Expressive Completeness

a basic model-theoretic concern
in varied (modal) settings

Martin Otto TU Darmstadt

expressive completeness

generic setting:

want **concrete & effective syntax** for

some **class of structural properties**

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examples:

- FO-properties preserved under extensions;
  corresponding to $\exists^*$-FO $\subseteq$ FO (Łos–Tarski)

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examples:

- FO-properties preserved under extensions; corresponding to $\exists^*$-FO $\subseteq$ FO (Łos–Tarski)
- FO-properties preserved under bisimulation; corresponding to ML $\subseteq$ FO (van Benthem)

$$\mathbf{FO}/\!\sim\; \equiv \mathbf{ML}$$

expressive completeness

generic setting:

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some **class of structural properties**
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remarks:

- not to be confused with deductive completeness
  as familiar from modal correspondence theory

expressive completeness

generic setting:

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remarks:

- **undecidability vs. effective syntax (!)**

motivation – from classical model theory

- correspondences between semantic and syntactic features
  universal algebra + logic
- the non-trivial parts of classical 'preservation theorems'
- usefull syntactic normal forms
- logical transfer phenomena ($\rightarrow$ upgrading, below)

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some classical preservation theorems:

pres. in hom. images	—	positive FO
pres. under homs	—	positive-existential FO (Lyndon–Tarski)
pres. in extensions	—	$\exists^*$-FO (Łos–Tarski)
monotonicity	—	positivity

motivation – from finite model theory (fmt)

(A) same motivation — fewer positive results

classical expressive completeness proofs invariably fail

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some survive – with new proofs that give new insights

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(B) new motivation & ramifications

- **other classes** of interest besides 'just finite'
- **complexity** as another semantic constraint

motivation – from modal model theory

a different sense of correspondence

variation of the underlying class of frames/models
familiar from classical modal correspondence theory

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varying the domain of (model-theoretic) discourse

rather than sticking with basic modal logic ML
as the (syntactic) background logic, can look at

semantic criterion of bisimulation invariance
over specific classes of frames/models

motivation – from descriptive complexity

complexity is a semantic constraint

e.g., the class of all Ptime recognisable
properties of finite structures (!)

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  why?

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- not known to possess a syntactic characterisation
  the long-open logic-for-Ptime issue

finding a logic for Ptime is an expressive completeness issue

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remark: natural positive solution for
Ptime *properties of linearly ordered finite structures:*
least fixed-point logic LFP (Immermann, Vardi)

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trivial $\simeq$

plan

- model-theoretic upgrading & model constructions
- specific constructions/issues in the modal setting
- specific constructions/issues in the guarded setting
- on descriptive complexity in these settings

a general line

classical lemma (based on compactness)

for fragment $L \subseteq \mathrm{FO}$ (closed under $\wedge, \vee$)
and $\varphi \in \mathrm{FO}$ t.f.a.e.

- $\varphi \equiv \varphi' \in L$
- φ preserved under L-transfer, $\Rightarrow_L$

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non-classical substitute (based on Ehrenfeucht–Fraïssé)

for natural fragments $L \subseteq \mathrm{FO}$

can typically replace $\Rightarrow_L$
by finite index approximants $\Rightarrow_L^\ell$
for *some* $\ell \in \mathbb{N}$ (which $\ell = \ell(\varphi)$? extra insight: $\varphi' \in L^\ell$)

model-theoretic upgrading

the technical key to expressive completeness results

upgrading example:

model-theoretic upgrading

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upgrading example: Łos–Tarski thm

for $\varphi \in$ FO, equivalence of $\begin{cases} \varphi \text{ pres. under extensions} \\ \varphi \text{ pres. under } \exists^*\text{-transfer, } \Rightarrow_\exists \\ \varphi \text{ formalisable in } \exists^*\text{-FO} \end{cases}$ • •

