# The Largest Possible Initial Value Space for Local Strong Solutions of the Navier-Stokes Equations in General Domains

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Consider the Navier-Stokes system with initial value $u_0 \in L^2_\sigma(\Omega)$ and vanishing external force in a general (bounded or unbounded, smooth or nonsmooth) domain $\Omega \subseteq \mathbb{R}^3$ and a time interval $[0,T)$, $0 < T \leq \infty$. Our aim is to characterize the largest possible space of initial values $u_0$ yielding a unique strong solution $u$ in Serrin's class $L^8\big(0,T;L^4(\Omega)\big)$. As the main result we prove that the additional condition $\int_0^\infty \|e^{-tA}\,u_0\|_4^8\,dt < \infty$ is necessary and sufficient for the existence of such a local strong solution $u$ with $u(0) = u_0$; here $A$ denotes the Stokes operator on $L^2_\sigma(\Omega)$ generating the analytic semigroup $e^{-tA}$, $t \geq 0$. This assumption on $u_0$ is strictly weaker than the well-known $\mathcal{D}(A^{1/4})$-condition of Fujita and Kato (1964) and holds for general open connected subsets of $\mathbb{R}^3$.

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## 1 Introduction

We consider the instationary Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= 0, \qquad \operatorname{div} u = 0 \\ u\big|_{\partial\Omega} &= 0, \qquad u(0) = u_0 \end{aligned} \tag{1.1}$$

in a general domain $\Omega \subseteq \mathbb{R}^3$ – by definition an open connected subset – with boundary $\partial\Omega$ on a time interval $[0,T)$, $0 < T \leq \infty$, and with initial value $u_0$. First we recall the definitions of weak and strong solutions to (1.1) and introduce some notations before describing the main results.

**Definition 1.1.** Given $u_0 \in L^2_\sigma(\Omega)$ a vector field

$$u \in L^\infty\big(0,T;L^2_\sigma(\Omega)\big) \cap L^2_{\text{loc}}\big([0,T);\, W^{1,2}_{0,\sigma}(\Omega)\big) \tag{1.2}$$

is called a *weak solution in the sense of Leray-Hopf* of the Navier-Stokes system (1.1) with initial value $u(0) = u_0$ if the relation

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega \tag{1.3}$$

holds for each test function $w \in C_0^\infty\big([0,T);\, C_{0,\sigma}^\infty(\Omega)\big)$, and additionally the *energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\, d\tau \le \frac{1}{2}\|u_0\|_2^2 \tag{1.4}$$

is satisfied for all $t \in [0,T)$.

Such a weak solution $u$ is called a *strong solution* of (1.1) if additionally *Serrin's condition*

$$u \in L^s_{\text{loc}}\big([0,T);\, L^q(\Omega)\big) \tag{1.5}$$

is satisfied with exponents $2 < s < \infty$, $3 < q < \infty$ where $\frac{2}{s} + \frac{3}{q} \le 1$.

In this definition we used the Lebesgue spaces $L^q(\Omega)$, $1 < q < \infty$, with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and pairing $\langle \cdot, \cdot \rangle_\Omega$ on $L^q(\Omega) \times L^{q'}(\Omega)$ where $q' = \frac{q}{q-1}$. By analogy, Bochner spaces on $\Omega \times (0,T)$ are denoted by $L^s\big(0,T; L^q(\Omega)\big)$, $1 < s,\, q < \infty$, with norm

$$\|\cdot\|_{L^s(0,T;L^q(\Omega))} = \Big( \int_0^T \|\cdot\|_q^s\, d\tau \Big)^{1/s} = \|\cdot\|_{q,s,T}$$

and pairing $\langle \cdot, \cdot \rangle_{\Omega,T}$. Further, we need the usual Sobolev spaces $\big(W^{k,q}(\Omega),\, \|\cdot\|_{k,q}\big)$, $k \in \mathbb{N}$, $1 < q < \infty$, and $W_0^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,2}}$. To deal with solenoidal vector fields we introduce the spaces $C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0\}$, $L^2_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_2}$ and $W_{0,\sigma}^{1,2}(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{1,2}}$. Note that for a solenoidal vector field $u = (u_1, u_2, u_3)$ on $\Omega$

