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NUMERICAL ANALYSIS OF THE OSEEN-TYPE PETERLIN VISCOELASTIC MODEL BY THE STABILIZED LAGRANGE–GALERKIN METHOD PART II: A NONLINEAR SCHEME

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Abstract. A nonlinear stabilized Lagrange–Galerkin scheme for the Oseen-type Peterlin viscoelastic model is presented. Error estimates with the optimal convergence order are proved without any relation between the time increment and the mesh size. The result is valid for both the diffusive and the non-diffusive conformation tensor. The theoretical convergence order is confirmed by the numerical experiments. The scheme is a combination of the method of characteristics and Brezzi–Pitkäranta's stabilization method for the conforming linear elements, which yields an efficient computation with a small number of degrees of freedom.

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1. Introduction

This is the continuation of our paper on numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method. In the previous paper [8], Part I, we have dealt with a linear scheme. Here, in Part II, we present a nonlinear scheme and prove the optimal convergence order.

Many mathematical models have been proposed and analysed in order to understand the so-called non-Newtonian fluids. One of the most famous models is the Oldroyd-B model, cf., e.g., [14,15], which is based on a simple dumbbell model representing a polymer molecule as two beads connected by a spring. There is a broad literature on both analytical and numerical studies of the Oldroyd-B model and its diffusive version. We refer to the bibliography in Part I and references therein.

Here we study numerically the Peterlin viscoelastic model, the same model as is described in Part I. As for the diffusive Peterlin model Lukáčová-Medviďová et al. have proved the global existence of a weak solution and the uniqueness of regular solutions [9]. In this paper, we treat both the diffusive and the non-diffusive cases. As a starting point of the numerical analysis of this problem, we deal with the Oseen-type model, where the velocity of the convective terms is replaced by a known one. The numerical analysis of the original model will be a future work.

Keywords and phrases: Error estimates, Peterlin viscoelastic model, Lagrange–Galerkin method, Pressure-stabilization.

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The linear scheme proposed in Part I consists of the method of characteristics and Brezzi–Pitkäranta's stabilization method [3] for the conforming linear elements. This class of the stabilized Lagrange–Galerkin method has been studied for the Oseen, the Navier–Stokes, and natural convection problems in our papers by Notsu and Tabata [11–13]. The nonlinear scheme to be presented in this paper also belongs to the same class, and has the common advantages of schemes in this class, the robustness in convection-dominated problems and the small number of degrees of freedom. While a relation between the time and space discretization parameters is required for the error estimates in the linear scheme, no condition is necessary in the nonlinear scheme. We note that the error estimates remain true even in the case $\varepsilon = 0$, i.e., for the non-diffusive Peterlin model. Furthermore, under the condition $\Delta t = O(1/(1 + |\log h|))$ for $\varepsilon > 0$ and $\Delta t = O(h)$ for $\varepsilon = 0$, the uniqueness of the solution of the nonlinear scheme is ensured. Two-dimensional numerical experiments are shown in order to confirm the theoretical convergence order.

The paper is organized as follows. In Section 2 the mathematical formulation of the Oseen-type Peterlin viscoelastic model is described. In Section 3 a nonlinear stabilized Lagrange–Galerkin scheme is presented. The main result on the convergence with optimal error estimates is stated in Section 4, and proved in Section 5. In Section 6 the result on the uniqueness is presented and proved. The theoretical order of convergence is confirmed by numerical experiments in Section 7.

2. The Oseen-type Peterlin viscoelastic model

The function spaces and the notation to be used throughout the paper are as follows. Let Ω be a bounded domain in $\mathbb{R}^2$, $\Gamma := \partial\Omega$ the boundary of Ω, and T a positive constant. For $m \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$ we use the Sobolev spaces $W^{m,p}(\Omega)$, $W_0^{1,\infty}(\Omega)$, $H^m(\Omega)\,(= W^{m,2}(\Omega))$, $H_0^1(\Omega)$ and $L_0^2(\Omega) := \{q \in L^2(\Omega); \int_\Omega q\,dx = 0\}$. Furthermore, we employ function spaces $H^m_{sym}(\Omega) := \{\mathbf{D} \in H^m(\Omega)^{2\times2};\ \mathbf{D} = \mathbf{D}^T\}$ and $C^m_{sym}(\bar\Omega) := C^m(\bar\Omega)^{2\times2} \cap H^m_{sym}(\Omega)$, where the superscript T stands for the transposition. For any normed space S with norm $\|\cdot\|_S$, we define function spaces $H^m(0,T;S)$ and $C([0,T];S)$ consisting of S-valued functions in $H^m(0,T)$ and $C([0,T])$, respectively. We use the same notation $(\cdot,\cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. The dual pairing between S and the dual space S' is denoted by $\langle\cdot,\cdot\rangle$. The norms on $W^{m,p}(\Omega)$ and $H^m(\Omega)$ and their seminorms are simply denoted by $\|\cdot\|_{m,p}$ and $\|\cdot\|_m\,(= \|\cdot\|_{m,2})$ and by $|\cdot|_{m,p}$ and $|\cdot|_m\,(= |\cdot|_{m,2})$, respectively. The notations $\|\cdot\|_{m,p}$, $|\cdot|_{m,p}$, $\|\cdot\|_m$ and $|\cdot|_m$ are employed not only for scalar-valued functions but also for vector- and matrix-valued ones. We also denote the norm on $H^{-1}(\Omega)^2$ by $\|\cdot\|_{-1}$. For t_0 and $t_1 \in \mathbb{R}$ we introduce the function space,

$$Z^m(t_0,t_1) := \left\{\psi \in H^j(t_0,t_1;H^{m-j}(\Omega));\ j = 0,\ldots,m,\ \|\psi\|_{Z^m(t_0,t_1)} < \infty\right\}$$

with the norm

$$\|\psi\|_{Z^m(t_0,t_1)} := \left\{\sum_{j=0}^{m} \|\psi\|^2_{H^j(t_0,t_1;H^{m-j}(\Omega))}\right\}^{1/2},$$

and set $Z^m := Z^m(0,T)$. We often omit $[0,T]$, Ω, and the superscripts 2 and 2×2 for the vector and the matrix if there is no confusion, e.g., we shall write $C(L^\infty)$ in place of $C([0,T];L^\infty(\Omega)^{2\times2})$. For square matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbb{R}^{2\times2}$ we use the notation $\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij}B_{ij}$.

We consider the system of equations describing the unsteady motion of an incompressible viscoelastic fluid,

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \mathrm{div}\,\big(2\nu \mathrm{D}(\mathbf{u})\big) + \nabla p = \mathrm{div}\,[(\mathrm{tr}\,\mathbf{C})\mathbf{C}] + \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{1a}$$

$$\mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega \times (0,T), \tag{1b}$$

$$\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t} - \varepsilon\Delta\mathbf{C} = (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T - (\mathrm{tr}\,\mathbf{C})^2\,\mathbf{C} + (\mathrm{tr}\,\mathbf{C})\mathbf{I} + \mathbf{F} \quad \text{in } \Omega \times (0,T), \tag{1c}$$

$$\mathbf{u} = \mathbf{0}, \quad \varepsilon \frac{\partial \mathbf{C}}{\partial \mathbf{n}} = \mathbf{0}, \qquad \text{on } \Gamma \times (0,T), \tag{1d}$$

$$\mathbf{u} = \mathbf{u}^0, \quad \mathbf{C} = \mathbf{C}^0, \qquad \text{in } \Omega, \text{ at } t = 0, \tag{1e}$$

where $(\mathbf{u}, p, \mathbf{C}) : \Omega \times (0,T) \to \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{2\times 2}_{sym}$ are the unknown velocity, pressure and conformation tensor, $\nu > 0$ is a fluid viscosity, $\varepsilon \in [0,1]$ is an elastic stress viscosity, $(\mathbf{f}, \mathbf{F}) : \Omega \times (0,T) \to \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ is a pair of given external forces, $\mathrm{D}(\mathbf{u}) := (1/2)[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$ is the symmetric part of the velocity gradient, $\mathbf{I}$ is the identity matrix, $\mathbf{n} : \Gamma \to \mathbb{R}^2$ is the outward unit normal, $(\mathbf{u}^0, \mathbf{C}^0) : \Omega \to \mathbb{R}^2 \times \mathbb{R}^{2\times 2}_{sym}$ is a pair of given initial functions, and $\mathrm{D}/\mathrm{D}t$ is the material derivative defined by

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla,$$

where $\mathbf{w} : \Omega \times (0,T) \to \mathbb{R}^2$ is a given velocity.

Remark 1. *(i) In this paper we pay attention to the dependency on ε to include the degenerate case $\varepsilon = 0$. The upper bound 1 of ε is not essential but replaced by any positive constant ε_0, i.e., $\varepsilon \in [0, \varepsilon_0]$. The upper bound is needed in choosing the constants h_0, Δt_0 and $c_\dagger$ independent of ε in Theorem 1 below, where it is used for the estimate* (14g) *in Lemma 8.*

(ii) When $\varepsilon > 0$, the problem (1) *is the same system that is described in Part I [8]. Under regularity condition on* $\mathbf{w}$ *the global existence of a weak solution of* (2) *below can be proved in a similar way to the fully nonlinear case [9].*

(iii) When $\varepsilon = 0$, there is neither the diffusion term in (1c) *nor the boundary condition on* $\mathbf{C}$ *in* (1d)*. Because of the loss of the ellipticity,* $\mathbf{C}(t)$ *does not belong to* $H^1(\Omega)^{2\times 2}$ *in general. If there exists a solution satisfying Hypothesis 2 below, then we can show the convergence of the finite element solution to the exact one in Theorem 1.*

We set an assumption for the given velocity $\mathbf{w}$.

Hypothesis 1. *The function* $\mathbf{w}$ *satisfies* $\mathbf{w} \in C([0,T]; W^{1,\infty}_0(\Omega)^2)$.

Let $V := H^1_0(\Omega)^2$, $Q := L^2_0(\Omega)$ and $W := H^1_{sym}(\Omega)$. We define the bilinear forms a_u on $V \times V$, b on $V \times Q$, $\mathcal{A}$ on $(V \times Q) \times (V \times Q)$ and a_c on $W \times W$ by

$$a_u(\mathbf{u}, \mathbf{v}) := 2\big(\mathrm{D}(\mathbf{u}), \mathrm{D}(\mathbf{v})\big), \qquad b(\mathbf{u}, q) := -(\operatorname{div} \mathbf{u}, q), \qquad \mathcal{A}\big((\mathbf{u}, p), (\mathbf{v}, q)\big) := \nu a_u(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, q) + b(\mathbf{v}, p),$$
$$a_c(\mathbf{C}, \mathbf{D}) := (\nabla \mathbf{C}, \nabla \mathbf{D}),$$

respectively. We present the weak formulation of the problem (1); find $(\mathbf{u}, p, \mathbf{C}) : (0,T) \to V \times Q \times W$ such that for $t \in (0,T)$

$$\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(t), \mathbf{v}\right) + \mathcal{A}\big((\mathbf{u}, p)(t), (\mathbf{v}, q)\big) = -\left(\operatorname{tr} \mathbf{C}(t)\, \mathbf{C}(t), \nabla \mathbf{v}\right) + (\mathbf{f}(t), \mathbf{v}), \tag{2a}$$

$$\left(\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t}(t), \mathbf{D}\right) + \varepsilon a_c\big(\mathbf{C}(t), \mathbf{D}\big) = 2\big((\nabla \mathbf{u}(t))\mathbf{C}(t), \mathbf{D}\big) - \big((\operatorname{tr} \mathbf{C}(t))^2 \mathbf{C}(t), \mathbf{D}\big) + (\operatorname{tr} \mathbf{C}(t) \mathbf{I}, \mathbf{D}) + (\mathbf{F}(t), \mathbf{D}), \tag{2b}$$

$$\forall (\mathbf{v}, q, \mathbf{D}) \in V \times Q \times W,$$

with $(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}^0, \mathbf{C}^0)$.

3. A nonlinear stabilized Lagrange–Galerkin scheme

The aim of this section is to present a nonlinear stabilized Lagrange–Galerkin scheme for (1).

Let Δt be a time increment, $N_T := \lfloor T/\Delta t \rfloor$ the total number of time steps and $t^n := n\Delta t$ for $n = 0, \ldots, N_T$. Let $\mathbf{g}$ be a function defined in $\Omega \times (0,T)$ and $\mathbf{g}^n := \mathbf{g}(\cdot, t^n)$. For the approximation of the material derivative we employ the first-order characteristics method,

$$\frac{\mathrm{D}\mathbf{g}}{\mathrm{D}t}(x, t^n) = \frac{\mathbf{g}^n(x) - \left(\mathbf{g}^{n-1} \circ X_1^n\right)(x)}{\Delta t} + O(\Delta t), \tag{3}$$

where $X_1^n : \Omega \to \mathbb{R}^2$ is a mapping defined by

$$X_1^n(x) := x - \mathbf{w}^n(x)\Delta t,$$

and the symbol $\circ$ means the composition of functions,

$$(\mathbf{g}^{n-1} \circ X_1^n)(x) := \mathbf{g}^{n-1}(X_1^n(x)).$$

For the details on deriving the approximation (3) of $\mathrm{D}\mathbf{g}/\mathrm{D}t$, see, e.g., [12]. The point $X_1^n(x)$ is called the upwind point of x with respect to $\mathbf{w}^n$. The next proposition, which is a direct consequence of [16] and [18], presents sufficient conditions to ensure that all upwind points defined by X_1^n are in Ω and that its Jacobian $J^n := \det(\partial X_1^n/\partial x)$ is around 1.

