SPATIAL ASYMPTOTIC PROFILES OF SOLUTIONS OF THE NAVIER-STOKES SYSTEM IN A ROTATING FRAME

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The 3-dimensional Navier-Stokes equations with a constant Coriolis force in the whole space are considered at large distances. We investigate the solvability of the corresponding integral equations of these equations in weighted L^∞-spaces. Furthermore, we establish the leading terms of the asymptotic profile of the solution far from the axis of rotation.

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1. Introduction

In this paper we study the 3-dimensional rotating Navier-Stokes equations

$$\text{(NSC)}\begin{cases} u_t - \Delta u + u\cdot\nabla u + \Omega e_3 \times u + \nabla p &= 0 \quad \text{in} \quad \mathbb{R}^3\times(0,T),\\ \operatorname{div} u &= 0 \quad \text{in} \quad \mathbb{R}^3\times(0,T),\\ u(0) &= u_0 \quad \text{in} \quad \mathbb{R}^3, \end{cases}$$

with a given constant Coriolis parameter $\Omega \neq 0$ and initial data u_0. The unknowns $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote the velocity vector field and the pressure of the fluid at the point $(x,t) \in \mathbb{R}^3 \times [0,T)$, respectively. Here e_3 denotes the unit vector $(0,0,1)^T$ and the term $\Omega e_3 \times u$ restricted to divergence free vector fields is called *Coriolis operator*.

One of the most important features that distinguishes most flows in fluid dynamics from those in ocean and atmospherical dynamics is the rotation of the earth. The equations (NSC) describe the motion of rotating fluids influenced by the Coriolis force. Almost all of the models of oceanography and meteorology dealing with large-scale phenomena include a Coriolis force. In this case of low speed of rotation, i.e. Ω is sufficiently small, it is reasonable to neglect the centrifugal force.

The investigation of the spatial behaviour of the velocity field at large distances and of the leading asymptotic term is an important research topic, e.g. in the error analysis of numerical approximations. Bae, Brandolese and Vigneron found out the leading terms in the non-rotating case, see [2], [3]. For the spatial behaviour of the Boussinesq system including heat convection see [8].

To investigate the spatial behaviour of the solutions of the rotating Navier-Stokes system it will be helpful to consider the solvability of these equations in weighted L^∞-spaces. In the case of slow decay of u_0 the solution decreases

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almost in the same way as the initial velocity. But already Brandolese, Vigneron and Bae, see [2] and [3], proved in the case of the non-rotating Navier-Stokes equations that in general we cannot expect a faster decay behaviour than $\sim \frac{1}{|x|^4}$ or even $\sim \frac{1}{|x|^3}$ if the flow is influenced by a external force.

The present Navier-Stokes equations in a rotational frame have been investigated by several authors, see e.g. [1], [4], [9], [10], [11], [12], [14]. However, up to now the spatial asymptotics is almost disregarded.

But there is also numerous research in recent years on the Navier-Stokes flow around a rotating obstacle which leads to an additional linear term not subordered to the Laplacian. In particular Farwig, Galdi, Hishida and Kyed considered the asymptotic structure of stationary solutions, see e.g. [7], [6], [5], [13] and [15].

Let us introduce some elementary concepts. Using the Riesz transforms $\mathcal{R}_j = \partial_j(-\Delta)^{-\frac{1}{2}}$, $1 \leq j \leq n$, the Helmholtz projection is given by $\mathbb{P} = (\delta_{j,h} + \mathcal{R}_j\mathcal{R}_h)_{j,h=1}^n$. Furthermore, with the matrix

$$J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

characterising the linear map $J : \mathbb{R}^3 \to \mathbb{R}^3, Jv = e_3 \times v$ and the Coriolis operator $\mathcal{C} = \mathbb{P}J\mathbb{P}$ are transform the first line of (NSC) into the abstract equation

$$u_t + A_\Omega u + \mathbb{P}(u \cdot \nabla u) = 0 \qquad \text{in } \mathbb{R}^3 \times [0, T);$$

here $A_\Omega := -\mathbb{P}\Delta + \Omega\mathcal{C}$ denotes the combined operator summarised by the Stokes operator and the Coriolis operator. We also define the Riesz symbol

$$\mathbf{R}(\xi) = \big(\mathbf{R}_{i,j}(\xi)\big)_{i,j=1}^3 = \frac{1}{|\xi|}\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \tag{1.1}$$

and note $\mathbf{R}_{i,j}(\xi) = (1 - \delta_{i,j})(-1)^{\max\{1+i-j,j-i\}} \frac{\xi_{6-i-j}}{|\xi|}$. The symbol $\mathcal{C}(\xi)$ of $\mathcal{C}$ is nothing but $\frac{\xi_3}{|\xi|}\mathbf{R}(\xi)$ and thus

$$\mathcal{C} = \mathcal{R}_3 \begin{pmatrix} 0 & \mathcal{R}_3 & -\mathcal{R}_2 \\ -\mathcal{R}_3 & 0 & \mathcal{R}_1 \\ \mathcal{R}_2 & -\mathcal{R}_1 & 0 \end{pmatrix}.$$

Note that we used the Fourier transform, e.g. of a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^3)$, in the form

$$\mathcal{F}(\phi)(\xi) := \int_{\mathbb{R}^3} \phi(x) e^{-2\pi i x\cdot\xi}\, dx.$$

However, this leads to the integral equation

$$u(t) = e^{-tA_\Omega}u_0 - \int_0^t e^{-(t-\tau)A_\Omega}\mathbb{P}(u \cdot \nabla u)(\tau) d\tau, \tag{1.2}$$

where the semigroup e^{-tA_Ω} generated by A_Ω is given by the symbol

$$e^{-4\pi^2 t|\xi|^2}\left(\cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} - \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi)\right), \tag{1.3}$$

see [10]. A solution u of (1.2) is called a *mild solution.*

2. Main Results

In this paper we assume that the initial data u_0 belongs to weighted L^∞-spaces. The Banach space $L^\infty_\mu(\mathbb{R}^3)$, $\mu > 0$, is defined as the set of all measurable functions f on $\mathbb{R}^3$ such that

$$\|f\|_{L^\infty_\mu} := \operatorname{ess\,sup}_{x\in\mathbb{R}^3}(1+|x|)^\mu |f(x)| < \infty.$$

Using Banach's fixed point theorem we get the following existence theorem of mild solutions in spaces of weakly-* continuous functions in time with values in weighted L^∞-spaces.