$$u \cdot \nabla u = \sum_{j=1}^3 u_j \partial_j u = \operatorname{div}(uu), \quad uu = (u_i u_j)_{i,j=1}^3\,;$$

more generally, for a matrix-valued field $F = (F_{ij})_{i,j=1}^3 = (F_i)_{i=1}^3$ we define $\operatorname{div} F = (\operatorname{div} F_1, \operatorname{div} F_2, \operatorname{div} F_3)$.

For properties of weak and strong solutions to (1.1) we refer to [1], [2], [10], [11], [12], [13], [16]; for corresponding results in general domains, see e.g. [14, Chapters V.1, V.3 and V.4]. Given a weak solution $u$ we may assume without loss of generality that $u : [0,T) \to L^2_\sigma(\Omega)$ is weakly continuous; in this sense the initial value $u(0) = u_0$ is attained. Moreover, there exists an associated pressure $p$ in $\Omega \times (0,T)$, a distribution, such that

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = 0$$

in the sense of distributions. Serrin's condition (1.5) yields the following regularity result: If $\partial\Omega$ is of class $C^\infty$, then

$$u \in C^\infty\big(\overline{\Omega} \times (0,T)\big), \quad p \in C^\infty\big(\overline{\Omega} \times (0,T)\big);$$

therefore, a strong solution is also called a regular solution. For these results, in particular the existence of weak solutions, we refer to [14, Theorems V.1.3.1, V.1.8.2, V.3.1.1 and Chapter V.1.7].

However, the uniqueness of weak solutions and the existence of strong solutions to (1.1) for at least a sufficiently small interval $[0,T)$, $0 < T \le \infty$, requires a stronger assumption on $u_0$ as in Definition 1.1. It is not known up to now whether a strong solution $u$ exists for each (sufficiently smooth) $u_0$ and each given interval $[0,T)$. As long as this problem is open we try to extend the class of local strong solutions and to lower the assumptions on the set of initial values as far as possible.

In this paper we construct and characterize the largest possible space of initial values for local strong solutions in Serrin's class $L^8\big([0,T);\ L^4(\Omega)\big)$. To explain our main theorem we have to introduce the Helmholtz projection $P = P_2 : L^2(\Omega) \to L^2_\sigma(\Omega)$ and the Stokes operator

$$A = A_2 = -P\Delta :\ \mathcal{D}(A) \to L^2_\sigma(\Omega)$$

with domain

$$\mathcal{D}(A) = \{v \in W^{1,2}_{0,\sigma}(\Omega) :\ \exists f \in L^2_\sigma(\Omega) : \langle \nabla v, \nabla\varphi\rangle_\Omega = \langle f, \varphi\rangle_\Omega \quad \forall\ \varphi \in W^{1,2}_{0,\sigma}(\Omega)\}$$

such that $Av = -P\Delta v = f$, $v \in \mathcal{D}(A)$. It is known that for any domain $\Omega \subseteq \mathbb{R}^3$ the operator $A$ is self-adjoint and generates a bounded analytic semigroup $e^{-tA}$, $t \ge 0$, on $L^2_\sigma(\Omega)$. Further, we may define the fractional powers $A^\alpha : \mathcal{D}(A^\alpha) \to L^2_\sigma(\Omega)$, $-1 \le \alpha \le 1$, such that $\mathcal{D}(A) \subset \mathcal{D}(A^\alpha) \subset L^2_\sigma(\Omega)$ for $\alpha \in (0,1)$, see e.g. [14, Chapters III.2.2 and IV.1].

Now our main result reads as follows:

**Theorem 1.2.** *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain and let $u_0 \in L^2_\sigma(\Omega)$.*

(1) *The condition*

$$\int_0^\infty \|e^{-tA}u_0\|_4^8\, dt < \infty \tag{1.6}$$

*is necessary and sufficient for the existence of a unique strong solution*

$$u \in L^8\big(0,T; L^4(\Omega)\big) \tag{1.7}$$

*of the Navier-Stokes system* (1.1) *with $u(0) = u_0$ in some time interval* $[0,T)$, $0 < T \le \infty$.

