})/\omega \notin 2\pi\mathrm{i}\mathbb{Z} \iff \lambda + \mathrm{i}\gamma\xi_1 + \nu s^2 \neq k\mathrm{i}\omega \ \text{ for all } k \in \mathbb{Z} \iff \boldsymbol{\xi} \notin D.$$

For $\boldsymbol{\xi} \notin D$ we get the unique solution of (3.15)

$$\boldsymbol{\Psi}(\phi) \;=\; \frac{1}{\omega}\,\frac{1}{1 - \mathrm{e}^{2\pi a(\boldsymbol{\xi})/\omega}}\ \int_0^{2\pi} \mathrm{e}^{a(\boldsymbol{\xi})\phi'/\omega}\boldsymbol{\psi}(\phi' + \phi)\,\mathrm{d}\phi',$$

cf. (3.10) with ω replaced by $-\omega$. Since $\operatorname{supp}\boldsymbol{\psi} \subset \mathbb{R}^3 \setminus D$, we obtain that $\boldsymbol{\Psi} \in C_0^\infty(\mathbb{R}^3 \setminus D)^3$.

Now we use (3.14) to get that for all $\boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^3 \setminus D)^3$

$$\langle \widehat{\boldsymbol{w}}, \boldsymbol{\psi} \rangle = \langle \widehat{\boldsymbol{w}}, -\omega\partial_\phi \boldsymbol{\Psi} - a(\boldsymbol{\xi})\boldsymbol{\Psi} \rangle = \langle \omega\partial_\phi \widehat{\boldsymbol{w}} - (\lambda + \mathrm{i}\gamma\xi_1 + \nu s^2)\widehat{\boldsymbol{w}} \ , \boldsymbol{\Psi} \rangle = 0 \,.$$

This identity proves that $\operatorname{supp}\widehat{\boldsymbol{w}} \subset D$. $\square$

Continuation of the proof of Lemma 3.4. The set D can be written as the union of finitely many disjoint sets of type

$$D_k := \{\boldsymbol{\xi} \in \mathbb{R}^3;\ \alpha + \nu s^2 = 0,\ \beta + \gamma\xi_1 = \omega k\}, \quad k \in \mathbb{Z}\,;$$

each non–void set D_k defines a circle in $\mathbb{R}^3$ parallel to the $\xi_2\xi_3$–plane with center $(\xi_1^{(k)}, 0, 0)$ where $\xi_1^{(k)} = (\omega k - \beta)/\gamma$. At least the set D_0 is non–void since $\lambda \in \Lambda_\gamma^\omega$. Using a suitable partition of unity with respect to the variable ξ_1 we may write $\widehat{\boldsymbol{w}} \in \mathcal{S}'(\mathbb{R}^3)^3$, a solution of (3.14), as a finite linear combination of tempered distributions $\widehat{\boldsymbol{w}}_k$ with $\operatorname{supp}\widehat{\boldsymbol{w}}_k \subset D_k$. Since $\boldsymbol{w} \in L^q(\mathbb{R}^3)^3$, also $\boldsymbol{w}_k \in L^q(\mathbb{R}^3)^3$ for each k. Moreover, since by elliptic regularity theory, $\boldsymbol{w} \in C^\infty(\mathbb{R}^3)^3$, also $\boldsymbol{w}_k \in C^\infty(\mathbb{R}^3)^3$. We shall further need the assertion:

Assertion 3.2. *Let $h \in \mathcal{S}'(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ satisfy $\operatorname{supp}\widehat{h} \subset \{0\} \times \mathbb{R}^2$. Then h is a polynomial with respect to the variable x_1.*

Proof of Assertion 3.2. Given $h \in \mathcal{S}'(\mathbb{R}^3)$ we find $N \in \mathbb{N}$ and a constant $c > 0$ such that

$$|\langle \widehat{h}, \psi \rangle| \ \le \ c \Big\| (1 + |\,.\,|)^N \sum_{0 \le |\mu| \le N} |\partial^\mu \psi| \Big\|_\infty$$

for all test functions $\psi \in \mathcal{S}(\mathbb{R}^3)$; here $\mu \in \mathbb{N}^3$ denotes multi-indices and $\|\cdot\|_\infty$ the supremum norm for functions on $\mathbb{R}^3$. Let $\eta \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\eta(\xi_1) = 1$ for ξ_1 in a neighborhood of 0, and let $\eta_\epsilon(\xi_1) = \eta(\xi_1/\epsilon)$ for $\epsilon > 0$. Note that $\|\partial_1^m \eta_\epsilon\|_\infty \le c_m \epsilon^{-m}$ for all $m \in \mathbb{N}$. Since $\operatorname{supp}\widehat{h} \subset \{0\} \times \mathbb{R}^2$ we get that $\langle \widehat{h}, \psi \rangle = \langle \widehat{h}, \eta_\epsilon \psi \rangle$ for all $\epsilon > 0$. Hence, with a constant $c > 0$ independent of $\epsilon \in (0, 1)$,

$$\big|\big\langle \xi_1^{N+1}\widehat{h},\, \psi \big\rangle\big| \ = \ \big|\big\langle \widehat{h},\, \xi_1^{N+1}\eta_\epsilon \psi \big\rangle\big| \ \le \ c\,\epsilon \Big\| (1 + |\,.\,|)^N \sum_{0 \le |\mu| \le N} |\partial^\mu \psi| \Big\|_\infty .$$

Consequently, $\xi_1^{N+1}\widehat{h} = 0$. Now we conclude that $\partial_1^{N+1} h = 0$ and that $h \in C^\infty(\mathbb{R}^3)$ is a polynomial with respect to the variable x_1 of order at most N. $\square$

Completion of the proof of Lemma 3.4. To complete the proof of Lemma 3.4, let $\boldsymbol{v} \in D(A_\gamma^\omega)$ satisfy the equation $A_\gamma^\omega \boldsymbol{v} - \lambda \boldsymbol{v} = \boldsymbol{0}$. By the above arguments it suffices to show that each $\boldsymbol{w}_k$ in the partition of $\boldsymbol{w}$ vanishes. Here $\boldsymbol{w}_k \in L^q(\mathbb{R}^3)^3 \cap C^\infty(\mathbb{R}^3)^3$ has the property $\operatorname{supp}\widehat{\boldsymbol{w}}_k \subset D_k$, i.e. $\operatorname{supp}\widehat{\boldsymbol{w}}_k \subset \{\xi_1^{(k)}\} \times \mathbb{R}^2$ with $\xi_1^{(k)} = (\omega k - \beta)/\gamma$. Then the function $x \mapsto \mathrm{e}^{\mathrm{i}x_1\xi_1^{(k)}} \boldsymbol{w}_k(\boldsymbol{x})$, which satisfies the assumptions of Assertion 3.2, is a polynomial with respect to the variable x_1. Since this function is contained in $L^q(\mathbb{R}^3)^3$, it must vanish identically. Now we proved that $\boldsymbol{w} = \boldsymbol{0}$ and also $\boldsymbol{v} = \boldsymbol{0}$. $\square$