Theorem 2.1: (Existence and Uniqueness of Mild Solutions) *Let $\varepsilon \in (0, \frac{1}{3})$. For every initial velocity $u_0 \in L^\infty_{\mu+\varepsilon}(\mathbb{R}^3)^3$ with* div $u_0 = 0$ *and and arbitrary external force $f \in L^\infty([0,\infty); L^\infty_{\mu+\varepsilon}(\mathbb{R}^3)^3)$, $\mu \in (0, 3]$, there exists a constant $T_0 > 0$ and a unique solution*

$$u \in t^{-\kappa}L^\infty([0,T]; L^\infty_\mu(\mathbb{R}^3)^3) \cap C_\omega\left([\delta, T_0]; L^\infty_\mu(\mathbb{R}^3)^3\right)$$

to the integral equation (1.2) for all $\delta \in (0, T_0)$, where $\kappa := \frac{3\varepsilon^2}{2(1+\varepsilon^2)} \in (0, \frac{1}{4})$. In particular, with the bound $C_0 = C_0(\Omega)$ for the operator norm in Lemma 5.2.2, any $T_0 > 0$ satisfying

$$8C_0(T^{\frac{1}{2}-\kappa} + T^6)\left[\frac{4}{3}C_0\|u_0\|_{L^\infty_{\mu+\varepsilon}} + \left(\|u_0\|_{L^\infty_{\mu+\varepsilon}} + \sup_{0\le t\le T}\|f(t)\|_{L^\infty_{\mu+\varepsilon}}\right)^{\frac{1}{2}}\right] < 1$$

is possible.

The space $C_\omega\left([0,T]; L^\infty_\mu\right)$ denotes all L^∞_μ-valued weakly-* continuous functions $v(t)$ defined in $[0,T]$, i.e., $v(t')$ converges to $v(t)$ in the distributional sense as $t' \to t$ for all $t \in [0,T]$. The necessity for working in the space C_ω lies in the fact that already $e^{-t\Delta}f$, with $f \in L^\infty_\mu$, does not converge to f in L^∞_μ as $t \searrow 0$, but only weakly-*. But at least, since the Stokes operator $-\mathbb{P}\Delta$ generates a bounded analytic semigroup $e^{-t\Delta}$ on $L^p_\sigma(\mathbb{R}^3)$ for all $p \in (1,\infty)$ by perturbation theory, see [17, Chapter 3], we conclude that the operator A_Ω generates also an analytic semigroup e^{-tA_Ω} on $L^p_\sigma(\mathbb{R}^3)$, since the Coriolis operator $\mathcal{C} = \mathbb{P}J\mathbb{P}$ is bounded on $L^p_\sigma(\mathbb{R}^3)$.

In view of the result $u(t) \in L^\infty_\mu(\mathbb{R}^3)^3$ with $\mu \in (0,3]$ for mild solutions in Theorem 2.1 it is an interesting question whether the upper bound 3 for μ is optimal in some sense. Actually, the decay $|x|^{-4}$ is optimal for solutions to the non-rotating Navier-Stokes equations without external forces; see Brandolese and Vigneron [3] who proved that the result of Theorem 2.1 cannot be true for $\mu > 4$. Can we expect in general that the solution $u(t)$ belongs to $L^\infty_\mu(\mathbb{R}^3)^3$ if $\mu > 3$?

For our purpose it is usefull to introduce Bessel functions J_ν which can be represented as the series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+\nu)!} \left(\frac{z}{2}\right)^{2n+\nu} \qquad \text{for all } \nu \in \mathbb{N}_0 \text{ and } z \in \mathbb{R}, \tag{2.1}$$

see [18, chapter 3]. Especially, we are interested in the functions J_0 and J_1. In our main result these both functions play a crucial role by describing the asymptotic profile. Moreover, by x' we denote the vector $(x_1, x_2, 0)$.

Theorem 2.2: (Spatial Asymptotics Far from the Axis of Rotation) *Let $\varepsilon > 0$ and $\delta := \frac{2\varepsilon}{1+\varepsilon}$. For $\mu > 4$ and an initial velocity $u_0 \in L^\infty_\mu(\mathbb{R}^3)^3$ with* div $u_0 = 0$*, let u be the mild solution of Theorem 2.1. Then the following profile holds for almost all $|x| \gg \sqrt{t}$ and $|x_3|^{1+\varepsilon} \leq |x|$:*

$$\begin{aligned} u(x,t) = \frac{\Omega t}{2\pi |x|^3} \Big[& J_0(\Omega t) e_3 \times \int_{\mathbb{R}^3} u_0(y)dy - \frac{1}{2} J_1(\Omega t) \int_{\mathbb{R}^3} u_0(y)dy \\ & - 3\frac{x_3}{|x|^2} J_0(\Omega t) x' \times \int_{\mathbb{R}^3} u_0(y)dy \Big] + \mathcal{O}_t(|x|^{-\min\{3+\delta,4\}}). \end{aligned}$$

We note that due to the restriction $|x_3|^{1+\varepsilon} \leq |x|$ only the first two summands of our asymptotic profile above are leading terms of first order, i.e.

$$\frac{\Omega t}{2\pi |x|^3} \Big[J_0(\Omega t) e_3 \times \int_{\mathbb{R}^3} u_0(y)dy - \frac{1}{2} J_1(\Omega t) \int_{\mathbb{R}^3} u_0(y)dy \Big]$$

decays as $|x|^{-3}$ indeed, but we have

$$\frac{3x_3 \Omega t}{2\pi |x|^5} J_0(\Omega t) x' \times \int_{\mathbb{R}^3} u_0(y)dy = \mathcal{O}_t(|x|^{-3-\frac{\delta}{2}}).$$

Thus this third summand is in some sense a second order leading term.

As long as the inital data u_0 belongs to L^∞_μ, with $\mu > 4$, but has non-zero mean this theorem shows that in general we expect an $|x|^{-3}$-decay of the velocity. In particular, this implies no matter how small the Coriolis parameter Ω is, it has a significant effect at large distances. Thus the Coriolis force causes the velocity of the fluid to decrease less fast in the far-field.