(2) *There exists an absolute constant $\varepsilon_* > 0$ (independent of the domain) with the following property: If*

$$\int_0^T \|e^{-tA}u_0\|_4^8\, dt \le \varepsilon_* \quad \textit{for some } 0 < T \le \infty, \tag{1.8}$$

*then the Navier-Stokes system* (1.1) *has a unique strong solution $u$ on* $[0,T)$ *with $u(0) = u_0$ satisfying* (1.7).

To interpret the results of Theorem 1.2 and to compare them with known results we summarize some facts on the Stokes operator. The fractional powers $A^\alpha$ of $A$ satisfy the interpolation inequality

$$\|A^\alpha v\|_2 \le \|Av\|_2^\alpha\,\|v\|_2^{1-\alpha}, \quad v \in \mathcal{D}(A),\, 0 \le \alpha \le 1, \tag{1.9}$$

and the embedding estimate

$$\|v\|_q \le C\|A^\alpha v\|_2, \quad v \in \mathcal{D}(A^\alpha),\; 0 \le \alpha \le \frac{1}{2},\; 2\alpha + \frac{3}{q} = \frac{3}{2}\,, \tag{1.10}$$

with a constant $C = C(\alpha) > 0$ independent of $\Omega$. Moreover,

$$\|A^{1/2}v\|_2 = \|\nabla v\|_2, \quad v \in W^{1,2}_{0,\sigma}(\Omega) = \mathcal{D}(A^{1/2}). \tag{1.11}$$

Concerning the Stokes semigroup we mention that

$$\|A^\alpha e^{-tA}v\|_2 \le t^{-\alpha}\|v\|_2, \quad v \in L^2_\sigma(\Omega),\, 0 \le \alpha \le 1, \tag{1.12}$$

and that

$$\|A^{1/s}\, e^{-tA}v\|_{2,s;T} \le \|v\|_2, \quad v \in L^2_\sigma(\Omega),\, 2 \le s < \infty; \tag{1.13}$$

see in particular [14, Chapters III.2 and IV.1].

Theorem 1.2 should be compared with similar, but more general results in [6] where the authors analyzed the same problem in smooth bounded domains. The condition (1.6) is replaced by the more general condition

$$\int_0^\infty \|e^{-tA_2}u_0\|_q^s < \infty, \quad \frac{2}{s} + \frac{3}{q} = 1,\; 2 < s < \infty,\; 3 < q < \infty,$$

on $u_0 \in L^2_\sigma(\Omega)$ using the $L^q$-theory of the Stokes operator in bounded domains. However, in general unbounded domains and also in bounded domains with non-smooth boundary only an $L^2$-theory is available leading to the exponents $s = 8$, $q = 4$ in (1.5).

**Remark 1.3.** (1) The initial condition (1.6) is not only sufficient, but also necessary for the existence of a strong solution $u \in L^8\big(0, T; L^4(\Omega)\big)$ of (1.1) in some interval $[0, T)$, $T > 0$. Therefore, the condition (1.6) yields within $L^2_\sigma(\Omega)$ the largest possible initial value space for the existence of such unique local strong solutions u.

(2) The constant $\varepsilon_* > 0$ in (1.8) is a so-called absolute constant. In particular, $\varepsilon_*$ does not depend on the domain $\Omega$. Therefore, if for each domain $\Omega \subseteq \mathbb{R}^3$ an initial value $u_0 = u_0(\Omega) \in L^2_\sigma(\Omega)$ is given and satisfies (1.8) uniformly with respect to $\Omega$ with some fixed $0 < T \le \infty$, then there exists a strong solution $u_\Omega \in L^8\big(0, T; L^4(\Omega)\big)$ with the same interval of existence $[0, T)$ for all domains $\Omega$.