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for $\varphi \in \mathrm{FO}$, equivalence of $\begin{cases} \varphi \text{ pres. under extensions} & \bullet \\ \varphi \text{ pres. under } \exists^*\text{-transfer, } \Rrightarrow_{\exists} & \bullet \\ \varphi \text{ formalisable in } \exists^*\text{-FO} \end{cases}$

crux: if $\varphi \in \mathrm{FO}$ is preserved under extensions,

then $\boxed{\mathfrak{A} \Rrightarrow_{\exists} \mathfrak{B} \quad \text{implies} \quad \mathfrak{A} \Rrightarrow_{\varphi} \mathfrak{B}}$

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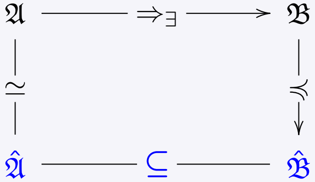
compactness argument
yields this upgrading:



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upgrading example: van Benthem's thm

for $\varphi \in \mathrm{FO}$, equivalence of $\begin{cases} \varphi \text{ pres. under bisimulation} & \bullet \\ \varphi \text{ pres. under ML-transfer} & \bullet \\ \varphi \text{ formalisable in ML} & \end{cases}$

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(e.g. modal saturation)
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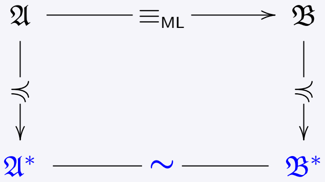

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upgrading example: van Benthem–Rosen thm, recast

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for $\varphi \in \mathrm{FO}$, equivalence of
$$\begin{cases} \varphi \text{ pres. under bisimulation} & \bullet \\ \varphi \text{ pres. under } \sim^{\ell} \text{ for some } \ell & \bullet \\ \varphi \text{ expressible in ML} \end{cases}$$

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for $\varphi \in \mathrm{FO}$, equivalence of
$$\begin{cases} \varphi \text{ pres. under bisimulation} & \bullet \\ \varphi \text{ pres. under } \sim^{\ell} \text{ for } \ell = 2^{\mathrm{qr}(\varphi)} & ! \\ \varphi \text{ expressible in ML}^{\ell} \text{ for } \ell = 2^{\mathrm{qr}(\varphi)} & \end{cases}$$

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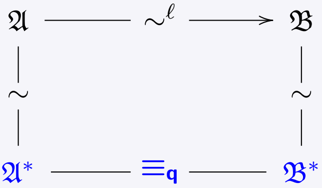
game argument
and model construction
provides this upgrading,
classically and fmt:



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cf. classical case:
via compactness

the modal and guarded worlds

modal logic

Kripke structures:
coloured **graphs**

modal bisimulation:
graph bisimulation

$\longrightarrow$ classically:
tree unfolding,
tree models

guarded logic

relational structures:
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modal model theory = model theory of bisimulation invariance

guarded model theory = model theory of guarded bisimulation invariance

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modal logic	guarded logic
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modal bisimulation: **graph bisimulation**	guarded bisimulation: **hypergraph bisimulation**
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?? fmt ??

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specific model constructions for upgrading

the classical modal example

for van Benthem–Rosen, it suffices to show:

$\varphi(x) \in \mathrm{FO}\ \sim\text{-inv.} \;\Rightarrow\; \varphi\ \ell\text{-local for } \ell = 2^{\mathrm{qr}(\varphi)}$ (hence $\sim^{\ell}$-inv.)

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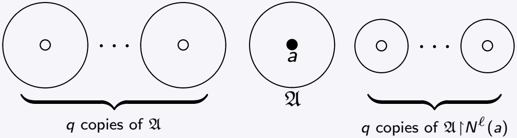
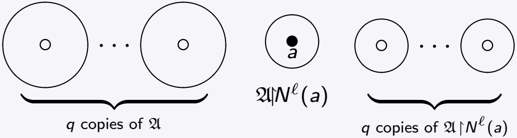


versus



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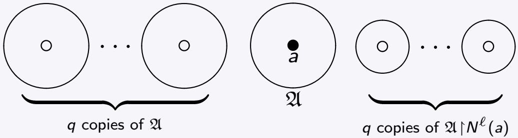
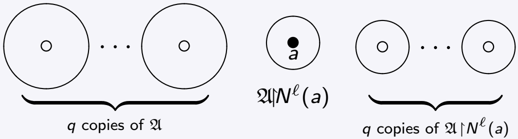