3. Preliminaries

In this section we study the convolution kernel $(K_{i,j})_{i,j=1}^3$ with the components

$$K_{i,j}(x,t) := \int_{\mathbb{R}^3} e^{-4\pi^2 t|\xi|^2 + 2\pi i x\cdot\xi} \left[\cos(\frac{\xi_3}{|\xi|}\Omega t)\delta_{i,j} - \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}_{i,j}(\xi) \right] d\xi. \tag{3.1}$$

which corresponds to the operator e^{-tA_Ω}, see (1.3). In [16, Prop. 11.1] the operator $e^{t\Delta}\mathbb{P}$ which deals with the non-rotating Navier-Stokes equations was treated as a pseudo-differential operator and the derivatives of the corresponding convolution kernel satisfy some decaying properties. Unfortunately, in our case of a rotating frame not even the semigroup e^{-tA_Ω}

does belong to $L^1(\mathbb{R}^3)$, since the symbol (1.3) is not continuous. Already Giga et al. [10] proved that the convolution kernel corresponding to e^{-tA_Ω} decays like $|x|^{-3}$ and thus lies in $L^p(\mathbb{R}^3)$, $p \in (1,\infty]$, since the symbol (1.3) is integrable. But just as in [16] we need a more precise investigation about the derivatives of this kernel which yields the next Lemma:

Lemma 3.1: *For $1 \le h,k \le 3$ and $t>0$, the kernel $\mathcal{K}_{h,k,t}$ which belongs to the pseudo-differential operator $\mathcal{R}_h\mathcal{R}_k e^{-tA_\Omega}$ by $\mathcal{R}_h\mathcal{R}_k e^{-tA_\Omega} f = \mathcal{K}_{h,k,t} * f$ satisfies $\mathcal{K}_{h,k,t}(x) = t^{-\frac{3}{2}}\mathcal{K}_{h,k}\left(\frac{x}{\sqrt{t}}, \Omega t\right)$, where the smooth functions $\mathcal{K}_{i,j}(.,\Omega t)$ satisfy*

$$\left(1+(|\Omega|t)^{4+|\alpha|}\right)^{-1}(1+|x|)^{3+|\alpha|}\,\partial^\alpha \mathcal{K}_{h,k}(.,\Omega t) \in L^\infty(\mathbb{R}^3\times\mathbb{R}_+)$$

for all $\alpha \in \mathbb{N}_0^3$.

Since $e^{-tA_\Omega} = -\sum_{j=1}^3 \mathcal{R}_j^2 e^{-tA_\Omega}$ the above Lemma yields that the derivatives of the corresponding kernel $(K_{i,j}(x,t))_{i,j=1}^3 = -\sum_{h=1}^3 \mathcal{K}_{h,h,t}(x)$ of e^{-tA_Ω} decay similarly like those of $\mathcal{K}_{h,k}$. However, this Lemma also leads to

$$\partial^\alpha \mathcal{K}_{h,k,t}(x) \le t^{-\frac{3}{2}}\partial^\alpha \mathcal{K}_{h,k}\left(\frac{x}{\sqrt{t}}, \Omega t\right) \lesssim (1+|\Omega|t)^{4+|\alpha|}\left(\sqrt{t}+|x|\right)^{-3-|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^3$ and $\kappa > 0$.

To construct a unique mild solution of (1.2) for a given initial data $u_0 \in L^\infty_\mu(\mathbb{R}^3)^3$ it is useful to study the bilinear integral operator

$$\mathcal{B}(u_1,u_2) := -\int_0^t e^{-(t-s)A_\Omega}\mathbb{P}\nabla\cdot(u_1\otimes u_2)(s)\,ds. \tag{3.2}$$

Lemma 3.2: *Let $T>0$, $\mu \in (0,3]$. Then the operator*

$$\mathcal{B}(.,.) : C_\omega\left([0,T];L^\infty_\mu\right) \to C_\omega\left([0,T];L^\infty_\mu\right),$$

see (3.2), is continuous with operator norm $\mathcal{O}(\sqrt{T}+T^6)$.

Thanks to the previous result the estimate for $\mathcal{B}$ can be proved in the same way as in [16, Prop. 25.1].

Sketch of the proof of Theorem 2.1: The existence and uniqueness of mild solutions to (1.2) base on the abstract formulation of a solution u as a fixed point of the coupled system

$$u(t) = e^{-tA_\Omega}u_0 + \mathcal{B}(u,u)(t)$$

in the Banach space $C_\omega\left([0,T];L^\infty_\mu\right)$. With the help of Lemmata 3.1 and 3.2 the result is easily proved by Banach's fixed point theorem. □

Similarly to [3], [8] we proceed to get an asymptotic profile of the solutions of the rotating Navier-Stokes equations and have to handle mainly the terms of the integral equation (1.2). But due the symbol (1.3) of e^{-tA_Ω} the present issue exacerbates this method by dealing with an infinite sum of Riesz operators applied to the heat kernel

$$\mathcal{G}_t(x) := \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}}.$$

The next statements are useful to manage this difficulty. At first we easily obtain by induction the following Lemma.

Lemma 3.3: *Let $\mathcal{R} := (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ be the vector of the Riesz operators and α be any multi index of even order, i.e. $|\alpha| = 2n$ for some $n \in \mathbb{N}$. Then for all $x \in \mathbb{R}^3$ and $t > 0$ there holds*

$$\mathcal{R}^\alpha \mathcal{G}_t(x) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \nabla^\alpha \mathcal{G}_s(x) ds.$$

The purpose of the Lemmata below is nothing but to ascertain the leading terms of e^{-tA_Ω}. Since the series

$$e^{-4\pi^2 t|\xi|^2} \left(\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} (\frac{\xi_3}{|\xi|}\Omega t)^{2n} \mathbf{I} - \sum_{n=0}^\infty \frac{(-1)^{n+1}}{(2n+1)!} (\frac{\xi_3}{|\xi|}\Omega t)^{2n+1} \mathbf{R}(\xi) \right)$$

converges to the symbol of (1.3) in L^1 the corresponding inverse Fourier transform of this series converges uniformly to the kernel $(K_{i,j})_{i,j=1}^3$, see (3.1). Thanks to Lemma 3.3 we get for all $x \in \mathbb{R}^3$