Lemma 3.5. *Let $1 < q < \infty$. Then $\sigma_{\mathrm{r}}(A_\gamma^\omega) = \emptyset$.*

Proof. Lemma 3.4 and duality arguments yield the assertion. □

Lemma 3.6. *Let* $1 < q < \infty$. *Then*

$$\sigma(A^\omega_\gamma) = \sigma_{\rm c}(A^\omega_\gamma) = \sigma_{\rm ess}(A^\omega_\gamma) \;=\; \Lambda^\omega_\gamma.$$

Proof. Lemmas 3.3–3.5 imply that $\sigma(A^\omega_\gamma) = \sigma_{\rm c}(A^\omega_\gamma) \subset \Lambda^\omega_\gamma$. Thus, we need to show the opposite inclusion, i.e. $\Lambda^\omega_\gamma \subset \sigma_{\rm c}(A^\omega_\gamma)$.

Suppose at first that $\lambda \in (\Lambda^\omega_\gamma)^\circ$, the interior of Λ^ω_γ. It means that there exist α, $\beta \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $\lambda = \alpha + {\rm i}\beta + {\rm i}k\omega$ and $\alpha < -\nu\beta^2/\gamma^2$. The number α can be written in the form $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = -\nu\beta^2/\gamma^2$ and $\alpha_2 < 0$. Assume that $k \neq 0$. The procedure in the case $k = 0$ would be analogous.

We shall explicitly define a sequence of functions $\{\boldsymbol{v}^n\}$ in $D(A^\omega_\gamma)$ such that $\|\boldsymbol{v}^n\|_q = 1$ and $\|(A^\omega_\gamma - \lambda I)\boldsymbol{v}^n\|_q \to 0$ as $n \to \infty$. It will imply that ${\rm nul}'(A^\omega_\gamma - \lambda I) > 0$.

Let us denote by v^n_1, v^n_r and v^n_φ the cylindrical components of $\boldsymbol{v}^n$. Put

$$\begin{aligned} v^n_1(x_1, r, \varphi) \;&:=\; 0, \\ v^n_r(x_1, r, \varphi) \;&:=\; \kappa_n\, U^n(x_1)\, V^n(r)\, {\rm e}^{{\rm i}k\varphi}, \\ v^n_\varphi(x_1, r, \varphi) \;&:=\; -\frac{1}{{\rm i}k}\, \partial_r\, [r\, v^n_r(x_1, r, \varphi)] \;=\; -\frac{1}{{\rm i}k}\, \kappa_n\, U^n(x_1) \Big[V^n(r) + r\, \frac{{\rm d}V^n(r)}{{\rm d}r} \Big]\, {\rm e}^{{\rm i}k\varphi}. \end{aligned}$$

Here the function U^n has the form

$$U^n(x_1) \;:=\; \eta^n_1(x_1)\, Y(x_1), \quad \text{where} \quad Y(x_1) = {\rm e}^{{\rm i}ax_1},\; a = -\frac{\beta}{\gamma}\,,$$

and where η^n_1 is an infinitely differentiable function on $(-\infty, +\infty)$ such that $0 \leq \eta^n_1 \leq 1$,

$$\eta^n_1(x_1) \;=\; \begin{cases} 0 & \text{for} \;\; x_1 \leq -n - n^2 \;\; \text{and} \;\; n + n^2 \leq x_1, \\ 1 & \text{for} \;\; -n^2 \leq x_1 \leq n^2. \end{cases}$$

The identity $\alpha_1 = -\nu\beta^2/\gamma^2$ guarantees that the characteristic equation $\nu\zeta^2 - \gamma\zeta - (\alpha_1 + {\rm i}\beta) = 0$, corresponding to the equation (3.16) below, has the root $\zeta_1 = {\rm i}a$. Thus, the function Y is a bounded non–trivial solution of the ordinary differential equation

$$\nu\, Y''(x_1) - \gamma\, Y'(x_1) - (\alpha_1 + {\rm i}\beta)\; Y(x_1) \;=\; 0 \tag{3.16}$$

in the interval $(-\infty, +\infty)$. The function V^n is supposed to have the form

$$V^n(r) \;:=\; \eta^n_2(r)\, {\rm e}^{{\rm i}br}, \quad b = \sqrt{-\frac{\alpha_2}{\nu}}\,,$$

where η^n_2 is an infinitely differentiable function on $[0, +\infty)$ such that $0 \leq \eta^n_2 \leq 1$ and

$$\eta^n_2(r) \;=\; \begin{cases} 0 & \text{for} \;\; 0 \leq r \leq n \;\; \text{and} \;\; 3n + n^2 \leq r, \\ 1 & \text{for} \;\; 2n \leq r \leq 2n + n^2. \end{cases}$$

Both the functions η^n_1 and η^n_2 can be chosen so that their derivatives are of the order $1/n$. The definition of V^n guarantees that it satisfies

$$\nu\, \frac{{\rm d}^2}{{\rm d}r^2}\, V^n(r) - \alpha_2\, V^n(r) \;=\; 0 \tag{3.17}$$

for $2n < r < 2n+n^2$. Finally, the constant κ_n is chosen so that $\|\boldsymbol{v}^n\|_q = 1$.

