Spectral Analysis of a Stokes–Type Operator Arising from Flow around a Rotating Body

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Abstract

We consider the spectrum of the Stokes operator with and without rotation effect for the whole space and exterior domains in L^q–spaces. Based on similar results for the Dirichlet–Laplacian on $\mathbb{R}^n$, $n \geq 2$, we prove in the whole space case that the spectrum as a set in $\mathbb{C}$ does not change with $q \in (1, \infty)$, but it changes its type from the residual to the continuous or to the point spectrum with q. The results for exterior domains are less complete, but the spectrum of the Stokes operator with rotation mainly is an essential one, consisting of infinitely many equidistant parallel half–lines in the left complex half–plane. The tools are strongly based on Fourier analysis in the whole space case and on stability properties of the essential spectrum for exterior domains.

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1 Introduction

We will study spectral properties of a Stokes type operator which arises from the problem of viscous fluid flow around a rotating body. To be more precise, the starting point is the non–stationary Navier–Stokes system modelling viscous incompressible fluid flow around a rotating obstacle in $\mathbb{R}^3$ with angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$; this Navier–Stokes system is formulated in a time–dependent exterior domain $\Omega(t)$, $t \geq 0$. Then, introducing a new coordinate system attached to the rotating body, see e.g. [7], [10], [23], and assuming that the velocity satisfies the no–slip boundary condition on the surface of the body and tends

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to zero at infinity, we get for the modified velocity $\mathbf{u}$ and pressure p the non–stationary Navier–Stokes–type problem

$$\begin{aligned} \partial_t\mathbf{u} - \nu\Delta\mathbf{u} - (\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega\times(0,\infty), \\ \operatorname{div}\mathbf{u} &= 0 && \text{in} \quad \Omega\times(0,\infty), \\ \mathbf{u}(\mathbf{x},\,.\,) &\to \mathbf{0} && \text{as} \quad |\mathbf{x}|\to\infty, \\ \mathbf{u}(\mathbf{x},\,.\,) &= \boldsymbol{\omega}\wedge\mathbf{x} && \text{for} \quad \mathbf{x}\in\partial\Omega, \\ \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) && \text{for} \quad \mathbf{x}\in\Omega. \end{aligned} \tag{1.1}$$

Here $\mathbf{f}$ denotes the modified external force density, and $\Omega\subset\mathbb{R}^3$ is the time–independent unbounded domain exterior to the obstacle.

In the linearized stationary case, i.e., in the linearized and time–periodic case of the original system, we are led to the Stokes–type problem

$$\begin{aligned} -\nu\Delta\mathbf{u} - (\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in} \quad \Omega, \\ \mathbf{u}(\mathbf{x}) &\to \mathbf{0} && \text{as} \quad |\mathbf{x}|\to\infty, \\ \mathbf{u}(\mathbf{x}) &= \boldsymbol{\omega}\wedge\mathbf{x} && \text{for} \quad \mathbf{x}\in\partial\Omega. \end{aligned} \tag{1.2}$$

The linear problem (1.2) has been analyzed in L^q–spaces, $1<q<\infty$, in [10], proving the existence of a strong solution $(\mathbf{u},p)$ satisfying the estimate

$$\|\nu\nabla^2\mathbf{u}\|_q + \|(\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla\mathbf{u} - \boldsymbol{\omega}\wedge\mathbf{u}\|_q + \|\nabla p\|_q \;\le\; C\,\|\mathbf{f}\|_q \tag{1.3}$$

with a constant $C=C(q)>0$ independent of $\mathbf{f}$, $\boldsymbol{\omega}$ and of the coefficient of viscosity ν. Similar results were obtained in the case of a rotating body with constant translational velocity $\mathbf{u}_\infty$ parallel to $\boldsymbol{\omega}$, leading to an Oseen system like (1.2) in which the term $\mathbf{u}_\infty\cdot\nabla\mathbf{u}$ has to be added in the equation of the balance of momentum, see [7, 8]. For related L^q–results on weak solutions we refer to [22], for the investigation of several auxiliary linear problems to [30, 31], and for weak solutions to an Oseen system of type (1.2) in L^2 with anisotropic weights see [26]; for results in L^q–spaces see [27, 28]. Pointwise estimates, even for solutions of the nonlinear Navier–Stokes equations, can be found in [16]; indeed, there exists a stationary strong solution $\mathbf{u}_s$ satisfying the estimate $|\mathbf{u}_s(\mathbf{x})|\le C/|\mathbf{x}|$. On the one hand, this result must be considered with regard to the fact that the corresponding fundamental solution $\Gamma(\mathbf{x},\mathbf{y})$ of (1.2) cannot be dominated uniformly by $|\mathbf{x}-\mathbf{y}|^{-1}$, see [10]. On the other hand, these pointwise estimates suggest to discuss (1.2) in weak L^q–spaces ($L^{3/2,\infty}$ and $L^{3,\infty}$) as done in [9, 22]. Extensions of the pointwise decay estimates and also representation formulae can be found in [3, 4]. Stability estimates in the L^2–setting are proved in [18], and in the $L^{3,\infty}$–setting in [24].

In this paper, we denote spaces of vector–valued functions by boldface letters. Otherwise we preserve the standard notation for Lebesgue and Sobolev spaces. As usually, $\mathbf{L}^q_\sigma(\Omega)$ denotes the closure of the space of all infinitely differentiable divergence–free vector fields in Ω, with compact support in Ω, in $\mathbf{L}^q(\Omega)$. The Helmholtz projection of $\mathbf{L}^q(\Omega)$, $1<q<\infty$, onto $\mathbf{L}^q_\sigma(\Omega)$ is denoted by P_q. The spectrum of an operator is denoted by σ, the essential spectrum by σ_{ess}, the point spectrum by σ_{p}, the continuous spectrum by σ_{c} and the residual spectrum by σ_{r}.

Assuming that $\omega := |\boldsymbol{\omega}| \neq 0$ and, say, $\boldsymbol{\omega} || \mathbf{e}_3$, and we analyze the Stokes–type operator

$$\mathcal{L}_q^\omega \mathbf{u} := \ P_q\big[-\nu\Delta\mathbf{u} - (\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{u}\big]$$

in the space $\mathbf{L}^q_\sigma(\Omega)$, $1 < q < \infty$. The domain of the operator $\mathcal{L}_q^\omega$ is

$$\mathcal{D}(\mathcal{L}_q^\omega) \ := \ \big\{\mathbf{u}\in\mathbf{W}^{2,q}(\Omega)\cap\mathbf{W}_0^{1,q}(\Omega)\cap\mathbf{L}^q_\sigma(\Omega);\ (\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla\mathbf{u}\in\mathbf{L}^q(\Omega)\big\}.$$

We consider $\mathcal{D}(\mathcal{L}_q^\omega)$ to be equipped by the norm

$$\|\mathbf{v}\|_{\mathcal{D}(\mathcal{L}_q^\omega)} \ := \ \|\mathbf{v}\|_{2,q} + \|(\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla\mathbf{v}\|_q, \tag{1.4}$$

equivalent to the graph norm $\|\mathbf{v}\|_q + \|\mathcal{L}_q^\omega\mathbf{v}\|_q$, to yield a Banach space since $\mathcal{L}_q^\omega$ is a closed operator; here $\|\,.\,\|_{2,q}$ denotes the norm in $\mathbf{W}^{2,q}(\Omega)$. From [23], we know that the semigroup generated by $\mathcal{L}_2^\omega$ for the whole space does not map $L^2_\sigma(\mathbb{R}^3)$ into the domain $D(\mathcal{L}_2^\omega)$ for $t > 0$, so that the semigroup $\mathrm{e}^{-t\mathcal{L}_2^\omega}$, $t \geq 0$, is not analytic. The same result holds in L^q–spaces and for exterior domains, see [19]. Hence the analysis of the spectrum of $\mathcal{L}_q^\omega$ is an interesting problem.

We know from [14] that the adjoint operator to $\mathcal{L}_q^\omega$ equals $\mathcal{L}_{q'}^{-\omega}$ (with $q' = q/(q-1)$) so that

$$(\mathcal{L}_q^\omega)^*\mathbf{u} = P_{q'}\big[-\Delta + (\boldsymbol{\omega}\wedge\mathbf{x})\cdot\nabla\mathbf{u} - \boldsymbol{\omega}\wedge\mathbf{u}\big] \quad \text{for} \quad \mathbf{u}\in\mathcal{D}(\mathcal{L}_{q'}^{-\omega}) = \mathcal{D}((\mathcal{L}_q^\omega)^*).$$

In [12] the first and third author proved in the whole space case and for $q = 2$ that

$$\sigma(-\mathcal{L}_q^\omega) \ = \ \sigma_{\mathrm{ess}}(-\mathcal{L}_q^\omega) \ = \ \sigma_{\mathrm{c}}(-\mathcal{L}_q^\omega) \ = \ \mathfrak{S}^\omega \ := \ \bigcup_{k=-\infty}^{+\infty}\big\{(-\infty,0] + \mathrm{i}\omega k\big\}, \tag{1.5}$$

i.e., the spectrum of $-\mathcal{L}_2^\omega$ is a purely continuous one and equals a countable set of equidistant half–lines in the left complex half–plane. This result was extended via detailed technical arguments by the authors to the case L^q, $q \neq 2$, see [13]. Exactly the same result holds for an exterior domain $\Omega \subset \mathbb{R}^3$ which is rotationally symmetric with respect to the axis of rotation. However, if Ω is not rotationally symmetric, then the above result holds for the essential spectrum only, i.e., $\sigma_{\mathrm{ess}}(-\mathcal{L}_q^\omega) = \mathfrak{S}^\omega$; the question of existence of eigenvalues in the left complex half–plane is open up to now.

The spectrum of a corresponding linear Oseen–type operator, i.e. the operator $-\mathcal{L}_q^\omega + \gamma\partial_3$, $\gamma \neq 0$, was studied in [14]: The essential spectrum consists of a countable set of overlapping parabolic regions in the left half–plane of the complex plane. Moreover, the full spectrum coincides with the essential and continuous one if $\Omega = \mathbb{R}^3$.

The aim of this paper is not only to present a new, functional analytic proof of the result (1.5) for all $q \in (1,\infty)$, but also to determine whether the spectrum is a continuous or residual one, or whether it consists of eigenvalues, and to prove similar results for the classical Stokes operator and also the Laplacian in all dimensions $n \geq 2$. Actually, our methods are based on techniques from harmonic analysis developed for the Laplacian on $\mathbb{R}^n$, see Theorem 3.1, and the Stokes operator on $\mathbb{R}^n$, see Theorem 3.2. Our result on the operator $\mathcal{L}_q^\omega$ on the whole space $\mathbb{R}^3$ shows that the character of the spectrum (not the set itself) strictly depends on q changing from a residual spectrum for $1 < q < \frac{3}{2}$ to a continuous one for $\frac{3}{2} \leq q \leq 3$ and to a pure point spectrum for $q > 3$. To be precise, we prove the theorem:

Theorem 1.1. *Let $1 < q < \infty$ and $\Omega = \mathbb{R}^3$. Then the spectrum $\sigma(-\mathcal{L}_q^\omega)$ of the operator $(-\mathcal{L}_q^\omega)$ is the set $\mathfrak{S}^\omega$, cf. (1.5). For each $\lambda \in \mathfrak{S}^\omega$ the range $\mathcal{R}(\lambda + \mathcal{L}_q^\omega)$ is not closed, which implies that $\sigma(-\mathcal{L}_q^\omega) = \sigma_{\rm ess}(-\mathcal{L}_q^\omega)$. Furthermore, $\mathrm{i}\omega\mathbb{Z} \subset \sigma_{\rm c}(-\mathcal{L}_q^\omega)$ and*

(i) if $1 < q < \frac{3}{2}$ then $\mathfrak{S}^\omega \setminus \mathrm{i}\omega\mathbb{Z} = \sigma_{\rm r}(-\mathcal{L}_q^\omega)$ and for each λ in this set the codimension of $\overline{\mathcal{R}(\lambda + \mathcal{L}_q^\omega)}$ equals infinity,

(ii) if $\frac{3}{2} \le q \le 3$ then $\mathfrak{S}^\omega \setminus \mathrm{i}\omega\mathbb{Z} = \sigma_{\rm c}(-\mathcal{L}_q^\omega)$,

(iii) if $3 < q < \infty$ then $\mathfrak{S}^\omega \setminus \mathrm{i}\omega\mathbb{Z} = \sigma_{\rm p}(-\mathcal{L}_q^\omega)$ and the geometric multiplicity of each eigenvalue is infinite.

Moreover, we estimate the resolvent operator $\lambda + \mathcal{L}_q^\omega$, $q = 2$, for $\lambda = \alpha + \mathrm{i}\beta$, $\alpha < 0$, $\beta \notin \omega\mathbb{Z}$, lying between two half–lines and going to infinity, i.e. $\alpha \to -\infty$, see Theorem 3.4. Finally, we describe the spectrum of $\mathcal{L}_q^\omega$ in the case of an exterior domain $\Omega \subset \mathbb{R}^3$, see Section 4.

2 Preliminaries

Recall that $1 < q < \infty$ and $\omega = |\boldsymbol{\omega}| \neq 0$ throughout the whole paper. Consider the spectral problem

$$\begin{aligned} \lambda \mathbf{u} - \nu\Delta\mathbf{u} - (\boldsymbol{\omega} \wedge \mathbf{x}) \cdot \nabla\mathbf{u} + \boldsymbol{\omega} \wedge \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}(\mathbf{x}) &\to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty \end{aligned} \tag{2.1}$$

when $\Omega = \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^3$ is an exterior domain; in the latter case the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ is added to (2.1). To solve (2.1) explicitly when $\Omega = \mathbb{R}^3$ we use the Fourier transform and multiplier operators. For simplicity we assume that the axis of rotation is parallel to the third unit vector $\mathbf{e}_3$, the angular velocity is equal to one, and that $\nu = 1$; hence

$$\boldsymbol{\omega} = \mathbf{e}_3, \quad \omega = |\boldsymbol{\omega}| = 1 \quad \text{and} \quad \nu = 1.$$

In order to recall this assumption, we use the notation $\mathcal{L}_q^1$ (instead of $\mathcal{L}_q^\omega$). Due to the geometry of the problem, it is reasonable to use cylindrical coordinates attached to $\mathbf{e}_3$ in $\mathbf{x}$-space and also in the space of the Fourier variable $\boldsymbol{\xi}$. In particular, let θ and φ denote the angular variables in $\mathbf{x}$– and $\boldsymbol{\xi}$–space, respectively. Note that

$$(\mathbf{e}_3 \wedge \mathbf{x}) \cdot \nabla_x = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} = \partial_\theta \tag{2.2}$$

is an angular derivative and that the Fourier transform of $(\mathbf{e}_3 \wedge \mathbf{x}) \cdot \nabla \mathbf{u}$ equals $(\mathbf{e}_3 \wedge \boldsymbol{\xi}) \cdot \nabla_\xi \widehat{\mathbf{u}} = \partial_\varphi \widehat{\mathbf{u}}$. Working at first formally or in the space $\mathcal{S}'(\mathbb{R}^3)$, we apply the Fourier transform $\mathcal{F}$, denoted by $\widehat{\ }$, to (2.1); see e.g. [20] for the definition and properties of the space $\mathcal{S}(\mathbb{R}^3)$ of Schwartz functions and the space $\mathcal{S}'(\mathbb{R}^3)$ of tempered distributions. With the Fourier variable $\boldsymbol{\xi} \in \mathbb{R}^3$ and its Euclidean length $s = |\boldsymbol{\xi}|$ we get from (2.1)

$$(\lambda + s^2)\, \widehat{\mathbf{u}} - \partial_\varphi \widehat{\mathbf{u}} + \mathbf{e}_3 \wedge \widehat{\mathbf{u}} + \mathrm{i}\boldsymbol{\xi}\, \widehat{p} = \widehat{\mathbf{f}}, \qquad \mathrm{i}\boldsymbol{\xi} \cdot \widehat{\mathbf{u}} = 0. \tag{2.3}$$

Since $\mathrm{i}\boldsymbol{\xi} \cdot \widehat{\mathbf{u}} = 0$ implies $\mathrm{i}\boldsymbol{\xi} \cdot \big(\partial_\varphi \widehat{\mathbf{u}} - \mathbf{e}_3 \times \widehat{\mathbf{u}}\big) = \mathbf{0}$, the unknown pressure p is given by $-|\boldsymbol{\xi}|^2 \widehat{p} = \mathrm{i}\boldsymbol{\xi} \cdot \widehat{\mathbf{f}}$. Then the Hörmander multiplier theorem yields the estimate

$$\|\nabla p\|_q \le C\, \|\mathbf{f}\|_q, \tag{2.4}$$

where $C = C(q) > 0$. In particular, $\nabla p \in \mathbf{L}^q(\mathbb{R}^3)$, cf. (1.3) when $\lambda = 0$.