$$\begin{aligned} \mathcal{F}^{-1}\left(\cos\left(\frac{\xi_3}{|\xi|}\Omega t \right) e^{-4\pi^2 t|\xi|^2} \right)(x) &= \sum_{n=0}^\infty \frac{(\Omega t)^{2n}}{(2n)!} \mathcal{R}_3^{2n} \mathcal{G}_t(x) \\ &= \mathcal{G}_t(x) + \sum_{n=1}^\infty \frac{(\Omega t)^{2n}}{(2n)!(n-1)!} \int_t^\infty (s-t)^{n-1} \partial_3^{2n} \mathcal{G}_s(x) ds \end{aligned} \tag{3.3}$$

as well as

$$\begin{aligned} &\mathcal{F}^{-1}\left(\frac{\xi_{6-i-j}}{|\xi|} \sin\left(\frac{\xi_3}{|\xi|}\Omega t \right) e^{-4\pi^2 t|\xi|^2} \right)(x) \\ &= \sum_{n=0}^\infty \frac{(\Omega t)^{2n+1}}{(2n+1)! n!} \int_t^\infty (s-t)^n \partial_{6-i-j} \partial_3^{2n+1} \mathcal{G}_s(x) ds. \end{aligned} \tag{3.4}$$

The integrals in (3.3) and (3.4) will be computed in the next Lemma. But for this we need to handle high order derivatives of the heat kernel. Thus it is reasonable to deal with *Hermite polynomials*:

$$H_n(y) := e^{y^2} \frac{d^n}{dy^n} e^{-y^2},$$

for all $y \in \mathbb{R}$, $n \in \mathbb{N}_0$. The Hermite ploynomials can be written explicitly as

$$H_n(y) = n! \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{l+n}}{l!(n-2l)!} (2y)^{n-2l} \tag{3.5}$$

by using the floor function $\lfloor . \rfloor$. Substituting $y = \frac{x_3}{\sqrt{4t}}$ in the identity $\frac{d^n}{dy^n} e^{-y^2} = H_n(y) e^{-y^2}$ we can represent each derivative of the heat kernel with respect to the last component as follows:

$$\partial_3^n \mathcal{G}_t(x) = \frac{1}{(4t)^{\frac{n}{2}}} H_n\left(\frac{x_3}{\sqrt{4t}}\right) \cdot \mathcal{G}_t(x). \tag{3.6}$$

This equation is crucial to obtain the leading terms of the asymptotic profile. Furthermore, we introduce the functions

$$\begin{aligned} L_{3,n}(x) &:= \pi^{-\frac{3}{2}}(2n)! \sum_{l=0}^{n} \frac{(-1)^l \Gamma(n-l+\frac{3}{2})}{4^l l!(2n-2l)!} \left(\frac{x_3}{|x|}\right)^{2(n-l)}, \\ L_{i+j,n}(x) &:= \pi^{-\frac{3}{2}}(2n-1)! \frac{x_{6-i-j}}{|x|} \sum_{l=0}^{n-1} \frac{(-1)^l \Gamma(n-l+\frac{3}{2})}{4^l l!(2n-1-2l)!} \left(\frac{x_3}{|x|}\right)^{2n-1-2l} \end{aligned} \tag{3.7}$$

for all $i+j \neq 3$ and establish the next result as a first step to our purpose.

Lemma 3.4: *Let $n \in \mathbb{N}$, $i, j = 1, 2, 3$ and $|x|^2 \gg t$. Then there holds*

$$\int_t^\infty (s-t)^{n-1} \partial_{6-i-j} \partial_3^{2n-1} \mathcal{G}_s(x) ds = |x|^{-3} L_{i+j,n}(x) + |x|^{-3} \Psi_{i+j,n}\left(\frac{x}{\sqrt{t}}\right)$$

with a remainder term $\Psi_{i+j,n}(x) = \mathcal{O}(|x|^{-2})$. The function $L_{i+j,n}(x)$ corresponding to the leading term is defined above.

Unfortunately, due to the complicated structure of the functions $L_{i+j,n}(x)$, see (3.7), on the whole space we have to be content with an asymptotic profile which still hides its shape in some sense. In the following there appears three distinct series which we have to manage, thus we define

$$\begin{aligned} V^{(1)}(x,t) &:= \pi^{-\frac{3}{2}} \sum_{n=1}^{\infty} \sum_{l=0}^{n} \frac{(-1)^{n+l} \Gamma(l+\frac{3}{2})}{4^{n-l}(n-1)!(n-l)!(2l)!} (\Omega t)^{2n} \left(\frac{x_3}{|x|}\right)^{2l} \\ V^{(2)}(x,t) &:= \pi^{-\frac{3}{2}} \sum_{n=0}^{\infty} \sum_{l=0}^{n+1} \frac{(-1)^{n+l} \Gamma(l+\frac{3}{2})(2n+2)}{4^{n+1-l} n!(n+1-l)!(2l)!} (\Omega t)^{2n+1} \left(\frac{x_3}{|x|}\right)^{2l} \\ V^{(3)}(x,t) &:= \pi^{-\frac{3}{2}} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{(-1)^{n+l} \Gamma(l+\frac{5}{2})}{4^{n-l} n!(n-l)!(2l+1)!} (\Omega t)^{2n+1} \left(\frac{x_3}{|x|}\right)^{2l} \end{aligned} \tag{3.8}$$

and analyse each of these series below. We note that the power series $V^{(j)}$, $j = 1, 2, 3$, is analytic and converges absolutely with respect to t, Ω and $\frac{x_3}{|x|}$. We can estimate this series easily

$$\begin{aligned} \left|V^{(j)}(x,t)\right| &\leq \frac{1+\Omega t}{\pi^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{2n+2}{4^n n!} \left[\sum_{l=0}^{n} \frac{(l+2)!}{(n-l)!(2l)!} + 4^n \frac{(n+2)!}{(2n+2)!}\right] (\Omega t)^{2n+1} \\ &\lesssim (1+|\Omega| t)|\Omega| t e^{(\Omega t)^2}, \end{aligned} \tag{3.9}$$

$j = 1, 2, 3$, and note that they increase at most exponentially in time. Due to the crucial Lemma 3.4 we obtain the spatial asymptotics of the considered kernel as follows:

Lemma 3.5: *Let $i, j = 1, 2, 3$ such that $i + j \neq 3$ and $|x|^2 \gg t$. Then the convolution kernel $K_{i,j}$, see (3.1), satisfies the equation $K_{i,j}(x,t) = K^*_{i,j}(x,t) + |x|^{-3}\Psi_{i,j}(x,t)$, where the leading term $K^*_{i,j}(x,t)$ is defined as*

$$\tfrac{1}{|x|^3}\left[\delta_{i,j}V^{(1)}(x_3,t) + (1-\delta_{i,j})(-1)^{\max\{1+i-j,j-i\}}\tfrac{x_{6-i-j}x_3}{|x|^2}V^{(3)}(x_3,t)\right].$$

Otherwise, if $i + j = 3$ we have

$$K^*_{i,j}(x,t) = (-1)^{\max\{i-j,1+j-i\}}\tfrac{1}{|x|^3}V^{(2)}(x_3,t).$$

In particular the remainder $\Psi_{i,j}$ satisfies $\Psi_{i,j}(x,t) = \mathcal{O}_t(|x|^{-2})$.