One can easily check that $\boldsymbol{v}^n$ satisfies the condition of incompressibility

$$\nabla \cdot \boldsymbol{v}^n \;\equiv\; \partial_1 v_1^n + \frac{1}{r}\,\partial_r(r v_r^n) + \frac{1}{r}\,\partial_\varphi v_\varphi^n \;=\; 0.$$

The support of $\boldsymbol{v}^n$ is a subset of

$$S^n \;:=\; \big\{\boldsymbol{x} = [x_1, r, \varphi] \in \mathbb{R}^3;\; |x_1| \le n + n^2,\; n \le r \le 3n + n^2,\; 0 \le \varphi < 2\pi\big\}. \tag{3.18}$$

Considering the norm of $\boldsymbol{v}^n$, we observe that for large n the decisive contribution comes from the integral of $|v_\varphi^n|^q$, namely of its part $\big|(-1/\mathrm{i}k)\,\kappa_n\,U^n\,r\,(\mathrm{d}V^n/\mathrm{d}r)\,\mathrm{e}^{\mathrm{i}k\varphi}\big|^q$, on the region

$$S_0^n \;:=\; \big\{\boldsymbol{x} = [x_1, r, \varphi] \in \mathbb{R}^3;\; |x_1| < n^2,\; 2n < r < 2n + n^2,\; 0 < \varphi < 2\pi\big\}.$$

The integrals of all other parts on other regions are of a lower order in n. Calculating the integral of $\big|(-1/\mathrm{i}k)\,\kappa_n\,U^n\,r\,(\mathrm{d}V^n/\mathrm{d}r)\,\mathrm{e}^{\mathrm{i}k\varphi}\big|^q$ on the domain S_0^n, we obtain

$$\begin{aligned}
&\int_{-n^2}^{n^2}\int_{2n}^{2n+n^2}\int_0^{2\pi}\left|\frac{\kappa_n}{\mathrm{i}k}\,U^n(x_1)\,r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\right|^q 2\,r\;\mathrm{d}\varphi\,\mathrm{d}r\,\mathrm{d}x_1\\
&=\; 2\pi\,\frac{\kappa_n^q}{|k|^q}\int_{-n^2}^{n^2}|U^n(x_1)|^q\,\mathrm{d}x_1\int_{2n}^{2n+n^2} r^{q+1}\left|\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\right|^q\mathrm{d}r\\
&=\; 2\pi\,\frac{\kappa_n^q}{|k|^q}\,2n^2\,\frac{b^q}{q+2}\Big((2n+n^2)^{q+2} - (2n)^{q+2}\Big).
\end{aligned}$$

Here we have used the equalities $\eta_1^n(x_1) = \eta_2^n(r) = 1$, hence $|U^n(x_1)| = |V^n(r)| = 1$ for $(x_1, r, \varphi) \in S_0^n$. Thus, there exist $n_0 \in \mathbb{N}$ and positive constants c_6 and c_7 (independent of n) such that

$$\forall\, n \in \mathbb{N},\; n \ge n_0 \;:\quad \frac{c_6}{n^{2+6/q}} \;\le\; \kappa_n \;\le\; \frac{c_7}{n^{2+6/q}}. \tag{3.19}$$

We can naturally use the form (3.1) of the operator A_γ^ω. Moreover, if we identify $\boldsymbol{v}$ with the triplet of its cylindrical coordinates: $\boldsymbol{v} \triangleq [v_1, v_r, v_\varphi]^T$ then we can verify that

$$A_\gamma^\omega \boldsymbol{v} \;=\; \nu\Delta\boldsymbol{v} + \omega\,\partial_\varphi\boldsymbol{v} - \gamma\partial_1\boldsymbol{v},$$

see [11]. Hence

$$(A_\gamma^\omega - \lambda I)\boldsymbol{v}^n \;=\; (\nu\Delta + \omega\,\partial_\varphi - \gamma\partial_1 - \lambda I)\begin{bmatrix} 0 \\ \kappa_n\,U^n(x_1)\,V^n(r)\,\mathrm{e}^{\mathrm{i}k\varphi} \\ -\dfrac{1}{\mathrm{i}k}\,\kappa_n\,U^n(x_1)\left[V^n(r) + r\,\dfrac{\mathrm{d}V^n(r)}{\mathrm{d}r}\right]\mathrm{e}^{\mathrm{i}k\varphi}\end{bmatrix}.$$

Calculating now the norm of $(A_\gamma^\omega - \lambda I)\boldsymbol{v}^n$ in $L_\sigma^q(\mathbb{R}^3)$, we observe that the contributions coming from $\mathbb{R}^3 \smallsetminus S_0^n$ tend to zero as $n \to \infty$ because they represent q–roots of integrals of functions bounded by $C\kappa_n^q r^q$ on $S^n \smallsetminus S_0^n$. Due to (3.19), this contribution is of the order $n^{-1/q}$. Concerning the integral on S_0^n, the decisive part again comes from $(\nu\Delta + \omega\partial_\varphi - \gamma\partial_1 - \lambda I)v_\varphi^n$, namely from $(\nu\Delta + \omega\partial_\varphi - \gamma\partial_1 - \lambda I)$ applied to the term $(-1/\mathrm{i}k)\,\kappa_n\,U^n\,r\,(\mathrm{d}V^n/\mathrm{d}r)\,\mathrm{e}^{\mathrm{i}k\varphi}$

because of the factor r inside this term. Note that $\lambda = \alpha_1 + \alpha_2 + \mathrm{i}\beta + \mathrm{i}k\omega$ and due to (3.16) and (3.17), we have

$$\begin{aligned}
&\big(\nu\Delta + \omega\partial_\varphi - \gamma\partial_1 - \lambda I\big)\left(\frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\, r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\, \mathrm{e}^{\mathrm{i}k\varphi}\right) \\
&= \frac{\kappa_n}{\mathrm{i}k}\left(\nu\partial_1^2 + \nu\partial_r^2 + \frac{\nu}{r}\,\partial_r + \frac{\nu}{r^2}\,\partial_\varphi^2 + \omega\,\partial_\varphi - \gamma\,\partial_1 - \lambda I\right)\left(U^n(x_1)\, r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\, \mathrm{e}^{\mathrm{i}k\varphi}\right) \\
&= \frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\left(\nu\, \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\nu}{r}\, \frac{\mathrm{d}}{\mathrm{d}r} - \alpha_2 I\right)\Big[r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big]\mathrm{e}^{\mathrm{i}k\varphi} \\
&\quad + \frac{\kappa_n}{\mathrm{i}k}\left(\nu Y''(x_1) - \gamma Y'(x_1) - [\alpha_1 + \mathrm{i}\beta]\, Y(x_1)\right)\Big[r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big]\mathrm{e}^{\mathrm{i}k\varphi} \\
&\quad - \frac{\kappa_n}{\mathrm{i}k}\, \frac{\nu k^2}{r^2}\, U^n(x_1)\ r\ \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\, \mathrm{e}^{\mathrm{i}k\varphi} \\
&= \frac{\kappa_n}{\mathrm{i}k}\bigg\{U^n(x_1)\ r\ \frac{\mathrm{d}}{\mathrm{d}r}\Big[\nu\, \frac{\mathrm{d}^2V^n(r)}{\mathrm{d}r^2} - \alpha_2\, V^n(r)\Big] + U^n(x_1)\ 2\nu\ \frac{\mathrm{d}^2V^n(r)}{\mathrm{d}r^2} \\
&\quad + U^n(x_1)\ \frac{\nu}{r}\ \frac{\mathrm{d}}{\mathrm{d}r}\Big[r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big] - \frac{\kappa_n}{\mathrm{i}k}\ U^n(x_1)\ \frac{\nu k^2}{r}\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\bigg\}\, \mathrm{e}^{\mathrm{i}k\varphi} \\
&= \frac{\nu\kappa_n}{\mathrm{i}k}\left(-3b^2 + \frac{\mathrm{i}b}{r} - \frac{k^2\,\mathrm{i}b}{r}\right)\mathrm{e}^{\mathrm{i}(ax_1+br)}\, \mathrm{e}^{\mathrm{i}k\varphi}
\end{aligned}$$

