Inductive Construction of Repletion

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Introduction

In [2] Martin Hyland has proposed the notion of "S-replete object" relative to a given object S as the appropriate generalisation of *predomain* for the purposes of *Synthetic Domain Theory* (SDT). The aim of SDT is to provide a constructive logical framework for reasoning about domains and thus about meanings of functional programs where all functions between predomains are continuous and all endofunctions on domains have least fixpoints. Furthermore, SDT allows one to consider predomains as a *full reflective subcategory* of the ambient category of constructive sets.

For this reason the ambient category is assumed to be a model of some sufficiently strong *impredicative constructive type theory* which will be used as the *internal language* for the ambient category of constructive sets in order to replace complicated external arguments by simpler proofs in the language of type theory.

For a given object S of the ambient category Hyland has defined an object A to be *S-replete* iff A^e is an isomorphism (i.e. one-to-one and onto) whenever e is *S-equable*, i.e. S^e is an isomorphism. As it stands this definition cannot be expressed in the internal language of the ambient category as it involves quantification over all S-equable e which is a hopelessly external collection of maps.

With some effort, however, one can show, cf. [2], that A is S-replete iff the map $\eta_A : A \to S^2(A) = S^{(S^A)}$ with $\eta_A(a) = \lambda p : S(A).\, p(a)$ is monic (i.e. one-to-one) and any S-equable e is an isomorphism provided e factors into η_A via some mono m, i.e. $\eta_A = m \circ e$.

A distinguishing feature of SDT is that predomains form a *full reflective subcategory* of the ambient category, i.e. for any object X there is a morphism $e_X : X \to R(X)$ such that $R(X)$ is S-replete and for any S-replete object A and map $f : X \to A$ there exists a unique extension $\overline{f} : R(X) \to A$ of f along e_X, i.e. $f = \overline{f} \circ e_X$ for a unique $\overline{f} : R(X) \to A$. In [2] it has been shown that $R(X)$ can be defined in the internal language as the *greatest* subobject $m_P : P \hookrightarrow S^2(X)$ such that P contains $\eta_X(X)$ and the unique e_P with $\eta_X = m_P \circ e_P$ is S-equable. This object $R(X)$ which is unique up to isomorphism is called the *S-repletion of X*.

However, under the assumption of a type-theoretic [1] universe Set with $S \in Set$ one may define $A \in Set$ to be S-replete iff A^e is one-to-one and onto for all S-equable $e \in Set$. For this *internal* notion of S-repleteness we will give in this paper an alternative construction of the repletion of $X \in Set$, namely as an *inductively defined subset* of $S^2(X)$. We will define the predicate characterising $R(X)$ as a subset of $S^2(X)$ as the *least closed predicate on $S^2(X)$ containing $\eta_X(X)$*, i.e. the *least closed predicate on $S^2(X)$ containing all neighbourhood filters of points in X*

[1] From the categorical perspective such a type-theoretic universe Set has to be closed especially under *small* limits, i.e. *finite limits* and *small internal products*, i.e. *internal products of families indexed over a type in Set*.

where $P \subseteq S^2(X)$ is called *closed* iff all S-equable $e \in Set$ are *orthogonal* to the inclusion map $m_P : P \hookrightarrow S^2(X)$. Such a least predicate is guaranteed to exist because closed predicates are closed under arbitrary intersections as are predicates containing all neighbourhood filters.

Actually, in realisability models (which we consider as the *intended models*) one may assume the universe Set to be closed also under arbitrary internal products and not only small ones which amounts to having Set as a *full internal internally complete subcategory of the ambient category.* This stronger assumption is reasonable as it is part of the assumption that the ambient category is a model of *impredicative* constructive type theory, cf. [9, 10].

Of course, employing the type-theoretic universe Set with $S \in Set$ the property "S-replete" can be expressed by a formula of the internal language of the ambient category of constructive sets. Hyland's abovementioned construction of repletion together with its verification could be formalised immediately in the internal language of the ambient category.

But we think that our inductive construction of repletion is conceptually simpler because it is a "construction from below" which starting from neighbourhood filters cumulatively adds "limit points". In this sense our inductive construction of repletion is analogous to the construction of the reflection to the category of ω-cpo's, i.e. S-posets closed under limits of ω-chains. In case of repletion the only difference is that a much wider class of limit processes has to be taken into account.

Discussion of Related Work.

After having found our simple inductive construction of repletion we have been pointed out that a related internal construction of repletion has been mentioned without proof by Hyland and Moggi, cf. their remark "(internal version)" at end of p. 108 of [3], albeit in the more specialised framework of quasi–toposes. Moreover, their internal version is different from ours in the respect that they cannot express repleteness (of subobjects of $S^2(X)$) directly in the internal language by quantifying over S-equable maps in some appropriate universe Set as in general the latter need not exists in arbitrary quasi–toposes.

An *external* construction of repletion similar to our's is given in Theorem 32 of [3] as an *external intersection* of all replete $P \subseteq S^2(X)$ containing $\eta_X(X)$. (Of course, one may perform this external construction for each fiber thus providing a reflection from I-indexed families of objects to I-indexed families of S-replete objects for every I in the ambient category.)

But notice that our internal inductive construction of repletion works independently from any external completeness assumptions. On the other hand our assumption of an impredicative universe $Prop$, i.e. $Prop$ being closed under arbitrary (internal) products, does not entail Hyland and Moggi's external completeness assumption (that in each fiber a distinguished class of monos is closed under arbitrary external intersections). Thus, in the end our internal and Hyland and Moggi's external inductive construction of repletion are quite incomparable despite some obvious formal analogies.

However, we think that our *strictly internal approach* is more in accordance with mathematical practice where there is not made any reference to external collections simply because those are not expressible in the language of mathematics (as e.g. set theory). Instead, one prefers to extend one's language by introducing "universes" whose subcollections can be referred to in the extended language by the same linguistic means. In set theory such universes are called "Grothendieck universes" (having been used for quantifying over e.g. all "small" structures of a certain kind as e.g. groups, rings, spaces ...). In type theory the rôle of universes is even more important as it is necessary for quantifying over predicates. However, though appropriate universes are readily available in realisability models of type theory (and of set theory as well) the existence of such universes in Grothendieck toposes is not known (rather one knows that *non-trivial impredicative universes* do *not* exist in Grothendieck toposes).

1 The Ambient Logic

We work in impredicative constructive type theory, see e.g. [9, 10], where besides an *impredicative* universe $Prop$ of proof-irrelevant propositions (i.e. all proofs of a proposition are equal as objects) we assume a further universe Set that contains $Prop$ as a subtype but *not* as an element and that is closed under small dependent sums and (small) dependent products.

Notice that impredicativity of $Prop$ is necessary for expressing repletion in the internal language whereas impredicativity of Set is not needed at all !

Intended models for such type theories are categories of realisability sets over arbitrary partial combinatory algebras (pca-s) $\mathcal{A}$ where Set is interpreted as the collection $PER_{\mathcal{A}}$ of partial equivalence relations (*pers*) on $\mathcal{A}$ and $Prop$ by the subtype of those pers R where $\mathcal{A}/R$ contains at most one element (which are in 1-1-correspondence with $\mathcal{P}(\mathcal{A})$, the collection of subsets of $\mathcal{A}$). Evidently, under this interpretation we have that $Prop \subseteq Set$ but bf not $Prop \in Set$ as $Prop$ is not (isomorphic to) a per.

We assume some S to be fixed about which we assume nothing else but $S \in Set$. For the rest of this paper let $S \in Set$ be arbitrary, but fixed.

Notice that our subsequent treatment of repletion can be fully formalised in such a type theory without any reference to a model. Actually, we have performed such a formalisation which has been checked mechanically using the *LEGO* proof assistant, cf. [5].

Accordingly, all our subsequent definitions and theorems have to be understood as internal to some (model of) type theory satisfying the requirements above.

2 Basic Definitions and Lemmas

We first define some basic notions.