Proposition 1. *Suppose Hypothesis 1 holds. Then, we have the following for $n \in \{0, \ldots, N_T\}$.*
(i) Under the condition $\Delta t|\mathbf{w}|_{C(W^{1,\infty})} < 1$, $X_1^n : \Omega \to \Omega$ is bijective.
(ii) Furthermore, under the condition

$$\Delta t|\mathbf{w}|_{C(W^{1,\infty})} \leq 1/4, \tag{4}$$

the estimate $1/2 \leq J^n \leq 3/2$ holds.

For the sake of simplicity we suppose that Ω is a polygonal domain. Let $\mathcal{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ $(= \bigcup_{K \in \mathcal{T}_h} K)$, h_K the diameter of $K \in \mathcal{T}_h$ and $h := \max_{K \in \mathcal{T}_h} h_K$ the maximum element size. We consider a regular family of subdivisions $\{\mathcal{T}_h\}_{h \downarrow 0}$ satisfying the inverse assumption [4], i.e., there exists a positive constant α_0 independent of h such that

$$\frac{h}{h_K} \leq \alpha_0, \quad \forall K \in \mathcal{T}_h, \ \forall h.$$

We define the discrete function spaces X_h, V_h, M_h, Q_h and W_h by

$$\begin{aligned} X_h &:= \left\{\mathbf{v}_h \in C(\bar{\Omega})^2;\ \mathbf{v}_{h|K} \in P_1(K)^2, \forall K \in \mathcal{T}_h\right\}, & V_h &:= X_h \cap V, \\ M_h &:= \left\{q_h \in C(\bar{\Omega});\ q_{h|K} \in P_1(K), \forall K \in \mathcal{T}_h\right\}, & Q_h &:= M_h \cap Q, \\ W_h &:= \left\{\mathbf{D}_h \in C_{sym}(\bar{\Omega});\ \mathbf{D}_{h|K} \in P_1(K)^{2\times 2}, \forall K \in \mathcal{T}_h\right\}, \end{aligned}$$

respectively, where $P_1(K)$ is the polynomial space of linear functions on $K \in \mathcal{T}_h$.

Let δ_0 be a small positive constant fixed arbitrarily and $(\cdot,\cdot)_K$ the $L^2(K)^2$ inner product. We define the bilinear forms $\mathcal{A}_h$ on $(V \times H^1(\Omega)) \times (V \times H^1(\Omega))$ and $\mathcal{S}_h$ on $H^1(\Omega) \times H^1(\Omega)$ by

$$\mathcal{A}_h\left((\mathbf{u},p),(\mathbf{v},q)\right) := \nu a_u\left(\mathbf{u},\mathbf{v}\right) + b(\mathbf{u},q) + b(\mathbf{v},p) - \mathcal{S}_h(p,q), \qquad \mathcal{S}_h(p,q) := \delta_0 \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K.$$

For $\mathbf{D} \in \mathbb{R}^{2\times 2}_{sym}$ let $\mathbf{D}^\# \in \mathbb{R}^{2\times 2}_{sym}$ be the adjugate matrix of $\mathbf{D}$ defined by

$$\mathbf{D}^\# := \begin{pmatrix} D_{22} & -D_{12} \\ -D_{12} & D_{11} \end{pmatrix}.$$

Let $(\mathbf{f}_h, \mathbf{F}_h) := (\{\mathbf{f}_h^n\}_{n=1}^{N_T}, \{\mathbf{F}_h^n\}_{n=1}^{N_T}) \subset L^2(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ and $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ be given. A nonlinear stabilized Lagrange–Galerkin scheme for (1) is to find $(\mathbf{u}_h, p_h, \mathbf{C}_h) := \{(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h \times W_h$ such that, for $n = 1, \ldots, N_T$,

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\big((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)\big) = -\big((\operatorname{tr} \mathbf{C}_h^n)\mathbf{C}_h^n, \nabla \mathbf{v}_h\big) + (\mathbf{f}_h^n, \mathbf{v}_h), \tag{5a}$$

$$\begin{aligned}\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon a_c\left(\mathbf{C}_h^n, \mathbf{D}_h\right) = 2\big((\nabla \mathbf{u}_h^n)\mathbf{C}_h^n, \mathbf{D}_h\big) + \big(\operatorname{div} \mathbf{u}_h^n (\mathbf{C}_h^n)^\#, \mathbf{D}_h\big) - \big((\operatorname{tr} \mathbf{C}_h^n)^2 \mathbf{C}_h^n, \mathbf{D}_h\big)& \\ + \big((\operatorname{tr} \mathbf{C}_h^n)\mathbf{I}, \mathbf{D}_h\big) + (\mathbf{F}_h^n, \mathbf{D}_h),& \qquad (5b)\\ \forall (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h.&\end{aligned}$$

4. The main result

In this section we present the main result on error estimates with the optimal convergence order of scheme (5).

We use c to represent a generic positive constant independent of the discretization parameters h and Δt. We also use constants c_w and c_s independent of h and Δt but dependent on $\mathbf{w}$ and the solution $(\mathbf{u}, p, \mathbf{C})$ of (2), respectively, and c_s often depends on $\mathbf{w}$ additionally. c, c_w and c_s may be dependent on ν but are independent of ε. The symbol "$\prime$ (prime)" is sometimes used in order to distinguish two constants, e.g., c_s and c_s', from each other. We use the following notation for the norms and seminorms, $\|\cdot\|_V = \|\cdot\|_{V_h} := \|\cdot\|_1$, $\|\cdot\|_Q = \|\cdot\|_{Q_h} := \|\cdot\|_0$,

$$\|(\mathbf{u}, \mathbf{C})\|_{Z^2(t_0,t_1)} := \left\{\|\mathbf{u}\|_{Z^2(t_0,t_1)}^2 + \|\mathbf{C}\|_{Z^2(t_0,t_1)}^2\right\}^{1/2}, \qquad \|\mathbf{u}\|_{\ell^\infty(X)} := \max_{n=0,\ldots,N_T} \|\mathbf{u}^n\|_X,$$

$$\|\mathbf{u}\|_{\ell^2(X)} := \left\{\Delta t \sum_{n=1}^{N_T} \|\mathbf{u}^n\|_X^2\right\}^{1/2}, \qquad |\mathbf{u}|_{\ell^2(X)} := \left\{\Delta t \sum_{n=1}^{N_T} |\mathbf{u}^n|_X^2\right\}^{1/2},$$

$$|p|_h := \left\{\sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla p)_K\right\}^{1/2}, \qquad |p|_{\ell^2(|.|_h)} := \left\{\Delta t \sum_{n=1}^{N_T} |p^n|_h^2\right\}^{1/2},$$

for $X = L^2(\Omega)$ or $H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by $\overline{D}_{\Delta t} u^n := (u^n - u^{n-1})/\Delta t$.

The existence of the solution of scheme (5) is guaranteed by the next proposition whose proof is given in the next section.

Proposition 2 (existence)**.** *Suppose Hypothesis 1 holds. For any $h > 0$ and $\Delta t \in (0, 1/2)$ satisfying* (4)*, there exists a solution $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ of scheme* (5)*.*

We state the main result after preparing a projection and a hypothesis.

Definition 1 (Stokes projection)**.** *For $(\mathbf{u}, p) \in V \times Q$ we define the* Stokes *projection $(\hat{\mathbf{u}}_h, \hat{p}_h) \in V_h \times Q_h$ of $(\mathbf{u}, p)$ by*

$$\mathcal{A}_h\left((\hat{\mathbf{u}}_h, \hat{p}_h), (\mathbf{v}_h, q_h)\right) = \mathcal{A}\left((\mathbf{u}, p), (\mathbf{v}_h, q_h)\right), \quad \forall (\mathbf{v}_h, q_h) \in V_h \times Q_h. \tag{6}$$

The Stokes projection derives an operator $\Pi_h^{\mathrm{S}} : V \times Q \to V_h \times Q_h$ defined by $\Pi_h^{\mathrm{S}}(\mathbf{u}, p) := (\hat{\mathbf{u}}_h, \hat{p}_h)$. The first component of $\Pi_h^{\mathrm{S}}(\mathbf{u}, p)$ is denoted by $[\Pi_h^{\mathrm{S}}(\mathbf{u}, p)]_1$. Let $\Pi_h : L^2(\Omega) \to M_h$ be the Clément interpolation operator [5]. The operators on $L^2(\Omega)^2$ and $L^2(\Omega)^{2\times 2}$ are denoted by the same symbol Π_h.

Remark 2. *While we introduced a* Poisson *projection for $\mathbf{C}$ in Part I [8], here we use the* Clément *interpolation operator Π_h, which is sufficient for the proof in the nonlinear scheme. The required regularity on $\mathbf{C}$ in Hypothesis 2 becomes a little weaker. We note that the* Clément *operator can be replaced by the* Lagrange *interpolation operator, when the function belongs to $C(\bar{\Omega})$.*

Hypothesis 2. *The solution* $(\mathbf{u}, p, \mathbf{C})$ *of* (2) *satisfies* $\mathbf{u} \in Z^2(0,T)^2 \cap H^1(0,T; V \cap H^2(\Omega)^2) \cap C([0,T]; W^{1,\infty}(\Omega)^2)$, $p \in H^1(0,T; Q \cap H^1(\Omega))$ *and*

$$\mathbf{C} \in \begin{cases} Z^2(0,T)^{2\times 2} \cap L^2(0,T;W) \cap C([0,T]; H^2(\Omega)^{2\times 2}) & (\varepsilon > 0), \\ Z^2(0,T)^{2\times 2} \cap L^2(0,T;W) \cap C([0,T]; L^\infty(\Omega)^{2\times 2}) & (\varepsilon = 0). \end{cases}$$

We now impose the conditions

$$(\mathbf{u}_h^0, \mathbf{C}_h^0) = ([\Pi_h^{\mathrm{S}}(\mathbf{u}^0, 0)]_1, \Pi_h \mathbf{C}^0), \quad (\mathbf{f}_h, \mathbf{F}_h) = (\mathbf{f}, \mathbf{F}). \tag{7}$$

Theorem 1 (error estimates)**.** *Suppose Hypotheses 1 and 2 hold. Then, there exist positive constants* h_0, Δt_0 *and* $c_\dagger$ *independent of* ε *such that, for any pair* $(h, \Delta t)$ *satisfying*

$$h \in (0, h_0], \quad \Delta t \in (0, \Delta t_0], \tag{8}$$

and any solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ *of scheme* (5) *with* (7)*, it holds that*

$$\begin{aligned} &\|\mathbf{u}_h - \mathbf{u}\|_{\ell^\infty(L^2)},\ \sqrt{\nu}\|\mathbf{u}_h - \mathbf{u}\|_{\ell^2(H^1)},\ |p_h - p|_{\ell^2(|.|_h)}, \\ &\|\mathbf{C}_h - \mathbf{C}\|_{\ell^\infty(L^2)},\ \sqrt{\varepsilon}|\mathbf{C}_h - \mathbf{C}|_{\ell^2(H^1)},\ \big\|\mathrm{tr}\,(\mathbf{C}_h - \mathbf{C})(\mathbf{C}_h - \mathbf{C})\big\|_{\ell^2(L^2)} \le c_\dagger(h + \Delta t). \end{aligned} \tag{9}$$

Remark 3. *(i) The estimates* (9) *hold even for* $\varepsilon = 0$*. Then, of course, the fifth term of the left-hand side of* (9) *vanishes.*
(ii) Here we do not need the uniqueness of the solution of scheme (5)*. The uniqueness is discussed in Proposition 3 below.*

5. Proofs

In what follows we prove Proposition 2 and Theorem 1.

5.1. Preliminaries

Let us list lemmas directly employed below in the proofs. Although some of those lemmas have been already used in Part I [8], we list them again here for the self-containment. In the lemmas, α_i, $i = 1, \ldots, 4$, are numerical constants. They are independent of h, Δt, ν and ε but may depend on Ω.