where in the last step we used the simple forms of the functions U^n and V^n on S_0^n, i.e. $U^n(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$ and $V^n(r) = \mathrm{e}^{\mathrm{i}br}$. Hence

$$\begin{aligned}
&\left[\int_{-n^2}^{n^2}\int_{2n}^{2n+n^2}\int_0^{2\pi}\left|\Big(\nu\Delta + \omega\partial_\varphi - \gamma\partial_1 - \lambda I\Big)\left(\frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\, r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\, \mathrm{e}^{\mathrm{i}k\varphi}\right)\right|^q r\; \mathrm{d}\varphi\, \mathrm{d}r\, \mathrm{d}x_1\right]^{1/q} \\
&\leq\ C(\nu,k,b)\, \kappa_n\left[\int_{-n^2}^{n^2}\int_{2n}^{2n+n^2} r\; \mathrm{d}r\, \mathrm{d}x_1\right]^{1/q} \\
&=\ C(\nu,k,b)\, \kappa_n\left\{2n^2\left[(2n+n^2)^2 - (2n)^2\right]\right\}^{1/q}.
\end{aligned}$$

The last term tends to zero as $n \to \infty$ due to (3.19). In this way, we are led to the convergence $\|(A_\gamma^\omega - \lambda I)\boldsymbol{v}^n\|_q \to 0$ as $n \to \infty$.

We have proved that $\mathrm{nul}'(A_\gamma^\omega - \lambda I) > 0$. Hence $\mathrm{nul}\,(A_\gamma^\omega - \lambda I) \neq \mathrm{nul}'(A_\gamma^\omega - \lambda I)$. It means that the range $R(A_\gamma^\omega - \lambda I)$ is not closed in $L^q_\sigma(\mathbb{R}^3)$. Consequently, $\mathrm{nul}'(A_\gamma^\omega - \lambda I) = \mathrm{def}'(A_\gamma^\omega - \lambda I) = +\infty$ and $\lambda \in \sigma_{\mathrm{ess}}(A_\gamma^\omega)$. Since $\sigma_{\mathrm{ess}}(A_\gamma^\omega)$ is closed, Λ_γ^ω is a subset of $\sigma_{\mathrm{ess}}(A_\gamma^\omega)$. Due to Lemmas 3.4 and 3.5, we have the inclusion $\Lambda_\gamma^\omega \subset \sigma_{\mathrm{c}}(A_\gamma^\omega)$. □

If $q = 2$ then $L^q_\sigma(\mathbb{R}^3) \equiv L^2_\sigma(\mathbb{R}^3)$ is a Hilbert space and it is natural to ask whether the operator A_γ^ω is normal. The answer is given by the next lemma.

Lemma 3.7. *Let $q = 2$. Then A_γ^ω is a normal operator in $L^2_\sigma(\mathbb{R}^3)$.*

Proof. Using the cylindrical coordinates (x_1, r, φ) as in the proof of Lemma 3.1, we can express the operators A_γ^ω and $(A_\gamma^\omega)^*$ in accordance with (3.1), (3.2) and (3.6) as

$$A_\gamma^\omega \boldsymbol{v} \;=\; \nu\Delta\boldsymbol{v} + \omega\,\partial_\varphi\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v} - \gamma\partial_1\boldsymbol{v},$$

$$(A^\omega_\gamma)^*\boldsymbol{v} \;=\; \nu\Delta\boldsymbol{v} - \omega\partial_\varphi\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} + \gamma\partial_1\boldsymbol{v},$$

where $D(A^\omega_\gamma) = D((A^\omega_\gamma)^*)$. We need to show that $A^\omega_\gamma(A^\omega_\gamma)^* = (A^\omega_\gamma)^*A^\omega_\gamma$, i.e.,

a) $D\big(A^\omega_\gamma(A^\omega_\gamma)^*\big) = D\big((A^\omega_\gamma)^*A^\omega_\gamma\big)$,

b) $A^\omega_\gamma(A^\omega_\gamma)^*\boldsymbol{u} = (A^\omega_\gamma)^*A^\omega_\gamma\boldsymbol{u}$ for all $\boldsymbol{u}\in D\big((A^\omega_\gamma)^*A^\omega_\gamma\big)$.

Let us begin with part a). Suppose that $\boldsymbol{u}\in D(A^\omega_\gamma(A^\omega_\gamma)^*)$, i.e. $\boldsymbol{u}\in D((A^\omega_\gamma)^*)$ and $(A^\omega_\gamma)^*\boldsymbol{u}\in D(A^\omega_\gamma)$. In order to show that $\boldsymbol{u}\in D((A^\omega_\gamma)^*A^\omega_\gamma)$, we treat the scalar product $(A^\omega_\gamma\boldsymbol{u}, A^\omega_\gamma\boldsymbol{v})_2$ of $A^\omega_\gamma\boldsymbol{u}$ and $A^\omega_\gamma\boldsymbol{v}$ in $L^2_\sigma(\mathbb{R}^3)$ for $\boldsymbol{v}\in D(A^\omega_\gamma)$ as follows:

$$\begin{aligned}
(A^\omega_\gamma\boldsymbol{u}, A^\omega_\gamma\boldsymbol{v})_2 \;&=\; \big((A^\omega_\gamma)^*\boldsymbol{u}, A^\omega_\gamma\boldsymbol{v}\big)_2 + 2\left(\omega\partial_\varphi\boldsymbol{u} - \boldsymbol{\omega}\times\boldsymbol{u} - \gamma\partial_1\boldsymbol{u},\, A^\omega_\gamma\boldsymbol{v}\right)_2\\
&=\; \big((A^\omega_\gamma)^*\boldsymbol{u}, A^\omega_\gamma\boldsymbol{v}\big)_2 + 2\left(\omega\partial_\varphi\boldsymbol{u} - \boldsymbol{\omega}\times\boldsymbol{u} - \gamma\partial_1\boldsymbol{u},\, \nu\Delta\boldsymbol{v}\right)_2\\
&\quad + 2\left(\omega\partial_\varphi\boldsymbol{u} - \boldsymbol{\omega}\times\boldsymbol{u} - \gamma\partial_1\boldsymbol{u},\, \omega\partial_\varphi\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v} - \gamma\partial_1\boldsymbol{v}\right)_2.
\end{aligned}\tag{3.20}$$

Let us first assume that $\boldsymbol{v}$ has a compact support. Then $(\partial_\varphi\boldsymbol{u}, \Delta\boldsymbol{v})_2 = -(\Delta\boldsymbol{u}, \partial_\varphi\boldsymbol{v})_2$ and

$$\big(-\boldsymbol{\omega}\times\boldsymbol{u} - \gamma\partial_1\boldsymbol{u},\, \nu\Delta\boldsymbol{v}\big)_2 \;=\; \big(\nu\Delta\boldsymbol{u},\, \boldsymbol{\omega}\times\boldsymbol{v} + \gamma\partial_1\boldsymbol{v}\big)_2.$$

Substituting these identities into (3.20), we obtain

$$\begin{aligned}
(A^\omega_\gamma\boldsymbol{u}, A^\omega_\gamma\boldsymbol{v})_2 \;&=\; \big((A^\omega_\gamma)^*\boldsymbol{u}, A^\omega_\gamma\boldsymbol{v}\big)_2 - 2\left(\nu\Delta\boldsymbol{u},\, \omega\partial_\varphi\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v} - \gamma\partial_1\boldsymbol{v}\right)_2\\
&\quad + 2\left(\omega\partial_\varphi\boldsymbol{u} - \boldsymbol{\omega}\times\boldsymbol{u} - \gamma\partial_1\boldsymbol{u},\, \omega\partial_\varphi\boldsymbol{v} - \boldsymbol{\omega}\times\boldsymbol{v} - \gamma\partial_1\boldsymbol{v}\right)_2\\
&=\; \big((A^\omega_\gamma)^*\boldsymbol{u}, A^\omega_\gamma\boldsymbol{v}\big)_2 + 2\left((A^\omega_\gamma)^*\boldsymbol{u},\, -\omega\partial_\varphi\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} + \gamma\partial_1\boldsymbol{v}\right)_2\\
&=\; \big((A^\omega_\gamma)^*\boldsymbol{u},\, (A^\omega_\gamma)^*\boldsymbol{v}\big)_2 \;=\; \big(A^\omega_\gamma(A^\omega_\gamma)^*\boldsymbol{u},\, \boldsymbol{v}\big)_2.
\end{aligned}\tag{3.21}$$

In fact, (3.21) holds for all $\boldsymbol{v}\in D(A^\omega_\gamma)$ because by Lemma 2.3 the set $D_0(A^\omega_\gamma) = \{\boldsymbol{v}\in D(A^\omega_\gamma);\ \boldsymbol{v}$ has a compact support in $\mathbb{R}^3\}$ is a core of A^ω_γ. Now, (3.21) shows that for fixed $\boldsymbol{u}$, $(A^\omega_\gamma\boldsymbol{u}, A^\omega_\gamma\boldsymbol{v})_2$ can be extended to a continuous linear functional of $\boldsymbol{v}\in L^2_\sigma(\mathbb{R}^3)$. Thus, $\boldsymbol{u}\in D((A^\omega_\gamma)^*A^\omega_\gamma)$.

We have proved the inclusion $D(A^\omega_\gamma(A^\omega_\gamma)^*) \subset D((A^\omega_\gamma)^*A^\omega_\gamma)$. The opposite inclusion can be proved in the same way.

Concerning part b), (3.21) implies that

$$\big((A^\omega_\gamma)^*A^\omega_\gamma\boldsymbol{u},\, \boldsymbol{v}\big)_2 \;=\; \big(A^\omega_\gamma(A^\omega_\gamma)^*\boldsymbol{u},\, \boldsymbol{v}\big)_2$$

for all $\boldsymbol{v}\in D(A^\omega_\gamma)$ and even for all $\boldsymbol{v}\in L^2_\sigma(\mathbb{R}^3)$ due to the density of $D(A^\omega_\gamma)$ in $L^2_\sigma(\mathbb{R}^3)$. Hence the operators A^ω_γ and $(A^\omega_\gamma)^*$ commute. □

Theorem 1.2 resumes the results of Lemmas 3.3–3.7.

4 The case of an exterior domain Ω

In this section, we assume that $\Omega\subset\mathbb{R}^3$ is an exterior domain, different from $\mathbb{R}^3$, with boundary of class $C^{1,1}$.

Lemma 4.1. $\Lambda^\omega_\gamma \subset \sigma_{\mathrm{ess}}(A^\omega_\gamma)$.