Notice that we will follow the convention that for any A we may write $A(\_)$ instead of $A^{(\_)}$ whenever it increases readability. Typically, we will write $A^n(\_)$ as a shorthand for the n-times application of the functor $A(\_)$ to an object of morphism.

Definition 2.1 *A map $e : Y \to X$ in Set is called S-**epic** (or an S-**epi**) iff S^e is monic, i.e. one–to–one in the sense of the internal logic, and e is called S-**equable** (or an S-**iso**) iff S^e is an isomorphism, i.e. one–to–one and onto in the sense of the internal logic. As S is clear from the context we often simply say equable instead of S-equable.*
*A type A in Set is S-**replete** or simply **replete** iff A^e is an isomorphism for all equable e in Set.*
We write $\mathcal{R}_S$ or simply $\mathcal{R}$ for the full internal subcategory of Set on S-replete objects.

The motivating idea behind the definition of "replete" is that an equable morphism represents a *generalised limit process (g.l.p.)* and that a type A in Set is "replete" iff *A is closed under all generalised limit processes* because the requirement that A^e is an isomorphism, i.e. that A^e is one-to-one and onto, for equable $e : Y \to X$ means that for any $f : Y \to A$ there exists a unique extension of f along

e, i.e. a unique $\overline{f} = (A^e)^{-1}(f) : X \to A$ with $A^e(\overline{f}) = \overline{f} \circ e = f$ as illustrated in the following diagram.

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "f"'] & X \arrow[dl, dotted, "\overline{f} = (A^e)^{-1}(f)"] \\ A & \end{tikzcd}$$

An "prototypical" such limit process is given by the inclusion map $\iota : \omega \hookrightarrow \overline{\omega}$ from the initial into the terminal lift algebra, cf. [2, 4]. From an external perspective the order type of ω is the ordinal ω and that of $\overline{\omega}$ the ordinal $\omega + 1$ containing an extra limit point ∞ as the greatest element of $\overline{\omega}$.

The reason for identifying the S-equable morphisms as the *most general class of limit processes* is that *S itself should be closed under all g.l.p.-s* and therefore any such g.l.p. e must satisfy the requirement that S^e is one-to-one and onto. Thus any g.l.p. must be an S-equable morphism and we may take all of them as g.l.p.-s because this maximal choice will work and we are looking for a class of limit processes which is *as general as possible.*

The replete objects satisfy the following pleasant closure properties.

Lemma 2.1 *The replete objects are closed under finite products and equalisers and all internal products that live in Set.*

Thus, if the universe Set is impredicative, i.e. closed under all internal products, then the replete objects are closed under all internal limits, too.

Proof: That replete objects are closed under arbitrary internal products can be seen as follows. Let $(A_i)_{i \in I}$ be a family of replete objects such that $\Pi i{:}I.A_i$ exists in Set. Suppose that $e : Y \to X$ is equable and $f : Y \to \Pi i{:}I.A_i$. Then $\overline{f} : X \to \Pi i{:}I.A_i$ can be defined pointwise as $\overline{f}(x)(i) := A_i(e)^{-1}(\lambda y : Y.\, f(y)(i))(x)$. It is immediate to verify that $\overline{f} \circ e = f$ and it is unique w.r.t. this property.

The closure under finite products is shown analogously.

For the closure under equalisers assume that $h, k : A \to B$ are functions between replete objects. Let $m : E \hookrightarrow A$ be the equaliser of h and k. Let $e : Y \to X$ be equable and $f : Y \to E$. As A is replete there exists a unique $g : X \to A$ with $m \circ f = g \circ e$. As $h \circ g \circ e = h \circ m \circ f = k \circ m \circ f = k \circ g \circ e$ and B is replete we get $h \circ g = k \circ g$. Therefore there exists a unique $\overline{f} : X \to E$ with $g = m \circ \overline{f}$. It can be seen easily that $\overline{f}$ is the unique extension of f along e. $\square$

Next we define when a subobject as given by a mono is *"closed under generalised limit processes"*.

For this purpose the following notion of *orthogonality* will be helpful.

Definition 2.2 *A morphism $f : Y \to X$ is* orthogonal to *a morphism $g : V \to U$ (for which we write $f \perp g$) iff for all $k : Y \to V$ and $h : X \to U$ with $g \circ k = h \circ f$ there exists a* unique *$\alpha : X \to V$ with $k = \alpha \circ f$ and $h = g \circ \alpha$ as illustrated in the following diagram.*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[d, "h"] \arrow[dl, dotted, "\alpha"] \\ V \arrow[r, "g"'] & U \end{tikzcd}$$

Notice that the orthogonality relation between morphisms is *not symmetric* as one is used to from geometry.

Definition 2.3 *A subobject $m_P : P \hookrightarrow Z$ in Set is* **closed** *(under g.l.p.-s) iff $e \perp m_P$ for all equable e in Set.*

More explicitly, this means that for any equable $e : Y \to X$ and arbitrary $f : X \to Z$ such that $f \circ e$ factors through m_P via a unique $g : Y \to P$, i.e. $m_P \circ g = f \circ e$, there exists a unique $h : X \to P$ with $f = m_P \circ h$.

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "g"'] & X \arrow[d, "f"] \arrow[dl, dotted, "h"] \\ P \arrow[r, "m_P"'] & Z \end{tikzcd}$$

As m_P is required to be monic the further requirement $g = h \circ e$ follows from $m_P \circ g = f \circ e = m_P \circ h \circ e$.

If A is a replete object then a mono $m_P : P \hookrightarrow A$ is "closed" iff for any g.l.p. $e : Y \to X$ and "generalised e-chain in A" $c : Y \to A$ that factors through m_P via a (unique) $g : Y \to P$, i.e. $c = m_P \circ g$, the "unique extension f of c along e" factors through m_P as well (via a unique h). That means that "all limit points" added by the "unique extension f of c along the g.l.p. e" are themselves contained in P.

If for e we take the "prototypical" limit process as given by the inclusion $\iota : \omega \hookrightarrow \overline{\omega}$ (of the initial lift algebra into the terminal lift algebra) that means that for any chain $c : \omega \to A$ factoring through m_P its extension $\overline{c} : \overline{\omega} \to A$ factors through m_P as well, i.e. the limit point $\overline{c}(\infty)$ is contained in P provided all "approximating" chain elements are contained in P.

Next we prove a lemma relating the notions "replete" and "closed" in a way most useful for the proof of our main theorem.

Lemma 2.2 *If A is replete and $m_P : P \hookrightarrow A$ is a subobject of A then P is replete iff m_P is closed.*

Proof: Consider the diagram

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "g"'] & X \arrow[d, "f"] \arrow[dl, dotted, "h"] \\ P \arrow[r, "m_P"'] & A \end{tikzcd}$$

where e is assumed to be equable.

For the implication from left to right assume that P is replete. If the above square commutes and e is equable then there exists a unique h with $h \circ e = g$ as P is replete by assumption. Then we have also $m_P \circ h \circ e = m_P \circ g = f \circ e$ which implies that $m_P \circ h = f$ as e is equable and A is replete. The morphism h is unique w.r.t. the required property as m_P is monic.

For the reverse direction assume that $e \perp m_P$. For showing that P is replete let $g : Y \to P$ be an arbitrary map and $e : Y \to X$ be equable. As A is replete there exists a unique extension $f : X \to A$ of $m_P \circ g$ along e, i.e. $m_P \circ g = f \circ e$.

Due to the assumption $e \perp m_P$ there exists a unique $h : X \to P$ with $g = h \circ e$ (and $m_P \circ h = f$). Thus h extends g along e. For uniqueness suppose $g = k \circ e$. Then $m_P \circ k \circ e = m_P \circ g = m_P \circ h \circ e$. As A is replete and e is equable we get $m_P \circ k = m_P \circ h$ and therefore $k = h$ as m_P is a mono. $\square$

We finish this section with a useful characterisation of equable maps due to Paul Taylor, cf. [11].