Lemma 1 ([6])**.** *Let* Ω *be a bounded domain with a* Lipschitz*-continuous boundary. Then, the following inequalities hold.*

$$\|\mathrm{D}(\mathbf{v})\|_0 \le \|\mathbf{v}\|_1 \le \alpha_1 \|\mathrm{D}(\mathbf{v})\|_0, \qquad \forall \mathbf{v} \in H_0^1(\Omega)^2.$$

We introduce the function

$$D(h) := (1 + |\log h|)^{1/2}, \tag{10}$$

which is used in the sequel.

Lemma 2 ([1, 4, 5])**.** *The following inequalities hold.*

$$\begin{aligned} \|\Pi_h \mathbf{g}\|_{0,\infty} &\le \|\mathbf{g}\|_{0,\infty}, & &\forall \mathbf{g} \in L^\infty(\Omega)^s, \\ \|\Pi_h \mathbf{g}\|_{1,\infty} &\le \alpha_{20} \|\mathbf{g}\|_{1,\infty}, & &\forall \mathbf{g} \in W^{1,\infty}(\Omega)^s, \\ \|\Pi_h \mathbf{g} - \mathbf{g}\|_0 &\le \alpha_{21} h \|\mathbf{g}\|_1, & &\forall \mathbf{g} \in H^1(\Omega)^s \cap L^\infty(\Omega)^s, \end{aligned}$$

$$
\begin{aligned}
\|\Pi_h \mathbf{g} - \mathbf{g}\|_1 &\le \alpha_{22} h \|\mathbf{g}\|_2, && \forall \mathbf{g} \in H^2(\Omega)^s,\\
\|\mathbf{g}_h\|_{0,\infty} &\le \alpha_{23} h^{-1} \|\mathbf{g}_h\|_0, && \forall \mathbf{g}_h \in S_h,\\
\|\mathbf{g}_h\|_{0,\infty} &\le \alpha_{24} D(h) \|\mathbf{g}_h\|_1, && \forall \mathbf{g}_h \in S_h,\\
\|\mathbf{g}_h\|_{1,\infty} &\le \alpha_{25} h^{-1} \|\mathbf{g}_h\|_1, && \forall \mathbf{g}_h \in S_h,\\
\|\mathbf{g}_h\|_1 &\le \alpha_{26} h^{-1} \|\mathbf{g}_h\|_0, && \forall \mathbf{g}_h \in S_h,
\end{aligned}
$$

where $s = 2$ *or* 2×2 *and* $S_h = V_h$ *or* W_h*.*

Lemma 3 ([2])**.** *Assume* $(\mathbf{u}, p) \in (V \cap H^2(\Omega)^2) \times (Q \cap H^1(\Omega))$*. Let* $(\hat{\mathbf{u}}_h, \hat{p}_h) \in V_h \times Q_h$ *be the* Stokes *projection of* $(\mathbf{u}, p)$ *by* (6)*. Then, the following inequalities hold,*

$$\|\hat{\mathbf{u}}_h - \mathbf{u}\|_1, \quad \|\hat{p}_h - p\|_0, \quad |\hat{p}_h - p|_h \le \alpha_3 h \|(\mathbf{u}, p)\|_{H^2 \times H^1}.$$

Lemma 4 ([8])**.** *Under Hypothesis 1 and the condition* (4) *the following inequality holds for any* $n \in \{0, \ldots, N_T\}$

$$\|\mathbf{g} \circ X_1^n\|_0 \le (1 + \alpha_4 |\mathbf{w}^n|_{1,\infty} \Delta t) \|\mathbf{g}\|_0, \qquad \forall \mathbf{g} \in L^2(\Omega)^s,$$

where $s = 2$ *or* 2×2*.*

We present a key lemma in order to deal with the nonlinear terms.

Lemma 5. *For* $\mathbf{v} \in \mathbb{R}^2$ *and* $\mathbf{D} \in \mathbb{R}^{2\times 2}_{sym}$ *it holds that*

$$\big((\operatorname{tr} \mathbf{D})\mathbf{D}, \nabla \mathbf{v}\big) - \big((\nabla \mathbf{v})\mathbf{D}, \mathbf{D}\big) - \frac{1}{2}\big(\operatorname{div} \mathbf{v} (\mathbf{D})^{\#}, \mathbf{D}\big) = 0.$$

Proof. The direct calculation yields the desired result. □

Lemma 6 ([17])**.** *Let* a_i*,* $i = 1, 2$*, be non-negative number,* Δt *a positive number, and* $\{x^n\}_{n\ge 0}$*,* $\{y^n\}_{n\ge 1}$ *and* $\{b^n\}_{n\ge 1}$ *non-negative sequences. Assume* $\Delta t \in (0, 1/(2a_0)]$ *for* $a_0 \ne 0$*. Suppose*

$$\overline{D}_{\Delta t} x^n + y^n \le a_0 x^n + a_1 x^{n-1} + b^n, \quad \forall n \ge 1.$$

Then, it holds that

$$x^n + \Delta t \sum_{i=1}^{n} y^i \le \exp[(2a_0 + a_1) n \Delta t] \bigg(x^0 + \Delta t \sum_{i=1}^{n} b^i \bigg), \quad \forall n \ge 1.$$

Lemma 7 ([19, Chap. II, Lemma 1.4], [7, Chap. I, Lemme 4.3])**.** *Let* X *be a finite dimensional* Hilbert *space with inner product* $(\cdot, \cdot)_X$ *and norm* $\|\cdot\|_X$ *and let* $\mathcal{P}$ *be a continuous mapping from* X *into itself such that* $(\mathcal{P}(\xi), \xi)_X > 0$ *for* $\|\xi\|_X = \rho_0 > 0$*. Then, there exists* $\xi \in X$*,* $\|\xi\|_X \le \rho_0$*, such that* $\mathcal{P}(\xi) = 0$*.*

5.2. **Proof of Proposition 2**

We apply Lemma 7 for the proof. Let $n \in \{1, \ldots, N_T\}$ be a fixed number and $(\mathbf{u}_h^{n-1}, \mathbf{C}_h^{n-1}) \in V_h \times W_h$ a pair of given functions. We set $\mu_0 := (1 - 2\Delta t)/2 > 0$. We define a finite dimensional inner product space $X := V_h \times Q_h \times W_h$ equipped with the inner product,

$$\big((\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{v}_h, q_h, \mathbf{D}_h)\big)_X := \frac{1}{\Delta t}(\mathbf{u}_h, \mathbf{v}_h) + 4\nu\big(\mathrm{D}(\mathbf{u}_h), \mathrm{D}(\mathbf{v}_h)\big)$$

$$+ 2\delta_0 \sum_{K\in\mathcal{T}_h} h_K^2 (p_h, q_h)_K + \frac{\mu_0}{\Delta t}(\mathbf{C}_h, \mathbf{D}_h) + \varepsilon(\nabla \mathbf{C}_h, \nabla \mathbf{D}_h),$$

which induces the norm $\|\cdot\|_X$ for any $\varepsilon \geq 0$. Let $\mathcal{P} : V_h \times Q_h \times W_h \to V_h \times Q_h \times W_h$ be a mapping defined by

$$\begin{aligned}
\big(\mathcal{P}(\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{v}_h, q_h, \mathbf{D}_h)\big)_X = {} & \left(\frac{\mathbf{u}_h - \mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\big((\mathbf{u}_h, p_h), (\mathbf{v}_h, -q_h)\big) + \big((\operatorname{tr} \mathbf{C}_h)\mathbf{C}_h, \nabla \mathbf{v}_h\big) \\
& - (\mathbf{f}_h^n, \mathbf{v}_h) + \frac{1}{2}\left(\frac{\mathbf{C}_h - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \frac{\varepsilon}{2} a_c(\mathbf{C}_h, \mathbf{D}_h) - \big((\nabla \mathbf{u}_h)\mathbf{C}_h, \mathbf{D}_h\big) \\
& - \frac{1}{2}\big((\operatorname{div} \mathbf{u}_h)\mathbf{C}_h^{\#}, \mathbf{D}_h\big) + \frac{1}{2}\big((\operatorname{tr} \mathbf{C}_h)^2 \mathbf{C}_h, \mathbf{D}_h\big) - \frac{1}{2}\big((\operatorname{tr} \mathbf{C}_h)\mathbf{I}, \mathbf{D}_h\big) \\
& - \frac{1}{2}(\mathbf{F}_h^n, \mathbf{D}_h), \qquad \forall (\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h. \qquad (11)
\end{aligned}$$

Obviously $\mathcal{P}$ is continuous. Substituting $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in (11) and using the inequality $\|\operatorname{tr} \mathbf{C}_h\|_0 \leq \sqrt{2}\|\mathbf{C}_h\|_0$, we have

$$\begin{aligned}
& \big(\mathcal{P}(\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{u}_h, p_h, \mathbf{C}_h)\big)_X \\
& = \left(\frac{\mathbf{u}_h - \mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{u}_h\right) + 2\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + \delta_0 |p_h|_h^2 - (\mathbf{f}_h^n, \mathbf{u}_h) \\
& \quad + \frac{1}{2}\left(\frac{\mathbf{C}_h - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{C}_h\right) + \frac{\varepsilon}{2}|\mathbf{C}_h|_1^2 + \frac{1}{2}\|(\operatorname{tr} \mathbf{C}_h)\mathbf{C}_h\|_0^2 - \frac{1}{2}\|\operatorname{tr} \mathbf{C}_h\|_0^2 - \frac{1}{2}(\mathbf{F}_h^n, \mathbf{C}_h) \\
& \geq \frac{1}{\Delta t}\big(\|\mathbf{u}_h\|_0^2 - \|\mathbf{u}_h^{n-1} \circ X_1^n\|_0 \|\mathbf{u}_h\|_0\big) + 2\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + \delta_0|p_h|_h^2 - \|\mathbf{f}_h^n\|_0\|\mathbf{u}_h\|_0 \\
& \quad + \frac{1}{2\Delta t}\big(\|\mathbf{C}_h\|_0^2 - \|\mathbf{C}_h^{n-1} \circ X_1^n\|_0\|\mathbf{C}_h\|_0\big) + \frac{\varepsilon}{2}|\mathbf{C}_h|_1^2 - \|\mathbf{C}_h\|_0^2 - \frac{1}{2}\|\mathbf{F}_h^n\|_0\|\mathbf{C}_h\|_0 \\
& \geq \frac{1}{2\Delta t}\Big\{2\|\mathbf{u}_h\|_0^2 - \beta_0\|\mathbf{u}_h\|_0^2 - \frac{1}{\beta_0}\|\mathbf{u}_h^{n-1} \circ X_1^n\|_0^2 + \|\mathbf{C}_h\|_0^2 - \beta_1\|\mathbf{C}_h\|_0^2 - \frac{1}{4\beta_1}\|\mathbf{C}_h^{n-1} \circ X_1^n\|_0^2\Big\} \\
& \quad + 2\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + \delta_0|p_h|_h^2 - \frac{\beta_2}{2\Delta t}\|\mathbf{u}_h\|_0^2 - \frac{\Delta t}{2\beta_2}\|\mathbf{f}_h^n\|_0^2 + \frac{\varepsilon}{2}|\mathbf{C}_h|_1^2 - \|\mathbf{C}_h\|_0^2 - \frac{\beta_3}{2\Delta t}\|\mathbf{C}_h\|_0^2 - \frac{\Delta t}{8\beta_3}\|\mathbf{F}_h^n\|_0^2 \\
& \geq \frac{1}{2\Delta t}\Big\{(2 - \beta_0 - \beta_2)\|\mathbf{u}_h\|_0^2 + (1 - \beta_1 - 2\Delta t - \beta_3)\|\mathbf{C}_h\|_0^2\Big\} + 2\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + \delta_0|p_h|_h^2 \\
& \quad + \frac{\varepsilon}{2}|\mathbf{C}_h|_1^2 - \frac{1}{2\beta_0\Delta t}\|\mathbf{u}_h^{n-1} \circ X_1^n\|_0^2 - \frac{1}{8\beta_1\Delta t}\|\mathbf{C}_h^{n-1} \circ X_1^n\|_0^2 - \frac{\Delta t}{2\beta_2}\|\mathbf{f}_h^n\|_0^2 - \frac{\Delta t}{8\beta_3}\|\mathbf{F}_h^n\|_0^2
\end{aligned}$$

for any $\beta_i > 0$. Choosing $\beta_0 = \beta_2 = 1/2$ and $\beta_1 = \beta_3 = \mu_0/2$, we get

$$\begin{aligned}
\big(\mathcal{P}(\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{u}_h, p_h, \mathbf{C}_h)\big)_X \geq {} & \frac{1}{2}\Bigg[\Big\{\frac{1}{\Delta t}\|\mathbf{u}_h\|_0^2 + 4\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + 2\delta_0|p_h|_h^2 + \frac{\mu_0}{\Delta t}\|\mathbf{C}_h\|_0^2 + \varepsilon|\mathbf{C}_h|_1^2\Big\} \\
& - \Big\{\frac{2\|\mathbf{u}_h^{n-1} \circ X_1^n\|_0^2}{\Delta t} + \frac{\|\mathbf{C}_h^{n-1} \circ X_1^n\|_0^2}{2\mu_0\Delta t} + 2\Delta t\|\mathbf{f}_h^n\|_0^2 + \frac{\Delta t\|\mathbf{F}_h^n\|_0^2}{2\mu_0}\Big\}\Bigg] \\
= {} & \frac{1}{2}\Big[\|(\mathbf{u}_h, p_h, \mathbf{C}_h)\|_X^2 - \beta_*^2\Big],
\end{aligned}$$

where

$$\beta_* := \left\{\frac{2\|\mathbf{u}_h^{n-1} \circ X_1^n\|_0^2}{\Delta t} + \frac{\|\mathbf{C}_h^{n-1} \circ X_1^n\|_0^2}{2\mu_0\Delta t} + 2\Delta t\|\mathbf{f}_h^n\|_0^2 + \frac{\Delta t\|\mathbf{F}_h^n\|_0^2}{2\mu_0}\right\}^{1/2}.$$

The right-hand side is, therefore, positive on the sphere of radius $\rho_0 = \beta_* + 1$. From Lemma 7 there exists an element $(\mathbf{u}_h, p_h, \mathbf{C}_h) \in V_h \times Q_h \times W_h$ such that $\mathcal{P}(\mathbf{u}_h, p_h, \mathbf{C}_h) = 0$, which is nothing but a solution of equations (5). □

5.3. A system of equations for the error and the estimate of remainder terms

In this subsection we prepare a system of equations for the error and a lemma for the estimate of remainder terms in the system before starting the proof of Theorem 1.

