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Inquisitive Bisimulation

????? In Modal Logics ¿¿¿¿¿

Inquisitive Logic
Amsterdam 2018

Martin Otto joint work with Ivano Ciardelli

content

- **bisimulation:**
  the quintessential back&forth
- **inquisitive modal & epistemic logic InqML:**
  one level up from standard Kripke models with a built-in team semantic level on top of modal logic
- **inquisitive bisimulation & Ehrenfeucht-Fraïssé**
  back&forth somewhere between FO and MSO
- **characterisation theorems InqML $\equiv$ FO$/\sim$**
  expressive completeness results over two-sorted relational structures

bisimulation (background)

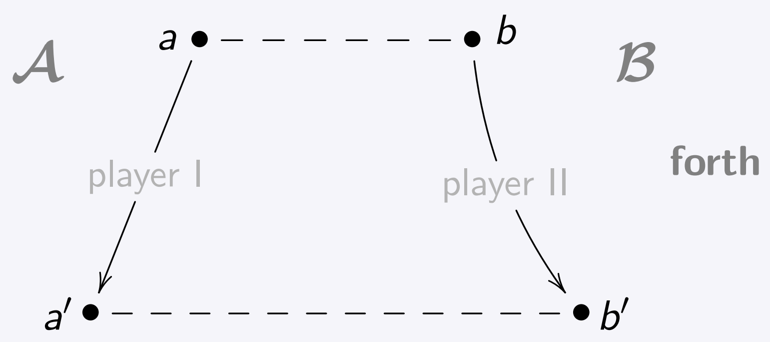
- bisimulation: back&forth/zig-zag
- bisimulation invariance: the hallmark of modal semantics

modal model theory
= **model theory of bisimulation invariance**

inquisitive bisimulation: bisimulation inquisitive modal logic inquisitive bisimulation expressive completeness

bisimulation: back&forth

game protocol for testing equivalence
between pointed Kripke models $\mathcal{A}, a$ and $\mathcal{B}, b$



player I: challenge equivalence (move along accessibility edge)
player II: respond & maintain propositional equivalence

- II has strategy in unbounded game: $\mathcal{A}, a \sim \mathcal{B}, b$
- II has strategy for ℓ rounds: $\mathcal{A}, a \sim^{\ell} \mathcal{B}, b$

bisimulation: modal Ehrenfeucht–Fraïssé

a special case in the tradition of back&forth equivalences in classical logic, viz. its adaptation to $\Box/\Diamond$ quantification:

modal Ehrenfeucht–Fraïssé thm

for any two pointed Kripke models *in a finite signature* and $\ell \in \mathbb{N}$:

$$\mathcal{A}, a \sim^{\ell} \mathcal{B}, b \quad \Leftrightarrow \quad \mathcal{A}, a \equiv^{\ell}_{\mathsf{ML}} \mathcal{B}, b$$

in particular (& w/o restriction on signature):

semantics of ML is invariant under bisimulation

$$\mathcal{A}, a \sim \mathcal{B}, b \quad \Rightarrow \quad \mathcal{A}, a \equiv_{\mathsf{ML}} \mathcal{B}, b$$

inquisitive bisimulation: bisimulation inquisitive modal logic inquisitive bisimulation expressive completeness

bisimulation: characteristic formulae

a special case in the tradition of back&forth equivalences in classical logic, viz. its adaptation to $\Box/\Diamond$ quantification:

modal Ehrenfeucht–Fraïssé thm (refined)

for any pointed Kripke model $\mathcal{A}, a$ *in a finite signature* and $\ell \in \mathbb{N}$ there is a *characteristic formula* $\chi^{\ell}_{\mathcal{A},a} \in \mathsf{ML}_{\ell}$ such that

$$\mathcal{B}, b \models \chi^{\ell}_{\mathcal{A},a} \quad \Leftrightarrow \quad \mathcal{B}, b \equiv^{\ell}_{\mathsf{ML}} \mathcal{A}, a$$

$\rightsquigarrow$ disjunctions of $\chi^{\ell}_{\mathcal{A},a}$ as normal form for $\begin{cases} \sim^{\ell}\text{-closed properties} \\ \mathsf{ML}_{\ell}\text{-formulae} \end{cases}$

bisimulation: expressive completeness

van Benthem–Rosen thm $\mathrm{FO}/\!\sim\; \equiv \mathrm{ML}$

classically and in fmt,
t.f.a.e. for $\varphi(x) \in \mathrm{FO}$:

(i) φ $\sim$-invariant
(ii) $\varphi \equiv \varphi' \in \mathrm{ML}$
(iii) $\varphi \equiv \varphi' \in \mathrm{ML}_\ell$ for some $\ell \in \mathbb{N}$
(iv) φ $\sim^\ell$-invariant for some $\ell \in \mathbb{N}$

compactness property
for $\sim$-invariance

with many variations for other classes of (finite) frames

Janin–Walukiewicz thm $\mathrm{MSO}/\!\sim\; \equiv \mathrm{L}_\mu$

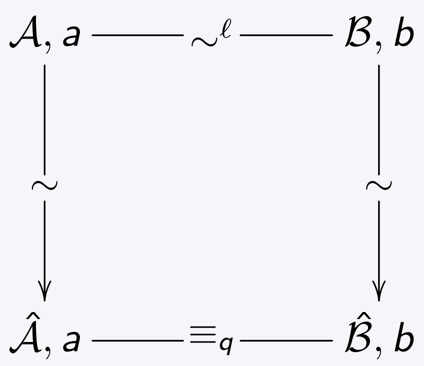
remains notoriously open in fmt !

expressive completeness through upgrading

for $\mathrm{FO}/\!\sim\; \subseteq \mathrm{ML}$ over (non-elementary classes) $\mathcal{C}$:

upgrading argument for
compactness property

from $\sim$ to $\sim^\ell$
for $\varphi \in \mathrm{FO}_q$



the rest is Ehrenfeucht–Fraïssé!

from modal to inquisitive modal (background)

standard modal models/Kripke structures
for the semantics of basic modal logic ML

- set of possible worlds W
- propositional assignment $\rho\colon p \mapsto \rho(p) \in 2^W$
  globally assigning semantics to proposition p
- accessibility relation(s) $R \subseteq W \times W$
  or function(s) $\sigma : W \longrightarrow 2^W$, $w \longmapsto \sigma(w) := R[w]$

  locally assigning set(s) of accessible worlds: *information states*

  for semantics of modal $\Box/\Diamond$ at w: FO-quantification over $R[w]$

from modal to inquisitive modal

from **modal assignment** of sets of accessible worlds:

$$\sigma\colon w \quad \longmapsto \quad \sigma(w) \in 2^W$$

possible worlds

in epistemic reading: $\sigma(w)$ = (lack of) knowledge in w
the *information state* as the set
of equally possible worlds at w
for semantics of $\Box/\Diamond$

to an **inquisitive assignment** of sets of information states:

$$\Sigma\colon w \quad \longmapsto \quad \Sigma(w) \in 2^{2^W}$$

possible answers

in epistemic reading: $\Sigma(w)$ = possible updates in w
the *inquisitive state* as the set
of possible information updates at w
for semantics of inquisitive modalities $\boxplus/\diamondplus$

inquisitive models (functional format)

augment Kripke structures $\mathcal{K} = (W, \sigma, \rho)$
to inquisitive structures $\mathcal{K} = (W, \Sigma, \rho)$

- set of possible worlds W
- propositional assignment $\rho\colon p \longmapsto \rho(p) \in \rho(p) \in 2^W$
  for semantics of proposition p
- inquisitive assignment(s) $\Sigma\colon w \longmapsto \Sigma(w) \in 2^{2^W}$
  for semantics of inquisitive modalities $\boxplus/\diamondplus$

with

- induced modal assignment(s) $\sigma\colon u \longmapsto \bigcup \Sigma(u) \in 2^W$
  for semantics of modal $\Box/\Diamond$

inquisitive models (relational format, two-sorted!)

from Kripke structures $\mathcal{K} = (W, R, \rho)$ with $R \subseteq W \times W$
to inquisitive structures $\mathbb{K} = (W, E, \rho)$ with $E \subseteq W \times 2^W$

encode $\Sigma\colon w \longmapsto \Sigma(w) \in 2^{2^W}$ by its graph $E \subseteq W \times 2^W$
in a two-sorted relational structure with

- first sort: possible worlds, W
- second sort: information states, $S \subseteq 2^W$

linked by two mixed-sorted relations in $W \times S$:

- $E \subseteq W \times S$ (the graph of Σ)
- set-theoretic $\in \subseteq W \times S$ (built-in like $=$)

with induced modal accessibility relation(s)