$\sim$ $\mathfrak{A}, a$

versus $\qquad \cong_q$



$\sim$ $\mathfrak{A}\restriction N^{\ell}(a), a$

the modal and guarded worlds

modal logic

Kripke structures:
coloured **graphs**

modal bisimulation:
graph bisimulation

→ classically:
tree unfolding,
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fmt

guarded logic

relational structures:
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acyclic hypergraph models

acyclicity in (graph) covers

for upgrading $\sim^\ell$ (and its variants) to $\equiv_q$

more generally need $\left\{ \begin{array}{l} \quad \text{uniform degree of } \textbf{local acyclicity} \\ \& \text{ finite saturation w.r.t. multiplicities} \end{array} \right.$

modularity of FO Ehrenfeucht–Fraïssé game (locality of FO)
then guarantees upgrading

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local acyclicity = 'local uncluttering'

local normalisation up to $\sim$
replacing (infinite) tree unfolding

question: does every finite Kripke structure possess a finite bisimilar companion without any short undirected cycles?

acyclicity in finite bisimilar graph covers

bisimilar cover $\pi : \hat{\mathfrak{A}} \xrightarrow{\sim} \mathfrak{A}$:

homomorphism with the *back*-property
= bisimulation induced by a function/projection
= bounded morphism

- bisimilar tree-unfoldings provide acyclic covers
- if $\mathfrak{A}$ has cycles, then any *acyclic cover* is infinite

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thm O_'04

every finite Kripke structure/frame admits bisimilar covers by finite ℓ-locally acyclic structures/frames

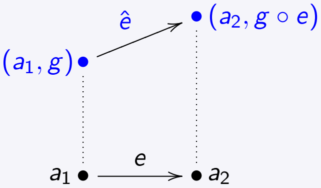
ℓ-local acyclicity: no (undirected) cycles $\begin{cases} \text{in } \ell\text{-neighbourhoods,} \\ \text{of length} \leqslant 2\ell + 1 \end{cases}$

generic construction in the modal world (graphs)

simple idea: natural product with Cayley group of large girth

given such G with generators $e \in E^{\mathfrak{A}}$:

lift edge $e = (a_1, a_2)$ in $\mathfrak{A}$
to edges $\hat{e} = ((a_1, g), (a_2, g \circ e))$
in cover with vertex set $A \times G$



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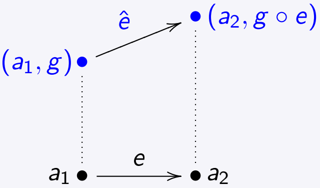

a combinatorial group construction (Biggs)

find finite Cayley groups of large girth
for any given finite set E of generators,

generated by group action on E-coloured trees

aside: Cayley groups of large girth

given: set E of involutive generators,
bound N on girth (length of shortest cycles)

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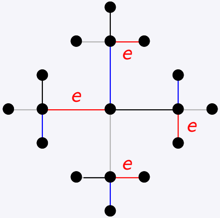
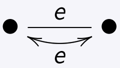


let $e \in E$ operate through
swaps of nodes in e-edges:



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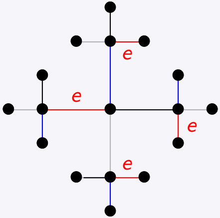

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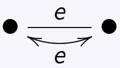

$\mathbf{G} := \langle \mathbf{E} \rangle^{\mathrm{Sym}(\mathbf{T})} \subseteq \mathrm{Sym}(\mathbf{T})$
subgroup generated by the permutations $e \in E$

no short cycles: $e_1 \circ e_2 \circ \cdots \circ e_k \neq 1$ for $k \leqslant N$

sample results for FO/$\sim$ and FO/$\sim_\forall$, FO/$\sim_{-,\forall}$

based on locally acyclic covers	O_'04, Dawar–O_'09
FO/$\sim_*$ $\equiv$ ML[$*$]	all (finite) frames
FO/$\sim$ $\equiv$ ML[$\forall$]	(finite) rooted frames
FO/$\sim_*$ $\equiv$ ML[$*$]	(finite) equivalence frames

sample results for $FO/{\sim}$ and $FO/{\sim_\forall}$, $FO/{\sim_{\_,\forall}}$

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based on tree interpretations Dawar–O_'09