Lemma 2.3 *An S-epi $e : Y \to X$ is S-equable iff e factors into $\eta_Y : Y \to S^2(Y)$ where η is the natural transformation from Id_{Set} to $S^2(\_) : Set \to Set$ whose component at Y is given by $\eta_Y(y) = \lambda p : S(Y).p(y)$.*

Proof: If e is S-equable then there is a unique $f : X \to S^2(Y)$ with $\eta_Y = f \circ e$ as $S^2(Y)$ is S-replete.

On the other hand if $\eta_Y = f \circ e$ for some (necessarily unique) map $f : X \to S^2(Y)$ then we have $S(\eta_Y) = S(e) \circ S(f)$. Straightforward computation shows that $S(\eta_Y) \circ \eta_{S(Y)} = id_{S(Y)}$. Therefore we get

$$S(e) \circ S(f) \circ \eta_{S(Y)} = id_{S(Y)}$$

(i.e. $S(e)$ is split epi) which by the assumption that $S(e)$ is monic implies that $S(e)$ is also an isomorphism.

Thus e is S-equable. $\square$

Now we have gathered all notions and lemmas necessary for proving our main result.

3 Inductive Construction of Repletion

The task of *repletion* is to define for every $X \in Set$ a so-called *repletion map* $e_X : X \to R(X)$ such that $R(X)$ is replete and for any $f : X \to A$ in Set with A replete there is a *unique* map $\overline{f} : R(X) \to A$ with $f = \overline{f} \circ e_X$. Of course, this e_X is unique up to isomorphism.

Our main theorem shows how for any $X \in Set$ such an $R(X)$ can be *constructed inductively* as a subobject of $S^2(X)$.

Theorem 3.1 *Let $X \in Set$. We define $R(X)$ as the intersection of all closed subobjects of $S^2(X)$ containing $\eta_X(X)$, i.e.*

$$R(X) =_{def} \bigcap \{ P \subseteq S^2(X) \mid P \ closed\ ,\ \eta_X(X) \subseteq P \}$$

which itself is a closed subobject of $S^2(X)$ and contains $\eta_X(X)$ and therefore S-replete.

Let $m_X : R(X) \hookrightarrow S^2(X)$ be the inclusion map for $P \subseteq S^2(X)$ and let $e_X : X \to R(X)$ send $x \in X$ to $\eta_X(x) \in R(X) \subset S^2(X)$.

Then e_X is S-equable and therefore a repletion map *for X, i.e. for any $f : X \to A$ with A replete there exists a unique extension $\overline{f}$ of f along e_X, i.e. $f = \overline{f} \circ e_X$.*

Proof: First we show that closed subobjects of a type A are closed under arbitrary internal intersections. Let $P : I \to Prop^A$ be an I-indexed family of predicates on A. Suppose that $P(i)$ is closed for all $i \in I$. We have to show that the predicate $\lambda a{:}A.\, \forall i{:}I.\, P(i)(a)$ is closed as well. Assume that $e : Y \to X$ is equable and $f : X \to A$ is a map with $\forall y{:}Y.\, \forall i{:}I.\, P(i)(f(e(y)))$ which is equivalent to $\forall i{:}I.\, \forall y{:}Y.\, P(i)(f(e(y)))$. As each $P(i)$ is closed we get that $\forall i{:}I.\, \forall x{:}X.\, P(i)(f(x))$

and therefore $\forall x{:}X.\, \forall i{:}I.\, P(i)(f(x))$. Thus we have shown that the predicate $\lambda a{:}A.\, \forall i{:}I.\, P(i)(a)$ is closed.

Therefore, $R(X) \subseteq S^2(X)$ is a closed predicate and contains $\eta_X(X)$ as it is an intersection of predicates containing $\eta_X(X)$. By Lemma 2.2 it follows that $R(X)$ is replete.

As by construction $e_X : X \to R(X)$ factors into η_X via m_X, i.e. $\eta_X = m_X \circ e_X$, it follows from Lemma 2.3 that e_X is equable iff e_X is S-epic, i.e. $S(e_X)$ is monic.

Let $p, q : R(X) \to S$ with $p \circ e_X = q \circ e_X$. Let $eq : E \hookrightarrow R(X)$ be the equaliser of p and q which by Lemma 2.1 is in $\mathcal{R}_S$ as p and q are in $\mathcal{R}_S$. As e_X by assumption equalises p and q it factors through eq, i.e. $e_X = eq \circ h$ for some unique $h : X \to E$. But then $m_X \circ eq$ is a closed subobject of $S^2(X)$ by Lemma 2.2 as E is replete and $m_X \circ eq$ contains η_X as $\eta_X = m_X \circ e_X = m_X \circ eq \circ h$. But then m_X is contained in $m_X \circ eq$ and therefore eq is an iso which implies that $p = q$.

Thus $S(e_X)$ is a mono, i.e. e_X is S-epic, which – as already remarked – entails that e_X is also S-equable.

But, if e_X is S-equable then for any $f : X \to A$ with A replete there exists a unique extension $\overline{f} : R(X) \to A$ of f along e_X, i.e. $f = \overline{f} \circ e_X$. $\square$

We think that, conceptually, the most important consequence of this construction of repletion is that $R(X)$, the *repletion* of X, is constructed as an *inductively defined subset* of $S^2(X)$. In the internal language this construction can be formalized as follows:

$$\phi \in R(X) \;:\Leftrightarrow \forall P : \mathcal{P}(S^2(X)).\, [\, closed(P) \wedge \eta_X(X) \subseteq S^2(X)\,] \;\Rightarrow\; P(\phi)$$

where

$$\begin{aligned} & closed(P : \mathcal{P}(S^2(X))) \;:\Leftrightarrow \\ & \quad \forall V, U : Set.\, \forall e : U^V.\, equable(e) \Rightarrow \\ & \quad\quad \forall f : S^2(X)^U.\, [\forall v : V.\, P(f(e(v)))] \Rightarrow \forall u : U.\, P(f(u)) \end{aligned}$$

and

$$equable(e : U^V) :\Leftrightarrow \forall g : S(V).\, \exists! f : S(U).\, g = f \circ e \;.$$

Notice that $closed(P) \wedge \eta_X(X) \subseteq S^2(X)$ means $\Phi(P) \subseteq P$ where Φ is the endofunction on $\mathcal{P}(S^2(X))$ defined as

$$\begin{aligned} \phi \in \Phi(P) :\Leftrightarrow\; & \exists x : X.\, \phi = \eta_X(x) \vee \\ & \exists V, U : Set, e : U^V, f : S^2(X)^U, u : U. \\ & \quad equable(e) \wedge [\forall v : V.\, P(f(e(v)))] \wedge x = f(u) \end{aligned}$$

which, obviously, is monotonic w.r.t. set inclusion, i.e. $\Phi(P) \subseteq \Phi(Q)$ whenever $P \subseteq Q$, and therefore we have

$$\phi \in R(X) \;\Leftrightarrow\; \forall P : \mathcal{P}(S^2(X)).\, \Phi(P) \subseteq P \;\Rightarrow P(\phi) \;.$$

Thus $R(X)$ *is an inductively defined subset of* $S^2(X)$*, namely the least fixpoint of* Φ. There are *two rules generating* $R(X)$ (corresponding to the two clauses of the disjunction defining Φ):

1. If $x \in X$ then the neighbourhood filter $\eta_X(x) \in R(X)$.
2. If $e : V \to U$ is equable, i.e. e is a generalised limit process, and $f : U \to S^2(X)$ is a map with $f(e(v)) \in R(X)$ for all $v \in V$ then $f(u) \in R(X)$ for all $u \in U$.

Thus $R(X)$ is the subset of $S^2(X)$ generated from neighbourhood filters of points in X by cumulatively adding limit points generated by generalised limit processes as given by equable maps in the universe Set.

Notice that in the above use of the internal language for the *purpose of forming propositions* we have allowed ourselves *quantification over elements of types not contained in* Set, e.g. when defining *equable*, *closed* and Φ, *but we have nowhere assumed that* Set *is closed under arbitrary products.* We only have assumed that Set is closed under function spaces and under equalisers (which is guaranteed by the assumptions that $Prop$ contains all identity types, $Prop$ is included in Set and Set is closed under small dependent sum and small dependent function spaces).