- $R = E \circ \in^{-1}$ (the graph of $\sigma\colon w \longmapsto \bigcup \Sigma(w)$)

inquisitive modal logic InqML $\supseteq$ ML

satisfaction relation, team semantic style, here: *support semantics*

linking | information states over $\mathbb{K}$, i.e. $s \in 2^W$ | and | formulae $\varphi \in$ InqML

read $\mathbb{K}, s \models \varphi$ as "s supports φ"

with $\mathbb{K}, \{w\} \models \varphi$ emulating $\mathbb{K}, w \models \varphi$ for $\varphi \in$ ML

semantic constraints on models:

- inquisitive assignments $\Sigma(w)$ *downward closed* in 2^W (!)

and for (multi-agent) epistemic setting:

- induced modal σ_a/R_a are S5 with classes $[w]_a = \sigma_a(w)$
- each Σ_a constant on its equivalence classes $[w]_a = \sigma_a(w)$

syntax and semantics for InqML $\supseteq$ ML

atoms $p, \bot$: $\mathbb{K}, s \models p$ if $s \subseteq \rho(p)$ — flat

$\mathbb{K}, s \models \bot$ iff $s = \emptyset$

strong disjunction $\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$: $\mathbb{K}, s \models \varphi_1 \mathbin{\rotatebox[origin=c]{90}{$\geqslant$}} \varphi_2$ if $\mathbb{K}, s \models \varphi_1$ or $\mathbb{K}, s \models \varphi_2$ — non-flat

team implication $\to$: $\mathbb{K}, s \models \varphi \to \psi$ if for all $s' \subseteq s$ $\mathbb{K}, s' \models \varphi \;\Rightarrow\; \mathbb{K}, s' \models \psi$ — non-flat

inquisitive modalities $\boxplus$:

$$\mathbb{K}, s \models \boxplus\varphi \quad \text{if} \quad \begin{cases} \mathbb{K}, s' \models \varphi \\ \text{for all } s' \in \Sigma(w), w \in s \end{cases}$$ flattening

induced plain modalities $\Box$:

$$\mathbb{K}, s \models \Box\varphi \quad \text{if} \quad \begin{cases} \mathbb{K}, \{v\} \models \varphi \\ \text{for all } v \in \sigma(w), w \in s \end{cases}$$ flattening

some examples (involving questions)

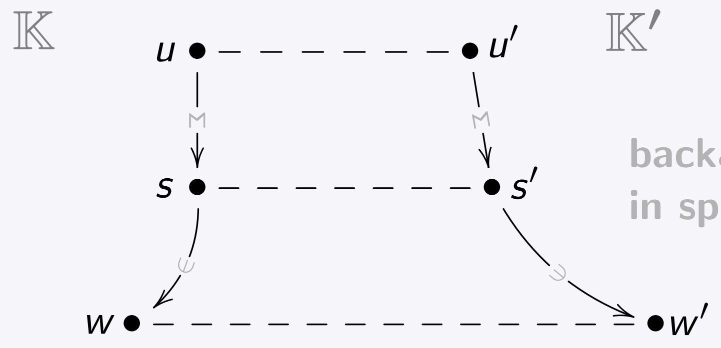
$?\varphi := \varphi \vee\!\!\!\vee \neg\varphi$ captures "question *whether* φ"
or whether φ is settled *either way*

crucially non-flat

	supported by s in $\mathbb{K}$ iff
$?\varphi$	"s settles φ"
$\boxplus\,?\,\varphi$	"every information update in s settles φ"
$\Box\,?\,\varphi$	"all information updates in s settle φ the same"
$\neg\Box\,?\,\varphi \wedge \boxplus\,?\,\varphi$	"the open question φ gets settled in s"

inquisitive bisimulation

testing inquisitive equivalence in back&forth game:



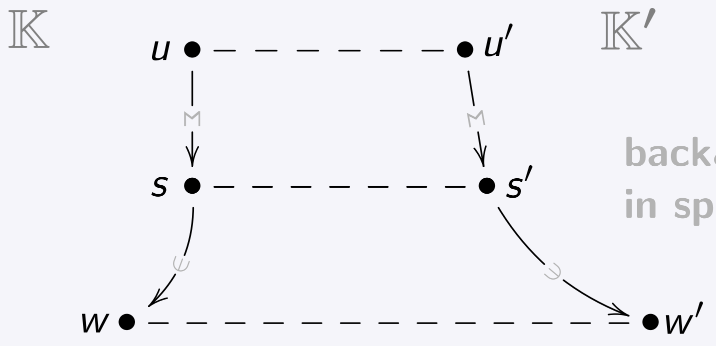
back&forth in split rounds

in interleaving challenge/response steps

from matching worlds (u, u')
to matching information states $(s, s') \in \Sigma(u) \times \Sigma(u')$
to matching worlds $(w, w') \in s \times s'$ ...