$FO/{\sim} \equiv ML$ $\begin{cases} \text{all transitive trees,} \\ \text{finite irreflexive transitive trees} \end{cases}$

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based on tree interpretations Dawar–O_'09

$\mathrm{FO}/\!\sim \;\equiv\; \mathrm{ML}$ — all transitive trees, finite irreflexive transitive trees

$\mathrm{FO}/\!\sim \;\equiv\; \mathrm{MSO}/\!\sim \;\equiv\; \mathrm{ML}[\Diamond^*]$ — finite transitive frames, transitive path-finite frames

for new modality $\Diamond^*$:
- not generally $\sim$-safe
- referring to types within E-clusters
- non-trivial in non-irreflexive case

the modal and guarded worlds

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modal bisimulation:
graph bisimulation

$\longrightarrow$ classically:
tree unfolding,
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$\longrightarrow$ for fmt:
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guarded bisimulation:
hypergraph bisimulation

$\longrightarrow$ classically:
guarded tree unfolding
acyclic hypergraph models

?? fmt ??

from graphs to hypergraphs

hypergraph bisimulation & covers

guarded bisimulation $\sim_g$ (hypergraph bisimulation)
the game equivalenve for guarded fragment GF

thm Andreka–van Benthem–Nemeti'98

$\mathbf{FO}/{\sim_g} \equiv \mathbf{GF}$

had been open in fmt since!

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hypergraph cover $\pi\colon \hat{\mathfrak{A}} \xrightarrow{\sim} \mathfrak{A}$

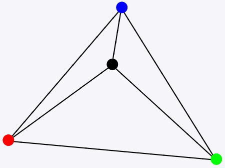
cover of relational structures (hypergraphs)
w.r.t. guarded bisimulation (hypergraph bisimulation)
= homomorphism with the *back*-property
= guarded bisimulation induced by a function/projection

acyclicity in finite bisimilar hypergraph covers

example: $\mathbf{H}_4^3$

the full width 3 hypergraph on 4 nodes;

$=$ tetrahedron with faces as hyperedges



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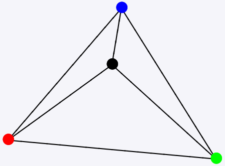

unfolds into acyclic hypergraph,
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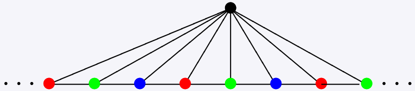

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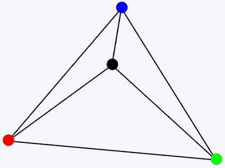

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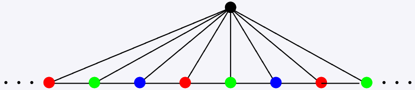
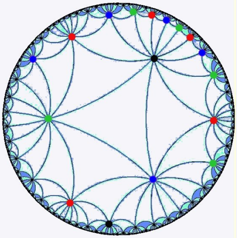


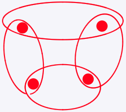
even 1-locally *infinite*,

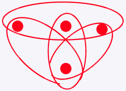
or into *locally finite* hypergraph without *short* chordless cycles



how much acyclicity in finite hypergraph covers?

hypergraph acyclicity = chordality + conformality

no 

no 

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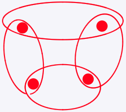

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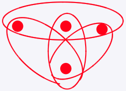

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every finite hypergraph admits a finite conformal cover

applications: reductions from CGF to GF for fmp
Herwig–Lascar–Hrushovski results

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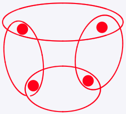
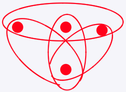
no  no 