Proof. We prove in almost the same way as in the proof of Lemma 3.6 that if $\lambda \in (\Lambda_\gamma^\omega)^\circ$ then $\mathrm{nul}'(A_\gamma^\omega - \lambda I) > 0$. Unfortunately, since an analogue to Lemma 3.4 is not available, we cannot deduce directly from this inequality that $\lambda \in \sigma_{\mathrm{ess}}(A_\gamma^\omega)$ as in the proof of Lemma 3.6. However, we will use that $\mathrm{supp}\, \boldsymbol{v}^n \subset S^n$, see (3.18). Thus, there exists $n_0 \in \mathbb{N}$ so large that $\boldsymbol{v}^n$ belongs to the domain of A_γ^ω (as an operator in $L^q_\sigma(\Omega)$) for $n \geq n_0$. We observe that any subsequence $\{S^{k_n}\}$ of $\{S^n\}$ has the intersection property $\cap_{n=1}^\infty S^{k_n} = \emptyset$. It implies that the sequence $\{\boldsymbol{v}^n\}_{n\geq n_0}$ is not compact in $L^q_\sigma(\Omega)$. Consequently, $\mathrm{nul}'(A_\gamma^\omega - \lambda I) = \infty$. We can prove in the same way that $\mathrm{nul}'\big((A_\gamma^\omega)^* - \overline{\lambda} I\big) = \infty$. Hence $\mathrm{def}'(A_\gamma^\omega - \lambda I) = \infty$, and the operator $A_\gamma^\omega - \lambda I$ is not semi–Fredholm. Thus, $\lambda \in \sigma_{\mathrm{ess}}(A_\gamma^\omega)$. The inclusion $\Lambda_\gamma^\omega \subset \sigma_{\mathrm{ess}}(A_\gamma^\omega)$ now follows from the closedness of $\sigma_{\mathrm{ess}}(A_\gamma^\omega)$. □

Lemma 4.2. $\sigma_{\mathrm{ess}}(A_\gamma^\omega) \subset \Lambda_\gamma^\omega$.

Proof. Let $\lambda \in \sigma_{\mathrm{ess}}(A_\gamma^\omega)$. Then $\mathrm{nul}'(A_\gamma^\omega - \lambda I) = \infty$. This information enables us to construct, by mathematical induction, a sequence $\{\boldsymbol{u}^n\}$ in $D(A_\gamma^\omega)$ satisfying $\|\boldsymbol{u}^n\|_q = 1$, $\|(A_\gamma^\omega - \lambda I)\boldsymbol{u}^n\|_q \to 0$ as $n \to \infty$ and

$$\mathrm{dist}\big(\boldsymbol{u}^n;\, \mathcal{L}_{n-1}\big) \;=\; 1 \tag{4.1}$$

for all $n \in \mathbb{N}$, where $\mathcal{L}_{n-1}$ denotes the linear hull of the functions $\boldsymbol{u}^1, \dots, \boldsymbol{u}^{n-1}$: Suppose that we have already constructed $\boldsymbol{u}^1, \dots, \boldsymbol{u}^k$ satisfying $\|(A_\gamma^\omega - \lambda I)\boldsymbol{u}^j\|_q \leq 1/j$ for $j = 1, \dots, k$ and (4.1) for all $n = 1, \dots, k$. To $\epsilon_{k+1} = 1/(k+1)$ there exists an infinite dimensional linear manifold M_{k+1} in $D(A_\gamma^\omega)$ such that $\|(A_\gamma^\omega - \lambda I)\boldsymbol{u}\|_q \leq \epsilon_{k+1}$ for all $\boldsymbol{u} \in M_{k+1}$. Then due to Lemma IV.2.3 in [20], we find $\boldsymbol{u}^{k+1} \in M_{k+1}$ such that $\|\boldsymbol{u}^{k+1}\|_q = 1$ and $\mathrm{dist}\big(\boldsymbol{u}^{k+1};\, \mathcal{L}_k\big) = 1$. The sequence $\{\boldsymbol{u}^n\}$ satisfies

$$\|(A_\gamma^\omega - \lambda I)\boldsymbol{u}^n\|_q \;\leq\; \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

Denote $\boldsymbol{f}^n := (A_\gamma^\omega - \lambda I)\boldsymbol{u}^n$. Lemma 2.2 yields the estimates

$$\begin{aligned} \|\boldsymbol{u}^n\|_{2,q} + \|(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}^n\|_q \;&\leq\; c_1\, \|\boldsymbol{f}^n\|_q + \big(c_2 + c_1\,|\lambda|\big)\, \|\boldsymbol{u}^n\|_q \\ &\leq\; c_1 + \big(c_2 + c_1\,|\lambda|\big) \;:=\; c_8 \end{aligned} \tag{4.3}$$

with a constant $c_8 > 0$ independent of $n \in \mathbb{N}$. Furthermore, there exists $\nabla p^n \in L^q(\Omega)^3$ such that

$$\nu\Delta\boldsymbol{u}^n + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}^n - \boldsymbol{\omega}\times\boldsymbol{u}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{u}^n - \nabla p^n = \boldsymbol{f}^n \tag{4.4}$$

in Ω and that by (4.3)

$$\|\nabla p^n\|_q \;\leq\; c_8. \tag{4.5}$$

The sequence $\{\boldsymbol{u}^n\}$ is bounded in the space $D(A_\gamma^\omega)$. Hence there exists a subsequence again denoted by $\{\boldsymbol{u}^n\}$ which is weakly convergent in $D(A_\gamma^\omega)$. This subsequence naturally preserves the property (4.2).

Put $\boldsymbol{v}^n := (\boldsymbol{u}^{n+1} - \boldsymbol{u}^n)/\delta_n$ where $\delta_n = \|\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\|_q$. Then $\{\boldsymbol{v}^n\}$ is a sequence in the unit sphere in $L^q_\sigma(\Omega)$. It converges weakly to zero in $D(A_\gamma^\omega)$ because $(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) \rightharpoonup \boldsymbol{0}$ in $L^q_\sigma(\Omega)$

as $n\to\infty$ and by (4.1) $\delta_n\geq 1$. Hence $\{\boldsymbol{v}^n\}$ converges strongly to $\mathbf{0}$ in $W^{1,q}(\Omega\cap B_R(\mathbf{0}))^3$ for each $R>0$. Note that the function $\boldsymbol{v}^n$ satisfies the equation

$$\begin{aligned}
&\nu\Delta\boldsymbol{v}^n+(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}^n-\boldsymbol{\omega}\times\boldsymbol{v}^n-\gamma\partial_1\boldsymbol{v}^n-\lambda\boldsymbol{v}^n-\frac{1}{\delta_n}\nabla(p^{n+1}-p^n)\\
&\quad=\ \frac{1}{\delta_n}\big(\boldsymbol{f}^{n+1}-\boldsymbol{f}^n\big) \qquad (4.6)
\end{aligned}$$

in Ω. This equation, together with the information on the weak convergence of $\{\boldsymbol{v}^n\}$ to zero in $D(A^\omega_\gamma)$, implies that the sequence $\{\nabla(p^{n+1}-p^n)\}$ weakly converges to zero in $L^q(\Omega)^3$. Thus, the functions p^n, which are given uniquely up to an additive constant, can be chosen so that $p^{n+1}-p^n\to 0$ strongly in $L^q(\Omega\cap B_R(\mathbf{0}))$ for each $R>0$.