(3) Using (1.10) with $\alpha = \frac{3}{8}$, $q = 4$ and (1.12) we observe that

$$\|e^{-tA}u_0\|_4 \le C\|A^{3/8}\,e^{-tA}u_0\|_2 \le Ct^{-3/8}\|u_0\|_2, \quad u_0 \in L^2_\sigma(\Omega). \tag{1.14}$$

Therefore, the condition (1.6) simply means the integrability of the (continuous) function $t \mapsto \|e^{-tA}u_0\|_4$ near $t = 0$. Moreover, the validity of (1.8) implies due to (1.14) that (1.6) is satisfied.

(4) Using (1.10) with $\alpha = \frac{3}{8}$, $q = 4$ and (1.13) with $s = 8$ we conclude for any $u_0 \in \mathcal{D}(A^{1/4})$ that

$$\int_0^T \|e^{-tA}u_0\|_4^8\,dt \le C\int_0^\infty \|A^{1/8}\,e^{-tA}\big(A^{1/4}u_0\big)\|_2^8\,dt \le C\|A^{1/4}u_0\|_2^8.$$

Hence the assumption $u_0 \in \mathcal{D}(A^{1/4})$ implies (1.6). Note that the condition $u_0 \in \mathcal{D}(A^{1/4})$ was used by H. Fujita and T. Kato [7] to guarantee the existence of a local strong solution in a bounded domain; this result was extended to general domains in [14, Theorem V.4.2.2]. Theorem 1.2 implies that the solution class defined by (1.6) is (strictly) larger than the class defined by $u_0 \in \mathcal{D}(A^{1/4})$.

It is not difficult to extend Theorem 1.2 to nonvanishing external forces $f = \operatorname{div} F$ with $F \in L^4\big(0,T;L^2(\Omega)\big)$. In this case we consider instead of (1.1) the Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + u\cdot\nabla u + \nabla p &= f, & \operatorname{div} u &= 0,\\ u\big|_{\partial\Omega} &= 0, & u(0) &= u_0. \end{aligned} \tag{1.15}$$

By definition a weak solution $u$ of (1.15) satisfies (1.2),

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T} \tag{1.16}$$

for all test functions $w \in C_0^\infty\big([0,T);\,C_{0,\sigma}^\infty(\Omega)\big)$ and the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, u\rangle_\Omega\,d\tau, \quad t \in (0,T), \tag{1.17}$$

instead of (1.4). Further, as in Definition 1.1, a weak solution of (1.15) is called a strong solution if additionally Serrin's condition (1.5) is satisfied. Then we get a more general result the proof of which will be omitted:

**Theorem 1.4.** *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain and let $u_0 \in L^2_\sigma(\Omega)$, $f = \operatorname{div} F$, $F \in L^4\big(0,T;L^2(\Omega)\big)$.*

(1) *The condition*

$$\int_0^\infty \|e^{-tA}u_0\|_4^8\,dt < \infty$$

*is necessary and sufficient for the existence of a unique strong solution $u \in L^8\big(0,T;L^4(\Omega)\big)$ of (1.15) in some time interval $[0,T)$, $0 < T \le \infty$.*

(2) *There exists an absolute constant* $\varepsilon_* > 0$ *with the following property: If for some* $0 < T \le \infty$

$$\int_0^T \|e^{-tA}u_0\|_4^8\,dt + \int_0^T \|F\|_2^4\,dt \le \varepsilon_*,$$

*then the Navier-Stokes system* (1.15) *has a unique strong solution* $u \in L^8\big(0,T;L^4(\Omega)\big)$ *in* $[0,T)$.

## 2 Proof of Theorem 1.2

To carry out the proof of Theorem 1.2 we need some preliminaries on linear problems.