Let $(\hat{\mathbf{u}}_h, \hat{p}_h)(t) := \Pi_h^{\mathrm{S}}(\mathbf{u}, p)(t) \in V_h \times Q_h$ and $\check{\mathbf{C}}_h(t) := \Pi_h \mathbf{C}(t) \in W_h$ for $t \in [0, T]$ and let

$$\mathbf{e}_h^n := \mathbf{u}_h^n - \hat{\mathbf{u}}_h^n, \qquad \epsilon_h^n := p_h^n - \hat{p}_h^n, \qquad \mathbf{E}_h^n := \mathbf{C}_h^n - \check{\mathbf{C}}_h^n, \qquad \boldsymbol{\eta}(t) := (\mathbf{u} - \hat{\mathbf{u}}_h)(t), \qquad \boldsymbol{\Xi}(t) := (\mathbf{C} - \check{\mathbf{C}}_h)(t).$$

Then, from (5), (6) and (2), we have for $n \geq 1$

$$\left(\frac{\mathbf{e}_h^n - \mathbf{e}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h \right) + \mathcal{A}_h\big((\mathbf{e}_h^n, \epsilon_h^n), (\mathbf{v}_h, q_h)\big) = -\big((\mathrm{tr}\, \mathbf{E}_h^n)\mathbf{E}_h^n, \nabla \mathbf{v}_h\big) + \langle \mathbf{r}_h^n, \mathbf{v}_h \rangle, \tag{12a}$$

$$\left(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h \right) + \varepsilon a_c(\mathbf{E}_h^n, \mathbf{D}_h) = 2\big((\nabla \mathbf{e}_h^n)\mathbf{E}_h^n, \mathbf{D}_h\big) + \big((\mathrm{div}\, \mathbf{e}_h^n)(\mathbf{E}_h^n)^{\#}, \mathbf{D}_h\big) + \langle \mathbf{R}_h^n, \mathbf{D}_h \rangle, \tag{12b}$$

$$\forall (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h,$$

where

$$\begin{aligned}
\mathbf{r}_h^n &:= \sum_{i=1}^{4} \mathbf{r}_{hi}^n \in V_h', \qquad \mathbf{R}_h^n := \sum_{i=1}^{11} \mathbf{R}_{hi}^n \in W_h', \\
\langle \mathbf{r}_{h1}^n, \mathbf{v}_h \rangle &:= \left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t} - \frac{\mathbf{u}^n - \mathbf{u}^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h \right), \\
\langle \mathbf{r}_{h2}^n, \mathbf{v}_h \rangle &:= \frac{1}{\Delta t} \left(\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1} \circ X_1^n, \mathbf{v}_h \right), \\
\langle \mathbf{r}_{h3}^n, \mathbf{v}_h \rangle &:= -\big((\mathrm{tr}\, \check{\mathbf{C}}^n)\mathbf{E}_h^n + (\mathrm{tr}\, \mathbf{E}_h^n)\check{\mathbf{C}}^n, \nabla \mathbf{v}_h \big), \\
\langle \mathbf{r}_{h4}^n, \mathbf{v}_h \rangle &:= \big((\mathrm{tr}\, \check{\mathbf{C}}^n)\boldsymbol{\Xi}^n + (\mathrm{tr}\, \boldsymbol{\Xi}^n)\mathbf{C}^n, \nabla \mathbf{v}_h \big), \\
\langle \mathbf{R}_{h1}^n, \mathbf{D}_h \rangle &:= \left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h2}^n, \mathbf{D}_h \rangle &:= \frac{1}{\Delta t} \left(\boldsymbol{\Xi}^n - \boldsymbol{\Xi}^{n-1} \circ X_1^n, \mathbf{D}_h \right), \\
\langle \mathbf{R}_{h3}^n, \mathbf{D}_h \rangle &:= \varepsilon a_c(\boldsymbol{\Xi}^n, \mathbf{D}_h), \\
\langle \mathbf{R}_{h4}^n, \mathbf{D}_h \rangle &:= 2\big((\nabla \hat{\mathbf{u}}_h^n)\mathbf{E}_h^n + (\nabla \mathbf{e}_h^n)\check{\mathbf{C}}^n, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h5}^n, \mathbf{D}_h \rangle &:= -2\big((\nabla \hat{\mathbf{u}}_h^n)\boldsymbol{\Xi}^n + (\nabla \boldsymbol{\eta}^n)\mathbf{C}^n, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h6}^n, \mathbf{D}_h \rangle &:= \big((\mathrm{div}\, \hat{\mathbf{u}}_h^n)(\mathbf{E}_h^n)^{\#} + (\mathrm{div}\, \mathbf{e}_h^n)(\check{\mathbf{C}}^n)^{\#}, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h7}^n, \mathbf{D}_h \rangle &:= -\big((\mathrm{div}\, \hat{\mathbf{u}}_h^n)(\boldsymbol{\Xi}^n)^{\#} + (\mathrm{div}\, \boldsymbol{\eta}^n)(\mathbf{C}^n)^{\#}, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h8}^n, \mathbf{D}_h \rangle &:= -\big([\mathrm{tr}\, (\mathbf{E}_h^n + \check{\mathbf{C}}^n)]^2 \mathbf{E}_h^n, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h9}^n, \mathbf{D}_h \rangle &:= -\big([\mathrm{tr}\, (\mathbf{E}_h^n + 2\check{\mathbf{C}}^n)](\mathrm{tr}\, \mathbf{E}_h^n)\check{\mathbf{C}}^n, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h10}^n, \mathbf{D}_h \rangle &:= \big((\mathrm{tr}\, \check{\mathbf{C}}^n)\boldsymbol{\Xi}^n + [\mathrm{tr}\, (\mathbf{C}^n + \check{\mathbf{C}}^n)](\mathrm{tr}\, \boldsymbol{\Xi}^n)\mathbf{C}^n, \mathbf{D}_h \big), \\
\langle \mathbf{R}_{h11}^n, \mathbf{D}_h \rangle &:= \left([\mathrm{tr}\, (\mathbf{E}_h^n - \boldsymbol{\Xi}^n)]\mathbf{I}, \mathbf{D}_h \right).
\end{aligned}$$

We note that

$$(\mathbf{e}_h^0, \mathbf{E}_h^0) = (\mathbf{u}_h^0, \mathbf{C}_h^0) - (\hat{\mathbf{u}}_h^0, \check{\mathbf{C}}_h^0) = ([\Pi_h^{\mathrm{S}}(\mathbf{0}, -p^0)]_1, \mathbf{0}). \tag{13}$$

The remainder terms are evaluated by the next lemma, whose proof is given in Subsection A.1.

Lemma 8. *Suppose Hypotheses 1 and 2 hold. Let $n \in \{1, \ldots, N_T\}$ be any fixed number. Then, under the condition* (4) *it holds that*

$$\|\mathbf{r}_{h1}^n\|_0 \le c_w \sqrt{\Delta t} \|\mathbf{u}\|_{Z^2(t^{n-1},t^n)}, \tag{14a}$$

$$\|\mathbf{r}_{h2}^n\|_0 \le \frac{c_w h}{\sqrt{\Delta t}} \|(\mathbf{u}, p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}, \tag{14b}$$

$$\|\mathbf{r}_{h3}^n\|_{-1} \le c_s \|\mathbf{E}_h^n\|_0, \tag{14c}$$

$$\|\mathbf{r}_{h4}^n\|_{-1} \le c_s h, \tag{14d}$$

$$\|\mathbf{R}_{h1}^n\|_0 \le c_w \sqrt{\Delta t} \|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}, \tag{14e}$$

$$\|\mathbf{R}_{h2}^n\|_0 \le \frac{c_w h}{\sqrt{\Delta t}} \|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^1)\cap L^2(t^{n-1},t^n;H^2)}, \tag{14f}$$

$$\left\langle \mathbf{R}_{h3}^n, \frac{1}{2}\mathbf{E}_h^n \right\rangle \le \frac{\varepsilon}{4} |\mathbf{E}_h^n|_1^2 + c_s h^2, \tag{14g}$$

$$\|\mathbf{R}_{h4}^n\|_0 \le c_s (\|\mathbf{e}_h^n\|_1 + \|\mathbf{E}_h^n\|_0), \tag{14h}$$

$$\|\mathbf{R}_{h5}^n\|_0 \le c_s h, \tag{14i}$$

$$\|\mathbf{R}_{h6}^n\|_0 \le c_s \big(\|\mathbf{e}_h^n\|_1 + \|\mathbf{E}_h^n\|_0\big), \tag{14j}$$

$$\|\mathbf{R}_{h7}^n\|_0 \le c_s h, \tag{14k}$$

$$\left\langle \mathbf{R}_{h8}^n, \frac{1}{2}\mathbf{E}_h^n \right\rangle \le -\frac{3}{8} \|(\operatorname{tr} \mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 + c_s \|\mathbf{E}_h^n\|_0^2, \tag{14l}$$

$$\left\langle \mathbf{R}_{h9}^n, \frac{1}{2}\mathbf{E}_h^n \right\rangle \le \frac{1}{8} \|(\operatorname{tr} \mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 + c_s \|\mathbf{E}_h^n\|_0^2, \tag{14m}$$

$$\|\mathbf{R}_{h10}^n\|_0 \le c_s h, \tag{14n}$$

$$\|\mathbf{R}_{h11}^n\|_0 \le c_s (\|\mathbf{E}_h^n\|_0 + h). \tag{14o}$$

5.4. Proof of Theorem 1

The constant h_0 can be chosen arbitrarily, say, $h_0 = 1$. We fix Δt_0 by

$$\Delta t_0 = \min\left\{ \frac{1}{4|\mathbf{w}|_{C(W^{1,\infty})}}, \frac{1}{2c_s} \right\}, \tag{15}$$

where c_s is the constant appearing in (18) below. We consider any pair $(h, \Delta t)$ satisfying (8) and any solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (5) with (7). We return to the argument in the previous subsection. Substituting $(\mathbf{e}_h^n, -\epsilon_h^n, \frac{1}{2}\mathbf{E}_h^n)$ into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in (12) and noting that

$$\left(\frac{\mathbf{e}_h^n - \mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{e}_h^n \right) \ge \frac{1}{2\Delta t}\Big[\|\mathbf{e}_h^n\|_0^2 - (1 + \alpha_4 |\mathbf{w}^n|_{1,\infty}\Delta t)^2 \|\mathbf{e}_h^{n-1}\|_0^2 \Big] \ge \overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2\Big) - c_w \|\mathbf{e}_h^{n-1}\|_0^2,$$

$$\mathcal{A}_h\big((\mathbf{e}_h^n, \epsilon_h^n), (\mathbf{e}_h^n, -\epsilon_h^n)\big) \ge \frac{2\nu}{\alpha_1^2} \|\mathbf{e}_h^n\|_1^2 + \delta_0 |p_h^n|_h^2,$$

$$\langle \mathbf{r}_h^n, \mathbf{e}_h^n \rangle \le \|\mathbf{r}_h^n\|_{-1} \|\mathbf{e}_h^n\|_1 \le \frac{\alpha_1^2}{4\nu} \|\mathbf{r}_h^n\|_{-1}^2 + \frac{\nu}{\alpha_1^2} \|\mathbf{e}_h^n\|_1^2,$$

$$\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t}, \frac{1}{2}\mathbf{E}_h^n\Big) \ge \overline{D}_{\Delta t}\Big(\frac{1}{4}\|\mathbf{E}_h^n\|_0^2\Big) - c_w\|\mathbf{E}_h^{n-1}\|_0^2,$$

$$\varepsilon a_c\Big(\mathbf{E}_h^n, \frac{1}{2}\mathbf{E}_h^n\Big) = \frac{\varepsilon}{2}|\mathbf{E}_h^n|_1^2,$$

and Lemma 5, we have

$$\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^n\|_0^2\Big) + \frac{\nu}{\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|\epsilon_h^n|_h^2 + \frac{\varepsilon}{2}|\mathbf{E}_h^n|_1^2 \le c_w(\|\mathbf{e}_h^{n-1}\|_0^2 + \|\mathbf{E}_h^{n-1}\|_0^2) + \frac{\alpha_1^2}{4\nu}\|\mathbf{r}_h^n\|_{-1}^2 + \Big\langle \mathbf{R}_h^n, \frac{1}{2}\mathbf{E}_h^n\Big\rangle. \tag{16}$$