Thus, when checking our construction and its verification within the LEGO proof checker we have used the impredicative universe $Prop$ provided by *LEGO* as our $Prop$ and the predicative universe $Type_0$ of *LEGO* as our Set, cf. [5]. But, as in our formalisation we never have used the *LEGO* assumption that $Prop \in Set$ our formalisation is consistent with the assumption that Set itself is impredicative (as in the modification of the *LEGO* system used by Bernhard Reus in his Thesis [7]).

We conclude this section by some alternative characterisations of repleteness and repletion.

First we get as a rather immediate consequence of Theorem 3.1 the following alternative characterisation of replete objects.

Lemma 3.1 *Let* $A \in Set$*. Then* A *is* S*-replete iff* η_A *is a closed monomorphism.*

Proof: If A is replete then e_A is an isomorphism where $e_A^{-1} = \overline{id_A}$ is the unique extension of id_A along e_A which exists as A is replete by assumption and e_A is equable by Theorem 3.1 (and therefore $\overline{id_A} \circ e_A = id_A$ and $e_A \circ \overline{id_A} = id_{R(A)}$ as we have $e_A \circ \overline{id_A} \circ e_A = e_A \circ id_A = id_{R(A)} \circ e_A$). Now as $\eta_A = m_A \circ e_A$ and e_A is an isomorphisms we get that η_A is isomorphic to m_A which is a closed monomorphism.

For the reverse direction assume that $\eta_A : A \hookrightarrow S^2(A)$ is a closed monomorphism. Then by Lemma 2.2 we get that A is replete (as $S^2(A)$ is replete). $\square$

Intuitively, that means that A is replete iff

- any point of $a \in A$ is uniquely determined by $\eta_A(a)$, its neighbourhood filter, and
- the set $\eta_A(A)$ of neighbourhood filters is *closed* in $S^2(A)$.

The following construction of repletion is due to Hyland [2].

Lemma 3.2 *Let* A *be a set and* $\eta_A = m \circ e$ *where* $m : P \hookrightarrow S^2(A)$ *and* $e : A \to P$ *is* S*-epic. Then* $P \subseteq R(A)$*, i.e. there exists an* $\alpha : P \to R(A)$ *with* $m = m_A \circ \alpha$*.*

Thus, $R(A)$ *is the greatest subobject* P *of* $S^2(A)$ *such that* P *contains* η_A *and this inclusion is* S*-epic.*

Proof: Suppose $m_A \circ \eta_A = m \circ e$ where $m : P \hookrightarrow S^2(A)$ and $e : A \to P$ is S-epic. By 2.3 the map e is equable as it factors into η_A. As $R(A)$ is replete there exists a unique $\alpha : P \to R(A)$ with $e_A = \alpha \circ e$. But now, as $m \circ e = m_A \circ e_A = m_A \circ \alpha \circ e$, e is S-epic and $S^2(A)$ is a power of S it follows that $m = m_A \circ \alpha$. $\square$

Notice that this construction of $R(A)$ does not refer to the collection of all S-equable maps. Instead, $R(A)$ is constructed as the em union of all $P \subseteq S^2(A)$ with $\eta_A(A)$ dense in P.

4 Restricting Generalised Limit Processes

In this section we will give characterisations of repleteness and constructions of repletion which use only a *restricted* class of generalised limit processes. As we will see it is sufficient to take into account only certain classes of equable subobjects, i.e. monomorphisms along which maps to S extend in a unique way.

First we show that $\mathcal{R}_S$ is the least full reflective subcategory of *Set* which contains S. Actually, this was the original motivation of the notion of "S-replete" in [2]).

Lemma 4.1 *Let $\mathcal{C}$ be a full reflective subcategory of Set which is closed under isomorphisms in Set. Let $R_{\mathcal{C}} : Set \to \mathcal{C}$ be the reflection functor, i.e. left adjoint to the inclusion of $\mathcal{C}$ into Set.*

If $S \in \mathcal{C}$ then $\mathcal{R}_S \subseteq \mathcal{C}$ contains $\mathcal{R}_S$ and $R_{\mathcal{C}}$ preserves S-equable maps.

Proof: Let A be S-replete. We have to show that $A \in \mathcal{C}$. As by assumption $\mathcal{C}$ is closed under isomorphisms in *Set* it suffices to show that the reflection map $r_A : A \to R_{\mathcal{C}}(A)$ is an isomorphism. As $S \in \mathcal{C}$ the reflection map r_A is S-equable. Thus, there is a unique $f : R_{\mathcal{C}}(A) \to R(A)$ with $f \circ r_A = e_A$. As A is replete the map e_A is an isomorphism and we have $e_A^{-1} \circ f \circ r_A = e_A^{-1} \circ e_A = id_A$. Thus, we also have that $r_A \circ e_A^{-1} \circ f \circ r_A = r_A \circ id_A = id_{R_{\mathcal{C}}(A)} \circ r_A$ which entails that $r_A \circ e_A^{-1} \circ f = id_{R_{\mathcal{C}}(A)}$ as r_A is a reflection map for A to $\mathcal{C}$. But then $e_A^{-1} \circ f$ is the desired inverse to r_A.

Let $e : Y \to X$ be an S-equable map in *Set*. For showing that $R_{\mathcal{C}}(e)$ is S-equable, too, let $g : R_{\mathcal{C}}(Y) \to S$. We must show that there exists a unique $f : R_{\mathcal{C}}(X) \to S$ with $g = f \circ R_{\mathcal{C}}(e)$. As $r_X \circ e$ is S-equable there exists a unique $f : R_{\mathcal{C}}(X) \to S$ with $f \circ r_X \circ e = g \circ r_Y$. As $r_X \circ e = R_{\mathcal{C}}(e) \circ r_Y$ we get that $g \circ r_Y = f \circ r_X \circ e = f \circ R_{\mathcal{C}}(e) \circ r_Y$. As r_Y is S-equable we conclude that $g = f \circ R_{\mathcal{C}}(e)$. The construction is visualized in the diagram below.

$$\begin{tikzcd}
Y \arrow[rr, "e"] \arrow[d, "r_Y"'] & & X \arrow[d, "r_X"] \\
R_{\mathcal{C}}(Y) \arrow[rr, "R_{\mathcal{C}}(e)"] \arrow[dr, "g"'] & & R_{\mathcal{C}}(X) \arrow[dl, "f"] \\
& S &
\end{tikzcd}$$

For uniqueness suppose that $g = f' \circ R_{\mathcal{C}}(e)$. But then $g \circ r_Y = f' \circ R_{\mathcal{C}}(e) \circ r_Y = f' \circ r_X \circ e$ which implies that $f' = f$ as $r_X \circ e$ is S-equable. $\square$

From this lemma we immediately get the following improved characterisation of repleteness.

Theorem 4.1 *Let $\mathcal{C}$ be a full reflective subcategory of Set with $S \in \mathcal{C}$. Then a set A is S-replete iff $A \in \mathcal{C}$ and $e \perp A$ for all equable e in $\mathcal{C}$.*

Proof: $\Rightarrow$: By Lemma 4.1 a replete set A is contained in $\mathcal{C}$ and as $e \perp A$ for all equable e in *Set* it especially holds that $e \perp A$ for all equable e in $\mathcal{C}$.
$\Leftarrow$: Suppose $A \in \mathcal{C}$ and $e \perp A$ for all equable e in $\mathcal{C}$. Now let $e : Y \to X$ be an equable map in *Set* and $f : Y \to A$. From Lemma 4.1 we know that $R_{\mathcal{C}}(e)$ is an equable map in $\mathcal{C}$ and $R_{\mathcal{C}}(e) \circ r_Y = r_X \circ e$ where r_Y and r_X are the reflection

maps to $\mathcal{C}$ for Y and X, respectively. Let $f' : R_{\mathcal{C}}(Y) \to A$ be the unique map with $f = f' \circ r_Y$. As $R_{\mathcal{C}}(e)$ is an equable map in $\mathcal{C}$, due to our assumption, there exists a unique map $g' : R_{\mathcal{C}}(X) \to A$ with $f' = g' \circ R_{\mathcal{C}}(e)$. Thus, for $g := g' \circ r_X$ we have

$$g \circ e = g' \circ r_X \circ e = g' \circ R_{\mathcal{C}}(e) \circ r_Y = f' \circ r_Y = f$$

i.e. g extends f along e.
For uniqueness of this extension suppose that $f = h \circ e$. Then there exists a unique $h' : R_{\mathcal{C}}(X) \to A$ with $h = h' \circ r_X$. For this h' we have $f' \circ r_Y = h' \circ R_{\mathcal{C}}(e) \circ r_Y$ from which it follows that $f' = h' \circ R_{\mathcal{C}}(e)$ as $A \in \mathcal{C}$ and r_Y is a reflection map to $\mathcal{C}$ for Y. But from $f' = h' \circ R_{\mathcal{C}}(e)$ we immediately get that $h' = g'$ (as $R_{\mathcal{C}}(e) \perp A$) and, therefore, that $h = g$. $\square$

The next lemma gives a simplified criterion for closedness of monomorphisms living in a full reflective subcategory $\mathcal{C}$ of Set with $S \in \mathcal{C}$.