Let $F = (F_{ij})_{i,j=1}^3 \in L^2(\Omega)$. Then there exists a unique vector field $\psi \in L^2_\sigma(\Omega)$, also denoted by $A^{-1/2}\,P\,\mathrm{div}\,F$, such that

$$\langle \psi, \varphi\rangle = -\langle F, \nabla A^{-1/2}\,\varphi\rangle \quad \text{for all } \varphi \in L^2_\sigma(\Omega).$$

In this sense, the operator $A^{-1/2}\,P\,\mathrm{div} \,:\, L^2(\Omega) \to L^2_\sigma(\Omega)$ is well-defined by the relation

$$\langle A^{-1/2}P\mathrm{div}\,F, \varphi\rangle = -\langle F, \nabla A^{-1/2}\varphi\rangle, \quad \varphi \in L^2_\sigma(\Omega)\,,$$

and satisfies the estimate

$$\|A^{-1/2}\,P\,\mathrm{div}\,F\|_2 \le \|F\|_2\,, \quad F \in L^2(\Omega), \tag{2.1}$$

see also [14, Lemma III.2.6.1].

**Lemma 2.1.** *On a general domain* $\Omega \subseteq \mathbb{R}^3$ *we consider the instationary Stokes system*

$$\begin{aligned} u_t - \Delta u + \nabla p &= f, \qquad \mathrm{div}\,u = 0 \quad \text{in } \Omega\times(0,T),\\ u|_{\partial\Omega} &= 0, \qquad u(0) = u_0. \end{aligned} \tag{2.2}$$

(1) *Assume* $f = \mathrm{div}\,F$, $F \in L^2\big(0,T;L^2(\Omega)\big)$, *and* $u_0 \in L^2_\sigma(\Omega)$. *Then* (2.2) *has a unique weak solution* $u$ *satisfying* (1.2) *(and* (1.16) *without the nonlinear term* $\langle uu, \nabla w\rangle_{\Omega,T}$*) and the energy inequality* (1.17). *Moreover,* $u$ *has the representation*

$$u(t) = e^{-tA}\,u_0 + \int_0^t A^{1/2}e^{-(t-\tau)A}\,A^{-1/2}P\,\mathrm{div}\,F\,d\tau, \quad 0 \le t < T. \tag{2.3}$$

(2) *Assume* $f \in L^s\big(0,T;L^2(\Omega)\big)$, $1 < s < \infty$, *and* $u_0 = 0$. *Then* (2.2) *has a unique weak solution* $u$ *which also may be interpreted as solution of the abstract evolution problem*

$$u_t + Au = Pf, \quad u(0) = 0. \tag{2.4}$$

*This solution $u$ has the representation*

$$u(t) = \int_0^t e^{-(t-\tau)A}\, Pf\, d\tau, \tag{2.5}$$

*admits the maximal regularity estimate*

$$\|u_t\|_{2,s;T} + \|Au\|_{2,s;T} \le C\|f\|_{2,s;T} \tag{2.6}$$

*with some constant $C = C(s) > 0$; in particular, $u_t,\ Au \in L^s\big(0,T;L^2(\Omega)\big)$. Moreover, for every $s \le r < \infty$ and $\alpha = 1 + \frac{1}{r} - \frac{1}{s}$,*

$$\|A^\alpha u\|_{2,r;T} \le C\|Pf\|_{2,s;T} \tag{2.7}$$

*with some constant $C = C(r,s) > 0$.*

*Proof.* For the proof of (1) cf. [14, Lemma IV.2.4.2], for the proof of (2) [14, Theorem IV.2.5.2]. Actually, (2.7) is a consequence of (1.12) and of the Hardy-Littlewood inequality: By (2.5)

$$\|A^\alpha u(t)\|_2 \le \int_0^t (t-\tau)^{-\alpha}\, \|Pf\|_2\, d\tau$$

and consequently $\|A^\alpha u\|_{2,r;T} \le C\|Pf\|_{2,s;T}$. □

*Proof of Theorem 1.2(2).* First we assume that the inequality

$$\int_0^T \|e^{-tA}u_0\|_4^8\, dt \le C$$

holds with $0 < T \le \infty$ and any given constant $C > 0$. Later on we will choose $C = \varepsilon_* > 0$ sufficiently small. Moreover, let us assume at the beginning of the proof that $u \in L^8\big(0,T;L^4(\Omega)\big)$ is a given strong solution of (1.1) with $u(0) = u_0$. Then we set $F = -uu$ and write (1.1) as a linear system in the form

$$\begin{aligned} u_t - \Delta u + \nabla p &= \operatorname{div} F, & \operatorname{div} u &= 0, \\ u\big|_{\partial\Omega} &= 0, & u(0) &= u_0. \end{aligned}$$