Since the condition (4) is satisfied, Lemma 8 implies that

$$\|\mathbf{r}_h^n\|_{-1}^2 \le c_s\|\mathbf{E}_h^n\|_0^2 + c_s'\Big[\Delta t\|\mathbf{u}\|_{Z^2(t^{n-1},t^n)}^2 + h^2\Big(\frac{1}{\Delta t}\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + 1\Big)\Big], \tag{17a}$$

$$\begin{aligned}\Big\langle \mathbf{R}_h^n, \frac{1}{2}\mathbf{E}_h^n\Big\rangle &\le c_s\|\mathbf{E}_h^n\|_0^2 + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \frac{\varepsilon}{4}|\mathbf{E}_h^n|_1^2 - \frac{1}{4}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2\\ &\quad + c_s'\Big[\Delta t\|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2 + h^2\Big(\frac{1}{\Delta t}\|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2 + 1\Big)\Big].\end{aligned} \tag{17b}$$

Combining (17) with (16), we obtain

$$\begin{aligned}&\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^n\|_0^2\Big) + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|\epsilon_h^n|_h^2 + \frac{\varepsilon}{4}|\mathbf{E}_h^n|_1^2 + \frac{1}{4}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2\\ &\le c_s\Big(\frac{1}{2}\|\mathbf{e}_h^{n-1}\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^{n-1}\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^n\|_0^2\Big)\\ &\quad + c_s'\Big[\Delta t\|(\mathbf{u},\mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2 + h^2\Big\{\frac{1}{\Delta t}\big(\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + \|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2\big) + 1\Big\}\Big].\end{aligned} \tag{18}$$

From (8) and (15) it holds that $\Delta t \in (0, 1/(2c_s)]$. By applying Lemma 6 to (18) and noting that

$$\|\mathbf{e}_h^0\|_0 \le \alpha_3 h\|(0,-p^0)\|_{H^2\times H^1} = \alpha_3 h\|p\|_{C(H^1)}, \quad \|\mathbf{E}_h^0\|_0 = 0,$$

from (13), there exists a positive constant

$$\tilde{c}_\dagger = c\exp(3c_sT/2)\big[\|p\|_{C(H^1)} + \sqrt{c_s'}\big(\|(\mathbf{u},\mathbf{C})\|_{Z^2} + \|(\mathbf{u},p)\|_{H^1(H^2\times H^1)} + \sqrt{T}\big)\big]$$

independent of ε such that

$$\|\mathbf{e}_h\|_{\ell^\infty(L^2)},\ \sqrt{\nu}\|\mathbf{e}_h\|_{\ell^2(H^1)},\ |\epsilon_h|_{\ell^2(|.|_h)},\ \|\mathbf{E}_h\|_{\ell^\infty(L^2)},\ \sqrt{\varepsilon}|\mathbf{E}_h|_{\ell^2(H^1)},\ \big\|(\mathrm{tr}\,\mathbf{E}_h)\mathbf{E}_h\big\|_{\ell^2(L^2)} \le \tilde{c}_\dagger(h+\Delta t). \tag{19}$$

Hence, we obtain (9) from (19) and the estimates,

$$\begin{aligned}\|\mathbf{u}_h^n - \mathbf{u}^n\|_s &\le \|\mathbf{e}_h^n\|_s + \|\boldsymbol{\eta}^n\|_1 \le \|\mathbf{e}_h^n\|_s + \alpha_3 h\|(\mathbf{u},p)\|_{C(H^2\times H^1)},\\ |p_h^n - p^n|_h &\le |\epsilon_h^n|_h + |\hat{p}_h^n - p^n|_h \le |\epsilon_h^n|_h + \alpha_3 h\|(\mathbf{u},p)\|_{C(H^2\times H^1)},\\ \|\mathbf{C}_h^n - \mathbf{C}^n\|_s &\le \|\mathbf{E}_h^n\|_s + \|\boldsymbol{\Xi}^n\|_s \le \|\mathbf{E}_h^n\|_s + \alpha_{2(s+1)} h\|\mathbf{C}\|_{C(H^{s+1})},\\ \|\mathrm{tr}\,(\mathbf{C}_h^n - \mathbf{C}^n)(\mathbf{C}_h^n - \mathbf{C}^n)\|_0 &= \|\mathrm{tr}\,(\mathbf{E}_h^n - \boldsymbol{\Xi}^n)(\mathbf{E}_h^n - \boldsymbol{\Xi}^n)\|_0 \le \|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0 + c_s h(\|\mathbf{E}_h^n\|_0 + 1),\end{aligned}$$

for $s = 0$ or 1.

When $\varepsilon = 0$, (9) is still valid, since $\mathbf{R}_{h3}^n$ vanishes and $c_\dagger$ is independent of ε. □

6. Uniqueness of the solution

In this section we present and prove the result on the uniqueness of the solution of scheme (5). Let us remind that the function $D(h)$ has been defined in (10).

Proposition 3 (uniqueness). *Suppose Hypotheses 1 and 2 hold. Then, for any pair $(h, \Delta t)$ satisfying the following condition (20) or (21), the solution of scheme (5) with (7) is unique.*
(i) When $\varepsilon > 0$,

$$h \in (0, h_\star], \quad \Delta t \le D(h)^{-2}, \tag{20}$$

where the constant $h_\star$ is defined by (33) below.
(ii) When $\varepsilon = 0$,

$$h \in (0, \bar{h}_\star], \quad \Delta t \le \bar{c}_\star h, \tag{21}$$

where the constants $\bar{h}_\star$ and $\bar{c}_\star$ are defined by (34) and (37) below.

The proof is given after preparing the next lemma.

Lemma 9. *Suppose Hypotheses 1 and 2 hold. Then, for any pair $(h, \Delta t)$ satisfying the following condition (23) or (24), any solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (5) with (7) satisfies*

$$\|\mathbf{C}_h\|_{\ell^\infty(L^\infty)} \le c_c, \qquad \|\mathbf{u}_h\|_{\ell^\infty(L^\infty)} \le c_u, \tag{22}$$

where c_c and c_u are positive constants independent of h and Δt defined just below.
(i) When $\varepsilon > 0$,

$$h \in (0, h_\dagger], \quad \Delta t \le D(h)^{-2}, \tag{23}$$

where $h_\dagger$ is defined by (25d) below. Furthermore, $c_c = c_{\dagger c}$ and $c_u = c_{\dagger u}$, which are defined by (25e) and (25f).
(ii) When $\varepsilon = 0$,

$$h \in (0, \bar{h}_\dagger], \quad \Delta t \le h, \tag{24}$$

where $\bar{h}_\dagger$ is defined by (25a) below. Furthermore, $c_c = \bar{c}_{\dagger c}$ and $c_u = \bar{c}_{\dagger u}$, which are defined by (25b) and (25c).

Proof. Let $n \in \{0, \dots, N_T\}$ be fixed arbitrarily, and let h_0, Δt_0 and $\tilde{c}_\dagger$ be the positive constants in the statement of Theorem 1 and in (19). We fix a positive constant $h_1 \in (0, 1]$ such that

$$h_1 \le D(h_1)^{-2} \le \Delta t_0.$$

We prepare the following constants to be used in the proof:

$$\bar{h}_\dagger := \min\{h_0, \Delta t_0\}, \tag{25a}$$
$$\bar{c}_{\dagger c} := 2\alpha_{23}\tilde{c}_\dagger + \|\mathbf{C}\|_{C(L^\infty)}, \tag{25b}$$
$$\bar{c}_{\dagger u} := \alpha_{23}\big[2\tilde{c}_\dagger + (\alpha_{21} + \alpha_3)\|(\mathbf{u}, p)\|_{C(H^2 \times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)}, \tag{25c}$$
$$c_1 := \tilde{c}_\dagger \max\{1, (T + \varepsilon^{-1})^{1/2}, \nu^{-1/2}\},$$
$$h_\dagger := \min\{\bar{h}_\dagger, h_1\}, \tag{25d}$$
$$c_{\dagger c} := \max\{2\alpha_{24}c_1 + \|\mathbf{C}\|_{C(L^\infty)}, \bar{c}_{\dagger c}\}, \tag{25e}$$
$$c_{\dagger u} := \max\{\alpha_{24}\big[2c_1 + (\alpha_{22} + \alpha_3)\|(\mathbf{u}, p)\|_{C(H^2 \times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)}, \bar{c}_{\dagger u}\}. \tag{25f}$$

Firstly, we prove (22) in case (ii). Since condition (24) implies (8), Theorem 1 ensures (19). Then, the boundedness of $\|\mathbf{C}_h^n\|_{0,\infty}$ is obtained as follows:

$$\begin{aligned}\|\mathbf{C}_h^n\|_{0,\infty} &\le \|\mathbf{E}_h^n\|_{0,\infty} + \|\check{\mathbf{C}}_h^n\|_{0,\infty} \le \alpha_{23} h^{-1}\|\mathbf{E}_h^n\|_0 + \|\mathbf{C}\|_{C(L^\infty)} \\ &\le \alpha_{23} h^{-1}\tilde{c}_\dagger(\Delta t + h) + \|\mathbf{C}\|_{C(L^\infty)} \le 2\alpha_{23}\tilde{c}_\dagger + \|\mathbf{C}\|_{C(L^\infty)} \\ &= \bar{c}_{\dagger c}.\end{aligned}$$

Let $\check{\mathbf{u}}_h(t) := (\Pi_h \mathbf{u})(t)$ for $t \in [0,T]$. The boundedness of $\|\mathbf{u}_h^n\|_{0,\infty}$ is obtained as follows:

$$\begin{aligned}\|\mathbf{u}_h^n\|_{0,\infty} &\le \|\mathbf{e}_h^n\|_{0,\infty} + \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_{0,\infty} + \|\check{\mathbf{u}}_h^n\|_{0,\infty} \le \alpha_{23} h^{-1}\big[\|\mathbf{e}_h^n\|_0 + \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_0\big] + \|\mathbf{u}\|_{C(L^\infty)} \\ &\le \alpha_{23} h^{-1}\big[\|\mathbf{e}_h^n\|_0 + \|\hat{\mathbf{u}}_h^n - \mathbf{u}^n\|_0 + \|\mathbf{u}^n - \check{\mathbf{u}}_h^n\|_0\big] + \|\mathbf{u}\|_{C(L^\infty)} \\ &\le \alpha_{23} h^{-1}\big[\tilde{c}_\dagger(\Delta t + h) + \alpha_3 h\|(\mathbf{u},p)\|_{C(H^2\times H^1)} + \alpha_{21} h\|\mathbf{u}\|_{C(H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)} \\ &\le \alpha_{23}\big[2\tilde{c}_\dagger + (\alpha_{21}+\alpha_3)\|(\mathbf{u},p)\|_{C(H^2\times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)} \\ &= \bar{c}_{\dagger u}.\end{aligned}$$

Secondly, we prove (22) in case (i). Since condition (23) implies (8), the estimates (19) and the definition of c_1 lead to

$$\|\mathbf{e}_h\|_{\ell^\infty(L^2)},\ \|\mathbf{e}_h\|_{\ell^2(H^1)},\ \|\mathbf{E}_h\|_{\ell^\infty(L^2)},\ \|\mathbf{E}_h\|_{\ell^2(H^1)} \le c_1(\Delta t + h).$$