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even 1-local acyclic covers may necessarily be infinite: $\mathbf{H}_4^3$

N-acyclicity: **no small cyclic sub-configurations**
relativisation to size N configurations
rather than localisation

N-acyclic guarded covers

thm O_'10

every finite hypergraph admits covers by finite
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fmp for GF on classes with forbidden cyclic configurations

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every finite hypergraph admits covers by finite *N*-acyclic hypergraphs

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fmt version of Andreka–van Benthem–Nemeti:

thm O_'10

$\mathbf{FO}/{\sim_{\mathrm{g}}} \equiv \mathbf{GF}$ over all finite structures

hypergraph covers and upgrading $\sim_g^\ell$ to $\equiv_q$

using more highly acyclic groups

- to unclutter hyperedges up to $\sim_g$
- for finitary saturation & freeness

stronger form of acyclicity necessary
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stronger form of acyclicity necessary
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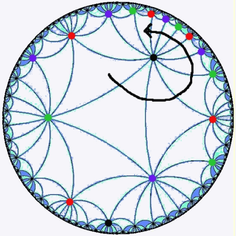
hyperedge transitions may or may not
contribute to progress along a cycle

short chordless cycles may correspond
to long generator sequences



other new results in the guarded world

weakly N-acyclic covers Barany–Gottlob–O_'10

a weaker notion of acyclic covers
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yield

- near-optimal small models for GF and CGF
- fmp for GF and CGF over classes with forbidden homomorphic embeddings
  $\longrightarrow$ **finite control over conjunctive queries/GF constraints**
- Ptime reconstruction of canonical finite models from abstract specification of their $\sim_{\mathrm{g}}$-class
  $\longrightarrow$ **canonisation & capturing** (next)

descriptive complexity: capturing modal/guarded Ptime

crux of capturing:

semantic constraint on (Ptime) machines
$\simeq$-invariance: Ptime $\longrightarrow$ **Ptime/$\simeq$**

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here look at
$$\begin{cases} \textbf{Ptime}/\sim & \textbf{modal Ptime} \\ \textbf{Ptime}/\sim_{\mathbf{g}} & \textbf{guarded Ptime} \end{cases}$$

Ptime in the modal and guarded worlds

how to enforce this (rougher) granularity?

capturing modal/guarded Ptime

generic pre-processing idea: Ptime canonisation as a filter

$$\mathfrak{A} \quad \stackrel{\mathbf{I}}{\longmapsto} \quad \mathbf{I}(\mathfrak{A}) = \mathbf{I}([\mathfrak{A}]_\sim) \quad \stackrel{\mathbf{F}}{\longmapsto} \quad \mathbf{F}(\mathbf{I}(\mathfrak{A})) \in [\mathfrak{A}]_\sim$$

structure — complete invariant/$\sim$ — canonical representative/$\sim$

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structure — complete invariant/$\sim$ — canonical representative/$\sim$

if in Ptime: $\mathbf{H} := \mathbf{F} \circ \mathbf{I}$ provides Ptime canonisation & filter

pre-processing with H enforces $\sim$-invariance

trivial for $\sim$, but not for $\sim_g$

Ptime canonisation and Ptime/$\sim$ and Ptime/$\sim_g$

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Ptime canonisation and Ptime/$\sim$ and Ptime/$\sim_{\mathbf{g}}$

in both cases, natural complete invariant: bisimulation quotient of associated game graph

canonisation through reconstruction

in the modal case: bisimulation quotient *is* canonical representative

$\longrightarrow$ **capturing Ptime/$\sim$** (O_'99)

in the guarded case: non-trivial Ptime construction of a model from this quotient

$\longrightarrow$ **capturing Ptime/$\sim_{\mathbf{g}}$** (Barany–Gottlob–O_'10)

yet another asset of the guarded world

summary & remarks

effectively capturing semantic phenomena
over interesting classes of structures

e.g., modal/guarded preservation properties

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interesting, non-trivial finite model theory
of modal and guarded logics

with many further worthwhile variations

summary & remarks

many open problems remain,

e.g., the status of the Janin–Walukiewicz thm

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The End