Since by Hölder's inequality

$$\|F\|_{2,4,T} = \|uu\|_{2,4,T} \le c\|u\|_{4,8;T}^2 < \infty$$

with some absolute constant $c > 0$, we obtain that

$$F = -uu \in L^4\big(0,T;L^2(\Omega)\big).$$

Lemma 2.1(1) yields the representation

$$u(t) = e^{-tA}\, u_0 - \int_0^t A^{1/2} e^{-(t-\tau)A}\, A^{-1/2} P \operatorname{div}(uu) d\tau \tag{2.8}$$

and the energy inequality (1.17) for all $t \in [0,T)$; in particular, $u$ is also a weak solution. First this holds if $T$ is finite since $F \in L^2\big(0,T; L^2(\Omega)\big)$ is needed in this lemma. However, if $T = \infty$, we obtain the same result by applying this lemma to all finite intervals.

Now (2.8) will be considered as a fixed point problem for the strong solution we are looking for. Let $X$ be the Banach space of vector fields,

$$X = \{u : (0,T) \to L^2_\sigma(\Omega) :\; (A^{-1/2}u)_t, A^{1/2}u \in L^4\big(0,T; L^2(\Omega)\big), (A^{-1/2}u)(0) = 0\},$$

equipped with the norm

$$\|u\|_X = \|(A^{-1/2}u)_t\|_{2,4;T} + \|A^{1/2}u\|_{2,4;T} < \infty.$$

Note that, since $(A^{-1/2}u)_t \in L^4\big(0,T; L^2(\Omega)\big)$, the map $t \mapsto A^{-1/2}u(t)$ is Hölder-continuous from $[0,T)$ to $L^2_\sigma(\Omega)$; in particular the initial condition $A^{-1/2}u(0) = 0$ is well-defined. Moreover, the interpolation estimate

$$\|u(t)\|_2 = \|A^{1/2}A^{-1/2}u(t)\|_2 \le \|A^{1/2}u(t)\|_2^{1/2}\, \|A^{-1/2}u(t)\|_2^{1/2},$$

see (1.9), implies that

$$u \in L^4_{\mathrm{loc}}\big([0,T); L^2_\sigma(\Omega)\big) \quad \text{for every } u \in X. \tag{2.9}$$

We claim that $X$ is continuously embedded into $L^8\big(0,T; L^4(\Omega)\big)$; more precisely,

$$\|u\|_{4,8;T} \le c\|u\|_X \quad \text{for all } u \in X \tag{2.10}$$

with an absolute constant $c > 0$. For its proof consider $u \in X$ and set $\tilde{u} = A^{-1/2}u$ and $\tilde{f} = A^{-1/2}u_t + A^{1/2}u \in L^4\big(0,T; L^2_\sigma(\Omega)\big)$. Obviously $\tilde{u}$ is a solution of the abstract evolution problem

$$\tilde{u}_t + A\tilde{u} = \tilde{f}, \quad \tilde{u}(0) = 0,$$

cf. (2.4). Then Lemma 2.1(2) yields the representation formulae

$$\begin{aligned} A^{-1/2}u(t) &= \int_0^t e^{-(t-\tau)A}\tilde{f}\, d\tau, \\ u(t) &= \int_0^t A^{1/2}e^{-(t-\tau)A}\tilde{f}\, d\tau, \quad 0 \le t < T, \end{aligned}$$

as well as the *a priori* estimate

$$\begin{aligned} \|u\|_{4,8;T} &\le c\|A^{3/8}u\|_{2,8;T} &&= \|A^{7/8}\tilde{u}\|_{2,8;T} \\ &\le c\|\tilde{f}\|_{2,4;T} &&\le c\|u\|_X; \end{aligned}$$

with absolute generic constants $c > 0$; for the proof of this estimate we used (1.10) with $\alpha = \frac{3}{8}$, $q = 4$ and (2.7) with $\alpha = \frac{7}{8}$, $r = 8$, $s = 4$. Now (2.10) is proved.

