Tree Unfoldings and Their Finite Counterparts

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tree unfoldings of graphs – (G)

tree/forest unfoldings, based on the set of all paths,

- preserve (two-way) bisimulation type
- avoid cycles
- lend themselves to automata based analysis
- are usually infinite

ω-unfoldings, additionally saturated w.r.t. branching degrees,

- make bisimilar structures isomorphic

  $\rightarrow$ canonical representation of $\sim$-classes
- collapse MSO to L_μ (Janin–Walukiewicz):

  $\rightarrow$ expressive completeness for $\mathrm{L}_\mu \equiv \mathrm{MSO}/\sim$

analogous unfoldings of hypergraphs/relational structures

based on tree unfoldings of intersection graphs

- preserve hypergraph/guarded bisimulation type

- avoid chordless cycles and unguarded cliques:

  $\rightarrow$ acyclic in the hypergraph sense (!)
  $\rightarrow$ tree-decomposable with guarded bags

- reduce much of the classical model theory of guarded logics to that of modal logics (and MSO) on trees, e.g.,

  $\rightarrow$ automata methods for SAT
  $\rightarrow$ expressive completeness $\mathrm{GSO}/\!\sim_{\mathrm{g}} \equiv \mu\mathrm{GF}$ (Grädel–Hirsch-O'02)

(G) for graph-like structures

(O'04)

every finite width 2 relational structure $\mathcal{A}$ admits,
for every $N \in \mathbb{N}$, a (guarded) bisimilar cover

$$\pi : \hat{\mathcal{A}} \xrightarrow{\sim_{\mathrm{g}}} \mathcal{A}$$

by some finite structure $\hat{\mathcal{A}}$ that has no cycles of length up to N

N-acyclicity: all substructures of size up to N are acyclic

neat method: direct product with Cayley group of large girth

finite approximations to the acyclicity in tree unfoldings

(H) for hypergraphs/relational structures

(Hodkinson–O'03)

every finite relational structure $\mathcal{A}$ admits,
a (guarded) bisimilar cover by some finite conformal $\hat{\mathcal{A}}$

$$\pi : \hat{\mathcal{A}} \xrightarrow{\sim_{\mathrm{g}}} \mathcal{A}$$

conformal: no unguarded Gaifman cliques
easy half of hypergraph acyclicity

use: reduction from CGF to GF for FINSAT

(H) for hypergraphs/relational structures

(O'10)

every finite relational structure $\mathcal{A}$ admits
a (guarded) bisimilar cover by some finite, conformal, N-chordal $\hat{\mathcal{A}}$

$$\pi \colon \hat{\mathcal{A}} \xrightarrow{\;\sim_{\mathrm{g}}\;} \mathcal{A}$$

N-chordal: no chordless cycles of lengths up to N
chordality is the other half of hypergraph acyclicity

N-acyclic: every substructure of size up to N is acyclic

method: technically non-trivial (& maybe not the final word)

finite approximations to ∞ branching degree in ω-unfoldings

(G) for graphs

can boost all degrees by factor n through
product with K_n prior to cover construction

lemma: in sufficiently highly branching and locally acyclic covers $\sim^{\mathbf{f(q)}}$ forces $\equiv^{\mathbf{q}}$

use: finite model theory analysis of FO/$\sim$

$$\begin{array}{ccc}
\mathcal{A} & \text{——} \sim^{\mathbf{f(q)}} \text{——} & \mathcal{B} \\
| & & | \\
\sim & & \sim \\
| & & | \\
\hat{\mathcal{A}} & \text{——} \equiv^{\mathbf{q}} \text{——} & \hat{\mathcal{B}}
\end{array}$$

(H) for hypergraphs/relational structures

(O'10)

from finite (N-acyclic) to finite (N-acyclic) n-free finite covers through reduced products with N-acyclic Cayley groups, which preserve N-acyclicity of the cover

n-freeness: up to $\sim_{\mathrm{g}}$, can boost distance between s and $\mathbf{a}$ in $\mathcal{A} \setminus (s \cap \mathbf{a})$ beyond n while preserving $s \cap \mathbf{a}$

lemma: in sufficiently free and sufficiently acyclic covers $\sim_{\mathbf{g}}^{\mathbf{f(q)}}$ forces $\equiv^{\mathbf{q}}$

use: expressive completeness $\mathbf{FO}/\sim_{\mathrm{g}} \equiv \mathbf{GF}$ **(fmt)**

N-acyclic group: large girth w.r.t. to reduced distance measure

corollary: linking finite and infinite covers

cheating FO about infinity // a curious finite model property

(H) general relational structures

for the infinite ω-unfolding $\mathcal{A}^{\omega*}$ and
any N-acyclic, n-free finite cover $\hat{\mathcal{A}}$
for sufficiently large N, n

$$\left.\right\} \quad \mathcal{A}^{\omega*} \equiv^{\mathrm{q}} \hat{\mathcal{A}}$$

(G) analogous, but simpler

corollary: $\mathcal{A} \sim_{\mathrm{g}} \boxed{\hat{\mathcal{A}} \equiv^{\mathrm{q}} \mathcal{A}^{\omega*}} \simeq \hat{\mathcal{A}}^{\omega*} \sim_{\mathrm{g}} \hat{\mathcal{A}} \sim_{\mathrm{g}} \mathcal{A}$

remark: need to avoid small $\left\{ \begin{array}{l} \text{chordless cycles} \\ \text{unguarded cliques} \\ \text{bounds on reduced distances} \end{array} \right.$

guarded negation and GNF

focus on negation & quantification pattern, alternative views:

- restrict negation to formulae in guarded free variables
  and allow unrestricted $\exists$
- allow $\exists$-pos constraints (cq) on guarded tuples
  and stick with guarded quantification of GF

(Barany–ten Cate–Segoufin'11)

GNF decidable, has FMP, generalised tree model property, ...

method: reduction to GF in the presence of forbidden homomorphisms (Barany–Gottlob–O'10)

guarded negation bisimulation

back&forth equivalence based on local homomorphisms at guarded tuples

game on $\mathcal{A}; \mathcal{B}$

positions: local isomorphisms between guarded tuples $\rho \colon \mathbf{a} \mapsto \mathbf{b}$

single round:

- player I proposes subset $A_0 \subseteq A$ (or $B_0 \subseteq B$)
- player II responds with $h \colon \mathcal{A} \restriction A_0 \xrightarrow{\text{hom}} \mathcal{B}$ compatible with ρ
- player I picks guarded tuple $\mathbf{a}' \in A_0$
  new position: $\rho' \colon \mathbf{a}' \mapsto h(\mathbf{a}')$

k-bounded version: restrict homomorphisms to size $|h| \leqslant k$

$\mathcal{A}, \mathbf{a} \sim_{\mathsf{gn}[\mathrm{k}]} \mathcal{B}, \mathbf{b}$ and $\mathcal{A}, \mathbf{a} \sim^{m}_{\mathsf{gn}[\mathrm{k}]} \mathcal{B}, \mathbf{b}$ defined as usual

guarded negation bisimulation and GNF

(Barany–ten Cate–Segoufin'11)

- guarded negation bisimulation preserves GNF
- $\sim_{\mathrm{gn[k]}}$ preserves k-GNF ($\exists$-pos assertions of width $\leqslant k$)
- expressive completeness FO$/\sim_{\mathrm{gn[k]}} \equiv \mathrm{k}$-GNF (classical)

caveat: semantics is over structures with guarded parameter tuples

goal here: FO$/\sim_{\mathrm{gn[k]}} \equiv$ k-GNF (fmt)

method: reduction to GF and $\sim_{\mathrm{g}}$

crux: preparation of models

preparation: yet another saturation condition

need to have exactly the same images of k-bounded structures under guarded homomorphisms (ghom)

$h\colon \mathcal{C} \xrightarrow{\text{ghom}} \mathcal{A}$: hom. that is injective on guarded tuples

ghom-images are only weak substructures $h(\mathcal{C}) \subseteq_{\mathsf{w}} \mathcal{A}$

here want to avoid new guarded subsets, $h(\mathcal{C}) \subseteq_{\mathsf{g}} \mathcal{A}$

$h(\mathcal{C}) \subseteq_{\mathsf{g}} \mathcal{A}$: $\mathbf{a}$ guarded in $\mathcal{A} \Rightarrow \mathbf{a} \restriction h(\mathcal{C})$ guarded in $h(\mathcal{C})$

k-richness

for every guarded homomorphism $(h\colon \mathcal{C} \xrightarrow{\text{ghom}} \mathcal{A}) \leqslant k$ at $\mathbf{a}$, there is an isomorphic embedding $h'\colon \mathcal{C} \xrightarrow{\simeq} \mathcal{A}$ at $\mathbf{a}$ such that

- $h'(\mathcal{C}) \subseteq_{\mathsf{g}} \mathcal{A}$
- $h \circ h'^{-1}$ preserves $\sim_{\mathsf{g}}$ on guarded tuples

reduction idea

want:

$$\mathcal{A} \sim^{m}_{\mathrm{gn[k]}} \mathcal{B} \quad \Rightarrow \quad \mathcal{A} \sim_{\mathrm{gn[k]}} \hat{\mathcal{A}} \equiv^{q} \hat{\mathcal{B}} \sim_{\mathrm{gn[k]}} \mathcal{B}'$$

for $m = f(q)$, all finite $\mathcal{A}, \mathcal{B}$ and suitable finite $\hat{\mathcal{A}}, \hat{\mathcal{B}}$

- using new k-ary relation R to guard all small $\subseteq_{\mathrm{g}}$-substructures in k-rich $\mathcal{A}$ and $\mathcal{B}$:

  $(*) \quad \mathcal{A} \sim^{m}_{\mathrm{gn[k]}} \mathcal{B} \quad \Leftrightarrow \quad (\mathcal{A}, R(\mathcal{A})) \sim^{m}_{\mathrm{g}} (\mathcal{B}, R(\mathcal{B}))$
- k-richness can be achieved in ω-unfoldings $\mathcal{A}^{\omega *}$ and $\mathcal{B}^{\omega *}$ for which $(*)$ implies $\mathcal{A}^{\omega *} \equiv^{q} \mathcal{B}^{\omega *}$, if m is large enough
- strong FMP for GF-theoris of regular tree-like models from Barany–Gottlob–O'10 yields finite counterparts of $\mathcal{A}^{\omega *}$, $\mathcal{B}^{\omega *}$
- sufficiently free and acyclic finite covers of these are thus finite, FO_q equivalent, and $\sim_{\mathrm{gn[k]}}$-equivalent to $\mathcal{A}$ and $\mathcal{B}$, rsp.

expressive completeness for GNF

theorem (O'12)

$\mathsf{FO}/\sim_{\mathsf{gn[k]}} \equiv \mathrm{k}\text{-}\mathsf{GNF}$, classically and in finite model theory

core: upgrading in finite covers that behave like trees