The sequence $\{\boldsymbol{v}^n\}$ does not contain any subsequence, convergent in $L^q_\sigma(\Omega)$. Indeed, assume that $\{\boldsymbol{v}^{k_n}\}$ is a convergent subsequence of $\{\boldsymbol{v}^n\}$ in $L^q_\sigma(\Omega)$. This subsequence has the same weak limit as $\{\boldsymbol{v}^n\}$, hence $\boldsymbol{v}^{k_n}\rightharpoonup\mathbf{0}$ in $L^q_\sigma(\Omega)$ as $n\to\infty$. Then even $\boldsymbol{v}^{k_n}\to\mathbf{0}$ in $L^q_\sigma(\Omega)$ as $n\to\infty$. However, this is impossible because $\|\boldsymbol{v}^{k_n}\|_q=1$.

Suppose that $R>0$ is so large that the domain $\{\boldsymbol{x}\in\mathbb{R}^3;\ |\boldsymbol{x}|>R\}$ is a subset of Ω. Let η be an infinitely differentiable cut–off function in Ω, with values in $[0,1]$, such that

$$\eta(\boldsymbol{x})\ =\ \begin{cases}0 & \text{for}\ \ |\boldsymbol{x}|\leq R,\\ 1 & \text{for}\ \ R+1\leq|\boldsymbol{x}|.\end{cases}$$

Put $K_R:=\{\boldsymbol{x}\in\mathbb{R}^3;\ R<|\boldsymbol{x}|<R+1\}$ and let $\mathfrak{B}:W^{1,q}_0(K_R)\mapsto W^{2,q}_0(K_R)^3$ be the Bogovskij operator, see the proof of Lemma 2.3. Then $\boldsymbol{V}^n:=\mathfrak{B}\big(\nabla\eta\cdot\boldsymbol{v}^n\big)$ belongs to $W^{2,q}_0(K_R)^3$. If we extend it by zero to $\Omega\smallsetminus K_R$, it can be considered as an element of $W^{2,q}_0(\Omega)^3$. Due to the continuity of the operator $\mathfrak{B}$ and the strong convergence of $\{\boldsymbol{v}^n\}$ to $\mathbf{0}$ in $W^{1,q}\big(\Omega\cap B_{R+1}(\mathbf{0})\big)^3$, we get that $\boldsymbol{V}^n\to\mathbf{0}$ in $W^{2,q}_0(\Omega)^3$.

Now we define $\boldsymbol{w}^n(\boldsymbol{x}):=\eta(\boldsymbol{x})\,\boldsymbol{v}^n(\boldsymbol{x})-\boldsymbol{V}^n(\boldsymbol{x})$. The function $\boldsymbol{w}^n$ belongs to $D(A^\omega_\gamma)$ and, due to (4.6), satisfies the equation

$$\begin{aligned}
&\nu\Delta\boldsymbol{w}^n+(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{w}^n-\boldsymbol{\omega}\times\boldsymbol{w}^n-\gamma\partial_1\boldsymbol{w}^n-\lambda\boldsymbol{w}^n-\frac{1}{\delta_n}\nabla\big[\eta(p^{n+1}-p^n)\big]\\
&\quad=\ \frac{\eta}{\delta_n}\,(\boldsymbol{f}^{n+1}-\boldsymbol{f}^n)-\frac{1}{\delta_n}\,\nabla\eta\,(p^{n+1}-p^n)+\nu\,(\Delta\eta)\,\boldsymbol{v}^n+2\nu\,\nabla\eta\cdot\nabla\boldsymbol{v}^n-\nu\Delta\boldsymbol{V}^n+\lambda\boldsymbol{V}^n\\
&\qquad+(\boldsymbol{\omega}\times\boldsymbol{x})\cdot(\nabla\eta\otimes\boldsymbol{v}^n)-(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{V}^n+\boldsymbol{\omega}\times\boldsymbol{V}^n-\gamma(\partial_1\eta)\boldsymbol{v}^n+\gamma\partial_1\boldsymbol{V}^n. \qquad (4.7)
\end{aligned}$$

The right hand side converges strongly to zero in $L^q(\Omega)^3$ as $n\to\infty$; this follows from the strong convergence of $\{\boldsymbol{v}^n\}$ to zero in $W^{1,q}(\Omega\cap B_{R+1}(\mathbf{0}))^3$, the strong convergence of $\{p^{n+1}-p^n\}$ to zero in $L^q(\Omega\cap B_{R+1}(\mathbf{0}))$, from the information on the support of $\nabla\eta$ and $\Delta\eta$ and from the strong convergence of $\{\boldsymbol{V}^n\}$ to zero in $W^{2,q}(\Omega)^3$. Hence

$$\|(A^\omega_\gamma-\lambda I)\boldsymbol{w}^n\|_q\ \longrightarrow\ 0\qquad\text{as}\ \ n\to\infty. \qquad (4.8)$$

Moreover, given $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that if $n\in\mathbb{N}$, $n\geq n_0$, then

$$\|\boldsymbol{w}^n\|_q\ \ \leq\ \ \bigg(\int_{|x|<R+1}|\eta\boldsymbol{v}^n-\boldsymbol{V}^n|^q\,\mathrm{d}\boldsymbol{x}\bigg)^{1/q}+\bigg(\int_{R+1<|x|}|\boldsymbol{v}^n|^q\,\mathrm{d}\boldsymbol{x}\bigg)^{1/q}\ \leq\ \epsilon+1,$$

$$\begin{aligned}\|\boldsymbol{w}^n\|_q &\geq \left(\int_{R+1<|x|} |\boldsymbol{v}^n|^q \,\mathrm{d}\boldsymbol{x}\right)^{1/q} \geq \left(\int_{\Omega} |\boldsymbol{v}^n|^q \,\mathrm{d}\boldsymbol{x}\right)^{1/q} - \left(\int_{|x|<R+1} |\boldsymbol{v}^n|^q \,\mathrm{d}\boldsymbol{x}\right)^{1/q} \\ &\geq 1-\epsilon\,.\end{aligned}$$