Returning to (2.8) set $v(t) = e^{-tA}u_0$, $U = v - u$ and $h = A^{-1/2}\, P \operatorname{div}(uu) \in L^4\big(0, T; L^2(\Omega)\big)$ so that

$$U(t) = \int_0^t A^{1/2} e^{-(t-\tau)A} h\, d\tau, \quad A^{-1/2} U(t) = \int_0^t e^{-(t-\tau)A} h\, d\tau.$$

Then the maximal regularity estimate (2.6) with $s = 4$ and (2.1) yield the inequality

$$\|U\|_X \le c_1 \|h\|_{2,4;T} \le c_1 \|uu\|_{2,4;T} \le c_2 \|u\|^2_{4,8;T} < \infty \tag{2.11}$$

with absolute constants $c_1, c_2 > 0$. Summarizing the previous estimates we see that

$$U = v - u \in X \subset L^8\big(0, T; L^4(\Omega)\big). \tag{2.12}$$

To solve the fixed point problem (2.8) in $X$ we define the nonlinear operator $\mathcal{F}$ by

$$\begin{aligned} \mathcal{F}(U)(t) &= \int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2}\, P \operatorname{div}(uu) d\tau \\ &= \int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2}\, P \operatorname{div}\big((v-U)(v-U)\big) d\tau. \end{aligned} \tag{2.13}$$

From (2.11) we conclude that $\mathcal{F} : X \to X$ and that

$$\|\mathcal{F}(U)\|_X \le c_2 \|v - U\|^2_{4,8;T} \le c_2 \big(\|U\|_{4,8;T} + \|v\|_{4,8;T}\big)^2. \tag{2.14}$$

Obviously, a solution $u \in L^8\big(0, T; L^4(\Omega)\big)$ of (2.8) is a fixed point $U \in X$ of $\mathcal{F}$ and vice versa when defining $U = v - u$. To find a fixed point $U \in X$ of $\mathcal{F}$ let $b = \|v\|_{4,8;T}$, use (2.10), write (2.14) in the form

$$\|\mathcal{F}(U)\|_X + b \le c\big(\|U\|_X + b\big)^2 + b$$

with an absolute constant $c > 0$, and consider the quadratic equation

$$y^2 - \frac{1}{c} y + \frac{b}{c} = 0 \quad \text{on } (0, \infty).$$

Next we choose the constant $\varepsilon_* > 0$ in (1.8) in such a way that $\varepsilon_* < \big(\frac{1}{4c}\big)^8$; this condition implies that the assumption $b = \|v\|_{4,8;T} < \varepsilon_*^{1/8}$ in (1.8) leads to $4cb < 1$ and that the above quadratic equation has a minimal positive root $y_1$ satisfying $y_1 < 2b$. Defining the closed ball $B = \{U \in X : \|U\|_X \le y_1 - b\}$ we get that for $U \in B$

$$\|\mathcal{F}(U)\|_X + b \le c\big(\|U\|_X + b)\big)^2 + b \le c y_1^2 + b = y_1$$

and consequently $\mathcal{F}(B) \subset B$.

To show that $\mathcal{F}$ is a strict contraction on $B$ consider $U, U' \in B$. Then we obtain

$$\big(\mathcal{F}(U) - \mathcal{F}(U')\big)(t) = \int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2} P \operatorname{div}\big((v-U)(U'-U) + (U'-U)(v-U)\big)\, d\tau$$

and the arguments which led to (2.14) now yield the inequality

$$\begin{aligned} \|\mathcal{F}(U) - \mathcal{F}(U')\|_X &\le c\big(\|U\|_X + b + \|U'\|_X + b\big)\, \|U - U'\|_X \\ &\le 2cy_1 \|U - U'\|_X \le 4cb\|U - U'\|_X. \end{aligned}$$

This proves that $\mathcal{F} : B \to B$ is a strict contraction, and Banach's fixed point theorem yields the existence of $U \in X$ satisfying $\mathcal{F}(U) = U$. Then we set $u = v - U$ and have to show that $u$ is the strong solution in $[0, T)$ we are looking for. For this purpose we need the following properties.