4. Proof of Lemmata 3.1, 3.4 and 3.5

First of all let us briefly introduce some notations of the so called Littlewood-Paley decomposition which are needed in the next proof. Let $\phi \in \mathcal{S}(\mathbb{R}^3)$ denote a non-negative function which is supported in the annulus $\{\xi \in \mathbb{R}^3 \mid \frac{1}{2} \leq |\xi| \leq 2\}$ such that $\sum_{k=-\infty}^{\infty}\phi(2^{-k}\xi) = 1$ for all $\xi \neq 0$. We define for all $j \in \mathbb{Z}$ a function φ_j as follows

$$\hat{\varphi}_j(\xi) := \phi(2^{-j}\xi) \text{ for all } \xi \neq 0.$$

The family of functions $\{\varphi_j\}_{j\in\mathbb{Z}}$ is called the *Littlewood-Paley decomposition.* Further we define

$$\psi_0(\xi) := \mathcal{F}^{-1}\left(1 - \sum_{j=0}^{\infty}\hat{\varphi}_j(\xi)\right) \text{ for all } \xi \in \mathbb{R}^3.$$

Proof of Lemma 3.1: We set

$$\mathcal{K}_{h,k}(.,\Omega t) := -\mathcal{F}^{-1}\left(\frac{\xi_h\xi_k}{|\xi|^2}g_{i,j}(\xi,\Omega t)e^{-4\pi^2|\xi|^2}\right)^3_{i,j=1}$$

with the bounded function

$$g_{i,j}(\xi,\Omega t) := \delta_{i,j}\cos(\frac{\xi_3}{|\xi|}\Omega t) - \mathbf{R}_{i,j}(\xi)\sin(\frac{\xi_3}{|\xi|}\Omega t).$$

By this definition and (1.3) there obviously holds

$$\mathcal{K}_{h,k,\Omega t}(x) = t^{-\frac{3}{2}}\mathcal{K}_{h,k}\left(\frac{x}{\sqrt{t}},\Omega t\right).$$

In the following we write shortly $\mathcal{K}_{h,k}$ instead of $\mathcal{K}_{h,k}(.,\Omega t)$ if we fixed Ωt. Because of the rapid decay of the symbol

$$\frac{\xi_h\xi_k}{|\xi|^2}g_{i,j}(\xi,\Omega t)e^{-4\pi^2|\xi|^2}$$

it is still integrable mutliplied by any polynomial and thus $\partial^\alpha \mathcal{K}_{h,k} \in L^\infty(\mathbb{R}^3)$ for all $\alpha \in \mathbb{N}^3$ and fixed Ωt. For $|x| \geq 1$ we use the Littlewood-Paley decomposition and write

$$\mathcal{K}_{h,k} = \mathcal{K}_{h,k} - \psi_0 * \mathcal{K}_{h,k} + \sum_{l<0} \varphi_l * \mathcal{K}_{h,k}$$

with $\mathcal{K}_{h,k} - \psi_0 * \mathcal{K}_{h,k} \in \mathcal{S}(\mathbb{R}^3)$. By substituting $\xi = 2^l \tilde{\xi}$ we have for all $l < 0$

$$\begin{aligned}\varphi_l * \mathcal{K}_{h,k}(.) &= \mathcal{F}^{-1}\left(\phi(2^{-l}\xi)\hat{\mathcal{K}}_{h,k}(\xi)\right)(.)\\ &= 2^{3l}\mathcal{F}^{-1}\left(\phi(\tilde{\xi})\hat{\mathcal{K}}_{h,k}(2^l\tilde{\xi})\right)(2^l.) \quad \in \mathcal{S}(\mathbb{R}^3).\end{aligned}$$

Since the functions $\mathcal{F}^{-1}\left(\phi(\tilde{\xi})\hat{\mathcal{K}}_{h,k}(2^l\tilde{\xi})\right)$ are a bounded set in $\mathcal{S}(\mathbb{R}^3)$ with respect to $l < N$, we get for every $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^3$

$$\left(1 + 2^l|x|\right)^N 2^{-l(3+|\alpha|)} \left|\partial^\alpha \left(\varphi_l * \mathcal{K}_{h,k}\right)(x)\right| \leq C_N \left(1 + (|\Omega|t)^N\right).$$

This gives for $N := 4 + |\alpha|$ and $|x| > 1$

$$\begin{aligned}|\partial^\alpha(\psi_0 * \mathcal{K}_{h,k})(x)| &\leq C\left(1 + (|\Omega|t)^N\right)\\ &\qquad \times \Big[\sum_{l:2^l|x|\leq 1} 2^{l(3+|\alpha|)} + \sum_{l:2^l|x|>1} 2^{l(3+|\alpha|-N)}|x|^{-N}\Big]\\ &\leq C\left(1 + (|\Omega|t)^N\right)|x|^{-3-|\alpha|}.\end{aligned}$$

Since $\mathcal{K}_{h,k} - \psi_0 * \mathcal{K}_{h,k} \in \mathcal{S}(\mathbb{R}^3)$ the Lemma is proved. □