When $\Delta t \le h$, we have $\|\mathbf{C}_h^n\|_{0,\infty} \le \bar{c}_{\dagger c} \le c_{\dagger c}$ and $\|\mathbf{u}_h^n\|_{0,\infty} \le \bar{c}_{\dagger u} \le c_{\dagger u}$ from the proof in case (ii) above. When $(D(h)^2h^2 \le)\ h \le \Delta t \le D(h)^{-2}$, we have

$$\begin{aligned}\|\mathbf{C}_h^n\|_{0,\infty} &\le \|\mathbf{E}_h^n\|_{0,\infty} + \|\mathbf{C}\|_{C(L^\infty)} \le \alpha_{24} D(h)\|\mathbf{E}_h^n\|_1 + \|\mathbf{C}\|_{C(L^\infty)} \le \alpha_{24} D(h)\Delta t^{-1/2}\|\mathbf{E}_h\|_{\ell^2(H^1)} + \|\mathbf{C}\|_{C(L^\infty)} \\ &\le \alpha_{24} c_1 D(h)(\Delta t^{1/2} + \Delta t^{-1/2}h) + \|\mathbf{C}\|_{C(L^\infty)} \le 2\alpha_{24}c_1 + \|\mathbf{C}\|_{C(L^\infty)} \\ &\le c_{\dagger c}, \\ \|\mathbf{u}_h^n\|_{0,\infty} &\le \|\mathbf{e}_h^n\|_{0,\infty} + \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_{0,\infty} + \|\check{\mathbf{u}}_h^n\|_{0,\infty} \le \alpha_{24} D(h)\big[\|\mathbf{e}_h^n\|_1 + \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_1\big] + \|\mathbf{u}\|_{C(L^\infty)} \\ &\le \alpha_{24} D(h)\big[\Delta t^{-1/2}\|\mathbf{e}_h\|_{\ell^2(H^1)} + \|\hat{\mathbf{u}}_h^n - \mathbf{u}^n\|_1 + \|\mathbf{u}^n - \check{\mathbf{u}}_h^n\|_1\big] + \|\mathbf{u}\|_{C(L^\infty)} \\ &\le \alpha_{24} D(h)\big[c_1(\Delta t^{1/2} + \Delta t^{-1/2}h) + (\alpha_{22}+\alpha_3)h\|(\mathbf{u},p)\|_{C(H^2\times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)} \\ &\le \alpha_{24}\big[2c_1 + (\alpha_{22}+\alpha_3)\|(\mathbf{u},p)\|_{C(H^2\times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)} \\ &\le c_{\dagger u}.\end{aligned}$$

Thus, we obtain (22). □

Proof of Proposition 3. The definitions (33), (34) and (37) below of the constants $h_\star$, $\bar{h}_\star$ and $c_\star$ imply $h_\star \le h_\dagger$, $\bar{h}_\star \le \bar{h}_\dagger$ and $\bar{c}_\star \le 1$. Hence any pair of $(h, \Delta t)$ in Proposition 3 satisfies the assumptions of Lemma 9 for $\varepsilon \ge 0$.

Suppose $(\tilde{\mathbf{u}}_h, \tilde{p}_h, \tilde{\mathbf{C}}_h)$ and $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ are any two solutions of scheme (5) with (7). Let $(\tilde{\mathbf{e}}_h, \tilde{\epsilon}_h, \tilde{\mathbf{E}}_h) := (\tilde{\mathbf{u}}_h, \tilde{p}_h, \tilde{\mathbf{C}}_h) - (\mathbf{u}_h, p_h, \mathbf{C}_h)$ be the difference. Since both of $(\tilde{\mathbf{u}}_h, \tilde{p}_h, \tilde{\mathbf{C}}_h)$ and $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ satisfy scheme (5) with (7), we have

$$\Big(\frac{\tilde{\mathbf{e}}_h^n - \tilde{\mathbf{e}}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\Big) + \mathcal{A}_h\big((\tilde{\mathbf{e}}_h^n, \tilde{\epsilon}_h^n), (\mathbf{v}_h, q_h)\big) = -\big((\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n, \nabla \mathbf{v}_h\big) + \langle \tilde{\mathbf{r}}_h^n, \mathbf{v}_h\rangle, \tag{26a}$$

$$\Big(\frac{\tilde{\mathbf{E}}_h^n - \tilde{\mathbf{E}}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\Big) + \varepsilon a_c(\tilde{\mathbf{E}}_h^n, \mathbf{D}_h) = 2\big((\nabla\tilde{\mathbf{e}}_h^n)\tilde{\mathbf{E}}_h^n, \mathbf{D}_h\big) + \big((\mathrm{div}\,\tilde{\mathbf{e}}_h^n)(\tilde{\mathbf{E}}_h^n)^\#, \mathbf{D}_h\big) + \langle \tilde{\mathbf{R}}_h^n, \mathbf{D}_h\rangle, \tag{26b}$$

$$\forall(\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h,$$

where

$$\tilde{\mathbf{r}}_h^n \in V_h', \qquad \tilde{\mathbf{R}}_h^n := \sum_{i=1}^{5} \tilde{\mathbf{R}}_{hi}^n \in W_h',$$
$$\langle \tilde{\mathbf{r}}_h^n, \mathbf{v}_h \rangle := -\big((\operatorname{tr} \mathbf{C}_h^n) \tilde{\mathbf{E}}_h^n + (\operatorname{tr} \tilde{\mathbf{E}}_h^n) \mathbf{C}_h^n, \nabla \mathbf{v}_h \big),$$
$$\langle \tilde{\mathbf{R}}_{h1}^n, \mathbf{D}_h \rangle := 2\big((\nabla \mathbf{u}_h^n) \tilde{\mathbf{E}}_h^n + (\nabla \tilde{\mathbf{e}}_h^n) \mathbf{C}_h^n, \mathbf{D}_h \big),$$
$$\langle \tilde{\mathbf{R}}_{h2}^n, \mathbf{D}_h \rangle := \big((\operatorname{div} \mathbf{u}_h^n)(\tilde{\mathbf{E}}_h^n)^{\#} + (\operatorname{div} \tilde{\mathbf{e}}_h^n)(\mathbf{C}_h^n)^{\#}, \mathbf{D}_h \big),$$
$$\langle \tilde{\mathbf{R}}_{h3}^n, \mathbf{D}_h \rangle := -\big([\operatorname{tr} (\tilde{\mathbf{E}}_h^n + \mathbf{C}_h^n)]^2 \tilde{\mathbf{E}}_h^n, \mathbf{D}_h \big),$$
$$\langle \tilde{\mathbf{R}}_{h4}^n, \mathbf{D}_h \rangle := -\big([\operatorname{tr} (\tilde{\mathbf{E}}_h^n + 2\mathbf{C}_h^n)](\operatorname{tr} \tilde{\mathbf{E}}_h^n) \mathbf{C}_h^n, \mathbf{D}_h \big),$$
$$\langle \tilde{\mathbf{R}}_{h5}^n, \mathbf{D}_h \rangle := \big((\operatorname{tr} \tilde{\mathbf{E}}_h^n) \mathbf{I}, \mathbf{D}_h \big),$$

and $(\tilde{\mathbf{e}}_h^0, \tilde{\mathbf{E}}_h^0) = (\mathbf{0}, \mathbf{0})$. Substituting $(\tilde{\mathbf{e}}_h^n, -\tilde{\epsilon}_h^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n)$ into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in (26) and using Lemma 5 and similar estimates in the derivation of (16), we have

$$\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\tilde{\mathbf{e}}_h^n\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2 \Big) + \frac{\nu}{\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \delta_0 |\tilde{\epsilon}_h^n|_h^2 + \frac{\varepsilon}{2}|\tilde{\mathbf{E}}_h^n|_1^2 \le c_w (\|\tilde{\mathbf{e}}_h^{n-1}\|_0^2 + \|\tilde{\mathbf{E}}_h^{n-1}\|_0^2) + \frac{\alpha_1^2}{4\nu}\|\tilde{\mathbf{r}}_h^n\|_{-1}^2 + \Big\langle \tilde{\mathbf{R}}_h^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle. \tag{27}$$

The functionals $\tilde{\mathbf{r}}_h^n$ and $\tilde{\mathbf{R}}_h^n$ are estimated as follows:

$$\|\tilde{\mathbf{r}}_h^n\|_{-1} \le c \|\mathbf{C}_h^n\|_{0,\infty} \|\tilde{\mathbf{E}}_h^n\|_0, \tag{28}$$

$$\Big\langle \tilde{\mathbf{R}}_{h1}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle, \Big\langle \tilde{\mathbf{R}}_{h2}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle \le c \|\tilde{\mathbf{E}}_h^n\|_0 \big(\|\mathbf{u}_h^n\|_{0,\infty} |\tilde{\mathbf{E}}_h^n|_1 + \|\mathbf{C}_h^n\|_{0,\infty} |\tilde{\mathbf{e}}_h^n|_1 \big), \tag{29a}$$

$$\Big\langle \tilde{\mathbf{R}}_{h3}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle \le -\frac{3}{8}\|(\operatorname{tr} \tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2 + c\|\mathbf{C}_h^n\|_{0,\infty}^2 \|\tilde{\mathbf{E}}_h^n\|_0^2, \tag{29b}$$

$$\Big\langle \tilde{\mathbf{R}}_{h4}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle \le \frac{1}{8}\|(\operatorname{tr} \tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2 + c\|\mathbf{C}_h^n\|_{0,\infty}^2 \|\tilde{\mathbf{E}}_h^n\|_0^2, \tag{29c}$$

$$\|\tilde{\mathbf{R}}_{h5}^n\|_0 \le c \|\tilde{\mathbf{E}}_h^n\|_0, \tag{29d}$$

where the estimates (29a) are proved in Subsection A.2, and the other estimates (28), (29b), (29c) and (29d) are obtained similarly to (14c), (14l), (14m) and (14o), respectively. Applying Lemma 9 to (28), we have

$$\|\tilde{\mathbf{r}}_h^n\|_{-1} \le c c_c \|\tilde{\mathbf{E}}_h^n\|_0. \tag{30}$$

We consider case (i). The estimates (29) and Lemma 9 lead to

$$\Big\langle \tilde{\mathbf{R}}_h^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle \le \frac{c}{\varepsilon}(c_c^2 + c_u^2 + 1)\|\tilde{\mathbf{E}}_h^n\|_0^2 + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \frac{\varepsilon}{4}|\tilde{\mathbf{E}}_h^n|_1^2 - \frac{1}{4}\|(\operatorname{tr} \tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2. \tag{31}$$

Combining (30) and (31) with (27), we have

$$\begin{aligned} &\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\tilde{\mathbf{e}}_h^n\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2 \Big) + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \delta_0 |\tilde{\epsilon}_h^n|_h^2 + \frac{\varepsilon}{4}|\tilde{\mathbf{E}}_h^n|_1^2 + \frac{1}{4}\|(\operatorname{tr} \tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2 \\ &\qquad \le \frac{c}{\varepsilon}(c_c^2 + c_u^2 + 1)\Big(\frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2 \Big) + c_w \Big(\frac{1}{2}\|\tilde{\mathbf{e}}_h^{n-1}\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^{n-1}\|_0^2 \Big). \end{aligned} \tag{32}$$

Let $\Delta t_\star := \varepsilon/[2c(c_c^2+c_u^2+1)]$, and we fix a positive constant $h_2 \in (0,1]$ such that $D(h_2)^{-2} \le \Delta t_\star$. We define $h_\star$ by

$$h_\star := \min\{h_\dagger, h_2\}. \tag{33}$$

Condition (20) implies $\Delta t \le D(h_2)^{-2} \le \varepsilon/[2c(c_c^2+c_u^2+1)]$ $(= \Delta t_\star)$. Applying Lemma 6 to (32) and using the fact $(\tilde{\mathbf{e}}_h^0, \tilde{\mathbf{E}}_h^0) = (\mathbf{0}, \mathbf{0})$, we get $(\tilde{\mathbf{e}}_h, \tilde{\epsilon}_h, \tilde{\mathbf{E}}_h) = (\mathbf{0}, 0, \mathbf{0})$.

We prove (ii). In place of (29a) we use the estimates,

$$\Big\langle \tilde{\mathbf{R}}_{h1}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle, \Big\langle \tilde{\mathbf{R}}_{h2}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle \le c\|\tilde{\mathbf{E}}_h^n\|_0 \big(\alpha_{26} h^{-1}\|\mathbf{u}_h^n\|_{0,\infty}\|\tilde{\mathbf{E}}_h^n\|_0 + \|\mathbf{C}_h^n\|_{0,\infty}|\tilde{\mathbf{e}}_h^n|_1\big). \tag{29a$'$}$$

We define $\bar{h}_\star$ by

$$\bar{h}_\star := \min\big\{\bar{h}_\dagger, 1/c_u, c_u/c_c^2\big\}. \tag{34}$$

For any $h \in (0, \bar{h}_\star]$ the estimates (29), Lemma 9 and (34) lead to

$$\begin{aligned}\Big\langle \tilde{\mathbf{R}}_h^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n \Big\rangle &\le c\Big(\frac{c_u}{h} + c_c^2 + 1\Big)\|\tilde{\mathbf{E}}_h^n\|_0^2 + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 - \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2 \\ &\le \frac{c' c_u}{h}\|\tilde{\mathbf{E}}_h^n\|_0^2 + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 - \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2. \end{aligned} \tag{35}$$

Combining (30) and (35) with (27), we have

$$\overline{D}_{\Delta t}\Big(\frac{1}{2}\|\tilde{\mathbf{e}}_h^n\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2\Big) + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \delta_0|\tilde{\epsilon}_h^n|_h^2 + \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2 \le \frac{cc_u}{h}\Big(\frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2\Big) + c_w\Big(\frac{1}{2}\|\tilde{\mathbf{e}}_h^{n-1}\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^{n-1}\|_0^2\Big). \tag{36}$$

We define $\bar{c}_\star$ by

$$\bar{c}_\star := \min\big\{1, 1/(2cc_u)\big\}. \tag{37}$$

Since condition (21) implies $\Delta t \le h/(2cc_u)$, applying Lemma 6 to (36) and using the fact $(\tilde{\mathbf{e}}_h^0, \tilde{\mathbf{E}}_h^0) = (\mathbf{0}, \mathbf{0})$, we obtain $(\tilde{\mathbf{e}}_h, \tilde{\epsilon}_h, \tilde{\mathbf{E}}_h) = (\mathbf{0}, 0, \mathbf{0})$, which completes the proof of (ii). □

7. Numerical experiments

In this section we present numerical results by scheme (5) in order to confirm the theoretical convergence order. For the detailed description of the algorithm we refer to [10]. The following example is the same that is employed in Part I [8, Example].