Hence $\mathbf{u}$ can be considered as a (solenoidal) solution of the reduced problem

$$\lambda \mathbf{u} - \Delta \mathbf{u} - \partial_\theta \mathbf{u} + \mathbf{e}_3 \wedge \mathbf{u} \;=\; \mathbf{f}' := \mathbf{f} - \nabla p \qquad \text{in } \mathbb{R}^3. \tag{2.5}$$

or – in Fourier's space – as a solution of the first order linear inhomogeneous ordinary differential equation

$$-\partial_\varphi \widehat{\mathbf{u}} + \mathbf{e}_3 \wedge \widehat{\mathbf{u}} + (\lambda + s^2)\widehat{\mathbf{u}} \;=\; \widehat{\mathbf{f}'} \tag{2.6}$$

with respect to φ for $\widehat{\mathbf{u}}(\boldsymbol{\xi})$ as a 2π–periodic function in φ. Next we will get rid of the term $\mathbf{e}_3 \wedge \widehat{\mathbf{u}}$ in (2.6) by introducing the matrix of rotation $O(t)$:

$$O(t) = \begin{pmatrix} \cos t, & -\sin t, & 0 \\ \sin t, & \cos t, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \tag{2.7}$$

and the new unknown $\mathbf{v}$ by

$$\widehat{\mathbf{v}} = O(\varphi)^T \widehat{\mathbf{u}}. \tag{2.8}$$

Since $\partial_\varphi \widehat{\mathbf{u}} = O(\varphi)\partial_\varphi \widehat{\mathbf{v}} + \mathbf{e}_3 \wedge (O(\varphi)\widehat{\mathbf{v}})$, we see that $\widehat{\mathbf{v}}$ satisfies the equation

$$-\partial_\varphi \widehat{\mathbf{v}} + (\lambda + s^2)\widehat{\mathbf{v}} \;=\; \widehat{\mathbf{F}} := O^T(\varphi)\widehat{\mathbf{f}'}. \tag{2.9}$$

This problem was solved explicitly in [8], [10] when $\lambda = 0$. However, replacing in the solution formulas of [10] the term νs^2 by $\lambda + s^2$, we easily get that

$$\widehat{\mathbf{v}}(\boldsymbol{\xi}) \;=\; \int_0^\infty \mathrm{e}^{-(\lambda+s^2)t}\, \widehat{\mathbf{F}}\big(O(t)\boldsymbol{\xi}\big)\, \mathrm{d}t \tag{2.10}$$

and, using the definition

$$D(\boldsymbol{\xi}) \;:=\; 1 - \mathrm{e}^{-2\pi(\lambda+s^2)}, \tag{2.11}$$

also

$$\widehat{\mathbf{v}}(\xi) \;=\; \frac{1}{D(\boldsymbol{\xi})} \int_0^{2\pi} \mathrm{e}^{-(\lambda+s^2)t}\, \widehat{\mathbf{F}}\big(O(t)\xi\big)\, \mathrm{d}t. \tag{2.12}$$

Hence due to (2.8), with $s = |\boldsymbol{\xi}|$ as before,

$$\widehat{\mathbf{u}}(\boldsymbol{\xi}) \;=\; \frac{1}{D(\boldsymbol{\xi})} \int_0^{2\pi} \mathrm{e}^{-(\lambda+s^2)t}\, O^T(t)\, \widehat{\mathbf{f}'}\big(O(t)\boldsymbol{\xi}\big)\, \mathrm{d}t. \tag{2.13}$$

We recall the following result from [13, Theorems 2.1 and 2.2]; we note that the L^q-estimate in Theorem 2.1 below is a straightforward consequence of multiplier theory.

Theorem 2.1. *Suppose that $\mathbf{f} \in \mathbf{L}^q_\sigma(\mathbb{R}^3)$, $1 < q < \infty$, and $\lambda = \alpha + \mathrm{i}\beta$, $\alpha, \beta \in \mathbb{R}$, where either $\alpha > 0$ or $\beta \notin \mathbb{Z}$. Then the resolvent equation $(\lambda + \mathcal{L}^1_q)\mathbf{u} = \mathbf{f}$ has a unique solution $\mathbf{u} \in \mathcal{D}(\mathcal{L}^1_q)$. There exists a real constant $C > 0$ depending only on λ and q such that the solution satisfies the estimate*

$$\|\mathbf{u}\|_q \;\le\; C(\lambda, q)\, \|\mathbf{f}\|_q.$$

In particular, λ belongs to the resolvent set of the operator $-\mathcal{L}^1_q$.

The second result in [13], stating that $\sigma(-\mathcal{L}_q^1) = \sigma_{\mathrm{ess}}(-\mathcal{L}_q^1) = \mathfrak{S}^\omega$, will be proved in this paper by other tools.

Recall that for a closed operator T on a Banach space X with dense domain $\mathcal{D}(T)$ and range $\mathcal{R}(T)$, the *essential spectrum* $\sigma_{\mathrm{ess}}(T)$ is defined as the set of those $\lambda \in \mathbb{C}$ for which the operator $\lambda - T$ is not semi–Fredholm, which is equivalent to the identities $\mathrm{nul}'(\lambda - T) = \mathrm{def}'(\lambda - T) = \infty$. Here $\mathrm{nul}'(\lambda - T)$ denotes the *approximate nullity* and $\mathrm{def}'(\lambda - T) := \mathrm{nul}'(\overline{\lambda} - T^*)$ is the *approximate deficiency* of $\lambda - T$; T^* denotes the adjoint operator to T. Note that for a closed operator T as above $\mathrm{nul}(T) = \mathrm{nul}'(T)$ and $\mathrm{def}(T) = \mathrm{def}'(T)$ if $\mathcal{R}(T)$ is closed, and that $\mathrm{nul}'(T) = \mathrm{def}'(T) = \infty$ if $\mathcal{R}(T)$ is not closed, see [25, Theorem IV.5.10]. Moroever, $\mathrm{nul}'(T) = \infty$ if and only if there exists a non–compact sequence $\{\mathbf{u}_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(T)$ such that $\|\mathbf{u}_k\|_q = 1$ for all $k \in \mathbb{N}$ and $\|T\mathbf{u}_k\|_q \to 0$ as $k \to \infty$, see [25, Theorem IV.5.11]. For further properties of these notions we refer to [25, Ch. IV.5].

The following definitions will be important in proofs in Section 3. For $\psi \in \mathcal{S}(\mathbb{R}^n)$, $n \geq 2$, let the *average operator* M be given by

$$(M\psi)(r) \;=\; \fint_{\partial B_1} \psi(r, \boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} \;:=\; \frac{1}{|\partial B_1|} \int_{\partial B_1} \psi(r, \boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha}, \quad r \geq 0,$$

where $(r, \boldsymbol{\alpha}) = \big(r, \mathbf{x}/|\mathbf{x}|\big) \in [0, \infty) \times \partial B_1$ denote the polar coordinates of $\mathbf{x} \in \mathbb{R}^n$, $\mathrm{d}\boldsymbol{\alpha}$ indicates integration with respect to the surface measure on $\partial B_1 = \{\mathbf{x} \in \mathbb{R}^n;\ |\mathbf{x}| = 1\}$ and $|\partial B_1| = \int_{\partial B_1} \mathrm{d}\boldsymbol{\alpha}$. A function $\Psi \in \mathcal{S}(\mathbb{R}^n)$ is called *radial* if and only if $\Psi(\mathbf{x})$ depends only on $r = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$. Equivalently, $M\Psi = \Psi$ or $\Psi \circ R = \Psi$ for all orthogonal $n \times n$ matrices R. Obviously, $M\psi \in \mathcal{S}(\mathbb{R}^n)$ is radial for any $\psi \in \mathcal{S}(\mathbb{R}^n)$. Moreover,

$$\widehat{M\psi} = M(\widehat{\psi}). \tag{2.14}$$

Actually,

$$\widehat{M\psi}(\boldsymbol{\xi}) \;=\; \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}\, (M\psi)(|\mathbf{x}|)\, \mathrm{d}\mathbf{x} \;=\; \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}} \left(\fint_{\partial B_1} \psi(|\mathbf{x}|, \boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} \right) \mathrm{d}\mathbf{x}.$$

Using also the expressions $\mathbf{x} \hat{=} (r, \boldsymbol{\beta})$ where $r = |\mathbf{x}|$ and $\boldsymbol{\beta} = \mathbf{x}/|\mathbf{x}| \in \partial B_1$ and $\boldsymbol{\xi} \hat{=} (s, \boldsymbol{\gamma})$ where $s = |\boldsymbol{\xi}|$ and $\boldsymbol{\gamma} = \boldsymbol{\xi}/|\boldsymbol{\xi}| \in \partial B_1$, we have $\mathbf{x} \cdot \boldsymbol{\xi} = rs\, \boldsymbol{\beta} \cdot \boldsymbol{\gamma}$ and therefore

$$\begin{aligned}
\widehat{M\psi}(\boldsymbol{\xi}) \;&=\; \widehat{M\psi}(s, \boldsymbol{\gamma}) \;=\; \int_0^\infty \int_{\partial B_1} \mathrm{e}^{-\mathrm{i}rs\,\boldsymbol{\beta}\cdot\boldsymbol{\gamma}} \left(\fint_{\partial B_1} \psi(r, \boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} \right) \mathrm{d}\boldsymbol{\beta}\; r^{n-1}\, \mathrm{d}r \\
&=\; \fint_{\partial B_1} \left[\int_0^\infty \psi(r, \boldsymbol{\alpha})\, r^{n-1} \left(\int_{\partial B_1} \mathrm{e}^{-\mathrm{i}rs\,\boldsymbol{\beta}\cdot\boldsymbol{\gamma}}\, \mathrm{d}\boldsymbol{\beta} \right) \mathrm{d}r \right] \mathrm{d}\boldsymbol{\alpha}.
\end{aligned}$$

The integral $\int_{\partial B_1} \mathrm{e}^{-\mathrm{i}rs\,\boldsymbol{\beta}\cdot\boldsymbol{\gamma}}\, \mathrm{d}\boldsymbol{\beta}$ is independent of $\boldsymbol{\gamma}$ and therefore equals $\int_{\partial B_1} \mathrm{e}^{-\mathrm{i}rs\,\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}\, \mathrm{d}\boldsymbol{\beta}$. Hence

$$\begin{aligned}
\widehat{M\psi}(\boldsymbol{\xi}) \;&=\; \fint_{\partial B_1} \left[\int_0^\infty \psi(r, \boldsymbol{\alpha})\, r^{n-1} \left(\int_{\partial B_1} \mathrm{e}^{-\mathrm{i}rs\,\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}\, \mathrm{d}\boldsymbol{\beta} \right) \mathrm{d}r \right] \mathrm{d}\boldsymbol{\alpha} \\
&=\; \fint_{\partial B_1} \left[\int_0^\infty \left(\int_{\partial B_1} \mathrm{e}^{-\mathrm{i}rs\,\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}\, \psi(r, \boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} \right) r^{n-1}\, \mathrm{d}r \right] \mathrm{d}\boldsymbol{\beta} \\
&=\; (M\widehat{\psi})(s) \;=\; (M\widehat{\psi})\big(|\boldsymbol{\xi}|\big).
\end{aligned}$$

This proves the identity (2.14).

The definition of the average operator M can be easily transferred from $\mathcal{S}(\mathbb{R}^n)$ to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$: For $f \in \mathcal{S}'(\mathbb{R}^n)$ define Mf by

$$\langle Mf, \psi \rangle \;:=\; \langle f, M\psi \rangle \quad \text{for } \psi \in \mathcal{S}(\mathbb{R}^n).$$

Note that this definition is consistent with the case when the distribution f is itself represented by a function from $\mathcal{S}(\mathbb{R}^n)$. Now (2.14) easily yields the identity

$$\widehat{Mf} \;=\; M(\widehat{f}), \quad \text{for } f \in \mathcal{S}'(\mathbb{R}^n). \tag{2.15}$$

A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is called *radial* if $Mf = f$. In this case $\langle f, \psi \rangle = \langle f, \psi \circ R \rangle$ for every orthogonal $n \times n$ matrix R and for all $\psi \in \mathcal{S}(\mathbb{R}^n)$. Moreover, Mf is radial.

The importance of the average operator M lies in the construction and classification of radial tempered distributions $\widehat{f}$ with support in the unit sphere $\partial B_1 \subset \mathbb{R}^3$: if in this case $f \in L^q(\mathbb{R}^3)$ for some $q \in (1, \infty)$, then it is a constant multiple of the function $(\sin|\mathbf{x}|)/|\mathbf{x}|$; similar results hold in $\mathbb{R}^n$, $n \geq 2$, see Lemma 2.3 below.

Lemma 2.2. *Let f be a radial tempered distribution on $\mathbb{R}^n$, $n \geq 2$, with* $\operatorname{supp} \widehat{f} \subset \partial B_1$. *Then there exists $m \in \mathbb{N}$ such that $(1+\Delta)^m f = 0$.*

Proof. Since $\widehat{f} \in \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp} \widehat{f} \subset \partial B_1$, there exists $m \in \mathbb{N}$ and a constant $C \geq 0$ such that

$$\big|\langle \widehat{f}, \widehat{\varphi} \rangle\big| \;\leq\; C \sum_{|\alpha|=0}^{m} \|D^\alpha \widehat{\varphi}\|_{L^\infty(B_2 \setminus B_{1/2})}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where $\alpha \in \mathbb{N}_0^n$ is a multi–index, D^α denotes the corresponding partial derivative of order $|\alpha|$ and $B_R = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < R\}$. Choose an even cut–off function $\eta \in C_0^\infty(\mathbb{R})$ with $\eta(r) = 1$ for $r \in (-1, 1)$, and define $\eta_\varepsilon(\mathbf{x}) = \eta\big((|\mathbf{x}|^2 - 1)/\varepsilon\big)$, $\mathbf{x} \in \mathbb{R}^n$, $\varepsilon > 0$. Since $\eta_\varepsilon(\mathbf{x})$ is radial, we will also write $\eta_\varepsilon(r)$ where $r = |\mathbf{x}|$. Then for every $j \in \mathbb{N}_0$ there exists $c_j > 0$ such that $|D^\alpha \eta_\varepsilon| \leq c_j \varepsilon^{-j}$ for all $\varepsilon > 0$ and α such that $|\alpha| = j$. Since f and consequently also $\widehat{f}$ are radial, we easily get that

$$\langle \widehat{f}, \widehat{\varphi} \rangle \;=\; \langle \widehat{f}, M\widehat{\varphi} \rangle \;=\; \langle \widehat{f}, \eta_\varepsilon M\widehat{\varphi} \rangle.$$

Hence for all $\varepsilon > 0$ sufficiently small

$$\begin{aligned} \langle \widehat{f}, \widehat{\varphi} \rangle \;&=\; \Big\langle \widehat{f}, \Big(M\widehat{\varphi} - \sum_{j=0}^{m} \frac{1}{j!}\, \partial_\rho^j (M\widehat{\varphi})(\rho)\big|_{\rho=1} (r-1)^j \Big) \eta_\varepsilon \Big\rangle \\ &\quad + \Big\langle \widehat{f}, \Big(\sum_{j=0}^{m} \frac{1}{j!}\, \partial_\rho^j (M\widehat{\varphi})(\rho)\big|_{\rho=1} (r-1)^j \Big) \eta_\varepsilon \Big\rangle. \end{aligned} \tag{2.16}$$