By the assumption (1.8) we know that $v \in L^8\big(0, T; L^4(\Omega)\big)$, and, since $u_0 \in L^2_\sigma(\Omega)$, (1.13) with $s = 2$ implies that $A^{1/2}v$, $\nabla v \in L^2\big(0, T; L^2(\Omega)\big)$ whereas the integrability $v \in L^\infty\big(0, T; L^2(\Omega)\big)$ is obvious. Hence

$$v \in L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0, T); W^{1/2}_{0,\sigma}(\Omega)\big).$$

Concerning $U \in X$ we conclude from (2.10) that $U \in L^8\big(0, T; L^4(\Omega)\big)$ so that also $u = v - U \in L^8\big(0, T; L^4(\Omega)\big)$ and

$$uu \in L^4\big(0, T; L^2(\Omega)\big).$$

Moreover, due to (2.9), $U \in L^4_{\mathrm{loc}}\big([0, T); L^2_\sigma(\Omega)\big)$ so that even $U \in L^2_{\mathrm{loc}}\big([0, T); W^{1,2}_{0,\sigma}(\Omega)\big)$. Hence also

$$u \in L^2_{\mathrm{loc}}\big([0, T); W^{1,2}_{0,\sigma}(\Omega)\big).$$

Since $U = \mathcal{F}(U)$, (2.13) yields for $u = v - U$ the representation

$$u(t) = e^{-tA}u_0 - \int_0^t A^{1/2} e^{-(t-\tau)A}\, A^{-1/2}\, P \operatorname{div}(uu)\, d\tau, \quad 0 \le t < T.$$

Here $F := -uu \in L^2_{\mathrm{loc}}\big([0, T; L^2(\Omega)\big)$, and Lemma 2.1(1) implies that $u$ is the well-defined weak solution of the Stokes system with right-hand side $f = \operatorname{div} F$. Therefore, $u$ satisfies the energy inequality (1.17) on $[0, T)$, but (1.2) only with $T$ replaced by any finite $T' \in (0, T]$. However, a direct calculation shows that

$$\langle F, \nabla u\rangle_\Omega = -\langle uu, \nabla u\rangle_\Omega = \langle u, \frac{1}{2}\nabla|u|^2\rangle_\Omega = -\frac{1}{2}\langle \operatorname{div} u, |u|^2\rangle_\Omega = 0$$

and that the energy inequality (1.17) is satisfied in the form

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\, d\tau \le \frac{1}{2}\|u_0\|_2^2, \quad 0 \le t < T,$$

which implies (1.2), see [14, Theorem V.1.4.1]. Now we conclude that $u$ is the strong and weak solution of (1.1) we are looking for.

The proof of Theorem 1.2(2) is complete. $\square$

*Proof of Theorem 1.2(1).* Suppose (1.6) is satisfied. Then we find $T > 0$ such that (1.8) holds, and Theorem 1.2(2) shows the existence of a unique strong solution $u \in L^8\big(0, T; L^4(\Omega)\big)$ of (1.1) with $u(0) = u_0$. Hence (1.6) is sufficient.

Conversely, assume that $u \in L^8\big(0, T; L^4(\Omega)\big)$ is a strong solution of (1.1) on $[0, T)$, $0 < T \leq \infty$. Then (2.8), (2.12) imply that $v - u \in L^8\big(0, T; L^4(\Omega)\big)$ where $v(t) = e^{-tA}u_0$. Hence

$$\int_0^T \|e^{-tA}u_0\|_4^8 \, dt < \infty.$$

Finally, due to (1.14), $\int_T^\infty \|e^{-tA}u_0\|_4^8 \, dt < \infty$, and $u_0$ satisfies (1.6). This completes the proof of Theorem 1.2. $\square$

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