Example. *In problem* (1) *we set* $\Omega = (0,1)^2$ *and* $T = 0.5$*, and we consider three cases for the pair of* ν *and* ε*,*

$$(\nu, \varepsilon) = (10^{-1}, 10^{-1}),\ (10^{-1}, 10^{-3}),\ (1, 0).$$

The functions $\mathbf{f}$, $\mathbf{F}$, $\mathbf{u}^0$ *and* $\mathbf{C}^0$ *are given such that the exact solution to* (1) *is as follows:*

$$\begin{aligned}
\mathbf{u}(x,t) &= \left(\frac{\partial \psi}{\partial x_2}(x,t), -\frac{\partial \psi}{\partial x_1}(x,t)\right), \quad p(x,t) = \sin\{\pi(x_1+2x_2+t)\}, \\
C_{11}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+t)\}+1, \\
C_{22}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_2+t)\}+1, \\
C_{12}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+x_2+t)\} \; (= C_{21}(x,t)), \\
\psi(x,t) &:= \frac{\sqrt{3}}{2\pi}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+x_2+t)\}.
\end{aligned} \tag{38}$$

Since Theorem 1 holds for any fixed positive constant δ_0, we simply fix $\delta_0 = 1$. Let N be the division number of each side of the square domain. We set $N = 32, 64, 128$ and 256, and (re)define $h := 1/N$. The time increment is set as $\Delta t = h/2$.

We solve Example by scheme (5) with (7). For the solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (5) and the exact solution $(\mathbf{u}, p, \mathbf{C})$ given by (38) we define the relative errors $Er\,i$, $i = 1, \ldots, 6$, by

$$Er\,1 = \frac{\|\mathbf{u}_h - \Pi_h\mathbf{u}\|_{\ell^\infty(L^2)}}{\|\Pi_h\mathbf{u}\|_{\ell^\infty(L^2)}}, \qquad Er\,2 = \frac{\|\mathbf{u}_h - \Pi_h\mathbf{u}\|_{\ell^2(H^1)}}{\|\Pi_h\mathbf{u}\|_{\ell^2(H^1)}},$$

$$Er\,3 = \frac{\|p_h - \Pi_h p\|_{\ell^2(L^2)}}{\|\Pi_h p\|_{\ell^2(L^2)}}, \qquad Er\,4 = \frac{|p_h - \Pi_h p|_{\ell^2(|\cdot|_h)}}{\|\Pi_h p\|_{\ell^2(L^2)}},$$

$$Er\,5 = \frac{\|\mathbf{C}_h - \Pi_h\mathbf{C}\|_{\ell^\infty(L^2)}}{\|\Pi_h\mathbf{C}\|_{\ell^\infty(L^2)}}, \qquad Er\,6 = \frac{\|\mathbf{C}_h - \Pi_h\mathbf{C}\|_{\ell^2(H^1)}}{\|\Pi_h\mathbf{C}\|_{\ell^2(H^1)}}.$$

In the following we show three pairs of table and figure. Table 1 summarizes the symbols used in the figures. Tables & Figures 1, 2 and 3 present the results for the cases $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$, $(10^{-1}, 10^{-3})$ and $(1, 0)$, respectively. In the tables the values of the errors and the slopes are presented, and in the figures the graphs of the errors versus h in logarithmic scale are shown. In each figure the slope of the triangle is equal to 1, which shows the convergence order $O(h)$.

We can see that all the errors except $Er\,6$ for $(\nu, \varepsilon) = (1, 0)$ are almost of the first order in h for all the cases. These results support Theorem 1. In the case of $(\nu, \varepsilon) = (1, 0)$ there is no diffusion for $\mathbf{C}$ in equation (1c) and the error estimate of the conformation tensor in $\ell^2(H^1)$-seminorm disappear from (9). It is, therefore, natural that the slope of $Er\,6$ does not attain 1. Although we do not have any theoretical result for $Er\,3$, scheme (5) has produced convergence results also in this norm.

Table 1. Symbols used in the figures.

$\mathbf{u}_h$		p_h		$\mathbf{C}_h$	
○	●	△	▲	□	■
$Er\,1$	$Er\,2$	$Er\,3$	$Er\,4$	$Er\,5$	$Er\,6$

h	$Er\,1$	slope	$Er\,2$	slope
$1/32$	2.07×10^{-2}	–	2.91×10^{-2}	–
$1/64$	8.29×10^{-3}	1.32	1.21×10^{-2}	1.27
$1/128$	3.72×10^{-3}	1.16	5.85×10^{-3}	1.05
$1/256$	1.77×10^{-3}	1.07	2.60×10^{-3}	1.17
h	$Er\,3$	slope	$Er\,4$	slope
$1/32$	6.73×10^{-2}	–	5.08×10^{-2}	–
$1/64$	2.06×10^{-2}	1.71	1.86×10^{-2}	1.45
$1/128$	6.80×10^{-3}	1.60	8.38×10^{-3}	1.15
$1/256$	2.59×10^{-3}	1.39	3.68×10^{-3}	1.19
h	$Er\,5$	slope	$Er\,6$	slope
$1/32$	1.12×10^{-2}	–	4.80×10^{-1}	–
$1/64$	4.33×10^{-3}	1.37	1.66×10^{-2}	1.54
$1/128$	1.92×10^{-3}	1.18	6.56×10^{-3}	1.34
$1/256$	9.09×10^{-4}	1.08	2.90×10^{-3}	1.18

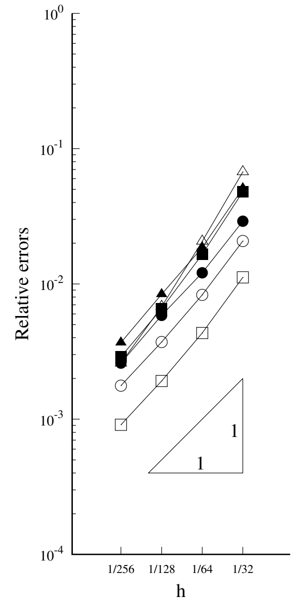

TABLE & FIGURE 1. Errors and slopes for $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$.

h	$Er\,1$	slope	$Er\,2$	slope
$1/32$	1.75×10^{-2}	–	2.71×10^{-2}	–
$1/64$	6.74×10^{-3}	1.37	1.12×10^{-2}	1.28
$1/128$	2.91×10^{-3}	1.21	5.49×10^{-3}	1.03
$1/256$	1.37×10^{-3}	1.09	2.44×10^{-3}	1.17
h	$Er\,3$	slope	$Er\,4$	slope
$1/32$	9.77×10^{-2}	–	6.56×10^{-2}	–
$1/64$	3.17×10^{-2}	1.62	2.22×10^{-2}	1.56
$1/128$	1.02×10^{-2}	1.63	9.01×10^{-3}	1.30
$1/256$	3.62×10^{-3}	1.50	3.78×10^{-3}	1.25
h	$Er\,5$	slope	$Er\,6$	slope
$1/32$	2.06×10^{-2}	–	2.76×10^{-1}	–
$1/64$	7.36×10^{-3}	1.49	1.16×10^{-1}	1.25
$1/128$	2.93×10^{-3}	1.33	4.40×10^{-2}	1.40
$1/256$	1.31×10^{-3}	1.17	1.51×10^{-2}	1.54

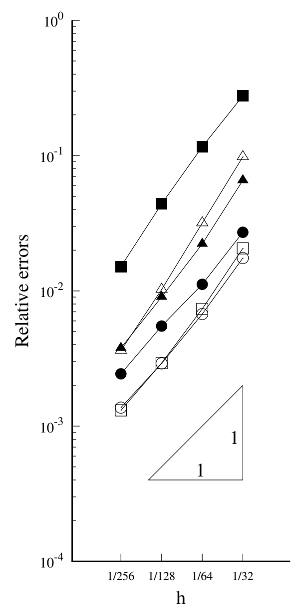

TABLE & FIGURE 2. Errors and slopes for $(\nu, \varepsilon) = (10^{-1}, 10^{-3})$.

h	$Er\,1$	slope	$Er\,2$	slope
1/32	1.36×10^{-2}	–	2.30×10^{-2}	–
1/64	4.26×10^{-3}	1.67	9.68×10^{-3}	1.25
1/128	1.40×10^{-3}	1.60	4.84×10^{-3}	1.00
1/256	5.15×10^{-4}	1.44	2.08×10^{-3}	1.22
h	$Er\,3$	slope	$Er\,4$	slope
1/32	2.03×10^{-1}	–	9.39×10^{-2}	–
1/64	6.98×10^{-2}	1.54	3.00×10^{-2}	1.65
1/128	2.16×10^{-2}	1.69	1.19×10^{-2}	1.34
1/256	6.86×10^{-3}	1.66	5.05×10^{-3}	1.23
h	$Er\,5$	slope	$Er\,6$	slope
1/32	2.13×10^{-2}	–	6.71×10^{-1}	–
1/64	7.64×10^{-3}	1.48	5.89×10^{-1}	0.19
1/128	2.81×10^{-3}	1.44	4.51×10^{-1}	0.38
1/256	1.11×10^{-3}	1.37	3.08×10^{-1}	0.55

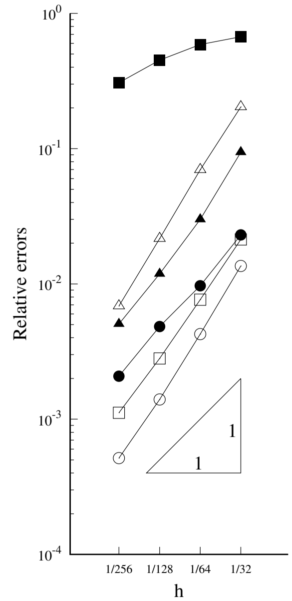

TABLE & FIGURE 3. Errors and slopes for $(\nu, \varepsilon) = (1, 0)$.

8. Conclusions

We have presented a nonlinear stabilized Lagrange–Galerkin scheme (5) for the Oseen-type Peterlin viscoelastic model. The scheme employs the conforming linear finite elements for all unknowns, velocity, pressure and conformation tensor, together with Brezzi–Pitkäranta's stabilization method. In Theorem 1 we have established error estimates with the optimal convergence order, which remain true even for $\varepsilon = 0$. We have also presented the result on the uniqueness of the solution of the scheme in Proposition 3. It is noted that any solution of the scheme converges to the exact solution without any relation between h and Δt, while the condition (20) or (21) is needed for the uniqueness of the solution. The theoretical convergence order has been confirmed by the two-dimensional numerical experiments.

In Part I [8] we have presented a linear scheme for the same model. There are no remarkable differences between the numerical results obtained by the linear scheme and the nonlinear scheme. While the argument discussed in the linear scheme can be extended to the three-dimensional problem, it is not so in the nonlinear scheme since Lemma 5 does not hold as it is in the three-dimensional space. On the other hand, while the convergence is proved in the nonlinear scheme including the non-diffusive case $\varepsilon = 0$, it is not straightforward to prove it in the non-diffusive case in the linear scheme since H^1-estimates of the conformation tensor are fully used in the proof in the diffusive case.

Although we have dealt with the stabilized scheme to reduce the number of degrees of freedom, the extension of the results to the combination of stable pairs for the velocity and the pressure, and conventional elements for the conformation tensor is straightforward, e.g., P2/P1/P2 element. We will extend the numerical analysis to the Peterlin viscoelastic model with the nonlinear convective terms in future.

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Appendix

A.1. **Proof of Lemma 8**

We prove only (14c), (14d), (14f), (14g), (14h), (14l) and (14m), since (14a) and (14b) are proved in [8, Appendix] and the other estimates are similarly obtained.