By the classical estimate of the remainder in Taylor's expansion of the smooth function $M\widehat{\varphi}(r)$, $r > 0$, we know that there exists $C > 0$ such that for $k = 0, \dots, m$

$$\Big| \partial_r^k \Big((M\widehat{\varphi})(r) - \sum_{j=0}^{m} \frac{1}{j!} \big[\partial_\rho^j (M\widehat{\varphi})(\rho)\big|_{\rho=1} \big] (r-1)^j \Big) \Big| \;\leq\; C\, |r-1|^{m+1-k}, \quad r \in \big(\tfrac{1}{2}, 2\big).$$

Thus for all $\varepsilon > 0$ sufficiently small and $k = 0, \ldots, m$ we are led to the estimate

$$\Big\| \partial_r^k \Big[\Big((M\widehat{\varphi})(r) - \sum_{j=0}^{m} \frac{1}{j!} \, \big[\partial_\rho^j (M\widehat{\varphi})(\rho) \big|_{\rho=1} \big] \, (r-1)^j \Big) \eta_\varepsilon(r) \Big] \Big\|_{L^\infty(B_2 \setminus B_{1/2})} \;\le\; C \, \varepsilon^{m+1-k}$$

with a constant $C > 0$. Consequently the first term on the right–hand side of (2.16) vanishes. Hence (2.16) may be rewritten in the form

$$\langle \widehat{f}, \widehat{\varphi} \rangle \;=\; \sum_{j=0}^{m} a_j \, \partial_\rho^j (M\widehat{\varphi})(\rho) \,\big|_{\rho=1} \tag{2.17}$$

where the constant $a_j \in \mathbb{C}$ equals $(1/j!) \, \langle \widehat{f}, \, (r-1)^j \eta_\varepsilon \rangle$, $j = 0, \ldots, m$ and $\varepsilon > 0$. Now, by Plancherel's theorem, (2.17) yields the identities

$$\begin{aligned} \big\langle (-1-\Delta)^{m+1} f, \, \varphi \big\rangle \;&=\; \big\langle \widehat{f}, \, (|\boldsymbol{\xi}|^2 - 1)^{m+1} \widehat{\varphi} \big\rangle \;=\; \sum_{j=0}^{m} a_j \, \partial_r^j \big[(r^2-1)^{m+1} \, (M\widehat{\varphi})(r) \big] \,\big|_{r=1} \\ &=\; \sum_{j=0}^{m} a_j \, ⨍_{\partial B_1} \partial_r^j \big((r^2-1)^{m+1} \widehat{\varphi}(r\boldsymbol{\beta}) \big) \big|_{r=1} \, \mathrm{d}\boldsymbol{\beta} \;=\; 0. \end{aligned}$$

Replacing m by $m-1$, the lemma is proved. □

In the next Lemma 2.3 below we will use the *Bessel functions of the first kind* J_μ, defined by the formula

$$J_\mu(r) \;:=\; \sum_{m=0}^{\infty} (-1)^m \, \frac{(r/2)^{\mu+m}}{m! \, \Gamma(\mu+m+1)}, \tag{2.18}$$

and *of the second kind* Y_μ, $\mu \in \mathbb{R}$; for definitions and the main properties see e.g. [33, Ch. III and VII]. Let us recall the following crucial items: If $\mathcal{C}_\mu$ denotes one of the Bessel functions, i.e. $\mathcal{C}_\mu = J_\mu$ or $\mathcal{C}_\mu = Y_\mu$, then

$$\Big[\partial_r^2 + \frac{1}{r} \, \partial_r + \Big(1 - \frac{\mu^2}{r^2} \Big) \Big] \, \mathcal{C}_\mu(r) \;=\; 0, \quad \text{for } \mu \in \mathbb{R}, \; r > 0. \tag{2.19}$$

The asymptotic behavior of $\mathcal{C}_\mu$ for large r is determined – up to the constant $\sqrt{2/\pi}$ – by

$$J_\mu(r) \;\sim\; \frac{1}{\sqrt{r}} \, \cos\Big(r - \frac{\mu\pi}{2} - \frac{\pi}{4} \Big), \quad Y_\mu(r) \;\sim\; \frac{1}{\sqrt{r}} \, \sin\Big(r - \frac{\mu\pi}{2} - \frac{\pi}{4} \Big) \quad \text{as } r \to \infty, \tag{2.20}$$

see [33, 7·21 (1), (2)]. Due to the recursion formula $\partial_r \mathcal{C}_\mu = \mathcal{C}_{\mu-1} - (\mu/r) \, \mathcal{C}_\mu$, see [33, 3·22 (3)], a similar behavior holds for all derivatives $\partial_r^k \mathcal{C}_\mu$, $k \in \mathbb{N}$; in particular, $\partial_r^k J_\mu$ and $\partial_r^k Y_\mu$ do decay as $r^{-1/2}$ and not faster as $r \to \infty$. Finally, by [33, 3·1 (8), 3·51 (3), 3·52 (3), 3·53 (1)], we know that up to the constant $[2^\mu \Gamma(\mu+1)]^{-1}$

$$J_\mu(r) \;\sim\; r^\mu \quad \text{as } r \to 0, \tag{2.21}$$

and, up to appropriate constants,

$$Y_\mu(r) \;\sim\; \begin{cases} r^{-\mu} & \text{if } \mu \in \frac{1}{2} + \mathbb{Z}, \\ \ln r & \text{if } \mu = 0, \\ r^{-\mu} & \text{if } \mu \in \mathbb{N} \end{cases} \qquad \text{as } r \to 0. \tag{2.22}$$

For later use we define the function

$$\mathcal{J}_n(r) = r^{-\frac{n-2}{2}}\ J_{\frac{n-2}{2}}(r). \tag{2.23}$$

Note that by (2.18) $\mathcal{J}_n$ is a smooth function on $[0,\infty)$, that $\partial_r^k \mathcal{J}_n$ (for $k \geq 0$, $n \geq 2$) decays as $r^{-\frac{n-1}{2}}$ when $r \to \infty$ and

$$\mathcal{J}_2(r) = J_0(r) \quad \text{and} \quad \mathcal{J}_3(r) = \frac{1}{\sqrt{r}}\ J_{\frac{1}{2}}(r) = \sqrt{\frac{2}{\pi}}\ \frac{\sin r}{r}.$$

Lemma 2.3. *Let $0 \neq f \in L^q(\mathbb{R}^n)$, $n \geq 2$, $q \in (1,\infty)$, be radial with* $\operatorname{supp} \widehat{f} = \partial B_1$.

(i) *If $n \geq 3$, then necessarily $q > \frac{2n}{n-1}$ and f is a multiple of $\mathcal{J}_n$. In particular, if $n = 3$, then f is a multiple of $(\sin r)/r$ where $r = |\mathbf{x}|$.*

(ii) *If $n = 2$ and $f \in W^{1,q}(\mathbb{R}^2)$, then $q > 4$ and f is a multiple of J_0.*

Proof. Consider a radial function $f \in L^q(\mathbb{R}^n)$, $q \in (1,\infty)$, with $\operatorname{supp} \widehat{f} = \partial B_1$. Then Lemma 2.2 implies that there exists $m \in \mathbb{N}$ such that $(1+\Delta)^m f = 0$. In the setting of radial solutions on $\mathbb{R}^n$ the operator $(1+\Delta)^m$ equals T_n^m where T_n is the second order ordinary differential operator

$$T_n \ = \ \partial_r^2 + \frac{n-1}{r}\ \partial_r + 1. \tag{2.24}$$

Let us determine a fundamental system of $2m$ solutions of the ordinary differential equation $T_n^m f(r) = 0$.

Assertion. *For each $m \in \mathbb{N}$ and dimension $n \geq 2$ there hold the identities*

$$T_n^m \big[r^{-\frac{n}{2}+m}\ J_{\frac{n}{2}+m-2}(r) \big] \ = \ T_n^m \big[r^{-\frac{n}{2}+m}\ Y_{\frac{n}{2}+m-2}(r) \big] \ = \ 0. \tag{2.25}$$

Proof. For $m = 1$ we use (2.19) for the Bessel functions $\mathcal{C}_\mu = J_\mu$ and $\mathcal{C}_\mu = Y_\mu$. Since

$$T_n\, \mathcal{C}_{\frac{n}{2}-1}(r) \ = \ \left[\frac{n-2}{r}\ \partial_r + \left(\frac{n}{2} - 1\right)^2 r^{-2} \right] \mathcal{C}_{\frac{n}{2}-1}(r),$$

we get that for every $n \geq 2$

$$\begin{aligned} T_n \big[r^{-\frac{n}{2}+1}\, \mathcal{C}_{\frac{n}{2}-1}(r) \big] \ &= \ r^{-\frac{n}{2}+1}\, T_n\, \mathcal{C}_{\frac{n}{2}-1}(r) \\ &\quad + \left[\left(-\frac{n}{2}+1\right)\left(-\frac{n}{2}\right) r^{-\frac{n}{2}-1} + 2\left(-\frac{n}{2}+1\right) r^{-\frac{n}{2}}\, \partial_r + \frac{n-1}{r} \left(-\frac{n}{2}+1\right) r^{-\frac{n}{2}} \right] \mathcal{C}_{\frac{n}{2}-1}(r) \\ &= \ 0. \end{aligned} \tag{2.26}$$

Assume that $m \geq 2$ and that identity (2.25) holds for $m-1$. At first we compute

$$T_n \big[r^{-\frac{n}{2}+m} \mathcal{C}_{\frac{n}{2}+m-2}(r) \big] \ = \ \left[T_{n+2m-2} + \frac{2-2m}{r}\ \partial_r \right] \left(r^{2m-2} \cdot r^{-(\frac{n}{2}+m-2)}\, \mathcal{C}_{\frac{n}{2}+m-2}(r) \right).$$

Since $T_{n+2m-2} \big[r^{-(\frac{n}{2}+m-2)}\, \mathcal{C}_{\frac{n}{2}+m-2}(r) \big] \ = 0$ by (2.26) (with n replaced by $n+2m$) and $\big[\frac{1}{2}n + m - 2 + r\,\partial_r \big]\, \mathcal{C}_{\frac{n}{2}+m-2}(r) \ = \ r\, \mathcal{C}_{\frac{n}{2}+m-3}(r)$, see [33, 3·9 (3)], a lengthy calculation implies that

$$T_n \big[r^{-\frac{n}{2}+m}\, \mathcal{C}_{\frac{n}{2}+m-2}(r) \big]$$

$$
\begin{aligned}
&= \Big[(2m-2)(2m-3)\, r^{2m-4} + 2(2m-2)\, r^{2m-3}\, \partial_r + \frac{n+2m-3}{r}\,(2m-2)\, r^{2m-3} \\
&\quad - (2m-2)^2\, r^{2m-4} + r^{2m-2}\, \frac{2-2m}{r}\, \partial_r\Big]\, \big[r^{-(\frac{n}{2}+m-2)}\, \mathcal{C}_{\frac{n}{2}+m-2}(r)\big] \\
&= (2m-2)\, r^{-\frac{n}{2}+m-2} \Big[\Big(\frac{n}{2}+m-2+r\,\partial_r\Big)\, \mathcal{C}_{\frac{n}{2}+m-2}(r)\Big] \\
&= (2m-2)\, r^{-\frac{n}{2}+m-1}\, \mathcal{C}_{\frac{n}{2}+m-3}(r).
\end{aligned}
$$

The validity of (2.25) for $m-1$ now implies that $T_n^{m-1}\, T_n\big[r^{-\frac{n}{2}+m}\, \mathcal{C}_{\frac{n}{2}+m-2}(r)\big] = 0$. $\square$

Continuation of the proof of Lemma 2.3. To obtain information on L^q-integrability properties of the functions

$$
r^{-\frac{n}{2}+k}\, J_{\frac{n}{2}+k-2}(r), \quad r^{-\frac{n}{2}+k}\, Y_{\frac{n}{2}+k-2}(r) \quad \text{for } n \geq 2,\ k \in \mathbb{N}, \tag{2.27}
$$

we look at the behavior of these functions for $r \to 0$ and $r \to \infty$. By (2.20) the fastest decaying functions in (2.27) are those with $k = 1$, they decay as $r^{-\frac{n-1}{2}}$ when $r \to \infty$. Hence the exponent q must satisfy $q > \frac{2n}{n-1}$. Moreover, by (2.21), (2.22), $\mathcal{J}_n(r)$ is the only integrable function in the kernel of T_n^m. To be more precise, it is integrable with all powers $q > \frac{2n}{n-1}$. If $n = 2$, then $J_0,\, Y_0 \in L^q(\mathbb{R}^2)$ for all $q > 4$, but only $J_0 \in W^{1,q}(\mathbb{R}^2)$.

Now let f be an arbitrary radial solution of $T_n^m f = 0$. To complete the proof it suffices to show that f is a linear combination of the $2m$ functions in (2.27) corresponding to $k = 1, \dots, m$, or in other words, that these functions are linearly independent. However, the linear independence is an immediate consequence of the decay properties (2.20) taking also into account the different behavior of the functions cos and sin. Now the proof of Lemma 2.3 is complete. $\square$

Finally we mention and prove a classical result which is important for the discussion concerning the spectral point $\lambda = 0$.

Lemma 2.4. *Let $n \geq 2$. For all $1 < q < \infty$ the set*

$$
\mathcal{S}_0(\mathbb{R}^n) \ :=\ \big\{u \in \mathcal{S}(\mathbb{R}^n);\ \operatorname{supp} \widehat{u} \cap \{\mathbf{0}\} = \emptyset\big\}
$$

is dense in $L^q(\mathbb{R}^n)$.

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be radial such that $\widehat{\eta}(\xi) = 1$ in a neighborhood of $\mathbf{0}$ and let $\eta_k(\mathbf{x}) := \eta(\mathbf{x}/k)/k^n$, $k \in \mathbb{N}$. Since $\|\eta_k\|_1$ is independent of $k \in \mathbb{N}$, the family of operators $\{T_k\}_{k\in\mathbb{N}}$, defined by $T_k f := \eta_k * f$, $f \in L^q(\mathbb{R}^n)$, is uniformly bounded. Moreover, if $f \in \mathcal{S}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ then $T_k f \in \mathcal{S}(\mathbb{R}^n)$ and the estimate

$$
\|T_k f\|_q \ \leq\ \|f\|_1\, \|\eta_k\|_q \ \leq\ C\, \|f\|_1\, k^{-n/q'}, \quad q' = \frac{q}{q-1},
$$

shows that $T_k f \to 0$ as $k \to \infty$ in this case. Hence the sequence $\{f - T_k f\}_{k\in\mathbb{N}}$ lies in the set $\mathcal{S}_0(\mathbb{R}^n)$ and converges to f in $L^q(\mathbb{R}^n)$. Finally, the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$ and the uniform boundedness of the operator family $\{T_k\}_{k\in\mathbb{N}}$ on $L^q(\mathbb{R}^n)$ proves that $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$. $\square$

As a first application we introduce the (two-dimensional) Riesz transforms R'_1, R'_2 defined on $\mathbb{R}^3$: $(R'_j f)(\mathbf{x}) := \mathcal{F}_{-1}\big[-\mathrm{i}\,(\xi_j/|\boldsymbol{\xi}'|)\,\widehat{f}(\boldsymbol{\xi})\big]$, $j = 1,2$, where $|\boldsymbol{\xi}'| = \sqrt{\xi_1^2+\xi_2^2}$. Thus, in the Fourier space, the transforms R'_1, R'_2 are determined by their multipliers

$$-\mathrm{i}\,\frac{\xi_1}{|\boldsymbol{\xi}'|} = -\mathrm{i}\,\cos\varphi, \qquad -\mathrm{i}\,\frac{\xi_2}{|\boldsymbol{\xi}'|} = -\mathrm{i}\,\sin\varphi,$$

respectively, where φ is the cylindrical coordinate in $\boldsymbol{\xi}$-space. Consequently, the multiplier of $\big[\mathrm{i}\,(R'_1 - \mathrm{i}R'_2)\big]^k$ equals $\mathrm{e}^{-\mathrm{i}k\varphi}$, $k \in \mathbb{Z}$. Obviously, R'_j, $j = 1,2$, is a bounded operator on $L^q(\mathbb{R}^3)$.