Lemma 4.2 *Let $\mathcal{C}$ be a full reflective subcategory of Set with $S \in \mathcal{C}$ and let $m_P : P \hookrightarrow Z$ be a monomorphism living in $\mathcal{C}$, i.e. with $Z, P \in \mathcal{C}$.*
Then m_P is closed iff $e \perp m_P$ for all equable e in $\mathcal{C}$.

Proof: The implication from left to right is trivial.
Suppose $m_P : P \hookrightarrow Z$ is a monomorphism living in $\mathcal{C}$ with $e \perp m_P$ for all equable e in $\mathcal{C}$. We have to show that $e \perp m_P$ for all equable e not necessarily living in $\mathcal{C}$. Let $e : Y \to X$ be an arbitrary equable map in Set and suppose that $f \circ e = m_P \circ g$. As $P, Z \in \mathcal{C}$ there exist unique $f' : R_{\mathcal{C}}(X) \to Z$ and $g' : R_{\mathcal{C}}(Y) \to P$ with $f' \circ r_X = f$ and $g' \circ r_Y = g$.

$$\begin{tikzcd}
Y \arrow[r, "e"] \arrow[d, "r_Y"'] & X \arrow[d, "r_X"] \\
R_{\mathcal{C}}(Y) \arrow[r, "R_{\mathcal{C}}(e)"] \arrow[d, "g'"'] & R_{\mathcal{C}}(X) \arrow[d, "f'"] \arrow[dl, dotted, "\alpha"] \\
P \arrow[r, "m_P"'] & Z
\end{tikzcd}$$

That the lower square in the above diagram commutes follows from

$$m_P \circ g' \circ r_Y = m_P \circ g = f \circ e = f' \circ r_X \circ e = f' \circ R_{\mathcal{C}}(e) \circ r_Y$$

as r_Y is a reflection map to $\mathcal{C}$ and Z is in $\mathcal{C}$.
Now due to our assumption about m_P there exists a unique $\alpha : R_{\mathcal{C}}(X) \to P$ with $g' = \alpha \circ R_{\mathcal{C}}(e)$ and $f' = m_P \circ \alpha$ as due to Lemma 4.1 the map $R_{\mathcal{C}}(e)$ is also equable. Putting $h := \alpha \circ r_X$ we get

$$g = g' \circ r_Y = \alpha \circ R_{\mathcal{C}}(e) \circ r_Y = \alpha \circ r_X \circ e = h \circ e$$

and

$$f = f' \circ r_X = m_P \circ \alpha \circ r_X = m_P \circ h\,.$$

Of course, h is unique with these properties as m_P is monic. $\square$

From this now we get the following simplified construction of repletion.

Theorem 4.2 *Let $\mathcal{C}$ be a full reflective subcategory of Set with $S \in \mathcal{C}$. Then for any $X \in Set$ the repletion $R(X)$ of X is obtained as the intersection of all subobjects $m_P : P \hookrightarrow S^2(X)$ such that*

- *$P \in \mathcal{C}$ and $e \perp m_P$ for all equable e in $\mathcal{C}$*
- *P contains $\eta_X(X)$, i.e. all neighbourhood filters of points in X .*

Proof: The monomorphism $m_X : R(X) \hookrightarrow S^2(X)$ lives in $\mathcal{R}_S$ and therefore in $\mathcal{C}$, and, furthermore, contains $\eta_X(X)$. As m_X is closed it is especially the case that $e \perp m_X$ for all equable e in $\mathcal{C}$.
On the other hand suppose that $P \subseteq S^2(X)$ with $P \in \mathcal{C}$ and $\eta_X(X) \subseteq P$ such that $e \perp m_P$ for all equable e in $\mathcal{C}$. From the latter assumption now it follows by Lemma 4.2 that m_P is closed as both P and $S^2(X)$ are in $\mathcal{C}$. Thus, P is a closed subobject of $S^2(X)$ containing $\eta_X(X)$ from which it follows that $R(X) \subseteq P$. $\square$

The intuitive meaning of the previous Theorem 4.2 is that in order to construct $R(X)$ one may form the intersection of a much smaller collection of subobjects of $S^2(X)$ containing $\eta_X(X)$, namely those living in $\mathcal{C}$ and closed under g.l.p.-s living in $\mathcal{C}$, whenever $\mathcal{C}$ is a full reflective subcategory of *Set* with $S \in \mathcal{C}$.

Typical such choices of $\mathcal{C}$ are *S-posets* and *extensional S-posets* which we are going to investigate next.

Definition 4.1 *An $A \in Set$ is called* S-separated *or an* S-poset *iff*

$$\forall a, a' : A.\, [\forall p : S(A).\, p(a) = p(a')] \Rightarrow a = a'$$

i.e. $\eta_A : A \hookrightarrow S^2(A)$. We write Sep_S for the full reflective subcategory of Set on S-separated objects.

Such objects are called "S-posets" because for such an A the "information ordering" on its elements

$$x \sqsubseteq_A y \;:\Leftrightarrow\; \forall p : S(A).\, p(x) \Rightarrow p(y)$$

is not only reflexive and transitive but also antisymmetric.

The following Lemma is easy to prove.

Lemma 4.3 *Sep_S is a full reflective subcategory of Set and closed under subobjects. Furthermore, S is S-separated and therefore $\mathcal{R}_S \subseteq Sep_S$.*

Proof: For $A \in Set$ its reflection to Sep_S is the image of η_A, i.e. the set of all neigbourhood filters in $S^2(A)$.
If $i : A' \hookrightarrow A$ and $\forall p : S(A').\, p(x) = p(y)$ then $\forall p : S(A).\, p(i(x)) = p(i(y))$. As A is S-separated we have $i(x) = i(y)$ and as i is monic we get $x = y$.
The set S is S-separated as if $\forall p{:}S(S).\, p(u) = p(v)$ for some $u, v \in S$ then by taking id_S for p we get that $u = id_S(u) = id_S(v) = v$, i.e. $u = v$.
Now by Lemma 4.1 it follows immediately that $\mathcal{R}_S \subseteq Sep_S$. $\square$

When choosing for $\mathcal{C}$ the full subcategory Sep_S of *Set* we obtain a simplified characterisation of repleteness and a simplified inductive construction of repletion.

But first notice the following fact which seems to have been overlooked in the literature on SDT.

Lemma 4.4 *If A is S-separated and $e : A \to X$ is an S-equable map in Set then e is monic.*

Proof: If e is S-equable then by Lemma 2.3 there exists a (unique) $f : X \to S^2(A)$ with $\eta_A = f \circ e$. As η_A is assumed to be monic it follows that e is monic as well. □

From this and the previous lemmas we get the following theorem.