inquisitive bisimulation

testing inquisitive equivalence in back&forth game:



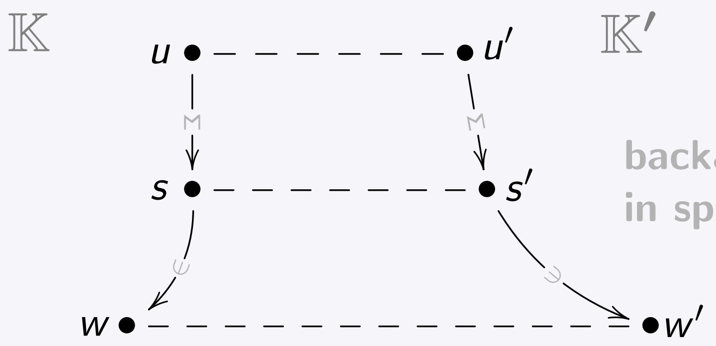
back&forth in split rounds

inquisitive bisimulation game $\rightsquigarrow$ natural notions of bisimilarity

$$\left.\begin{array}{l}\mathbb{K}, z \sim \mathbb{K}', z' \\ \mathbb{K}, z \sim^{\ell} \mathbb{K}', z'\end{array}\right\} \text{for world or state pairs } z, z'$$

inquisitive bisimulation

testing inquisitive equivalence in back&forth game:



back&forth in split rounds

inquisitive bisimulation game $\rightsquigarrow$ natural notions of bisimilarity

NB: these are symmetric *bi*-simulation equivalences
with built-in focus on $\downarrow$-closed state properties

NB: flattening ($s \to u$) vs. inquisitive expansion ($u \to s$)

inquisitive Ehrenfeucht–Fraïssé

inquisitive Ehrenfeucht–Fraïssé thm

for world- or state-pointed inquisitive models
in a finite signature and $\ell \in \mathbb{N}$:

$$\mathbb{K}, z \sim^{\ell} \mathbb{K}', z' \quad \Leftrightarrow \quad \mathbb{K}, z \equiv^{\ell}_{\mathsf{InqML}} \mathbb{K}', z'$$

in particular (& w/o restriction on signature):

InqML invariant under inquisitive bisimulation

$$\mathbb{K}, z \sim \mathbb{K}', z' \quad \Rightarrow \quad \mathbb{K}, z \equiv_{\mathsf{InqML}} \mathbb{K}', z'$$

inquisitive Ehrenfeucht–Fraïssé: characteristic formulae

inquisitive Ehrenfeucht–Fraïssé thm (refined)

for world- or state-pointed model $\mathbb{K}, z$
in a finite signature and $\ell \in \mathbb{N}$
there is a *characteristic formula* $\chi^{\ell}_{\mathcal{K},z} \in \mathsf{InqML}_{\ell}$ such that

$$\mathcal{K}', w' \models \chi^{\ell}_{\mathbb{K},w} \quad \Leftrightarrow \quad \mathbb{K}', z' \equiv^{\ell}_{\mathsf{InqML}} \mathbb{K}, z$$

$$\text{or} \quad \mathcal{K}', s' \models \chi^{\ell}_{\mathbb{K},s} \quad \Leftrightarrow \quad \mathbb{K}', s' \equiv^{\ell}_{\mathsf{InqML}} \mathbb{K}, t \;\text{ for some } t \subseteq s$$

$\rightsquigarrow$ **normal forms for (downward&) $\sim$-closed properties**

construct characteristic formulae by induction, in parallel for worlds/information states/inquisitive states

inquisitive Ehrenfeucht–Fraïssé: characteristic formulae

detail for experts: simultaneous induction $\ell \rightsquigarrow \ell+1$ for $\sim^\ell$-types of worlds/information states/inquisitive states

$$\chi^0_w \;=\; \text{propositional type of } w \in W \text{ (for } \ell = 0)$$

$$\chi^\ell_s = \bigvee \{ \chi^\ell_w : w \in s \} \qquad \sim^\ell\text{-type of } s \in 2^W \ (\downarrow)$$

$$\chi^\ell_\Pi = \mathbin{\rotatebox[origin=c]{180}{$\bigwedge$}}\!\!\!\bigvee \{ \chi^\ell_s : s \in \Pi \} \qquad \sim^\ell\text{-type of } \Pi \in 2^{2^W}$$