Lemma 2.5. *For $1 < q < \infty$ the operators R'_j ($j = 1,2$) are bounded linear operators on $\mathcal{D}(\mathcal{L}^1_q)$ (equipped with the norm (1.4)). In particular, for every $k \in \mathbb{Z}$ and $\mathbf{u} \in \mathcal{D}(\mathcal{L}^1_q)$*

$$(\lambda + \mathcal{L}^1_q)\big[\mathrm{i}\,(R'_1 - \mathrm{i}R'_2)\big]^k\mathbf{u} \;=\; \big[\mathrm{i}\,(R'_1 - \mathrm{i}R'_2)\big]^k\,(\lambda + \mathrm{i}k + \mathcal{L}^1_q)\,\mathbf{u}. \tag{2.28}$$

Proof. A straightforward calculation shows that R'_j, $j = 1,2$, commutes with $\lambda - \Delta$ and maps $\mathcal{D}(\mathcal{L}^1_q)$ to $\mathcal{D}(\Delta_q) \cap \mathbf{L}^q_\sigma(\mathbb{R}^3)$; here Δ_q denotes the Laplace operator in $\mathbf{L}^q(\mathbb{R}^3)$ with domain $\mathcal{D}(\Delta_q) = \mathbf{W}^{2,q}(\mathbb{R}^3)$. As to the operator $(\mathbf{e}_3 \wedge \mathbf{x}) \cdot \nabla = \partial_\theta$, see (2.2), note that for any $\mathbf{u} \in \mathcal{D}(\mathcal{L}^1_q)$ and any $\boldsymbol{\psi} \in \mathcal{S}_0(\mathbb{R}^3)^3$

$$\begin{aligned}
\big\langle (\mathrm{e}_3 \wedge \mathbf{x}) \cdot \nabla(\mathrm{i}R'_1\mathbf{u}),\, \boldsymbol{\psi}\big\rangle \;&=\; -\big\langle \mathbf{u},\, (\mathrm{i}R'_1)\,\partial_\theta\boldsymbol{\psi}\big\rangle \\
=\; -\big\langle \widehat{\mathbf{u}},\, \cos\varphi\,\partial_\varphi\widehat{\boldsymbol{\psi}}\big\rangle \;&=\; -\big\langle \widehat{\mathbf{u}},\, \partial_\varphi\big((\cos\varphi)\,\widehat{\boldsymbol{\psi}}\big) + (\sin\varphi)\,\widehat{\boldsymbol{\psi}}\big\rangle \\
=\; \big\langle (\mathrm{i}R'_1)\,(\mathbf{e}_3 \wedge \mathbf{x}) \cdot \nabla\mathbf{u},\, \boldsymbol{\psi}\big\rangle &- \big\langle \mathrm{i}R'_2\mathbf{u},\, \boldsymbol{\psi}\big\rangle.
\end{aligned}$$

A similar identity holds for $(\mathrm{i}R'_2)$. Since $\mathcal{S}_0(\mathbb{R}^3)^3$ is dense in $\mathbf{L}^q(\mathbb{R}^3)$, we conclude that $\mathrm{i}R'_j$, $j = 1,2$, are well–defined, bounded operators on $\mathcal{D}(\mathcal{L}^1_q)$, satisfying

$$(\mathbf{e}_3 \wedge \mathbf{x}) \cdot \nabla\big[\mathrm{i}(R'_1 - \mathrm{i}R'_2)\big]\,\mathbf{u} \;=\; \big[\mathrm{i}(R'_1 - \mathrm{i}R'_2)\big]\,\big[(\mathbf{e}_3 \wedge \mathbf{x}) \cdot \nabla - \mathrm{i}\big]\,\mathbf{u}.$$

From this identity and since $(R'_1 - \mathrm{i}R'_2)(R'_1 + \mathrm{i}R'_2) = -\mathrm{id}$, we get (2.28) for every $k \in \mathbb{Z}$. □

3 The spectrum of $-\mathcal{L}^1_q$ on $\mathbb{R}^3$

The aim of this section is to analyze the spectrum $\sigma(-\mathcal{L}^1_q)$ for every $1 < q < \infty$. We start with the corresponding problem for the Laplacian in $\mathbb{R}^n$, $n \geq 2$, proceed with the Stokes operator $A_q = -P_q\Delta$ and finish with the operator $\mathcal{L}^1_q$. We prove the remarkable result that the type of the spectrum (point spectrum, continuous spectrum, residual spectrum) changes with $q \in (1,\infty)$, but coincides with the essential one for all q. The results and ideas in the proofs for the Laplace and Stokes operators are needed in the sequel, but are also of their own interest.

We always denote by q' the conjugate exponent to q, i.e. $q' = q/(q-1)$.

Theorem 3.1. *Let $n \geq 2$. The Laplacian Δ_q in $L^q(\mathbb{R}^n)$ has the following spectral properties:*

$$\sigma(\Delta_q) = \sigma_{\mathrm{ess}}(\Delta_q) = (-\infty, 0], \qquad 0 \in \sigma_{\mathrm{c}}(\Delta_q),$$

for each $\lambda \le 0$ the range $\mathcal{R}(\lambda - \Delta_q)$ is not closed, and

$$(-\infty, 0) \subset \begin{cases} \sigma_{\mathrm{r}}(\Delta_q), & \text{if } 1 < q < \frac{2n}{n+1}, \\ \sigma_{\mathrm{c}}(\Delta_q), & \text{if } \frac{2n}{n+1} \le q \le \frac{2n}{n-1}, \\ \sigma_{\mathrm{p}}(\Delta_q), & \text{if } \frac{2n}{n-1} < q < \infty. \end{cases}$$

Moreover, if $1 < q < \frac{2n}{n+1}$ then for each $\lambda < 0$ the codimension of the closure of the range $\mathcal{R}(\lambda - \Delta_q)$ equals infinity. If $\frac{2n}{n-1} < q < \infty$ then the geometric multiplicity of each eigenvalue $\lambda < 0$ is infinite.

Proof. Without further proof we mention that the multiplier theory shows that $\lambda - \Delta_q$ has a bounded inverse on $L^q(\mathbb{R}^n)$ for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Hence $\sigma(\Delta_q) \subset (-\infty, 0]$. The fact that $0 \in \sigma(\Delta_q)$ is well–known, but also follows from the result $(-\infty, 0) \subset \sigma(\Delta_q)$ to be proved below. The assertion $0 \in \sigma_{\mathrm{c}}(\Delta_q)$ is a consequence of the inclusion $\mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{R}(\Delta_q)$ and Lemma 2.4. Indeed, given $f \in \mathcal{S}_0(\mathbb{R}^n)$, multiplier theory implies that the function u defined by $\widehat{u}(\boldsymbol{\xi}) := \widehat{f}(\boldsymbol{\xi})/|\boldsymbol{\xi}|^2$ is a solution of the equation $-\Delta_q u = f$ in $W^{2,q}(\mathbb{R}^n)$. Moreover, Δ_q is injective because there is no nontrivial harmonic function in $L^q(\mathbb{R}^n)$.

Now let $\lambda < 0$. Consider at first the case $1 < q \le 2$. Let $u \in W^{2,q}(\Omega) = \mathcal{D}(\Delta_q)$ satisfy $\lambda u - \Delta_q u = 0$. Using the Fourier transform we get that $(\lambda + |\boldsymbol{\xi}|^2)\widehat{u} = 0$ where $\widehat{u}$ is a function from $L^{q'}(\mathbb{R}^n)$. Hence $\widehat{u}$ vanishes almost everywhere, consequently $\widehat{u} = 0$ and also $u = 0$. This proves that $\sigma_{\mathrm{p}}(\Delta_q) = \emptyset$ for all $1 < q \le 2$. By duality, we conclude that $\sigma_{\mathrm{r}}(\Delta_q) = \emptyset$ when $2 \le q < \infty$.

Next let $q > \frac{2n}{n-1}$. Then a calculation shows that $(-1 - \Delta_q)\mathcal{J}_n = 0$ and $\mathcal{J}_n \in L^q(\mathbb{R}^n)$, see (2.23) and Lemma 2.3, in particular (2.25). Hence $-1 \in \sigma_p(\Delta_q)$. Moreover, since any partial derivative of $\mathcal{J}_n$ of any order is smooth and decays for $r \to \infty$ as fast as $\mathcal{J}_n(r)$ does, i.e. as $r^{-\frac{n-1}{2}}$, any non–zero linear combination of partial derivatives of $\mathcal{J}_n(r)$ is an eigenfunction of Δ_q with the eigenvalue -1 as well. To prove that the geometric multiplicity of this eigenvalue is infinite, consider a linear combination $v = \sum_{k=1}^m \alpha_k \, \partial_1^k \mathcal{J}_n$ with $(0, \dots, 0) \ne (\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m$. The Fourier transform of v is $\widehat{v}(\boldsymbol{\xi}) = \big(\sum_{k=1}^m \alpha_k \, \mathrm{i}^k \, \xi_1^k\big)\widehat{\mathcal{J}}_n(\boldsymbol{\xi})$, where $\widehat{\mathcal{J}}_n$ is a nonzero multiple of the surface measure $\mathrm{d}\boldsymbol{\alpha}$ of the unit sphere of $\mathbb{R}^n$, see [20, Appendix B.4]. Since the polynomial $p(t) = \sum_{k=1}^m \alpha_k \, \mathrm{i}^k \, t^k$ has at most m real roots and does not vanish identically, also v cannot vanish identically.

By analogy, if $q > \frac{2n}{n-1}$ and $\lambda < 0$, the function $\mathcal{J}_n\big(\sqrt{-\lambda} r\big) \in L^q(\mathbb{R}^n)$ is an eigenfunction corresponding to the eigenvalue λ of the operator Δ_q of infinite geometric multiplicity. By duality, we conclude for $1 < q < \frac{2n}{n+1}$ that $\sigma_{\mathrm{r}}(\Delta_q) = (-\infty, 0)$ and that the codimension of the closure of the range $\mathcal{R}(\lambda - \Delta_q)$ equals infinity.

Now let $\frac{2n}{n+1} \le q \le \frac{2n}{n-1}$. Assume that e.g. $-1 \in \sigma_{\mathrm{r}}(\Delta_q)$. Then, by definition, the range $\mathcal{R}(-1 - \Delta_q)$ is not dense in $L^q(\mathbb{R}^n)$, and Hahn–Banach's Theorem yields a nonzero $f \in L^{q'}(\mathbb{R}^n)$ such that $\big\langle (-1 - \Delta_q)u, f \big\rangle = 0$ for all $u \in W^{2,q}(\mathbb{R}^n)$. In Fourier terms we get that the tempered distribution $\widehat{f}$ satisfies

$$0 \;=\; \big\langle (-1 + |\boldsymbol{\xi}|^2)\widehat{u}, \, \widehat{f} \big\rangle \;=\; \big\langle \widehat{u}, \, (-1 + |\boldsymbol{\xi}|^2)\widehat{f} \big\rangle \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n). \tag{3.1}$$

Hence

$$\operatorname{supp} \widehat{f} \subset \partial B_1. \tag{3.2}$$

Our aim is to prove that this implies that either $f = 0$ or f is a non–vanishing multiple of $\mathcal{J}_n$ in $L^{q'}(\mathbb{R}^n)$; the latter case however is impossible since $q' \le \frac{2n}{n-1}$. Since the function f considered up to now is not necessarily a multiple of $\mathcal{J}_n$ unless it is radial, we will apply the average operator M defined in Section 2 to construct a radial function $f \in L^{q'}(\mathbb{R}^n)$ with the above properties (3.1), (3.2). By (3.1) for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$0 \;=\; \big\langle M\widehat{u},\, (-1+|\boldsymbol{\xi}|^2)\widehat{f}\,\big\rangle \;=\; \big\langle \widehat{u},\, M\big((-1+|\boldsymbol{\xi}|^2)\widehat{f}\,\big)\big\rangle \;=\; \big\langle \widehat{u},\, (-1+|\boldsymbol{\xi}|^2)M\widehat{f}\,\big\rangle,$$

i.e., even $M\widehat{f}$ instead of $\widehat{f}$ satisfies (3.1). Hence $\operatorname{supp} M\widehat{f} \subset \partial B_1$. Since $M\widehat{f} = \widehat{Mf}$ is radial, we conclude from Lemma 2.3 that $Mf = c\,\mathcal{J}_n$. If $c \ne 0$ then $Mf \notin L^{q'}(\mathbb{R}^n)$ for any $q' \le \frac{2n}{n-1}$; hence Mf vanishes. Now, denoting $u_{\mathbf{x}_0} := u(\cdot - \mathbf{x}_0)$ and $f_{\mathbf{x}_0} := f(\cdot - \mathbf{x}_0)$, we repeat the same argument with $u_{\mathbf{x}_0}$ instead of u for arbitrary $\mathbf{x}_0 \in \mathbb{R}^n$ and get that

$$0 \;=\; \big\langle \widehat{u}_{\mathbf{x}_0},\, (-1+|\boldsymbol{\xi}|^2)\widehat{f}\,\big\rangle \;=\; \big\langle \mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\widehat{u},\, (-1+|\boldsymbol{\xi}|^2)\widehat{f}\,\big\rangle \;=\; \big\langle \widehat{u},\, (-1+|\boldsymbol{\xi}|^2)\widehat{f}_{\mathbf{x}_0}\big\rangle.$$

Proceeding as before, i.e. replacing u by Mu, we conclude that $Mf_{\mathbf{x}_0}$ must vanish for arbitrary $\mathbf{x}_0 \in \mathbb{R}^n$. Hence $\int_{B_R(\mathbf{x}_0)} f\,\mathrm{d}\mathbf{x} = 0$ for all $\mathbf{x}_0 \in \mathbb{R}^n$ and all $R > 0$, and Lebesgue's Differentiation Theorem shows that $f = 0$ in $L^{q'}(\mathbb{R}^n)$.

We have seen for $\frac{2n}{n+1} \le q \le 2$ that $-1 \notin \sigma_{\mathrm{r}}(\Delta_q) \cup \sigma_{\mathrm{p}}(\Delta_q)$. To show that $-1 \in \sigma_{\mathrm{c}}(\Delta_q)$ we will find $f \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{R}(-1-\Delta_q)$. Indeed, consider any f such that $\hat{f} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ equals 1 in a neighborhood of $|\boldsymbol{\xi}| = 1$. If there exists $u \in \mathcal{D}(\Delta_q)$ satisfying $(-1-\Delta_q)u = f$, then $(-1+|\boldsymbol{\xi}|^2)\hat{u}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})$ where $\hat{u} \in L^{q'}(\mathbb{R}^n)$ since $q \le 2$. Consequently, for $|\boldsymbol{\xi}|$ close to 1, we see that

$$1 = |\hat{f}(\boldsymbol{\xi})| \le 4\big|1 - |\boldsymbol{\xi}|\big|\,|\hat{u}(\boldsymbol{\xi})|.$$

Hence $|\hat{u}(\boldsymbol{\xi})| \ge 1/\big(4|1-|\boldsymbol{\xi}||\big)$ for these $\boldsymbol{\xi}$; this contradicts the condition $\hat{u} \in L^{q'}(\mathbb{R}^n)$. This argument can be applied with any $\lambda < 0$, not only with $\lambda = -1$. Thus we proved $(-\infty, 0) \subset \sigma_{\mathrm{c}}(\Delta_q)$. If $2 < q \le \frac{2n}{n-1}$, the assumption $\lambda \in \sigma_{\mathrm{p}}(\Delta_q) \cap (-\infty, 0)$ would lead by duality to the assertion $\lambda \in \sigma_{\mathrm{r}}(\Delta_{q'})$ in $L^{q'}(\mathbb{R}^n)$ for $\frac{2n}{n+1} \le q' < 2$ which is impossible. By the Closed Range Theorem we conclude that $(-\infty, 0) \in \sigma_{\mathrm{c}}(\Delta_q)$ also in this case.