Proof of Lemma 3.4: First let $i + j = 3$. Then by (3.6) we have

$$\partial_3^{2n}\mathcal{G}_s(x) = \frac{1}{(4s)^n} H_{2n}\left(\frac{x_3}{\sqrt{4s}}\right) \cdot \mathcal{G}_s(x)$$

and thus by (3.5) and the substitution $\lambda = \frac{|x|}{\sqrt{4s}}$ we get

$$\begin{aligned}\int_t^\infty (s-t)^{n-1}\partial_3^{2n}\mathcal{G}_s(x)ds &= \int_t^\infty (s-t)^{n-1}\frac{1}{(4s)^n}H_{2n}\left(\frac{x_3}{\sqrt{4s}}\right)\cdot\mathcal{G}_s(x)ds\\ &= (2n)!\sum_{l=0}^n \frac{(-1)^l}{l!(2n-2l)!}\int_t^\infty \frac{(s-t)^{n-1}}{(4s)^n}\left(\frac{2x_3}{\sqrt{4s}}\right)^{2(n-l)}\cdot\mathcal{G}_s(x)ds\\ &= \frac{1}{2}\pi^{-\frac{3}{2}}(2n)!\sum_{l=0}^n \frac{(-1)^l}{l!(2n-2l)!}\\ &\quad \times \int_0^{\frac{|x|}{\sqrt{4t}}}\left(\frac{|x|^2}{4\lambda^2} - t\right)^{n-1}\left(\frac{\lambda}{|x|}\right)^{2n}\left(2\frac{x_3}{|x|}\lambda\right)^{2(n-l)}|x|^{-1}e^{-\lambda^2}\,d\lambda.\end{aligned}$$

Therefore, we obtain

$$\int_t^\infty (s-t)^{n-1}\partial_3^{2n}\mathcal{G}_s(x)ds$$
$$= \frac{(2n)!}{\pi^{\frac{3}{2}}|x|^3}\sum_{l=0}^{n}\frac{(-1)^l}{l!(2n-2l)!}\sum_{k=0}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix} 2^{1+2k-2l}\frac{t^k}{|x|^{2k}}\left(\frac{x_3}{|x|}\right)^{2(n-l)}$$
$$\times \int_0^{\frac{|x|}{\sqrt{4t}}}\lambda^{2+2n+2k-2l}e^{-\lambda^2}\,d\lambda.$$

(4.1)

By the definition of the Gamma function there holds

$$\int_0^{\frac{|x|}{\sqrt{4t}}}\lambda^{2+2n+2k-2l}e^{-\lambda^2}d\lambda$$
$$= \frac{1}{2}\Gamma\left(\frac{3+2n+2k-2l}{2}\right) - \int_{\frac{|x|}{\sqrt{4t}}}^\infty \lambda^{2+2n+2k-2l}e^{-\lambda^2}d\lambda$$

for all $k,l=0,...,n$. In (4.1) we fix $k=0$ and obtain for each $l=0,...,n$

$$\frac{(-1)^l(2n)!2^{1-2l}}{\pi^{\frac{3}{2}}l!(2n-2l)!|x|^3}\left(\frac{x_3}{|x|}\right)^{2(n-l)}\int_0^{\frac{|x|}{\sqrt{4t}}}\lambda^{2+2n-2l}e^{-\lambda^2}\,d\lambda$$
$$= \frac{(-1)^l(2n)!}{\pi^{\frac{3}{2}}l!(2n-2l)!|x|^3}\left[4^{-l}\Gamma(n-l+\frac{3}{2})\left(\frac{x_3}{|x|}\right)^{2(n-l)} + \Psi^{(1)}_{3,n,l}\left(\frac{x}{\sqrt{t}}\right)\right]$$

with the exponentially decaying function

$$\Psi^{(1)}_{3,n,l}(x) := -2^{1-2l}\left(\frac{x_3}{|x|}\right)^{2(n-l)}\int_{\frac{|x|}{2}}^\infty \lambda^{2+2n-2l}e^{-\lambda^2}d\lambda,$$

which satisfies the estimate

$$|\Psi^{(1)}_{3,n,l}(x)| \le 2^{1-2l}e^{-\frac{|x|^2}{8}}\int_{\frac{|x|}{2}}^\infty \lambda^{2+2n-2l}e^{-\frac{\lambda^2}{2}}d\lambda \le 2^{\frac{3}{2}+n-3l}(n-l+1)!e^{-\frac{|x|^2}{8}}.$$

To establish this estimate of the remainder function $\Psi^{(1)}_{3,n,l}$ we applied the equality

$$\int_0^\infty \tau^d e^{-\frac{\tau^2}{2}}d\tau = 2^{\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right), \tag{4.2}$$

which holds for all even natural numbers $d\in\mathbb{N}$ and can easily be proved by integration of parts. This yields the exponential decay of

$$\Psi^{(1)}_{3,n}(x) := \frac{(2n)!}{\pi^{\frac{3}{2}}}\sum_{l=0}^{n}\frac{(-1)^l}{l!(2n-2l)!}\Psi^{(1)}_{3,n,l}(x),$$

since

$$|\Psi^{(1)}_{3,n}(x)| \le \frac{4e^{\frac{1}{4}}}{\pi^{\frac{3}{2}}}(n+1)(2n)!e^{-\frac{|x|^2}{8}}.$$

In the same way we see that for all $k\neq 0$ and $l=0,...,n$ the term

$$|x|^{-3-2k}\left(\frac{x_3}{|x|}\right)^{2(n-l)}\int_0^{\frac{|x|}{\sqrt{4t}}}\lambda^{2+2n+2k-2l}e^{-\lambda^2}\,d\lambda$$

decays like $|x|^{-3-2k}$. Thus we put the left terms in the remainder $\Psi_{3,n}^{(2)}$ as follows:

$$\Psi_{3,n}^{(2)}\left(\frac{x}{\sqrt{t}}\right) := \frac{(2n)!}{\pi^{\frac{3}{2}}}\sum_{l=0}^{n}\frac{(-1)^l}{l!(2n-2l)!}\sum_{k=1}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix} 2^{1+2k-2l}\frac{(-t)^k}{|x|^{2k}}\left(\frac{x_3}{|x|}\right)^{2(n-l)} \times \int_0^{\frac{|x|}{\sqrt{4t}}} \lambda^{2+2n+2k-2l}e^{-\lambda^2}\, d\lambda$$

and since $|y| \geq 1$, where $y = \frac{x}{\sqrt{t}}$, we obtain by (4.2) the estimate:

$$\begin{aligned} \left|\Psi_{3,n}^{(2)}(x)\right| &\leq \pi^{-\frac{3}{2}}(2n)!|y|^{-2}\sum_{l=0}^{n}2^{-2+5n-3l}\frac{(2n-l)!}{l!(2n-2l)!} \\ &\leq 2^{-2+5n}\exp(\frac{n}{4})\pi^{-\frac{3}{2}}(2n)!|y|^{-2}. \end{aligned}$$