From Lemmas 2 and 3, (14c) and (14d) are obtained as follows:

$$\begin{aligned}
\|\mathbf{r}_{h3}^n\|_{-1} &\le \|(\mathrm{tr}\,\check{\mathbf{C}}^n)\mathbf{E}_h^n + (\mathrm{tr}\,\mathbf{E}_h^n)\check{\mathbf{C}}^n\|_0 \le c\|\check{\mathbf{C}}^n\|_{0,\infty}\|\mathbf{E}_h^n\|_0 \le c\|\mathbf{C}\|_{C(L^\infty)}\|\mathbf{E}_h^n\|_0,\\
\|\mathbf{r}_{h4}^n\|_{-1} &\le \|(\mathrm{tr}\,\check{\mathbf{C}}^n)\boldsymbol{\Xi}^n + (\mathrm{tr}\,\boldsymbol{\Xi}^n)\mathbf{C}^n\|_0 \le c\|\check{\mathbf{C}}^n\|_{0,\infty}\|\boldsymbol{\Xi}_h^n\|_0 \le c\|\mathbf{C}\|_{C(L^\infty)}\alpha_{21}h\|\mathbf{C}\|_{C(H^1)}.
\end{aligned}$$

We prove (14f). Let $y(x,s) := x - (1-s)\mathbf{w}^n(x)\Delta t$ and $t(s) := t^{n-1} + s\Delta t$ ($s \in [0,1]$). From the identity

$$\mathbf{R}_{h2}^n = \frac{1}{\Delta t}\Big[\boldsymbol{\Xi}\big(y(\cdot,s),t(s)\big)\Big]_{s=0}^1 = \int_0^1 \Big\{\Big(\frac{\partial}{\partial t} + \mathbf{w}^n(\cdot)\cdot\nabla\Big)\boldsymbol{\Xi}\Big\}\big(y(\cdot,s),t(s)\big)ds$$

and Proposition 1 we have

$$\begin{aligned}
\|\mathbf{R}_{h2}^n\|_0 &\le \int_0^1 \Big(\Big\|\frac{\partial\boldsymbol{\Xi}}{\partial t}\big(y(\cdot,s),t(s)\big)\Big\|_0 + c_w\big\|\nabla\boldsymbol{\Xi}\big(y(\cdot,s),t(s)\big)\big\|_0\Big)ds \le \sqrt{2}\int_0^1 \Big(\Big\|\frac{\partial\boldsymbol{\Xi}}{\partial t}\big(\cdot,t(s)\big)\Big\|_0 + c_w\big\|\nabla\boldsymbol{\Xi}\big(\cdot,t(s)\big)\big\|_0\Big)ds\\
&\le \sqrt{\frac{2}{\Delta t}}h\Big(\alpha_{21}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^1)} + c_w\alpha_{22}\|\mathbf{C}\|_{L^2(t^{n-1},t^n;H^2)}\Big),
\end{aligned}$$

which implies (14f).

The estimates (14g), (14l) and (14m) are obtained as follows:

$$\begin{aligned}
\Big\langle \mathbf{R}_{h3}^n, \frac{1}{2}\mathbf{E}_h^n\Big\rangle &\le \frac{\varepsilon}{2}|\boldsymbol{\Xi}^n|_1|\mathbf{E}_h^n|_1 \le \frac{\varepsilon}{4}(|\mathbf{E}_h^n|_1^2 + \alpha_3^2h^2\|\mathbf{C}\|_{C(H^2)}^2),\\
\Big\langle \mathbf{R}_{h8}^n, \frac{1}{2}\mathbf{E}_h^n\Big\rangle &= -\frac{1}{2}\big([(\mathrm{tr}\,\mathbf{E}_h^n)^2 + 2(\mathrm{tr}\,\mathbf{E}_h^n)(\mathrm{tr}\,\check{\mathbf{C}}^n) + (\mathrm{tr}\,\check{\mathbf{C}}^n)^2]\mathbf{E}_h^n, \mathbf{E}_h^n\big) \le -\frac{1}{2}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 - \big((\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n, (\mathrm{tr}\,\check{\mathbf{C}}^n)\mathbf{E}_h^n\big)\\
&\le -\frac{1}{2}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 + \frac{1}{8}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 + 2\|(\mathrm{tr}\,\check{\mathbf{C}}^n)\mathbf{E}_h^n\|_0^2 \le -\frac{3}{8}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 + c\|\mathbf{C}\|_{C(L^\infty)}^2\|\mathbf{E}_h^n\|_0^2,\\
\Big\langle \mathbf{R}_{h9}^n, \frac{1}{2}\mathbf{E}_h^n\Big\rangle &= -\frac{1}{2}\big((\mathrm{tr}\,\mathbf{E}_h^n)\check{\mathbf{C}}^n, (\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\big) - \big((\mathrm{tr}\,\check{\mathbf{C}}^n)(\mathrm{tr}\,\mathbf{E}_h^n)\check{\mathbf{C}}^n, \mathbf{E}_h^n\big) \le \frac{1}{8}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 + c\|\mathbf{C}\|_{C(L^\infty)}^2\|\mathbf{E}_h^n\|_0^2.
\end{aligned}$$

Let $\check{\mathbf{u}}_h(t) := (\Pi_h\mathbf{u})(t)$ for $t \in [0,T]$. The remaining estimate (14h) is proved as

$$\|\mathbf{R}_{h4}^n\|_0 \le 2(\|(\nabla\hat{\mathbf{u}}_h^n)\mathbf{E}_h^n\|_0 + \|(\nabla\mathbf{e}_h^n)\check{\mathbf{C}}^n\|_0) \le c(c_s\|\mathbf{E}_h^n\|_0 + \|\mathbf{C}\|_{C(L^\infty)}\|\nabla\mathbf{e}_h^n\|_0),$$

where we have used the boundedness of $\|\nabla\hat{\mathbf{u}}_h^n\|_{0,\infty}$ obtained by the estimate,

$$\begin{aligned}
\|\nabla\hat{\mathbf{u}}_h^n\|_{0,\infty} &\le \|\hat{\mathbf{u}}_h^n\|_{1,\infty} \le \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_{1,\infty} + \|\check{\mathbf{u}}_h^n\|_{1,\infty} \le \alpha_{25}h^{-1}\|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_1 + \alpha_{20}\|\mathbf{u}^n\|_{1,\infty}\\
&\le \alpha_{25}h^{-1}\big(\|\hat{\mathbf{u}}_h^n - \mathbf{u}^n\|_1 + \|\mathbf{u}^n - \check{\mathbf{u}}_h^n\|_1\big) + \alpha_{20}\|\mathbf{u}^n\|_{1,\infty}\\
&\le \alpha_{25}h^{-1}\big(\alpha_3h\|(\mathbf{u},p)^n\|_{H^2\times H^1} + \alpha_{22}h\|\mathbf{u}^n\|_2\big) + \alpha_{20}\|\mathbf{u}^n\|_{1,\infty}\\
&\le \alpha_{25}(\alpha_{22}+\alpha_3)\|(\mathbf{u},p)\|_{C(H^2\times H^1)} + \alpha_{20}\|\mathbf{u}\|_{C(W^{1,\infty})} \le c_s.
\end{aligned}$$

□

A.2. **Proofs of estimates** (29a)

We prove (29a) by the integration by parts as follows:

$$\begin{aligned}\Big\langle \tilde{\mathbf{R}}^n_{h1}, \frac{1}{2}\tilde{\mathbf{E}}^n_h \Big\rangle &= \big((\nabla \mathbf{u}^n_h)\tilde{\mathbf{E}}^n_h, \tilde{\mathbf{E}}^n_h\big) + \big((\nabla \tilde{\mathbf{e}}^n_h)\mathbf{C}^n_h, \tilde{\mathbf{E}}^n_h\big) = -\big(\mathbf{u}^n_h, \nabla(\tilde{\mathbf{E}}^n_h\tilde{\mathbf{E}}^n_h)\big) + \big((\nabla \tilde{\mathbf{e}}^n_h)\mathbf{C}^n_h, \tilde{\mathbf{E}}^n_h\big)\\
&\le c\big(\|\mathbf{u}^n_h\|_{0,\infty}\|\tilde{\mathbf{E}}^n_h\|_0|\tilde{\mathbf{E}}^n_h|_1 + \|\mathbf{C}^n_h\|_{0,\infty}|\tilde{\mathbf{e}}^n_h|_1\|\tilde{\mathbf{E}}^n_h\|_0\big),\\
\Big\langle \tilde{\mathbf{R}}^n_{h2}, \frac{1}{2}\tilde{\mathbf{E}}^n_h \Big\rangle &= \frac{1}{2}\big((\operatorname{div}\mathbf{u}^n_h)(\tilde{\mathbf{E}}^n_h)^{\#}, \tilde{\mathbf{E}}^n_h\big) + \frac{1}{2}\big((\operatorname{div}\tilde{\mathbf{e}}^n_h)(\mathbf{C}^n_h)^{\#}, \tilde{\mathbf{E}}^n_h\big)\\
&= -\frac{1}{2}\big(\mathbf{u}^n_h\nabla(\tilde{\mathbf{E}}^n_h)^{\#}, \tilde{\mathbf{E}}^n_h\big) - \frac{1}{2}\big((\tilde{\mathbf{E}}^n_h)^{\#}, \mathbf{u}^n_h\nabla\tilde{\mathbf{E}}^n_h\big) + \frac{1}{2}\big((\operatorname{div}\tilde{\mathbf{e}}^n_h)(\mathbf{C}^n_h)^{\#}, \tilde{\mathbf{E}}^n_h\big)\\
&\le c\big(\|\mathbf{u}^n_h\|_{0,\infty}|\tilde{\mathbf{E}}^n_h|_1\|\tilde{\mathbf{E}}^n_h\|_0 + \|\mathbf{C}^n_h\|_{0,\infty}|\tilde{\mathbf{e}}^n_h|_1\|\tilde{\mathbf{E}}^n_h\|_0\big).\end{aligned}$$

□

References

[1] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer, New York, 3rd edition, 2008.
[2] F. Brezzi and J. Douglas Jr. Stabilized mixed methods for the Stokes problem. *Numerische Mathematik*, 53:225–235, 1988.
[3] F. Brezzi and J. Pitkäranta. On the stabilization of finite element approximations of the Stokes equations. In W. Hackbusch, editor, *Efficient Solutions of Elliptic Systems*, pages 11–19, Wiesbaden, 1984. Vieweg.
[4] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.
[5] P. Clément. Approximation by finite element functions using local regularization. *RAIRO Analyse Numérique*, 9:77–84, 1975.
[6] G. Duvaut and J.L. Lions. *Inequalities in Mechanics and Physics.* Springer, Berlin, 1976.
[7] J.L. Lions. *Quelques Méthodes de Résolutiondes Problèmes aux Limites Non Linéaires.* Dunod et Gauthier-Villars, Paris, 1969.
[8] M. Lukáčová-Medviďová, H. Mizerová, H. Notsu, and M. Tabata. Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method, Part I: A linear scheme. arXiv:1603.01074 [math.NA].
[9] M. Lukáčová-Medviďová, H. Mizerová, and Š. Nečasová. Global existence and uniqueness result for the diffusive Peterlin viscoelastic model. *Nonlinear Analysis: Theory, Methods & Applications*, 120:154–170, 2015.
[10] H. Mizerová. Analysis and numerical solution of the Peterlin viscoelastic model. 2015. PhD thesis, University of Mainz, Germany.
[11] H. Notsu and M. Tabata. Error estimates of stable and stabilized Lagrange–Galerkin schemes for natural convection problems. arXiv:1511.01234 [math.NA].
[12] H. Notsu and M. Tabata. Error estimates of a pressure-stabilized characteristics finite element scheme for the Oseen equations. *Journal of Scientific Computing*, 65(3):940–955, 2015.
[13] H. Notsu and M. Tabata. Error estimates of a stabilized Lagrange–Galerkin scheme for the Navier–Stokes equations. *ESAIM: M2AN*, 50(2):361–380, 2016.
[14] M. Renardy. *Mathematical Analysis of Viscoelastic Flows.* CBMS-NSF Conference Series in Applied Mathematics 73. SIAM, New York, 2000.
[15] M. Renardy. Mathematical analysis of viscoelastic fluids. In *Handbook of Differential Equations: Evolutionary Equations*, volume 4, pages 229–265, Amsterdam, 2008. North-Holland.
[16] H. Rui and M. Tabata. A second order characteristic finite element scheme for convection-diffusion problems. *Numerische Mathematik*, 92:161–177, 2002.
[17] M. Tabata and D. Tagami. Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients. *Numerische Mathematik*, 100:351–372, 2005.
[18] M. Tabata and S. Uchiumi. A Lagrange–Galerkin scheme with a locally linearized velocity for the Navier–Stokes equations. arXiv:1505.06681 [math.NA].
[19] R. Temam. *Navier–Stokes Equations.* North-Holland, Amsterdam, 1984.