$$\chi^{\ell+1}_w = \chi^\ell_w \wedge \boxplus\chi^\ell_{\Sigma(w)} \wedge \bigwedge \{ \neg\boxplus\chi^\ell_\Pi : \Pi \subseteq \Sigma(w),\ \Pi \not\sim^\ell \Sigma(w) \}$$

$$\sim^{\ell+1}\text{-type of } w \in W$$

inquisitive bisimulation: bisimulation inquisitive modal logic inquisitive bisimulation expressive completeness

bisimulation invariance & compactness (1)

in relational format the actual extension of the second sort $S \subseteq 2^W$ in $\mathbb{K} = (W, S, E, \rho)$ is relatively free up to $\sim$

natural levels:

$S = 2^W$	full/maximal	×
$S \supseteq \bigcup_{u \in W} 2^{\sigma(u)}$	locally full	✓
$S \supseteq \bigcup_{s \in \Sigma(u)} 2^s$	minimal req.	✓

downward closure is a non-elementary condition (!)

compactness as a limitation:

over full/maximal format, FO/$\sim$ fails to satisfy compactness

InqML is known to be compact

failures of compactness in full/maximal scenario

in two-sorted models $\mathbb{K} = (W, E, \rho)$ with second sort $S = 2^W$, with induced standard Kripke structure $\mathcal{K} = (W, R, \rho)$:

$$\mathbf{FO}[\mathbb{K}, w] \supseteq \mathbf{MSO}[\mathcal{K}, w]$$

$$\mathbf{FO}/{\sim}[\mathbb{K}, w] \supseteq \mathbf{L}_\mu[\mathcal{K}, w]$$

e.g. can express "no infinite R-paths" (wellfoundedness of R^{-1})

which is incompatible with "no dead ends": $\{\Box^n \Diamond \top : n \in \mathbb{N}\}$

bisimulation invariance and compactness (2)

remaining natural levels:

~~$S = 2^W$~~	~~full/maximal~~	
$S \supseteq \bigcup_{u \in W} 2^{\sigma(u)}$	locally full	✓
$S \supseteq \bigcup_{s \in \Sigma(u)} 2^s$	min. req.	✓

over these non-elementary classes of two-sorted models:

InqML $\supseteq$ FO/$\sim$

is again equivalent to

compactness property for $\sim$-invariance

$\sim$-invariance $\Rightarrow$ $\sim^\ell$-invariance for some $\ell \in \mathbb{N}$

e.g. want $\mathbb{K}, w \sim^\ell \mathbb{K}', w' \quad \Rightarrow \quad \Big(\mathbb{K}, w \models \varphi \Leftrightarrow \mathbb{K}', w' \models \varphi \Big)$

Q: why so? ; for which ℓ?

characterisation theorems

⇝ expressive completeness for van Benthem–Rosen style characterisations of InqML:

$$\mathrm{FO}/\!\sim \;\equiv\; \mathrm{InqML} \text{ over } \mathcal{C} \quad \text{(classically and fmt)}$$

over remaining feasible classes $\mathcal{C}$ of two-sorted relational inquisitive structures

all these classes are non-elementary and combine FO and MSO features

- simpler case for basic InqML: local unfolding & stratification
- more challenging for multi-agent epistemic InqML with its extra constraints on S5 models

characterisation theorems

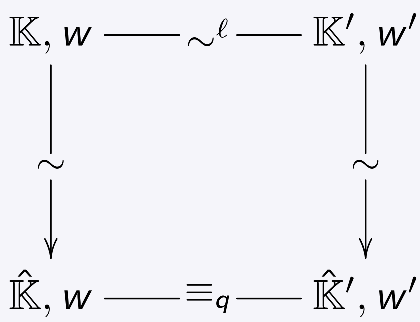
⇝ expressive completeness for van Benthem–Rosen style characterisations of InqML:

$$\mathrm{FO}/\!\sim \;\equiv\; \mathrm{InqML} \text{ over } \mathcal{C} \quad \text{(classically and fmt)}$$

for expressive completeness $\mathrm{FO}/\!\sim \;\subseteq\; \mathrm{InqML}$:

upgrading argument for compactness property

from $\sim$ to $\sim^{\ell}$ over $\mathcal{C}$ for $\varphi \in \mathrm{FO}_q$

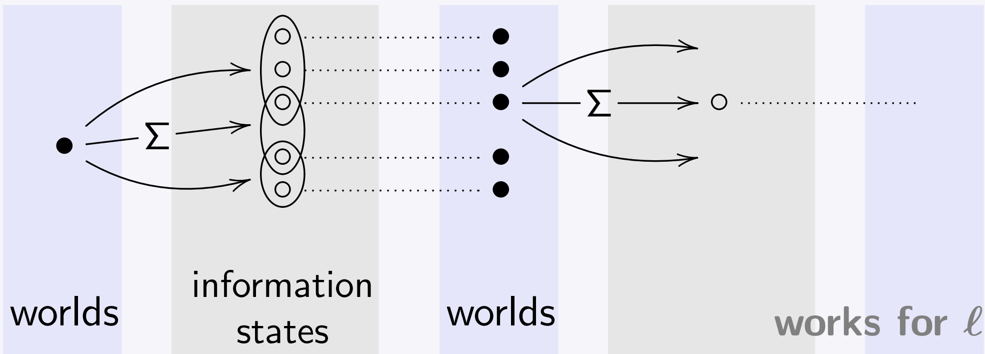


expressive completeness via upgrading (I): basic InqML

towards FO/$\sim$ $\subseteq$ InqML e.g. over the classes $\mathcal{C}/\mathcal{C}_{\mathrm{fin}}$ of locally full **relational inquisitive models**

- upgrading $\sim^{\ell}$ to $\equiv_q$ over $\mathcal{C}/\mathcal{C}_{\mathrm{fin}}$ using FO-locality:

  local unfolding & world/state-layer stratification
  with fresh worlds to instantiate information states



works for $\ell \approx 2^q$

expressive completeness via upgrading (II): S5 InqML

towards FO/$\sim$ $\subseteq$ InqML e.g. over the classes $\mathcal{C}/\mathcal{C}_{\mathrm{fin}}$ of locally full **relational inquisitive S5 models**

upgrading requires:

- local pre-processing of inquisitive assigments $\Sigma_a(w)$ in $[w]_a$

  need to boost multiplicities in $[w]_a$
  w.r.t. the relevant $\sim/\sim^{\ell}$-types (!)
  to escape MSO counting up to 2^q

- global pre-processing of overlap pattern between classes $[w]_a$

  want local tree-likeness to depth 2^q
  in hypergraph structure of the $[w]_a$
  to escape FO-detection of cycles

expressive completeness via upgrading (II): S5 InqML

towards $\mathrm{FO}/{\sim} \subseteq \mathrm{InqML}$ e.g. over the classes $\mathcal{C}/\mathcal{C}_{\mathrm{fin}}$ of locally full **relational inquisitive S5 models**

upgrading requires:

- local pre-processing of inquisitive assigments $\Sigma_a(w)$ in $[w]_a$
  $\rightsquigarrow$ simple lattice algebra & compositionality for unary MSO
- global pre-processing of overlap pattern between classes $[w]_a$
  $\rightsquigarrow$ treatment of S5 Kripke structures in Dawar–O_09

expressive completeness via upgrading (II): S5 InqML

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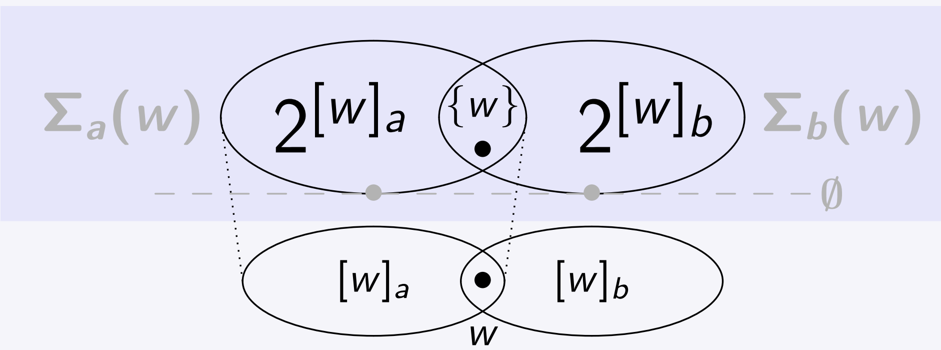

what makes this interesting . . .

- exploration of two-sortedness in a team semantic spirit
- find tame intermediate level between FO and MSO
- another case of locality analysis beyond FO
  cf. work with Felix Canavoi on ML[CK] in LICS 17

  with potential for further integration

$\longrightarrow$ Ciardelli–O_: results for basic InqML in TARK 17 &
draft journal paper arXiv:1803.03483