Theorem 4.3 *Let $A \in Set$. Then*

1. *A is replete iff A is an S-poset and $e \perp A$ for all* equable monomorphisms e *in Sep_S.*
2. *$R(A)$ is the intersection of all subobjects $m_P : P \hookrightarrow S^2(A)$ containing $\eta_A(A)$ such that $e \perp m_P$ for all* equable monomorphisms e *in Sep_S.*

Proof: By Lemma 4.3 the equable maps in Sep_S are precisely the equable subobjects in Sep_S. Therefore, when choosing $\mathcal{C}$ as Sep_S the first claim follows from Theorem 4.1 and the second claim follows from Theorem 4.2 as subobjects of $S^2(A)$ are always S-separated. □

Thus, *an S-poset is replete iff it is closed under g.l.p.-s that are monos in Sep_S.* Accordingly, when defining $R(A)$ inductively one starts with $\eta_A(A)$ and cumulatively adds new points obtained by extension along monic equables in Sep_S. This *drastically* reduces the number of generalised limit processes that have to be taken into account for defining repleteness and constructing repletion.

Before considering an even coarser class of equable subobjects we observe some simplifications which can be achieved by restricting attention to Sep_S. First we recall a characterisation of repleteness within Sep_S which is due to P. Taylor [11].

Lemma 4.5 *A set A is S-replete iff A is S-separated and any equable $e : A \to B$ in Sep_S is an isomorphism.*

Proof: $\Rightarrow$: Let $A \in Set$ be replete. Then A is S-separated as A appears as a subobject of $S^2(A)$. Suppose that $e : A \to B$ is equable and η_B is monic. As A is replete and e is equable there exists a map $f : B \to A$ with $id_A = f \circ e$. Then $e = e \circ f \circ e$ and also $\eta_B \circ e = \eta_B \circ e \circ f \circ e$. As $S^2(B)$ is a power of S and e is equable it follows that $eta_B = eta_B \circ e \circ f$. As η_B is monic we get $id_B = e \circ f$ and therefore e is an isomorphism.
$\Leftarrow$: If any equable $e : A \to B$ in Sep_S is an isomorphism then in particular $e_A : A \to R(A)$ is an isomorphism as $R(A) \subseteq S^2(A)$ is S-separated. Thus, A is replete. □

Hyland's version of repletion can be improved as follows.

Lemma 4.6 *Let A be S-separated. Then $e_A : A \hookrightarrow R(A)$ is the greatest extension of A in Sep_S such that the corresponding inclusion is equable.*

Proof: If A is S-separated then e_A is an equable monomorphism and $R(A)$ is S-separated. If $e : A \hookrightarrow B$ is equable with B in Sep_S then there exists a (unique) map $f : B \to R(A)$ with $e_A = f \circ e$. But, as $\eta_B \circ e = S^2(e) \circ \eta(A)$ by naturality of η we have $\eta_B \circ e = S^2(e) \circ m_A \circ e_A = S^2(e) \circ m_A \circ f \circ e$ from which irt follows that $\eta_B = S^2(e) \circ m_A \circ f$ as e is equable. Thus, $f : B \hookrightarrow R(A)$ as η_B is monic. □

This lemma makes precise the intuition that the *repletion of a set is its greatest S-separated extension inducing a bijection of "S-open subsets".*

As already mentioned there is an even more restricted class of generalised limit process sufficient for the purposes of characterising repleteness and inductive construction of repletion provided *we assume that equality on S is $\neg\neg$-closed*, i.e.

$$\forall u, v{:}S.\ \neg\neg\, u = v \Rightarrow u = v\ .$$

Equality on arbitrary types is $\neg\neg$-closed in all realisability models of constructive type theory independent from the underlying partial combinatory algebra. Thus, from our point of view this is a more than reasonable assumption, actually almost a logical assumption, as we consider realisability models as the indended models of SDT.

We first have to recall some terminology and results from [7] which gives an axiomatic treatment of the extensional pers of [1].

Definition 4.2 *A subobject $m_P : P \hookrightarrow X$ is $\neg\neg$-closed iff the predicate P satisfies $\forall x{:}X.\ \neg\neg P(x) \Rightarrow P(x)$ and $e : X \to Y$ is called a* $\neg\neg$-epi *iff the image of e is $\neg\neg$-dense in Y, i.e. $\forall y{:}Y. \neg\neg\exists x{:}X. y = e(x)$.*
An $A \in Set$ is called an extensional S-poset *iff $\eta_A : A \hookrightarrow S^2(A)$ is a $\neg\neg$-closed subobject of $S^2(A)$, i.e. the predicate*

$$\lambda\Phi{:}S^2(A).\, \exists x{:}A.\, \forall p{:}S(A).\, \Phi(p) = p(x)$$

on $S^2(A)$ is $\neg\neg$-closed. We write EP_S for the full subcategory of Set on extensional S-posets.

Thus an $A \in Set$ is an extensional S-poset iff any $x \in A$ is characterised uniquely by its neighbourhood filter and the neighbourhood filters form a $\neg\neg$-closed subset of $S^2(A)$.

We will need the following fact which is proved and discussed in detail in [7].

Fact 4.1 *The category EP_S of extensional S-posets is a full reflective subcategory of Set where reflection is given by*

$$R_{EP_S}(A) = \{\ \Phi \in S^2(A) \mid \neg\neg\exists x{:}A.\, \Phi = \eta_A(x)\ \}\ .$$

for any $A \in Set$. Furthermore, S is an extensional poset itself (as equality on S is $\neg\neg$-closed) and extensional S-posets are closed under $\neg\neg$-closed subobjects.
The $\neg\neg$-epis and $\neg\neg$-closed monomorphisms form a factorisation system in Set. If $f : X \to Y$ is a map in Set then $f = m \circ e$ where $m : \{\ y \in Y \mid \neg\neg\exists x{:}X.\ y = f(x)\ \} \hookrightarrow Y$.

From this we get the following important consequence.

Remark 4.1 *A subobject $m_P : P \hookrightarrow X$ in Set is $\neg\neg$-closed iff $e \perp m$ for all $\neg\neg$-epis e in Set.*

Now we may show that $R(A)$ is a $\neg\neg$-closed subobject of $S^2(A)$.

Lemma 4.7 *For any $A \in Set$ the subobject $m_A : R(A) \hookrightarrow S^2(A)$ is $\neg\neg$-closed.*

Proof: Let $m_A = m' \circ e'$ be the factorisation of $m_A : R(A) \hookrightarrow S^2(A)$ into a $\neg\neg$-epi e' followed by the $\neg\neg$-mono $m' : \neg\neg R(A) \hookrightarrow S^2(A)$.

As equality on S is $\neg\neg$-closed by assumption it follows that any $\neg\neg$-epi is also S-epic. Thus e' is S-epic and therefore $e' \circ e_A$ is an S-epi factoring into η_A via m'. By Lemma 2.3 it follows that $e' \circ e_A$ is also S-equable and therefore e' is equable, too. As $R(A)$ is replete there exists a (unique) $f : \neg\neg R(A) \to R(A)$ with $f \circ e' = id_{R(A)}$. Therefore, we have $m' \circ e' = m' \circ e' \circ f \circ e'$ from which it follows that $m' = m' \circ e' \circ f$ as e' is an S-epi. Thus, $id_{\neg\neg R(A)} = e'_A \circ f$ as m' is monic. Thus, e' is an isomorphism (with inverse f) by which $m_A : R(A) \hookrightarrow S^2(A)$ is isomorphic to the $\neg\neg$-closed $m' : \neg\neg R(A) \hookrightarrow S^2(A)$.

Thus, the subobject m_A is $\neg\neg$-closed, too. □

As EP_S is a full reflective subcategory of Set containing S and m_A is $\neg\neg$-closed we might apply Theorem 4.2 to obtain an inductive construction of repletion by taking for $R(A)$ the least $\neg\neg$-closed subobject $P \subseteq S^2(A)$ with $\eta_A(A) \subseteq P$ and $e \perp m_P$ for all equable e in EP_S. But, actually, we can do better using the following simplified characterisation of closed $\neg\neg$-closed subobjects of S-poset.