Finally $\Psi_{3,n} := \Psi_{3,n}^{(1)} + \Psi_{3,n}^{(2)}$ and (4.1) gives

$$\int_t^{\infty}(s-t)^{n-1}\partial_3^{2n}\mathcal{G}_s(x)ds = |x|^{-3}L_{3,n}(x) + |x|^{-3}\Psi_{3,n}\left(\frac{x}{\sqrt{t}}\right),$$

with

$$|\Psi_{3,n}(y)| \lesssim 42^n(2n)!|y|^{-2} \tag{4.3}$$

for all $n \in \mathbb{N}$ and $|x| \gg \sqrt{t}$. Now, let $i + j \neq 3$. In this case there holds

$$\partial_{6-i-j}\partial_3^{2n-1}\mathcal{G}_s(x) = -\frac{x_{6-i-j}}{2s}(4s)^{-\frac{2n-1}{2}}H_{2n-1}\left(\frac{x_3}{\sqrt{4s}}\right)\cdot\mathcal{G}_s(x).$$

Repeating the previous procedure we get similarly

$$\int_t^{\infty}(s-t)^{n-1}\partial_{6-i-j}\partial_3^{2n-1}\mathcal{G}_s(x)ds = |x|^{-3}L_{i+j,n}(x) + |x|^{-3}\Psi_{i+j,n}\left(\frac{x}{\sqrt{t}}\right),$$

where the right hand side of (4.3) limits again the remainder

$$\begin{aligned} \Psi_{i+j,n}(y) &:= -2\pi^{-\frac{3}{2}}(2n-1)!\frac{y_{6-i-j}}{|y|}\sum_{l=0}^{n-1}\frac{(-1)^{l+1}}{l!(2n-1-2l)!} \\ &\times \Big[\sum_{k=1}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix}(-1)^k 4^{k-l}|y|^{-3-2k}\left(\frac{y_3}{|y|}\right)^{2(n-l)-1}\int_0^{\frac{|y|}{2}}\lambda^{2+2n+2k-2l}e^{-\lambda^2}d\lambda \\ &\quad - 4^{-l}\left(\frac{y_3}{|y|}\right)^{2n-1-2l}\int_{\frac{|y|}{2}}^{\infty}\lambda^{2+2n-2l}e^{-\lambda^2}d\lambda\Big]. \end{aligned}$$

□

Proof of Lemma 3.5: Applying Lemma 3.4 to (3.3) or (3.4) leads to

$$\begin{aligned}
\mathcal{F}^{-1}\left(\cos(\frac{\xi_3}{|\xi|}\Omega t)e^{-4\pi^2 t|\xi|^2}\right) &= \mathcal{G}_t(x) + \sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!} \\
&\quad \times |x|^{-3}\left[L_{3,n}(x) + \Psi_{3,n}\left(\frac{x}{\sqrt{t}}\right)\right] \\
&= \pi^{-\frac{3}{2}}|x|^{-3}\sum_{n=1}^{\infty}\sum_{l=0}^{n}\frac{(-1)^l\Gamma(n-l+\frac{3}{2})}{4^l(n-1)!l!(2n-2l)!}(\Omega t)^{2n}\left(\frac{x_3}{|x|}\right)^{2(n-l)} \\
&\quad + \left[\mathcal{G}_t(x) + |x|^{-3}\sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}\Psi_{3,n}\left(\frac{x}{\sqrt{t}}\right)\right] \\
&= |x|^{-3}V^{(1)}(x_3,t) + |x|^{-3}\Psi^{(1)}(x,t)
\end{aligned}$$

with $\Psi^{(1)}(x,t) := |x|^3\mathcal{G}_t(x) + \sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}\Psi_{3,n}\left(\frac{x}{\sqrt{t}}\right)$ or

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\frac{\xi_{6-i-j}}{|\xi|}\sin(\frac{\xi_3}{|\xi|}\Omega t)e^{-4\pi^2 t|\xi|^2}\right) \\
&\quad = |x|^{-3}\sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\left[L_{i+j,n+1}(x) + \Psi_{i+j,n+1}\left(\frac{x}{\sqrt{t}}\right)\right],
\end{aligned}$$

respectively. We have to differ the cases if either $i+j=3$ or $i+j\neq 3$ to analyse the second term further. Let $i+j=3$ then we obtain

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\frac{\xi_{6-i-j}}{|\xi|}\sin(\frac{\xi_3}{|\xi|}\Omega t)e^{-4\pi^2 t|\xi|^2}\right) \\
&= \pi^{-\frac{3}{2}}|x|^{-3}\sum_{n=0}^{\infty}\sum_{l=0}^{n+1}\frac{(-1)^l\Gamma(n+1-l+\frac{3}{2})(2n+2)}{4^l n!l!(2n+2-2l)!}(\Omega t)^{2n+1}\left(\frac{x_3}{|x|}\right)^{2(n+1-l)} \\
&\quad + |x|^{-3}\sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\Psi_{i+j,n+1}\left(\frac{x}{\sqrt{t}}\right) \\
&= -|x|^{-3}V^{(2)}(x_3,t) + |x|^{-3}\Psi^{(2)}_{i+j}(x,t),
\end{aligned}$$

where we define $\Psi^{(2)}_{i+j}(x,t) := \sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\Psi_{i+j,n+1}\left(\frac{x}{\sqrt{t}}\right)$. Now let $i+j\neq 3$. This case implies

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\frac{\xi_{6-i-j}}{|\xi|}\sin(\frac{\xi_3}{|\xi|}\Omega t)e^{-4\pi^2 t|\xi|^2}\right) \\
&\quad = \pi^{-\frac{3}{2}}\frac{x_{6-i-j}x_3}{|x|^5}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(-1)^l\Gamma(n-l+\frac{5}{2})}{4^l n!l!(2n+1-2l)!}(\Omega t)^{2n+1}\left(\frac{x_3}{|x|}\right)^{2(n-l)} \\
&\qquad + |x|^{-3}\sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\Psi_{i+j,n+1}\left(\frac{x}{\sqrt{t}}\right) \\
&\quad = \frac{x_{6-i-j}x_3}{|x|^5}V^{(3)}(x_3,t) + |x|^{-3}\Psi^{(2)}_{i+j}(x,t).
\end{aligned}$$