Up to now we have proved that $\sigma(\Delta_q) = (-\infty, 0]$ for all $1 < q < \infty$. In particular, since the boundary of the resolvent set of Δ_q coincides with $\sigma(\Delta_q) = (-\infty, 0]$ which as a continuum does not have isolated points, the spectrum is a purely essential one, cf. [25, Problem IV.5.37].

Hence $\mathrm{nul}'(\lambda - \Delta_q) = \infty$ for all $1 < q < \infty$ and all $\lambda < 0$. When $1 < q \le \frac{2n}{n-1}$, we know that $\lambda - \Delta_q$ is injective, so that $\mathrm{nul}(\lambda - \Delta_q) = 0 \ne \mathrm{nul}'(\lambda - \Delta_q)$. Consequently, in this case, the range $\mathcal{R}(\lambda - \Delta_q)$ is not closed. Finally, the Closed Range Theorem implies even for $q > \frac{2n}{n-1}$ that $\mathcal{R}(\lambda - \Delta_q)$ is not closed. □

Theorem 3.2. *Let $n \ge 2$. The Stokes operator $A_q = -P_q\Delta$ on $\mathbf{L}^q_\sigma(\mathbb{R}^n)$ has the following spectral properties:*

$$\sigma(-A_q) \;=\; \sigma_{\mathrm{ess}}(-A_q) \;=\; (-\infty, 0], \qquad 0 \in \sigma_{\mathrm{c}}(-A_q),$$

for each $\lambda \le 0$ the range $\mathcal{R}(\lambda + A_q)$ is not closed, and

$$(-\infty, 0) \;\subset\; \begin{cases} \sigma_{\mathrm{r}}(-A_q), & \text{if } \; 1 < q < \frac{2n}{n+1}, \\ \sigma_{\mathrm{c}}(-A_q), & \text{if } \; \frac{2n}{n+1} \le q \le \frac{2n}{n-1}, \\ \sigma_{\mathrm{p}}(-A_q), & \text{if } \; \frac{2n}{n-1} < q < \infty. \end{cases}$$

Moreover, if $1 < q < \frac{2n}{n+1}$ *then for each* $\lambda < 0$ *the codimension of the closure of the range* $\mathcal{R}(\lambda + A_q)$ *equals infinity. If* $\frac{2n}{n-1} < q < \infty$ *then the geometric multiplicity of each eigenvalue* $\lambda < 0$ *is infinite.*

Proof. We follow the ideas of the proof of Theorem 3.1. As is well known, the Stokes operator $\lambda + A_q$ is boundedly invertible for all $\lambda \notin \mathbb{C} \setminus (-\infty, 0]$. As in the proof of Theorem 3.1, we show by means of Lemma 2.4 and multiplier theory that $0 \in \sigma_{\rm c}(-A_q)$.

If $1 < q \le 2$ and $\lambda < 0$, assume that $\mathbf{u} \in \mathcal{D}(A_q)$ and $\nabla p \in \mathbf{L}^q(\mathbb{R}^n)$ satisfy the equation $\lambda \mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{0}$ in the whole space $\mathbb{R}^3$. Since $\operatorname{div} \mathbf{u} = 0$, we conclude that $\Delta p = 0$ in $\mathcal{S}'(\mathbb{R}^n)$ and consequently $\nabla p = \mathbf{0}$. Using the Fourier transform, we deduce that $(\lambda + |\boldsymbol{\xi}|^2)\widehat{\mathbf{u}} = \mathbf{0}$, where $\widehat{\mathbf{u}}$ is a function from $\mathbf{L}^{q'}(\mathbb{R}^n)$. Hence $\mathbf{u} = \mathbf{0}$, cf. the proof of Theorem 3.1. This proves that $\sigma_{\rm p}(-A_q) = \emptyset$ for all $1 < q \le 2$. By duality arguments, in particular the fact that the dual space of $\mathbf{L}^q_\sigma(\mathbb{R}^n)$ is isomorphic to $\mathbf{L}^{q'}_\sigma(\mathbb{R}^n)$, we also obtain that $\sigma_{\rm r}(-A_q) = \emptyset$ when $2 \le q < \infty$.

Let $q > \frac{2n}{n-1}$. Recall that $\mathcal{J}_n\big(\sqrt{-\lambda} r\big)$ is an eigenvector of Δ_q corresponding to the eigenvalue $\lambda < 0$. Then for $i = 1, \dots, n$ let us define the solenoidal vector field

$$\mathbf{U}^{(i)}(\mathbf{x}) \ := \ P_q\big[\mathcal{J}_n\big(\sqrt{-\lambda} r\big)\, \mathbf{e}_i\big], \tag{3.3}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in $\mathbb{R}^n$. Since $\lambda - \Delta_q$ commutes with P_q, we get $(\lambda + A_q)\mathbf{U}^{(i)} = \mathbf{0}$. Considering partial derivatives of $\mathbf{U}^{(i)}$ of an arbitrary order, we see that the multiplicity of the eigenvalue λ is infinite. By duality we conclude that $(-\infty, 0) \subset \sigma_{\rm r}(-A_q)$ when $1 < q < \frac{2n}{n+1}$ and that the codimension of $\overline{\mathcal{R}(\lambda + A_q)}$ in $\mathbf{L}^q_\sigma(\mathbb{R}^n)$ is infinite for every $\lambda < 0$.

Now let $\frac{2n}{n+1} \le q \le 2$ and assume that $-1 \in \sigma_{\rm r}(-A_q)$, i.e., the closure of the range of $-1 + A_q$ is a proper subspace of $\mathbf{L}^q_\sigma(\mathbb{R}^n)$. Then Hahn–Banach's theorem yields a nonzero vector field $\mathbf{f} \in \mathbf{L}^{q'}_\sigma(\mathbb{R}^n)$ such that

$$0 \ = \ \big\langle (-1 + |\boldsymbol{\xi}|^2)\widehat{\mathbf{u}}, \widehat{\mathbf{f}} \big\rangle \ = \ \big\langle \widehat{\mathbf{u}}, (-1 + |\boldsymbol{\xi}|^2)\widehat{\mathbf{f}} \big\rangle \quad \text{for all } \mathbf{u} \in \mathbf{L}^q_\sigma(\mathbb{R}^n),$$

and, since $\mathbf{f}$ is solenoidal, even that $0 = \langle \widehat{\mathbf{u}}, (-1 + |\boldsymbol{\xi}|^2)\widehat{\mathbf{f}} \rangle$ for all $\mathbf{u} \in \mathbf{L}^q(\mathbb{R}^n)$. Replacing $\mathbf{u} \in \mathbf{L}^q(\mathbb{R}^n)$ by $M\mathbf{u}$, cf. the proof of Theorem 3.1, we also get that

$$0 \ = \ \big\langle \widehat{M\mathbf{u}}, (-1 + |\boldsymbol{\xi}|^2)\widehat{\mathbf{f}} \big\rangle \ = \ \big\langle \widehat{\mathbf{u}}, (-1 + |\boldsymbol{\xi}|^2)\widehat{M\mathbf{f}} \big\rangle.$$

Hence $\operatorname{supp} \widehat{M\mathbf{f}} \subset \partial B_1$ and, being radial, $M\mathbf{f} = \mathbf{c}\, \mathcal{J}_n$, where $\mathbf{c} \in \mathbb{R}^n$, see Lemma 2.3. Since $M\mathbf{f} \in \mathbf{L}^{q'}(\mathbb{R}^n)$ and $q' \le \frac{2n}{n-1}$, we conclude that $\mathbf{c} = \mathbf{0}$ and $M\mathbf{f} = \mathbf{0}$. Proceeding as in the proof of Theorem 3.1 we also derive that $M\mathbf{f}(\,.\, - \mathbf{x}_0)$ vanishes for every $\mathbf{x}_0 \in \mathbb{R}^n$ so that even $\mathbf{f} = \mathbf{0}$ a.e. in $\mathbb{R}^n$. This contradicts the assumption that $-1 \in \sigma_{\rm r}(-A_q)$. The same arguments can be used for any $\lambda < 0$.

To prove in this case that $(-\infty, 0) \subset \sigma_{\rm c}(-A_q)$ for $\frac{2n}{n+1} \le q \le 2$ we construct $\mathbf{f} \in L^q_\sigma(\mathbb{R}^n) \setminus \mathcal{R}(-1 + A_q)$. Consider any f with $\hat{f} \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$ such that $\hat{f}(\xi) = 1$ in a neighborhood of $|\xi| = 1$, and let $\mathbf{f}$ be defined by $\mathbf{f}(\xi) = (\xi_2 \hat{f}(\boldsymbol{\xi}), -\xi_1 \hat{f}(\boldsymbol{\xi}), 0, \dots, 0)^T$ so that $\mathbf{f} \in L^q_\sigma(\mathbb{R}^n)$. If there exists $\mathbf{u} \in \mathcal{D}(A_q)$ with $(-1 + A_q)\mathbf{u} = \mathbf{f}$, then $(-1 + |\boldsymbol{\xi}|^2)\hat{\mathbf{u}}(\boldsymbol{\xi}) = \hat{\mathbf{f}}(\boldsymbol{\xi})$ where $\hat{\mathbf{u}} \in L^{q'}(\mathbb{R}^n)$ since $q \le 2$. Consequently, for $|\boldsymbol{\xi}| \ge 1$ close to 1,

$$1 \le |\hat{\mathbf{f}}(\boldsymbol{\xi})| \le 4\big|1 - |\boldsymbol{\xi}|\big|\, |\hat{\mathbf{u}}(\boldsymbol{\xi})|.$$

As in the proof of Theorem 3.1 this inequality leads to a contradiction to the condition $\hat{\mathbf{u}} \in L^{q'}(\mathbb{R}^n)^n$. This argument also proves that $(-\infty, 0) \subset \sigma_c(-A_q)$ when $\frac{2n}{n+1} \leq q \leq 2$. Then duality arguments imply that $(-\infty, 0) \subset \sigma_c(-A_q)$ for $2 < q \leq \frac{2n}{n-1}$.

So far, we proved for all $1 < q < \infty$ that $\sigma(-A_q) = (-\infty, 0]$. Following the arguments at the end of the proof of Theorem 3.1 we conclude that the spectrum is a purely essential one and that $\mathrm{nul}'(\lambda + A_q) = \infty$ for all $\lambda < 0$. Moreover, for $1 < q \leq \frac{2n}{n-1}$, the operator $\lambda + A_q$ is injective and consequently $\mathrm{nul}(\lambda + A_q) = 0$; hence the range of $\lambda + A_q$ is not closed. Finally, the Closed Range Theorem yields the same result for $q > \frac{2n}{n-1}$. ☐

Now we are ready to discuss our first main result on the Stokes operator "with rotation" $\mathcal{L}_q^1$ on $L^q_\sigma(\mathbb{R}^3)$. Besides the set $\mathfrak{S}^1$, see (1.5), we do need the "relatively open" set

$$(\mathfrak{S}^1)^\circ \;:=\; \bigcup_{k=-\infty}^{\infty} \big\{(-\infty, 0) + \mathrm{i}k\big\} \;=\; \mathfrak{S}^1 \setminus \mathrm{i}\mathbb{Z}.$$

Theorem 3.3. *The operator $(-\mathcal{L}_q^1)$ on $L^q_\sigma(\mathbb{R}^3)$, $1 < q < \infty$, has the following spectral properties:*
The spectrum of the operator $-\mathcal{L}_q^1$ is $\sigma(-\mathcal{L}_q^1) = \sigma_{\mathrm{ess}}(-\mathcal{L}_q^1) = \mathfrak{S}^1$ and $\mathrm{i}\mathbb{Z} \subset \sigma_c(-\mathcal{L}_q^1)$, For every $\lambda \in \mathfrak{S}^1$ the range $\mathcal{R}(\lambda + \mathcal{L}_q^1)$ is not closed. Moreover,

(i) if $1 < q < \frac{3}{2}$, then $(\mathfrak{S}^1)^\circ = \sigma_r(-\mathcal{L}_q^1)$ and for each $\lambda \in (\mathfrak{S}^1)^\circ$ the codimension of $\overline{\mathcal{R}(\lambda + \mathcal{L}_q^1)}$ equals infinity,

(ii) if $\frac{3}{2} \leq q \leq 3$, then $\mathfrak{S}^1 = \sigma_c(-\mathcal{L}_q^1)$,

(iii) if $3 < q < \infty$, then $(\mathfrak{S}^1)^\circ = \sigma_p(-\mathcal{L}_q^1)$ and the geometric multiplicity of each eigenvalue $\lambda \in (\mathfrak{S}^1)^\circ$ is infinite.

Proof. We follow the ideas of the proof of Theorems 3.1 and 3.2. First we consider $\lambda = \mathrm{i}k$, $k \in \mathbb{Z}$, and assume that $\mathbf{u} \in \mathcal{D}(\mathcal{L}_q^1)$ is a solution of the equation $\big(\lambda + \mathcal{L}_q^1\big)\mathbf{u} = \mathbf{0}$. Then even $\big(\lambda - \Delta - \partial_\theta + \mathbf{e}_3 \wedge\big)\mathbf{u} = \mathbf{0}$. Using the Riesz transforms R_1', R_2' and Lemma 2.5, see (2.28), we may assume that $\lambda = 0$. Now [10, Theorem 1.1 (3)] yields that $\mathbf{u} \in \mathcal{D}(\mathcal{L}_q^1)$ must vanish. This shows that $\lambda = \mathrm{i}k$, $k \in \mathbb{Z}$, cannot be an eigenvalue. By duality arguments, $\lambda = \mathrm{i}k \notin \sigma_r(\mathcal{L}_q^1)$, $k \in \mathbb{Z}$, as well. Finally, since $(\mathfrak{S}^1)^\circ$ is a subset of the spectrum, see below, and the spectrum is closed, we still have to show that the range of $\mathrm{i}k + \mathcal{L}_q^1$ is dense in $\mathbf{L}^q_\sigma(\mathbb{R}^3)$ in order to conclude that $\lambda = \mathrm{i}k \in \sigma_c(\mathcal{L}_q^1)$. For simplicity let again $\lambda = 0$. Then the solution formula (2.13), see also [10, (2.4), (2.5)], multiplier theory and Lemma 2.4 imply that we find a dense subset of $\mathbf{L}^q_\sigma(\mathbb{R}^3)$ in the range of $\mathcal{L}_q^1$.