Let us now normalize the sequence $\{\boldsymbol{w}^n\}$ by dividing each of the functions $\boldsymbol{w}^n$ by its norm in $L^q_\sigma(\Omega)$. In order to preserve a simple notation, we denote the normalized functions again by $\boldsymbol{w}^n$. If we finally put $\boldsymbol{w}^n(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \mathbb{R}^3 \setminus \Omega$, we obtain a non–compact sequence in the unit sphere in $L^q_\sigma(\mathbb{R}^3)$, satisfying (4.8) with $\|\,.\,\|_q$ being the norm in $L^q_\sigma(\mathbb{R}^3)$. Let us denote, for a while, by $(A^\omega_\gamma)_{\mathbb{R}^3}$ the operator A^ω_γ, considered in $L^q_\sigma(\mathbb{R}^3)$. The existence of the sequence $\{\boldsymbol{w}^n\}$ with the above properties implies that $\mathrm{nul}'\big((A^\omega_\gamma)_{\mathbb{R}^3} - \lambda I\big) = \infty$. Hence $\lambda \notin \rho\big((A^\omega_\gamma)_{\mathbb{R}^3}\big)$, which, due to Theorem 1.1, yields $\lambda \in \Lambda^\omega_\gamma$. □

Lemma 4.3. *Let $\lambda \in \mathbb{C} \setminus \Lambda^\omega_\gamma$. Then either $\lambda \in \sigma_{\mathrm{p}}(A^\omega_\gamma)$ for all $1<q<\infty$ or $\lambda \in \rho(A^\omega_\gamma)$ for all $1<q<\infty$. Moreover, $\lambda \in \rho(A^\omega_\gamma)$ when* $\mathrm{Re}\,\lambda \geq 0$.

Proof. Assume that $1<q<\infty$ and that $\lambda \in \mathbb{C} \setminus \Lambda^\omega_\gamma$ is an eigenvalue of $(A^\omega_\gamma)_{\Omega,q}$ with nonzero eigenfunction $\boldsymbol{v} \in D\big((A^\omega_\gamma)_{\Omega,q}\big)$, where $(A^\omega_\gamma)_{\Omega,q}$ denotes the operator A^ω_γ on $L^q_\sigma(\Omega)$. Let p with $\nabla p \in L^q(\Omega)^3$ be a corresponding pressure function. Using the cut–off function η, the set K_R and Bogovskij's operator $\mathfrak{B} : W^{1,q}_0(K_R) \mapsto W^{2,q}_0(K_R)^3$ from the proof of Lemma 4.2, we get that $\boldsymbol{w}_1 := \eta\boldsymbol{v} - \boldsymbol{V}$ (where $\boldsymbol{V} = \mathfrak{B}(\nabla\eta\cdot\boldsymbol{v})$ in K_R, $\boldsymbol{V} = \boldsymbol{0}$ in $\mathbb{R}^3 \setminus K_R$) solves the equation, cf. (4.7),

$$\begin{aligned}&\nu\Delta\boldsymbol{w}_1 + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{w}_1 - \boldsymbol{\omega}\times\boldsymbol{w}_1 - \gamma\partial_1\boldsymbol{w}_1 - \lambda\boldsymbol{w}_1 - \nabla(\eta p) \\ &\quad = \boldsymbol{f} \quad := -\nabla\eta\, p + \nu\,(\Delta\eta)\,\boldsymbol{v} + 2\nu\,\nabla\eta\cdot\nabla\boldsymbol{v} - \nu\Delta\boldsymbol{V} + \lambda\boldsymbol{V} \\ &\qquad\qquad + (\boldsymbol{\omega}\times\boldsymbol{x})\cdot(\nabla\eta\otimes\boldsymbol{v}) - (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{V} + \boldsymbol{\omega}\times\boldsymbol{V} - \gamma(\partial_1\eta)\boldsymbol{v} + \gamma\partial_1\boldsymbol{V} \qquad (4.9)\end{aligned}$$

in $\mathbb{R}^3$.

First assume that $q \geq 3$. Due to the boundedness of the operator $\mathfrak{B}$ from $W^{2,q}_0(K_R)$ to $W^{3,q}_0(K_R)^3$ (see e.g. [12, p. 130]), the restriction of the vector field $\boldsymbol{V}$ to K_R belongs to $W^{3,q}_0(K_R)^3$. From this, we deduce that $\boldsymbol{f} \in W^{1,q}(K_R)$. Hence, by Sobolev's embedding theorem, $\boldsymbol{f} \in L^s(K_R)^3$ for $1<s<\infty$. Since $\boldsymbol{f}$ is supported in $\overline{K}_R$, we have that $\boldsymbol{f} \in L^s(\mathbb{R}^3)^3$ for $1<s<\infty$. Now we may apply Theorem 1.1 (with s instead of q) to the whole space problem (4.9) and conclude that $\boldsymbol{w}_1 \in D\big((A^\omega_\gamma)_{\mathbb{R}^3,s}\big) \subset W^{2,s}(\mathbb{R}^3)$ and $\nabla(\eta p) \in L^s(\mathbb{R}^3)^3$ because $\lambda \in \mathbb{C} \setminus \Lambda^\omega_\gamma$ belongs to the resolvent set of $(A^\omega_\gamma)_{\mathbb{R}^3,s}$. Similarly, we derive that $\boldsymbol{w}_2 := (1-\eta)\boldsymbol{v} + \boldsymbol{V}$, the solution of a problem analogous to (4.9) in the bounded domain $\Omega_{R+1} := \Omega \cap B_{R+1}(\boldsymbol{0})$, satisfies $\boldsymbol{w}_2 \in W^{2,s}(\Omega_{R+1})$. Consequently, $\boldsymbol{v} \equiv \boldsymbol{w}_1 + \boldsymbol{w}_2 \in D\big((A^\omega_\gamma)_{\Omega,s}\big)$ and λ is an eigenvalue of $(A^\omega_\gamma)_{\Omega,s}$.

If $1<q<3$ then we obtain the same result for $1<s<3q/(3-q)$. However, repeating finitely many times the same argument, we can extend the result to all $1<s<\infty$.

Finally, when $s=2$, a variational argument implies that $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma(A^\omega_\gamma)$, cf. [11, Theorem 1.1]. □

Now Theorem 1.2 is completely proved.

Remark 4.1. If $q=2$ then the interesting question occurs whether A^ω_γ is a normal operator in $L^2_\sigma(\Omega)$. We have proved in our previous papers [10] and [11] that

a) if $\gamma = 0$ and the domain Ω is axially symmetric with respect to the x_1–axis then A^ω_γ s normal,

b) if $\gamma \neq 0$ or the domain Ω is not axially symmetric then A^ω_γ is not normal.

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