Lemma 4.8 *Let A be an extensional S-poset and $m_P : P \hookrightarrow A$ be $\neg\neg$-closed. Then m_P is closed (in the sense of Definition 2.3) iff $e \perp m_P$ for all equable $\neg\neg$-closed subobjects e of extensional S-posets.*

Proof: The implication from left to right is obvious.

For the reverse direction assume that $e \perp m_P$ for all equable $\neg\neg$-closed subobjects e of extensional S-posets. As EP_S is a full reflective subcategory of Set and $S \in EP_S$ by Lemma 4.2 it suffices to show that $e \perp m_P$ for all equable e in EP_S.

Now suppose that $e : Y \to X$ is an equable map in EP_S with $f \circ e = m_P \circ g$. Let $e = m' \circ e'$ be a factorisation of e into a $\neg\neg$-epi $e' : Y \to Z$ followed by a $\neg\neg$-closed subobject $m' : Z \hookrightarrow X$. Then e' is an S-epi as it is a $\neg\neg$epi. Thus, $S(e')$ is monic and as $S(e) = S(e') \circ S(m')$ is an isomorphism it follows that $S(m')$ is an isomorphism as well, i.e. m' is equable.

We have to construct a map $h : X \to P$ with $h \circ e = g$. Notice that in this case the further equality $f = m_P \circ h$ then follows from e being equable and A being S-separated. Now consider the following diagram.

$$\begin{tikzcd}
Y \arrow[r, "e'"] \arrow[dr, "g"'] & Z \arrow[r, "m'"] \arrow[d, dotted, "\alpha"] & X \arrow[d, "f"] \arrow[dl, dotted, "h"] \\
& P \arrow[r, "m_P"'] & A
\end{tikzcd}$$

As e' is a $\neg\neg$-epi and $e' \perp m_P$ (by Remark 4.1 as m_P is $\neg\neg$-closed) there exists a unique map α with $g = \alpha \circ e'$ and $m_P \circ \alpha = f \circ m'$. As m' is an equable $\neg\neg$-closed subobject of the extensional S-poset X it follows from our assumption about m_P that $m' \perp m_P$ and therefore there exists a map $h : X \to P$ with $\alpha = h \circ m'$.

Thus, for h we have $g = h \circ m' \circ e' = h \circ e$. Uniqueness of h follows from e being equable and A being S-separated. $\square$

Now we can establish the following characterisation of repleteness and construction of repletion which refers only to the restricted class of generalised limit processes consisting of equable $\neg\neg$-closed subobjects of extensional S-posets.

Theorem 4.4 *For all $A \in Set$ we have that*

1. *A is replete iff A is an extensional S-poset and A^e is one-to-one and onto whenever e is an equable $\neg\neg$-closed subobject of some extensional S-poset.*

2. *$R(A)$ is the intersection of all $\neg\neg$-closed subobjects $P \subseteq S^2(A)$ such that $\eta_A(A) \subseteq P$ and $e \perp m_P$ for all equable $\neg\neg$-closed subobjects e of extensional S-posets.*

Proof: We first show the second claim. By Lemma 4.7 the monomorphism $m_A : R(A) \hookrightarrow S^2(A)$ is $\neg\neg$-closed and by its construction it is also closed which by Lemma 4.8 is equivalent to $e \perp m_A$ for all equable $\neg\neg$-closed subobjects e of extensional S-posets (as $S^2(A)$ is replete and therefore an extensional S-poset). Obviously, m_A is the least such subobject of $S^2(A)$ containing $\eta_A(A)$ (because m_A is even least among all closed subobjects of $S^2(A)$ containing $\eta_A(A)$).

Now we show the first claim. Suppose that A is replete. As A^e is one-to-one and onto for all equable e in Set this holds in particular for all equable e that are

$\neg\neg$-closed subobjects of some extensional S-poset. Furthermore, m_A is isomorphic to η_A as e_A is an isomorphism when A is replete. Therefore, η_A is a $\neg\neg$-closed subobject of $S^2(A)$ as m_A is $\neg\neg$-closed by Lemma 4.7. Thus, A is an extensional S-poset as by Fact 4.1 extensional S-posets are closed under $\neg\neg$-closed subobjects.

For the reverse direction assume that A is an extensional S-poset, i.e. η_A is a $\neg\neg$-closed subobject of $S^2(A)$, and A^e is one-to-one and onto for all equable $\neg\neg$-closed subobjects e of some extensional S-poset. We will show that $\eta_A : A \to S^2(A)$ is a closed subobject of $S^2(A)$ which by Lemma 3.1 is equivalent to A being replete. By the already established second claim of our Theorem it suffices to show that $e \perp \eta_A$ for all equable $\neg\neg$-closed subobjects e of some extensional S-poset. Suppose that $e : Y \hookrightarrow X$ is an equable $\neg\neg$-closed subobject of some extensional S-poset X and $f : X \to S^2(A)$ with $f \circ e$ factoring through η_A via some (unique) $g : Y \to A$, i.e. $f \circ e = \eta_A \circ g$. Due to our assumption about A there exists a unique $h : X \to A$ with $g = h \circ e$. As e is equable and $S^2(A)$ is replete we conclude $f = \eta_A \circ h$ from $f \circ e = \eta_A \circ g = \eta_A \circ h \circ e$. As $\eta_A \circ h \circ e = f \circ e = \eta_A \circ g$ and η_A is monic we also have $h \circ e = g$. The uniqueness of h w.r.t. its desired properties follows from the assumption that η_A is monic. $\square$

From B. Reus' Thesis [7] we know that the extensional S-posets are precisely the $\neg\neg$-closed subobjects of powers of S. Therefore, Theorem 4.4 tells us that it is sufficient to take into account as generalised limit processes just the *collection of all equable $\neg\neg$-closed subobjects of extensional S-posets*, i.e. all $D \subseteq_{\neg\neg} A$ where $A \subseteq_{\neg\neg} S(I)$ for some $I \in Set$ such that any map $p : D \to S$ uniquely extends to a map $\overline{p} : A \to S$ (i.e. $\overline{p}_{|D} = p$). As extensional S-posets are closed under $\neg\neg$-closed subobjects the previous theorem tells us how to characterise repleteness *completely inside* the full reflective subcategory EP_S of extensional S-posets. In that case one has to consider *only equable $\neg\neg$-closed subobjects of extensional S-posets* as generalised limit processes.

Maybe, this is not too big an advantage from the logical point of view (as the numerous double negations do not necessarily make life easier) but there is a great benefit when considering our intended models, namely realisability models. The point is that in realisability models the $\neg\neg$-closed subobjects correspond precisely to the *external subset* of realisability set inheriting the realisability relation from their supersets.

Thus, it suffices to take into account as generalised limit processes just those classical subspaces $D \subseteq_{\neg\neg} A$ where A is a classical subspace of some power of S and maps to S extend uniquely along the classical inclusion $D \hookrightarrow_{\neg\neg} A$. This is fairly concrete as any power of S arises as quotient of $S(\mathcal{A})$ by some $\neg\neg$-closed equivalence relation (where $\mathcal{A}$ is the realisability set whose underlying set of elements is the partial combinatory algebra $\mathcal{A}$ on which the realisability model under consideration is based and where any $a \in \mathcal{A}$ is realised precisely a).

Nevertheless, we are still far away from a good understanding (i.e. classification !) of equable $\neg\neg$-closed subobjects of extensional S-posets. But maybe it's hoping too much that such a classification is available for a class of things defined in such an indirect way. So there remains the problem that our understanding of repletion is only as concrete as our understanding of generalised limit processes. As we don't know how they look like in general we don't know concretely which limit points are actually added in the process of repletion.

5 Comparison with Reflection on Complete S-Posets

We will now explain in which sense our inductive construction of repletion is nothing else but a straightforward generalisation of the usual *reflection to ω-complete S-*

posets.

We recall some notion from [4] where it is claimed as an axiom that the inclusion $\iota : \omega \hookrightarrow \overline{\omega}$ of the initial into the terminal lift algebra is S-equable, i.e. that $S(\iota)$ is one-to-one and onto.