If $1 < q \leq 2$ and $\lambda \in (\mathfrak{S}^1)^\circ$, assume that $(\mathbf{u}, \nabla p) \in \mathcal{D}(\mathcal{L}_q^1) \times L^q(\mathbb{R}^3)$ satisfy

$$\lambda\mathbf{u} - \Delta\mathbf{u} - \partial_\theta\mathbf{u} + \mathbf{e}_3 \wedge \mathbf{u} + \nabla p \;=\; \mathbf{0} \quad \text{in } \mathbb{R}^3.$$

Hence in Fourier space, omitting the gradient of the pressure which will vanish, we see that in cylindrical coordinates (with $\boldsymbol{\xi} \equiv (|\boldsymbol{\xi}'|, \xi_3, \varphi)$ where $\boldsymbol{\xi}' = (\xi_1, \xi_2)$)

$$(\lambda + |\boldsymbol{\xi}|^2 - \partial_\varphi)\widehat{\mathbf{u}} + \mathbf{e}_3 \wedge \widehat{\mathbf{u}} \;=\; \mathbf{0}, \tag{3.4}$$

where $\widehat{\mathbf{u}} \in \mathbf{L}^{q'}(\mathbb{R}^3) \subset \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$. We multiply (3.4) by $\mathrm{e}^{-\mathrm{i}k\varphi}$, $k \in \mathbb{Z}$, and integrate with respect to $\varphi \in (0, 2\pi)$ to get for a.a. $|\xi'| = |(\xi_1, \xi_2)| > 0$ and $\xi_3 \in \mathbb{R}$ the identity

$$(\lambda - \mathrm{i}k + |\boldsymbol{\xi}|^2)\widehat{\mathbf{u}}^k + \mathbf{e}_3 \wedge \widehat{\mathbf{u}}^k \;=\; \mathbf{0}; \tag{3.5}$$

here
$$\widehat{\mathbf{u}}^k \;=\; \widehat{\mathbf{u}}^k(|\xi'|,\xi_3) \;=\; \int_0^{2\pi} \widehat{\mathbf{u}}(|\xi'|,\xi_3,\varphi)\, \mathrm{e}^{-\mathrm{i}k\varphi}\, \mathrm{d}\varphi$$

denotes the k–th Fourier coefficient (with respect to φ) of $\widehat{\mathbf{u}}(|\xi'|,\xi_3,\cdot) \in L^2(0,2\pi)$. Looking at the third component of the vector identity (3.5), where the term $\mathbf{e}_3 \wedge \widehat{\mathbf{u}}$ yields no contribution, we conclude that $\widehat{u}_3^k = 0$, $k \in \mathbb{Z}$, for a.a. $(|\xi'|,\xi_3)$. Thus $\widehat{u}_3(|\xi'|,\xi_3,\cdot) = 0$ for a.a. $(|\xi'|,\xi_3)$, and the L^2_{loc}–function $\widehat{u}_3$ vanishes. Consequently also $u_3 = 0$. The first two components of $\widehat{\mathbf{u}}^k$ are coupled in (3.5). An easy calculation yields the identity

$$\big[(\lambda - \mathrm{i}k + |\boldsymbol{\xi}|^2)^2 + 1\big]\, \widehat{u}_j^k = 0, \quad j = 1,2.$$

Now similar arguments as applied to u_3 above may be used to show that $u_1 = u_2 = 0$ as well. This proves that $\lambda \in (\mathfrak{S}^1)^\circ$ is not an eigenvalue, i.e. $\sigma_{\mathrm{p}}(-\mathcal{L}_q^1) = \emptyset$ when $1 < q \le 2$. By duality we get that $\sigma_{\mathrm{r}}(-\mathcal{L}_q^1) = \emptyset$ when $2 \le q < \infty$.

Next let $q > 3$ and $\lambda \in (\mathfrak{S}^1)^\circ$. For simplicity, we assume that $\lambda = -1 - \mathrm{i}k$, $k \in \mathbb{Z}$; the general case in which $Re\,\lambda < 0$ can be dealt similarly. Assume that $\mathbf{u} \in \mathcal{D}(\mathcal{L}_q^1)$ is an eigenfunction of $-\mathcal{L}_q^1$ with eigenvalue λ. Then its Fourier transform, a distribution $\widehat{\mathbf{u}} \in \mathcal{S}(\mathbb{R}^3)^3$, satisfies the equation

$$(\lambda + |\boldsymbol{\xi}|^2 - \partial_\varphi)\widehat{\mathbf{u}} + \mathbf{e}_3 \wedge \widehat{\mathbf{u}} \;=\; \mathbf{0}$$

so that

$$\begin{aligned}(\lambda + |\boldsymbol{\xi}|^2 - \partial_\varphi)\widehat{u}_3 \;&=\; 0\\ \big((\lambda + |\boldsymbol{\xi}|^2 - \partial_\varphi)^2 + 1\big)\widehat{u}_j \;&=\; 0, \quad j = 1,2.\end{aligned}$$

The following calculation is formal, since $\widehat{\mathbf{u}}$ cannot be assumed to be a function, but it will yield an idea how a possible eigenfunction $\mathbf{u}$ may look like. Interpreting the equation for $\widehat{u}_3$ as a linear homogeneous ordinary differential equation of the first order with respect to φ, we get for a.a. $(|\xi'|,\xi_3)$ that there exists a function $a_3 = a_3(|\xi'|,\xi_3)$ such that

$$\widehat{u}_3(|\xi'|,\xi_3,\varphi) \;=\; a_3(|\xi'|,\xi_3)\, \mathrm{e}^{(-1-\mathrm{i}k+|\boldsymbol{\xi}|^2)\varphi} \;=\; a_3(|\xi'|,\xi_3)\, \mathrm{e}^{-\mathrm{i}k\varphi}\, \mathrm{e}^{(|\boldsymbol{\xi}|^2-1)\varphi}.$$

Since $\widehat{u}_3(|\xi'|,\xi_3,\varphi)$ must be 2π–periodic in φ, we conclude that $\widehat{u}_3(\xi)$ vanishes unless $\boldsymbol{\xi} \in \partial B_1$. Therefore, using the characteristic function $\chi_{\partial B_1}$ of ∂B_1, let

$$\widehat{u}_3(|\xi'|,\xi_3,\varphi) \;=\; a_3(|\xi'|,\xi_3)\, \mathrm{e}^{-\mathrm{i}k\varphi}\, \chi_{\partial B_1}.$$

A formal calculation for the differential equation for u_1 implies the existence of functions $a_1 = a_1(|\xi'|,\xi_3)$, $a_2 = a_2(|\xi'|,\xi_3)$ such that

$$\widehat{u}_1(|\xi'|,\xi_3,\varphi) \;=\; a_1\, \mathrm{e}^{-\mathrm{i}(k-1)\varphi}\, \mathrm{e}^{(|\boldsymbol{\xi}|^2-1)\varphi} + a_2\, \mathrm{e}^{-\mathrm{i}(k+1)\varphi}\, \mathrm{e}^{(|\boldsymbol{\xi}|^2-1)\varphi}.$$

In order to get a 2π–periodic function in φ, we assume that

$$\widehat{u}_1(|\xi'|,\xi_3,\varphi) \;=\; a_1\, \mathrm{e}^{-\mathrm{i}(k-1)\varphi}\, \chi_{\partial B_1} + a_2\, \mathrm{e}^{-\mathrm{i}(k+1)\varphi}\, \chi_{\partial B_1}. \tag{3.6}$$

The second component u_2 has a similar form, but since $(\lambda + |\boldsymbol{\xi}|^2 - \partial_\varphi)\widehat{u}_1 - \widehat{u}_2 = 0$ and $(|\boldsymbol{\xi}|^2 - 1)\chi_{\partial B_1} = 0$ in $\mathcal{S}'(\mathbb{R}^3)$, we are led to the identity

$$\widehat{u}_2(|\xi'|,\xi_3,\varphi) \;=\; -a_1\, \mathrm{i}\, \mathrm{e}^{-\mathrm{i}(k-1)\varphi}\, \chi_{\partial B_1} + a_2\, \mathrm{i}\, \mathrm{e}^{-\mathrm{i}(k+1)\varphi}\, \chi_{\partial B_1}. \tag{3.7}$$

To satisfy the condition $\operatorname{div}\mathbf{u} = \mathbf{0}$, we have to require that

$$\begin{aligned} 0 \;=\; \boldsymbol{\xi}\cdot\widehat{\mathbf{u}} \;&=\; \big(a_1(\xi_1 - \mathrm{i}\xi_2)\,\mathrm{e}^{-\mathrm{i}(k-1)\varphi} + a_2(\xi_1 + \mathrm{i}\xi_2)\,\mathrm{e}^{-\mathrm{i}(k+1)\varphi} + \xi_3 a_3\,\mathrm{e}^{-\mathrm{i}k\varphi}\big)\,\chi_{\partial B_1}\\ &=\; \big((a_1 + a_2)\,|\boldsymbol{\xi}'| + \xi_3 a_3\big)\,\mathrm{e}^{-\mathrm{i}k\varphi}\,\chi_{\partial B_1}, \end{aligned}$$

since $\xi_1 \pm \mathrm{i}\xi_2 = |\xi'|\,\mathrm{e}^{\pm\mathrm{i}\varphi}$.

In view of (3.6), (3.7) let us choose $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$, $a_3 = 0$ and define the tempered distributions

$$\widehat{u}_1(\boldsymbol{\xi}) = \mathrm{i}\,\sin\varphi\,\mathrm{e}^{-\mathrm{i}k\varphi}\,\chi_{\partial B_1}, \quad \widehat{u}_2(\boldsymbol{\xi}) = -\mathrm{i}\,\cos\varphi\,\mathrm{e}^{-\mathrm{i}k\varphi}\,\chi_{\partial B_1}, \quad \widehat{u}_3(\boldsymbol{\xi}) = 0,$$

or, using the Riesz transforms R_1', R_2', see Lemma 2.5, and up to a multiplicative constant,

$$u_1 = -R_2'\big(\mathrm{i}\,(R_1' - \mathrm{i}R_2')\big)^k\,\mathcal{J}_3, \quad u_2 = R_1'\big(\mathrm{i}\,(R_1' - \mathrm{i}R_2')\big)^k\,\mathcal{J}_3, \quad u_3 = 0. \tag{3.8}$$

Since $\mathcal{J}_3 \in L^q(\mathbb{R}^3)$ for $q > 3$ and the Riesz transforms R_1', R_2' are bounded on $L^q(\mathbb{R}^3)$ (and on $L^q(\mathbb{R}^2)$), we see that $\mathbf{u} \in \mathbf{L}^q_\sigma(\mathbb{R}^3)$. Moreover, it is easy to check that $(-1-\mathrm{i}k+\mathcal{L}^1_q)\mathbf{u} = \mathbf{0}$. Hence $\lambda = -1 - \mathrm{i}k$ is an eigenvalue of $-\mathcal{L}^1_q$; its geometric multiplicity is infinite since by $\partial_3^j\mathbf{u}$, $j \in \mathbb{N}$, we are able to find infinitely many linearly independent eigenfunctions of $-\mathcal{L}^1_q$. This proves that $(\mathfrak{S}^1)^\circ = \sigma_{\mathrm{p}}(-\mathcal{L}^1_q)$ for $q > 3$. A duality argument implies that $(\mathfrak{S}^1)^\circ = \sigma_{\mathrm{r}}(-\mathcal{L}^1_q)$ for $1 < q < \frac{3}{2}$; moreover, for each $\lambda \in (\mathfrak{S}^1)^\circ$ the codimension of the closure of the range of $\lambda + \mathcal{L}^1_q$ is infinite.

Now let $\frac{3}{2} \le q \le 2$ and assume that $\lambda = -1 + \mathrm{i}k \in (\mathfrak{S}^1)^\circ$ lies in the residual spectrum of $-\mathcal{L}^1_q$. By (2.28) it suffices to consider $\lambda = -1$. Then Hahn–Banach's Theorem yields a non–vanishing $\mathbf{f} \in \mathbf{L}^{q'}_\sigma(\mathbb{R}^3)$ such that

$$\langle\,(-1 + |\boldsymbol{\xi}|^2 - \partial_\varphi + \mathbf{e}_3\wedge)\widehat{\mathbf{u}},\,\widehat{\mathbf{f}}\,\rangle \;=\; 0 \quad \text{for all } \mathbf{u} \in \mathcal{D}(\mathcal{L}^1_q).$$

Hence $\mathbf{f} \in \mathcal{D}((\mathcal{L}^1_q)^*) = \mathcal{D}(\mathcal{L}^{-1}_{q'})$ and

$$0 \;=\; \langle\,\widehat{\mathbf{u}},\,(-1 + |\boldsymbol{\xi}|^2 + \partial_\varphi - \mathbf{e}_3\wedge)\widehat{\mathbf{f}}\,\rangle \tag{3.9}$$

where, since $\operatorname{div}\mathcal{L}^{-1}_{q'}\mathbf{f} = 0$, $\mathbf{u}$ may run through all of $\mathbf{L}^q(\mathbb{R}^3)$. From (3.9) we conclude that

$$\operatorname{supp}\widehat{\mathbf{f}} \subset \partial B_1 \tag{3.10}$$

as follows: Actually, take any $\widehat{\mathbf{g}} \in \mathcal{S}(\mathbb{R}^3)^3$ with $\operatorname{supp}\widehat{\mathbf{g}} \cap \partial B_1 = \emptyset$. Then

$$\widehat{\mathbf{u}}(\xi) \;=\; \frac{1}{1 - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2-1)}}\int_0^{2\pi} \mathrm{e}^{-t(|\boldsymbol{\xi}|^2-1)}\,O(t)^T\,\widehat{\mathbf{g}}(O(t)\boldsymbol{\xi})\,\mathrm{d}t,$$

see (2.13) and [7, (2.7)], [10, p. 300] for related formulas, yields a solution of the equation $(-1 + |\boldsymbol{\xi}|^2 - \partial_\varphi + \mathrm{e}_3\wedge)\widehat{\mathbf{u}} = \widehat{\mathbf{g}}$; moreover, since $\operatorname{supp}\widehat{\mathbf{g}} \cap \partial B_1 = \emptyset$, also $\widehat{\mathbf{u}} \in \mathcal{S}(\mathbb{R}^3)^3$. Then (3.9) implies that $\langle\,\widehat{\mathbf{g}},\,\widehat{\mathbf{f}}\,\rangle = 0$ and proves (3.10).

To prove that $\mathbf{f} = \mathbf{0}$, let us generalize the last step, take any $\widehat{\mathbf{v}} \in \mathcal{S}(\mathbb{R}^3)^3$ with $\operatorname{supp}\widehat{\mathbf{v}} \cap \partial B_1 = \emptyset$, choose an arbitrary $\mathbf{x}_0 \in \mathbb{R}^3$, and let $\widehat{\mathbf{u}}$ solve the ordinary inhomogeneous linear first order equation

$$\big((-1 + |\boldsymbol{\xi}|^2) - \partial_\varphi + \mathbf{e}_3\wedge\big)\big(\mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\widehat{\mathbf{u}}\big) \;=\; \mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\big((-1 + |\boldsymbol{\xi}|^2) - \partial_\varphi + \mathbf{e}_3\wedge\big)\widehat{\mathbf{v}}.$$

As above we conclude that

$$\widehat{\mathbf{u}}(\boldsymbol{\xi}) \;=\; \frac{\mathrm{e}^{\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}}{1-\mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2-1)}}\int_0^{2\pi}\mathrm{e}^{-t(|\boldsymbol{\xi}|^2-1)}\;O(t)^T\;\mathbf{e}^{-\mathrm{i}\mathbf{x}_0\cdot O(t)\boldsymbol{\xi}}\;\cdot$$
$$\cdot\;\big((-1+|\boldsymbol{\xi}|^2)-\partial_\varphi+\mathrm{e}_3\wedge\big)\widehat{\mathbf{v}}(O(t)\boldsymbol{\xi})\;\mathrm{d}t$$

solves this equation and satisfies $\widehat{\mathbf{u}}\in\mathcal{S}(\mathbb{R}^3)^3$, $\operatorname{supp}\widehat{\mathbf{u}}\cap\partial B_1=\emptyset$, as $\widehat{\mathbf{v}}$ did. Hence considering (3.9) with $\widehat{\mathbf{u}}$ replaced by $\mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\widehat{\mathbf{u}}$, we are led to the identity

$$\begin{aligned}0 \;&=\; \big\langle \mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\widehat{\mathbf{u}},\,(-1+|\boldsymbol{\xi}|^2+\partial_\varphi-\mathbf{e}_3\wedge)\widehat{\mathbf{f}}\big\rangle\\ &=\; \big\langle \widehat{\mathbf{v}},\,(-1+|\boldsymbol{\xi}|^2+\partial_\varphi-\mathbf{e}_3\wedge)\big(\mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\widehat{\mathbf{f}}\big)\big\rangle.\end{aligned}$$

Finally, in the last equation, we replace $\widehat{\mathbf{v}}$ by $M\widehat{\mathbf{v}}$, note that $M\partial_\varphi=0$, and get that for all $\widehat{\mathbf{v}}\in\mathcal{S}(\mathbb{R}^3)$ with $\operatorname{supp}\widehat{\mathbf{v}}\cap\partial B_1=\emptyset$

$$\begin{aligned}0 \;&=\; \big\langle \widehat{\mathbf{v}},\,M(-1+|\boldsymbol{\xi}|^2+\partial_\varphi-\mathbf{e}_3\wedge)\big(\mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\widehat{\mathbf{f}}\big)\big\rangle\\ &=\; \big\langle \widehat{\mathbf{v}},\,(-1+|\boldsymbol{\xi}|^2-\mathbf{e}_3\wedge)M\big(\mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\widehat{\mathbf{f}}\big)\big\rangle.\end{aligned}$$

We conclude that for every $\mathbf{x}_0\in\mathbb{R}^3$ the radial distribution $M\big(\mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\widehat{\mathbf{f}}\big)$ has the property

$$\operatorname{supp}\,(-1+|\boldsymbol{\xi}|^2-\mathbf{e}_3\wedge)M\big(\mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\widehat{f}\big)\subset\partial B_1.$$

This fact immediately implies that also $\operatorname{supp}M\big(\mathrm{e}^{-\mathrm{i}\mathbf{x}_0\cdot\boldsymbol{\xi}}\,\widehat{\mathbf{f}}\big)\subset\partial B_1$. Hence by Lemma 2.3 $M\mathbf{f}(\cdot-\mathbf{x}_0)$ is a constant multiple of the function $\mathcal{J}_3\notin\mathbf{L}^{q'}(\mathbb{R}^3)$ when $2\le q'\le 3$. Hence the constant must vanish, $M\mathbf{f}(\,.-\mathbf{x}_0)=\mathbf{0}$ for all $\mathbf{x}_0\in\mathbb{R}^3$, and Lebesgue's Differentiation Theorem yields the contradiction $\mathbf{f}=0$. This proves that $\lambda=-1\notin\sigma_{\mathrm{r}}(-\mathcal{L}^1_q)$.