An ω-complete S-poset is an S-poset A such that $A(\iota)$ is an one-to-one and onto. For an ω-complete S-poset A we say that $m_P : P \hookrightarrow A$ is *closed under ω-chains* or simply *ω-complete* iff $\iota \perp m_P$, i.e. for any *ω-chain* $c : \omega \to A$ factoring through m_P its unique extension along ι, namely $\overline{c} := A(\iota)^{-1}(c) : \overline{\omega} \to S^2(A)$, does factor through m_P as well.

Whereas ω-completeness means "closure under limits of ω-chains" repleteness means "closure under all generalised limit processes" (similarly for "closed under ω-chains" vs. "closed"). Despite the delightful elegance of the notion of repleteness it is not clear up to now what would be the eventual benefit of working with replete objects instead of ω-complete S-posets. The reason is that we do not know much more equable maps than ι. Of course, for any $A \in Set$ its repletion map $e_A : A \to R(A)$ is equable but, alas, itself defined in a fairly indirect way. A rather big class of equable maps is given by the inclusions $B_D \subseteq D$where B_D is the basis of some effectively given domain D (of course, after identifying such a D as an object in Set where the ambient category is ω-Set, the Kleene realisability sets, cf. [6]).

Anyway, we will now show that ω-complete S-posets form a full reflective subcategory of Set.

For a given $X \in Set$ one constructs its reflection $C(X)$ as the least ω-complete subobject P of $S^2(X)$ containing $\eta_X(X)$. Of course, the intersection of ω-complete S-posets containing $\eta_X(X)$ is itself ω-complete and contains $\eta_X(X)$. Analogous to Lemma 2.2 one can prove that a subobject of an ω-complete S-poset is an ω-complete S-poset iff the corresponding inclusion map is closed under limits of ω-chains. Therefore $C(X)$ itself is ω-complete.

We write $i_X : C(X) \hookrightarrow S^2(X)$ for the obvious inclusion of $C(X)$ into $S^2(X)$ and let $r_X : X \to C(X)$ be the unique map with $\eta_X = i_X \circ r_X$. The reflection map r_X is an S-epi as if $p, q : C(X) \to S$ with $p \circ r_X = q \circ r_X$ then its equaliser is also an ω-closed subobject of $S^2(X)$ containing $\eta_X(X)$ (as r_X factors through the equaliser of p and q due to the assumption that $p \circ r_X = q \circ r_X$) and therefore the equaliser is an isomorphism and thus we get $p = q$. (As $\eta_X = i_x \circ r_X$ the map r_X is even equable.)

It remains to show that for any $f : X \to A$ with A an ω-complete S-poset there exists a unique $\overline{f} : C(X) \to A$ with $f = \overline{f} \circ r_X$. As $S^2(\_)$ is a natural transformation we have $S^2(f) \circ \eta_X = \eta_A \circ f$. As A is an ω-complete S-poset the inclusion map $\eta_A : A \hookrightarrow S^2(A)$ is an ω-closed subobject. As ω-closed subobjects are stable under pullbacks along arbitrary morphisms the pullback $p := S^2(f)^* \eta_A$ of η_A along $S^2(f)$ is also an ω-complete subobject. The situation is illustrated by the following diagram.

$$\begin{tikzcd}
X \arrow[d, "r_X"'] \arrow[dr, dotted, "\alpha"] \arrow[drr, "f"] & & \\
C(X) \arrow[r, dotted, "\beta"'] \arrow[dr, "i_X"'] & P \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta_A"] \\
& S^2(X) \arrow[r, "S^2(f)"'] & S^2(A)
\end{tikzcd}$$

We observe that p contains $\eta_X(X)$ (via the mediating arrow α). Therefore i_X is contained in p, i.e. there is a unique map $\beta : C(X) \to P$ with $i_X = p \circ \beta$.

It remains to show that $\alpha = \beta \circ r_X$ as the diagram suggests. We have

$$p \circ \alpha = \eta_X = i_X \circ r_X = p \circ \beta \circ r_X$$

and

$$\eta_A \circ q \circ \alpha = S^2(f) \circ p \circ \alpha = S^2(f) \circ i_X \circ r_X = S^2(f) \circ p \circ \beta \circ r_X = \eta_A \circ q \circ \beta \circ r_X$$

from which we get

$$q \circ \alpha = q \circ \beta \circ r_X$$

as η_A is monic. Thus α and $\beta \circ r_X$ are mediating arrows for the same cone and thus equal, i.e. we have the desired equality

$$\alpha = \beta \circ r_X \quad .$$

Now by putting $\overline{f} := q \circ \beta$ we get a map $\overline{f} : C(X) \to A$ satisfying $f = \overline{f} \circ r_X$ as the diagram above has been shown to commute.

The uniqueness of $\overline{f}$ follows from the fact that r_X is an S-epi and η_A is monic.

Thus we have shown that the ω-complete S-posets actually form a full reflective subcategory of *Set*. The construction is similar to our construction of repletion but - suprisingly - its verification is quite different from the one for the case of repletion.

6 Conclusion

Assuming a universe *Set* of small sets which is closed under small limits we have given a definition of S-repleteness in terms of the internal language of the ambient type theory *analogous* to the usual external definition. The only difference is that internally we quantify over equable maps living in *Set*. As a benefit we obtain an intuitive inductive construction of the repletion of some $X \in Set$: starting from the neighbourhood filters of points in X, i.e. the image of η_X, we cumulatively add limit points obtained by (unique) extensions of maps along generalised limit processes as given by equable maps in *Set*.

We think that this (re-)interpretation of equable maps as generalised limit processes has clarified their *intuitive* meaning whereas in the original papers on SDT the equable maps entered the scene like a *deus ex machina*. The best of motivation we could extract from these sources and discussions with their authors was something like "...the definition of repleteness resembles the definition of sheaves ...", see e.g. G. Rosolini's "Notes on Synthetic Domain Theory" [8].

Nevertheless, even under our (re-)interpretation the process of repletion is still very abstract in the sense that we do not know at all how equable maps look like in general. However, we have managed to restrict the class of generalised limit processes to those equable maps that are ($\neg\neg$-closed) subobjects of (extensional) S-posets. That means intuitively that it is enough to consider (classical) subspaces such that the corresponding inclusions satisfy a unique extension property for maps to S.

Nevertheless, it is not yet clear what "one has gained" by knowing that an object is replete and not simply an ω-closed S-poset. The reason is that we do not know enough examples of equable maps "occuring in nature". Therefore, it might be sufficient for a *logic of functional programs* to consider the wider class of ω-complete S-posets. As remarked in [4] the ω-complete S-posets constitute a *maximal* choice of a class of predomains whereas S-replete objects constitute a *minimal* such choice.

Anyway, one should emphasize that the definition of ω-complete S-poset and closedness under ω-chains does *not* require a universe like Set closed under small limits. The reason is that *one does not have to quantify over a class of generalised limit processes* but one has to consider only one limit process, namely the inclusion $\iota : \omega \hookrightarrow \overline{\omega}$. This has the consequence that everything can be expressed in the internal language *without any external quantification* even if there is no universe like Set available (as, seemingly, in the case of Grothendieck toposes).

Though we know that replete objects satisfy all the desired closure properties guaranteeing that all data types given by recursive domain equations will live in the world of replete objects there is still the problem left whether base types like Nat (natural numbers) and $Bool = 2 = 1 + 1$ (booleans) are replete. This definitely is the case for the model based on Kleene realisability (cf. [7] for a proof under rather general axiomatic assumptions). But for other realisability models, as e.g. the one based on the Böhm tree model for the untyped λ-calculus, Longley and Simpson have conjectured that Nat and $Bool$ are not replete.

These arguments suggest that ω-complete posets are a much more flexible notion than replete objects because they can be used in a wider class of models - especially when one insists on expressing everything inside their internal language. But for a lot of realisability models replete objects are very well-behaved, e.g. when based on recursion-theoretic or domain-theoretic partial combinatory algebras. However, even for the classical case of Kleene realisability they lead to a lot of difficult recursion-theoretic questions whose surface has just been scratched.

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