Since for $\frac{3}{2}\le q\le 2$ there are also no eigenvalues, we still have to prove that $(-\infty,0)\subset\sigma_{\mathrm{c}}(-\mathcal{L}^1_q)$. To this aim, let $f\in\mathcal{S}(\mathbb{R}^3)$ be defined by its Fourier transform $0\le\hat{f}\in C_0^\infty(\mathbb{R}^3)$ with support in the first octant $\{\boldsymbol{\xi};\ \xi_1>0,\,\xi_2>0,\,\xi_3>0\}$ such that $\hat{f}(\boldsymbol{\xi})=1$ in a neighborhood of the point $\boldsymbol{\xi}=\frac{1}{\sqrt{3}}(1,1,1)^T$. Then let $\mathbf{f}\in\mathbf{L}^q_\sigma(\mathbb{R}^3)$ be defined by $\hat{\mathbf{f}}(\boldsymbol{\xi})=(-\xi_3 f(\boldsymbol{\xi}),\,0,\,\xi_1\hat{f}(\boldsymbol{\xi}))^T$. Assuming that $\mathbf{f}\in\mathcal{R}(-1+\mathcal{L}^1_q)$ there exists $\mathbf{u}\in\mathcal{D}(\mathcal{L}^1_q)$ satisfying $(-1+\mathcal{L}^1_q)\mathbf{u}=\mathbf{f}$. Since $\mathbf{f}$ is solenoidal, we may ignore the Helmholtz projection in the definition of $\mathcal{L}^1_q$ and find for the third component u_3 of $\mathbf{u}$ the equation

$$(-1+|\boldsymbol{\xi}|^2-\partial_\varphi)\hat{u}_3=(\hat{\mathbf{f}})_3=\xi_1\hat{f}\,.$$

Now we apply the average operator M, note that $M\partial_\varphi=0$ and get for $\boldsymbol{\xi}$ close to the point $\frac{1}{\sqrt{3}}(1,1,1)^T$ the estimate $|(-1+|\boldsymbol{\xi}|^2)M\hat{u}_3(\boldsymbol{\xi})|=|M(\xi_1\hat{f}(\boldsymbol{\xi}))|\ge\alpha$ with $\alpha>0$. As in the proofs of Theorems 3.1, 3.2 this estimate will contradict the assumption that $\hat{u}_3\in L^{q'}(\mathbb{R}^3)$. Similarly, we may prove that $(-\infty,0)\subset\sigma_{\mathrm{c}}(-\mathcal{L}^1_q)$. Finally, Lemma 2.5 shows that $\mathrm{i}k+(-\infty,0)\subset\sigma_{\mathrm{c}}(-\mathcal{L}^1_q)$ for all $k\in\mathbb{Z}$, i.e., $(\mathfrak{S}^1)^\circ\cup\mathrm{i}\mathbb{Z}=\sigma_{\mathrm{c}}(-\mathcal{L}^1_q)$.

By duality, we also get that $\mathfrak{S}^1=\sigma_{\mathrm{c}}(-\mathcal{L}^1_q)$ when $2<q\le 3$.

We complete the proof by showing that in each case $\mathcal{R}(\lambda+\mathcal{L}^1_q)$ is not closed; here we follow the proofs of Theorems 3.1, 3.2. □

Finally we discuss the behavior of the resolvent $(\lambda+\mathcal{L}^1_q)^{-1}$ for $\lambda\notin\sigma(-\mathcal{L}^1_2)$, when $\alpha=Re\,\lambda\to-\infty$ and $\beta=Im\,\lambda$ is fixed. We do not consider the same result for other $q\in(1,\infty)$, $q\neq 2$, since our proof is strongly based on L^2-Fourier theory.

Theorem 3.4. *For $\lambda = \alpha + \mathrm{i}\beta$, $\alpha < 0$, $\beta \notin \mathbb{Z}$, the operator $\lambda + \mathcal{L}_2^1$ has the following properties: There exists a constant $C > 0$ independent of λ such that*

$$\|(\lambda + \mathcal{L}_2^1)^{-1}\|_2 \;\le\; \frac{C}{\operatorname{dist}(\beta, \mathbb{Z})}.$$

Moreover, for fixed $\beta \notin \mathbb{Z}$

$$(\lambda + \mathcal{L}_2^1)^{-1} \to 0 \quad \textit{strongly as} \quad \alpha \to -\infty$$

in the strong operator topology, i.e. $\|(\lambda + \mathcal{L}_2^1)^{-1}\mathbf{f}\|_2 \to 0$ for every $\mathbf{f} \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$. However, $(\lambda + \mathcal{L}_2^1)^{-1}$ does not converge to zero in the operator norm as $\alpha \to -\infty$.

Proof. For simplicity we fix $\beta \in [-\frac{1}{2}, \frac{1}{2}]$ and let $\mathbf{f} \in \mathbf{L}^2_\sigma(\Omega)$ so that $\mathbf{f}' = \mathbf{f} - \nabla p = \mathbf{f}$ in (2.5). Then (2.11), (2.13), Plancherel's Theorem, Fubini's Theorem, the inequality of Cauchy-Schwarz and the orthogonality of the matrix $O(t)$ imply for $\mathbf{u} = (\lambda + \mathcal{L}_2^1)^{-1}\mathbf{f}$ that

$$\begin{aligned}
\|\mathbf{u}\|_2^2 &= \|\widehat{\mathbf{u}}\|_2^2 = \int_{\mathbb{R}^3} \frac{1}{|D(\boldsymbol{\xi})|^2} \left| \int_0^{2\pi} \mathrm{e}^{-(|\boldsymbol{\xi}|^2+\lambda)t}\, O^T(t)\, \widehat{\mathbf{f}}(O(t)\boldsymbol{\xi})\, \mathrm{d}t \right|^2 \mathrm{d}\xi \\
&\le \int_{\mathbb{R}^3} \frac{1}{|D(\boldsymbol{\xi})|^2} \left(\int_0^{2\pi} \mathrm{e}^{-(|\boldsymbol{\xi}|^2+\alpha)t}\, \mathrm{d}t \right) \int_0^{2\pi} \mathrm{e}^{-(|\boldsymbol{\xi}|^2+\alpha)t}\, |\widehat{\mathbf{f}}(O(t)\boldsymbol{\xi})|^2\, \mathrm{d}t\, \mathrm{d}\boldsymbol{\xi} \\
&\le \int_{\mathbb{R}^3} \frac{1}{|D(\boldsymbol{\xi})|^2} \frac{1 - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2+\alpha)}}{|\boldsymbol{\xi}|^2 + \alpha} \int_0^{2\pi} \mathrm{e}^{-(|\boldsymbol{\xi}|^2+\alpha)t}\, |\widehat{\mathbf{f}}(O(t)\boldsymbol{\xi})|^2\, \mathrm{d}t\, \mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^3} \frac{1}{|D(\boldsymbol{\xi})|^2} \frac{1 - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2+\alpha)}}{|\boldsymbol{\xi}|^2 + \alpha} \left(\int_0^{2\pi} \mathrm{e}^{-(|\boldsymbol{\xi}|^2+\alpha)t}\, \mathrm{d}t \right) |\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2\, \mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^3} \left(\frac{1 - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2+\alpha)}}{(|\boldsymbol{\xi}|^2 + \alpha)\, |D(\boldsymbol{\xi})|} \right)^2 |\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2\, \mathrm{d}\boldsymbol{\xi}\,; \qquad (3.11)
\end{aligned}$$

here $D(\boldsymbol{\xi}) = 1 - \exp(-2\pi(\lambda + |\boldsymbol{\xi}|^2)$, cf. (2.11). To prove the first assertion it suffices to find a uniform estimate of the multiplier function

$$m(\boldsymbol{\xi}) \;=\; \frac{1 - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2+\alpha)}}{(|\boldsymbol{\xi}|^2 + \alpha)\, D(\boldsymbol{\xi})}.$$

If $\big||\boldsymbol{\xi}|^2 + \alpha\big| \le 1$, then we use the Taylor expansion of the exponential function to get that

$$\left| \frac{1 - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2+\alpha)}}{|\boldsymbol{\xi}|^2 + \alpha} \right| \le C \quad \text{and} \quad |D(\boldsymbol{\xi})| = \left|\mathrm{e}^{2\pi \mathrm{i}\beta} - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2+\alpha)}\right| \ge C\,|\beta| \qquad (3.12)$$

with a constant $C > 0$ not depending on λ, $\boldsymbol{\xi}$; hence $|m(\boldsymbol{\xi})| \le C/|\beta|$ for these $\boldsymbol{\xi}$. Next consider the case when $\big||\boldsymbol{\xi}|^2 + \alpha\big| > 1$. Now we find a constant $C > 0$ independent of λ, ξ such that

$$|m(\boldsymbol{\xi})| \;\le\; \frac{\left|1 - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2+\alpha)}\right|}{\left|\mathrm{e}^{2\pi \mathrm{i}\beta} - \mathrm{e}^{-2\pi(|\boldsymbol{\xi}|^2+\alpha)}\right|} \;\le\; C \;\le\; \frac{C}{|\beta|}.$$

Hence $\|\mathbf{u}\|_2 \le (C/|\beta|)\, \|\mathbf{f}\|_2$. Moreover, (3.11), (3.12) show that we can find functions $\mathbf{f}_\alpha \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$ satisfying $\operatorname{supp} \widehat{\mathbf{f}} = B_1(-\alpha)$, $\|\mathbf{f}_\alpha\|_2 = \text{const}$ and $\|(\lambda + \mathcal{L}_2^1)^{-1}\mathbf{f}_\alpha\|_2 \ge \text{const} \ne 0$. This implies that the operator family $(\lambda + \mathcal{L}_2^1)^{-1}$ does not converge to zero in the operator norm.

For the proof of the strong convergence of the operator family $(\lambda+\mathcal{L}_2^1)^{-1}$ as $Re\,\lambda \to -\infty$ it suffices due to the previous result to consider $\mathbf{f}$ in a dense subset of $\mathbf{L}^2_\sigma(\mathbb{R}^3)$, say, in the set of solenoidal vector fields $\mathbf{f}$ with compact support in Fourier's space. So let $\mathbf{f} \in \mathbf{L}^2_\sigma(\mathbb{R}^3)$ satisfy $\operatorname{supp}\widehat{\mathbf{f}} \subset B_R(\mathbf{0})$, $R > 0$. For $|\alpha| > 2R^2$ and $\boldsymbol{\xi} \in B_R(\mathbf{0})$, we find a constant $C > 0$ independent of λ, ξ such that $|m(\boldsymbol{\xi})| \leq C/\big||\boldsymbol{\xi}|^2 + \alpha\big| \leq 2C/|\alpha|$. Hence, by (3.11),

$$\|(\lambda + \mathcal{L}_2^1)^{-1}\mathbf{f}\|_2 \;\leq\; \int_{B_R(0)} |m(\boldsymbol{\xi})|^2\; |\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi} \;\leq\; \frac{C}{|\alpha|}\int_{B_R(0)} |\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi}.$$

This estimate proves that $\|(\lambda + \mathcal{L}_2^1)^{-1}\mathbf{f}\|_2$ decays as $1/|\alpha|$ for such a function $\mathbf{f}$. $\square$

4 The spectrum of $\mathcal{L}_q^\omega$ on an exterior domain

In this section, we assume that $\Omega \subset \mathbb{R}^3$ is an exterior domain with boundary of class $C^{1,1}$, different from $\mathbb{R}^3$.

Lemma 4.1. *For $1 < q < \infty$ it holds $\sigma_{\mathrm{ess}}(-\mathcal{L}_q^\omega) \subset \mathfrak{S}^\omega$.*

Proof. Let $\lambda \in \sigma_{\mathrm{ess}}(-\mathcal{L}_q^\omega)$. Then $\mathrm{nul}'(\lambda + \mathcal{L}_q^\omega) = \infty$. We will construct a sequence $\{\mathbf{U}_m\}$ in $\mathcal{D}(\mathcal{L}_q^\omega)$ satisfying $\|\mathbf{U}_m\|_q = 1$, $\|(\lambda + \mathcal{L}_q^\omega)\mathbf{U}_m\|_q \to 0$ as $m \to \infty$ and

$$\operatorname{dist}\big(\mathbf{U}_m;\, \mathcal{H}_{m-1}\big) \;=\; 1, \quad m \in \mathbb{N}, \tag{4.1}$$

where $\mathcal{H}_{m-1}$ denotes the linear hull of the functions $\mathbf{U}_1, \ldots, \mathbf{U}_{m-1}$: Suppose that we have already constructed $\mathbf{U}_1, \ldots, \mathbf{U}_k$ satisfying $\|(\lambda + \mathcal{L}_q^\omega)\mathbf{U}_j\|_q \leq 1/j$ for $j = 1,\ldots,k$ and (4.1) for all $m = 1,\ldots,k$. To $\epsilon_{k+1} = 1/(k+1)$ there exists an infinite dimensional linear manifold M_{k+1} in $\mathcal{D}(\mathcal{L}_q^\omega)$ such that $\|(\lambda + \mathcal{L}_q^\omega)\mathbf{u}\|_q \leq \epsilon_{k+1}\|\mathbf{u}\|_q$ for all $\mathbf{u} \in M_{k+1}$. Then due to Lemma IV.2.3 in [25], we find $\mathbf{U}_{k+1} \in M_{k+1}$ such that $\|\mathbf{U}_{k+1}\|_q = 1$ and $\operatorname{dist}\big(\mathbf{U}_{k+1};\, \mathcal{H}_k\big) = 1$. The sequence $\{\mathbf{U}_m\}$ satisfies

$$\|(\lambda + \mathcal{L}_q^\omega)\mathbf{U}_m\|_q \leq \frac{1}{m} \qquad \text{for all } m \in \mathbb{N}. \tag{4.2}$$

Denote $\mathbf{f}_m := (\lambda + \mathcal{L}_q^\omega)\mathbf{U}_m$. The function $\mathbf{U}_m$ satisfies the estimate

$$\|\mathbf{U}_m\|_{2,q} + \|(\boldsymbol{\omega} \times \mathbf{x}) \cdot \nabla\mathbf{U}_m\|_q \;\leq\; c_3\,\|\mathbf{f}_m\|_q + \big(c_4 + c_5\,|\lambda|\big)\,\|\mathbf{U}_m\|_q, \tag{4.3}$$

where the constants c_3, c_4, c_5 are independent of $\mathbf{U}_m$. This estimate was proved in [10] in the case when $\Omega = \mathbb{R}^3$ and its validity was later confirmed in the case of an exterior domain with a $C^{1,1}$–boundary in [14, Lemma 2.2]. Using (4.3), we observe that the sequence $\{\mathbf{U}_m\}$ is bounded in the space $\mathcal{D}(\mathcal{L}_q^\omega)$. Hence there exists a subsequence, again denoted by $\{\mathbf{U}_m\}$, which is weakly convergent in $\mathcal{D}(\mathcal{L}_q^\omega)$. The subsequence preserves the property (4.2).

Put $\mathbf{V}_m := (\mathbf{U}_{m+1} - \mathbf{U}_m)/\delta_m$ where $\delta_m = \|\mathbf{U}_{m+1} - \mathbf{U}_m\|_q$. Then $\{\mathbf{V}_m\}$ is a sequence in the unit sphere in $\mathbf{L}^q_\sigma(\Omega)$. The weak limit of this sequence in $\mathcal{D}(\mathcal{L}_q^\omega)$ must be zero because $(\mathbf{U}_{m+1} - \mathbf{U}_m) \rightharpoonup \mathbf{0}$ in $\mathbf{L}^q_\sigma(\Omega)$ as $m \to \infty$ and by (4.1) $\delta_m \geq 1$. Hence $\{\mathbf{V}_m\}$ converges strongly to $\mathbf{0}$ in $\mathbf{W}^{1,q}(\Omega_R)^3$ for each $R > 0$; here we denote $\Omega_R := \Omega \cap B_R(\mathbf{0})$. Note that $\|(\lambda + \mathcal{L}_q^\omega)\mathbf{V}_m\|_q \to 0$ as $m \to \infty$.

The sequence $\{\mathbf{V}_m\}$ does not contain any subsequence, convergent in $\mathbf{L}^q_\sigma(\Omega)$ as we will easily prove by contradiction: Assume that $\{\mathbf{V}_{k_m}\}$ is a convergent subsequence of $\{\mathbf{V}_m\}$ in $\mathbf{L}^q_\sigma(\Omega)$. This subsequence has the same weak limit as $\{\mathbf{V}_m\}$, hence $\mathbf{V}_{k_m} \rightharpoonup \mathbf{0}$ in $\mathbf{L}^q_\sigma(\Omega)$ as $m \to \infty$. Then the strong limit of the sequence $\{\mathbf{V}_{k_m}\}$ in $\mathbf{L}^q_\sigma(\Omega)$ must also be zero. However, this is impossible because $\|\mathbf{V}_{k_m}\|_q = 1$.

Further, we use a standard cut–off procedure combined with the so called Bogovskij operator, see [13, proof of Theorem 3.1] or [14, proof of Lemma 4.2] for more details. With these tools we modify the functions $\mathbf{V}_m$ in Ω_R (for a fixed $R > 0$ so large that $\mathbb{R}^3 \smallsetminus B_{R-1}(\mathbf{0}) \subset \Omega$) such that the new functions, denoted by $\tilde{\mathbf{V}}_m$, are equal to zero in $\Omega_{R-1} := \Omega \cap B_{R-1}(\mathbf{0})$, remain in some ball of $\mathbf{L}^q_\sigma(\Omega)$, and also satisfy $\|(\lambda + \mathcal{L}^\omega_q)\tilde{\mathbf{V}}_m\|_q \to 0$ as $m \to \infty$. Moreover, $\|\tilde{\mathbf{V}}_m\|_{1,q;\,\Omega_R} \le C\,\|\mathbf{V}_m\|_{1,q;\,\Omega_R}$, where the constant C is independent of m. Hence $\tilde{\mathbf{V}}_m \to \mathbf{0}$ strongly in $\mathbf{W}^{1,q}(\Omega_R)^3$.

The sequence $\{\tilde{\mathbf{V}}_m\}$ does not contain any subsequence convergent in $\mathbf{L}^q_\sigma(\Omega)$: Otherwise one can easily derive a contradiction with the facts that $\mathbf{V}_m \to \mathbf{0}$ strongly in $\mathbf{L}^q(\Omega_R)$ and the sequence $\{\mathbf{V}_m\}$ is non–compact in $\mathbf{L}^q_\sigma(\Omega)$.

Thus, the functions $\tilde{\mathbf{V}}_m$, extended by zero to $\mathbb{R}^3 \smallsetminus \Omega$, define a non–compact sequence in the unit sphere in $\mathbf{L}^q_\sigma(\mathbb{R}^3)$ such that $\big(\lambda + (\mathcal{L}^\omega_q)_{\mathbb{R}^3}\big)\tilde{\mathbf{V}}_m \to \mathbf{0}$ in $\mathbf{L}^q_\sigma(\mathbb{R}^3)$ as $m \to \infty$; here $(\mathcal{L}^\omega_q)_{\mathbb{R}^3}$ denotes the operator $\mathcal{L}^\omega_q$, acting on functions defined in the whole $\mathbb{R}^3$, i.e. the operator treated in Section 3. Hence $\mathrm{nul}'\big(\lambda + (\mathcal{L}^\omega_q)_{\mathbb{R}^3}\big) = \infty$, which means that $\lambda \in \sigma\big(-(\mathcal{L}^\omega_q)_{\mathbb{R}^3}\big)$. Since $\sigma\big(-(\mathcal{L}^\omega_q)_{\mathbb{R}^3}\big) = \mathfrak{S}^\omega$ by Theorem 1.1, we have proven that $\lambda \in \mathfrak{S}^\omega$. □

Lemma 4.2. *For $1 < q < \infty$ one has $\mathfrak{S}^\omega \subset \sigma_{\mathrm{ess}}(-\mathcal{L}^\omega_q)$.*

Proof. Let $\lambda \in \mathfrak{S}^\omega$. Then $\lambda \in \sigma_{\mathrm{ess}}\big(-(\mathcal{L}^\omega_q)_{\mathbb{R}^3}\big)$ by Theorem 1.1, where $(\mathcal{L}^\omega_q)_{\mathbb{R}^3}$ is the "whole space" operator defined in the proof of the previous Lemma 4.1. Thus, $\mathrm{nul}'\big(\lambda + (\mathcal{L}^\omega_q)_{\mathbb{R}^3}\big) = \infty$. Following the idea from the proof of Lemma 4.1, we choose $R > 0$ so large that $\mathbb{R}^3 \smallsetminus B_{R-1}(\mathbf{0}) \subset \Omega$ and we construct a non–compact sequence $\{\tilde{\mathbf{V}}_m\}$ in the unit sphere in $\mathbf{L}^q_\sigma(\mathbb{R}^3)$ such that $\tilde{\mathbf{V}}_m = \mathbf{0}$ in Ω_{R-1} and $(\lambda + (\mathcal{L}^\omega_q)_{\mathbb{R}^3})\tilde{\mathbf{V}}_m \to 0$ in $\mathbf{L}^q_\sigma(\mathbb{R}^3)$ as $m \to \infty$. However, if we denote the restriction of $\tilde{\mathbf{V}}_m$ to Ω again by $\tilde{\mathbf{V}}_m$, we get a non–compact sequence in $\mathbf{L}^q_\sigma(\Omega)$ such that $\|\tilde{\mathbf{V}}_m\|_{q;\Omega} = 1$ and $(\lambda + \mathcal{L}^\omega_q)\tilde{\mathbf{V}}_m \to 0$ in $\mathbf{L}^q_\sigma(\Omega)$ as $m \to \infty$. This means that $\mathrm{nul}'(\lambda + \mathcal{L}^\omega_q) = \infty$. We can prove in the same way that $\mathrm{nul}'\big(\overline{\lambda} + (\mathcal{L}^\omega_q)^*\big) = \infty$. Hence $\mathrm{def}'(\lambda + \mathcal{L}^\omega_q) = \infty$, and the operator $\lambda + \mathcal{L}^\omega_q$ is not semi–Fredholm. Thus, $\lambda \in \sigma_{\mathrm{ess}}(-\mathcal{L}^\omega_q)$. □

Lemmas 4.1 and 4.2 imply that $\sigma_{\mathrm{ess}}(-\mathcal{L}^\omega_q) = \mathfrak{S}^\omega$.

Lemma 4.3. *Let $\lambda \in \mathbb{C} \smallsetminus \mathfrak{S}^\omega$. Then either λ is an eigenvalue of $-\mathcal{L}^\omega_q$ for all $1 < q < \infty$, whose both algebraic and geometric multiplicities are finite and independent of q, or $\lambda \in \rho(\mathcal{L}^\omega_q)$ for all $1 < q < \infty$. Each eigenfunction and each generalized eigenfunction belongs to $\bigcap_{1<s<\infty} \mathcal{D}(\mathcal{L}^\omega_s)$. Moreover, $[\mathbb{C} \smallsetminus \mathfrak{S}^\omega] \cap \{\lambda \in \mathbb{C};\ Re\,\lambda \ge 0\} \subset \rho(\mathcal{L}^\omega_q)$.*

Proof. Since $\mathfrak{S}^\omega = \sigma_{\mathrm{ess}}(-\mathcal{L}^\omega_q)$, the set $\mathbb{C} \smallsetminus \mathfrak{S}^\omega$ is a subset of $\rho(-\mathcal{L}^\omega_q)$, with the possible exception of at most a countable set of isolated eigenvalues of $(-\mathcal{L}^\omega_q)$, which have finite algebraic multiplicities, see [25, p. 243].

Thus, assume that $\mathbf{w}$ is an eigenfunction of $-\mathcal{L}^\omega_q$, corresponding to an eigenvalue $\lambda \notin \mathfrak{S}^\omega$. The fact that then λ is an eigenvalue of $\mathcal{L}^\omega_s$ for all $1 < s < \infty$ is proven in [14]. The idea of the proof is as follows: Let $R > 0$ be so large that $\mathbb{R}^3 \smallsetminus B_{R-1}(\mathbf{0}) \subset \Omega$. We split

$\mathbf{w}$ by means of an appropriate cut–off function procedure and the Bogovskij operator to the sum $\mathbf{w}_1+\mathbf{w}_2$, where both $\mathbf{w}_1$ and $\mathbf{w}_2$ belong to $\mathcal{D}(\mathcal{L}_q^\omega)$, $\mathbf{w}_1$ is supported in $\mathbb{R}^3 \setminus B_{R-1}(\mathbf{0})$ and $\mathbf{w}_2$ is supported in $\Omega \cap B_R(\mathbf{0})$. Then $(\lambda + \mathcal{L}_q^\omega)\mathbf{w}_1 = \mathbf{f}_1$, where $\mathbf{f}_1$ can be explicitly calculated and is supported in $B_R(\mathbf{0}) \setminus B_{R-1}(\mathbf{0})$. Since $\mathbf{w} \in \mathcal{D}(\mathcal{L}_q^\omega)$, elliptic regularity theory and Sobolev's embedding theorem prove that $\mathbf{f}_1 \in \mathbf{L}^s_\sigma(\Omega)$ for all $1 < s < q^*$ where $q^* = \frac{nq}{n-q}$ if $q < n$ and $q^* = \infty$ if $q > n$. Extending $\mathbf{f}_1$ by zero to $\mathbb{R}^3 \setminus \Omega$, we get a function from $\mathbf{L}^s_\sigma(\mathbb{R}^3)$. Applying Theorem 2.1, we deduce that $\mathbf{w}_1$, extended by zero to $\mathbb{R}^3 \setminus \Omega$, belongs to $\mathbf{L}^s_\sigma(\mathbb{R}^3)$, $s < q^*$, as well. Applying further estimate (4.3), we obtain that $\mathbf{w}_1 \in \mathcal{D}((\mathcal{L}_s^\omega)_{\mathbb{R}^3})$. Furthermore, since $\mathbf{w}_2$ is supported in a bounded subdomain of Ω, we verify that $\mathbf{w}_2 \in \mathcal{D}(\mathcal{L}_s^\omega)$ for all $1 < s < q^*$. Repeating his step finitely many times, if necessary, we see that $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 \in \mathcal{D}(\mathcal{L}_s^\omega)$ for all $1 < s < \infty$. Therefore, $\mathbf{w}$ is an eigenfunction of $-\mathcal{L}_s^\omega$ to the eigenvalue λ for all $1 < s < \infty$.

Since the geometric multiplicity of λ is the maximum number of linearly independent associated eigenfunctions $\mathbf{w}$, and these eigenfunctions are independent of s, the geometric multiplicity of λ is also independent of s.

The algebraic multiplicity of λ, since it is finite, equals the sum of the lengths of all linearly independent chains of the so called generalized eigenfunctions, associated with the eigenvalue λ. If $\mathbf{w}^1, \dots, \mathbf{w}^m$ is such a chain, then $(\lambda + \mathcal{L}_q^\omega)\mathbf{w}^1 = \mathbf{0}$ and $(\lambda + \mathcal{L}_q^\omega)\mathbf{w}^k = \mathbf{w}^{k-1}$ for $k = 2, \dots, m$. By analogy with the eigenfunction $\mathbf{w}$ discussed above, one can successively show that all the functions $\mathbf{w}^1, \dots, \mathbf{w}^m$ also belong to $\mathcal{D}(\mathcal{L}_s^\omega)$ for all $1 < s < \infty$. Consequently, the algebraic multiplicity of λ, as an eigenfunction of $-\mathcal{L}_s^\omega$, is independent of s as well.

Finally, if $\lambda \in \mathbb{C} \setminus \mathfrak{S}^\omega$, $Re\,\lambda \geq 0$, then one can prove that the operator $\lambda + \mathcal{L}_2^\omega$ has a bounded inverse in $\mathbf{L}^2_\sigma(\Omega)$, just multiplying the resolvent equation $(\lambda + \mathcal{L}_2^\omega)\mathbf{u} = \mathbf{f}$ by $\mathbf{u}$ and integrating on Ω. Hence $\lambda \in \rho(-\mathcal{L}_2^\omega)$. Due to the explanation given above, λ is not an eigenvalue of $\mathcal{L}_q^\omega$ for any $q \in (1, \infty)$ and λ also cannot belong to $\sigma_{\mathrm{r}}(-\mathcal{L}_q^\omega)$. Hence $\lambda \in \rho(-\mathcal{L}_q^\omega)$, independently of q. □

Lemma 4.4. *Let domain Ω be axially symmetric about the x_3–axis and $\lambda \in \mathbb{C} \setminus \mathfrak{S}^\omega$. Then $\lambda \in \rho(\mathcal{L}_q^\omega)$.*

Proof. We have proved in [12] that if Ω is axially symmetric about the x_3–axis then $\sigma(\mathcal{L}_2^\omega) = \mathfrak{S}^\omega$. Hence all $\lambda \in \mathbb{C} \setminus \mathfrak{S}^\omega$ belong to $\rho(\mathcal{L}_2^\omega)$. Due to Lemma 4.3, these λ belong to $\rho(\mathcal{L}_q^\omega)$ for all $q \in (1, \infty)$ as well. □

The next theorem resumes the results of Lemmas 4.1–4.4.

Theorem 4.5. *Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary of class $C^{1,1}$. Then*

(i) $\sigma_{\mathrm{ess}}(-\mathcal{L}_q^\omega) = \mathfrak{S}^\omega$,

(ii) $\{z \in \mathbb{C};\ Re\,z \geq 0\} \setminus \{z = \mathrm{i}\omega k;\ k \in \mathbb{Z}\} \subset \rho(-\mathcal{L}_q^\omega)$,

(iii) each $\lambda \in \mathbb{C} \setminus \mathfrak{S}^\omega$ with $Re\,\lambda < 0$ is either an eigenvalue of $-\mathcal{L}_q^\omega$ for all $1 < q < \infty$, whose both algebraic and geometric multiplicities are finite and independent of q, or $\lambda \in \rho(-\mathcal{L}_q^\omega)$ for all $1 < q < \infty$; moreover, each eigenfunction and each generalized eigenfunction belongs to $\bigcap_{1<s<\infty} \mathcal{D}(\mathcal{L}_s^\omega)$,

(iv) if the domain Ω is axially symmetric about the x_3–axis, then $\rho(-\mathcal{L}_q^\omega) = \mathbb{C} \setminus \mathfrak{S}^\omega$.

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