However, considering (3.1) yields

$$K_{i,j}(x,t) = (-1)^{\max\{i-j,1+j-i\}}(1-\delta_{i,j})\frac{x_{6-i-j}x_3}{|x|^5}V^{(3)}(x_3,t) + |x|^{-3}\delta_{i,j}V^{(1)}(x_3,t) + |x|^{-3}\Psi_{i+j}(x,t)$$

if $i+j \neq 3$, and

$$K_{i,j}(x,t) = (-1)^{\max\{1+i-j,j-i\}}|x|^{-3}V^{(2)}(x_3,t) + |x|^{-3}\Psi_3(x,t)$$

if $i+j=3$. Here the remainder term Ψ_{i+j} is defined by

$$\Psi_{i+j}(x,t) := \delta_{i,j}\Psi^{(1)}(x,t) + (-1)^{\max\{i-j,1+j-i\}}(1-\delta_{i,j})\Psi^{(2)}_{i+j}(x,t).$$

It remains to verify the decay of this function. Therefore, due to (4.3) we obtain

$$\begin{aligned} |\Psi_{i+j}(x,t)| &\leq |\Psi^{(1)}(x,t)| + |\Psi^{(2)}_{i+j}(x,t)| \\ &\lesssim |x|^3\mathcal{G}_t(x) + |x|^{-2}t\,(1+|\Omega|t)\left[\sum_{n=1}^{\infty}\frac{42^n}{(n-1)!}(\Omega t)^{2n} + 1\right] \\ &\lesssim |x|^3\mathcal{G}_t(x) + |x|^{-2}t\,(1+|\Omega|t)^3\left[1+e^{42(\Omega t)^2}\right]. \end{aligned} \tag{4.4}$$

□

Note that due to (4.4) the remainder term $\Psi_{i,j}$ can blow up at most exponentially in time.

5. Proof of Theorem 2.2

Since a mild solution solves (1.2) we can rewrite each component of the velocity as

$$u_i(t) = \sum_{j=1}^{3}K_{i,j}(t)*u_{0,j} - \sum_{j,h=1}^{3}\int_0^t \partial_h(\delta_{i,j}+\mathcal{R}_i\mathcal{R}_j)K_{i,j}(t-s)*(u_ju_h)(s)ds,$$

where $K_{i,j}$ denotes the corresponding components of the convolution operator e^{-tA_Ω}, see (3.1). Assuming $\mu > 4$ we obtain with Lemma 3.1

$$\int_0^t \left(\partial_h(\delta_{i,j}+\mathcal{R}_i\mathcal{R}_j)K_{i,j}(t-s)*(u_ju_h)(s)\right)(x)ds \lesssim \left(\sqrt{t}+t^6\right)(1+|x|)^{-4}$$

for all $j,h=1,2,3$, i.e.

$$\int_0^t e^{-(t-\tau)A_\Omega}\mathbb{P}(u\cdot\nabla u)(\tau)d\tau = \mathcal{O}_t(|x|^{-4}). \tag{5.1}$$

Thus it suffices to analyse only the term $e^{-tA_\Omega}u_0$ more precisely. But the profile of e^{-tA_Ω} is already investigated in §3.

Let us define the auxiliary function v_j, $j=1,2,3$, as follows:

$$u_{0,j} = \mathcal{G}_1(x)\int_{\mathbb{R}^3}u_{0,j}(y)dy + v_j(x). \tag{5.2}$$

This leads us to

$$(K_{i,j}(t) * u_{0,j})(x) = (K_{i,j}(t) * \mathcal{G}_1)(x) \int_{\mathbb{R}^3} u_{0,j}(y)dy + (K_{i,j}(t) * v_j)(x)$$

and the Fourier transform yields

$$(K_{i,j}(t) * \mathcal{G}_1)(x) = \mathcal{F}^{-1}\left(e^{-4\pi^2(t+1)|\xi|^2}\left[\cos(\frac{\xi_3}{|\xi|}\Omega t)\delta_{i,j} - \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}_{i,j}(\xi)\right]\right).$$

Due to Lemma 3.4 we get the same leading term $L_{i+j,n}$ independent on time for the shifted integral

$$\int_{t+1}^{\infty}(s-t+1)^{n-1}\partial_{6-i-j}\partial_3^{2n-1}\mathcal{G}_s(x)ds = |x|^{-3}\left[L_{i+j,n}(x) + \Psi_{i+j,n}\left(\frac{x}{\sqrt{t+1}}\right)\right].$$

Dealing with $K_{i,j}(t) * \mathcal{G}_1$ instead of $K_{i,j}(t)$ thus only requires a slight modification on the remainder terms $\Psi_{i,j}$. To get the following profile it remains to handle the terms $K_{i,j}(t) * v_j$ which reveal a good behaviour.

Lemma 5.1: *For $\mu > 4$ and a initial velocity $u_0 \in L^\infty_4(\mathbb{R}^3)^3$ with $\operatorname{div} u_0 = 0$, let u be the mild solution of Theorem 2.1. Then the following profile holds for almost all $|x| \geq \sqrt{t}$:*

$$u(x,t) = |x|^{-3}\Big[V^{(1)}(x,t)\int_{\mathbb{R}^3} u_0(y)dy + V^{(2)}(x,t)e_3 \times \int_{\mathbb{R}^3} u_0(y)dy - \frac{x_3}{|x|^2}V^{(3)}(x,t)x' \times \int_{\mathbb{R}^3} u_0(y)dy\Big] + \mathcal{O}_t(|x|^{-4}).$$

Proof: The preceding definition (5.2) and Lemma 3.5 yield

$$(K_{i,j}(t) * u_{0,j})(x) = K^*_{i,j}(x,t)\int_{\mathbb{R}^3} u_{0,j}(y)dy + |x|^{-3}\tilde{\Psi}_{i,j}(x,t)\int_{\mathbb{R}^3} u_{0,j}(y)dy + (K_{i,j}(t) * v_j)(x).$$

Since

$$\tilde{\Psi}_{i,j}(x,t) := \delta_{i,j}|x|^3\mathcal{G}_{t+1}(x) + \delta_{i,j}\sum_{n=1}^{\infty}\frac{(\Omega t)^{2n}}{(2n)!(n-1)!}\Psi_{3,n}\left(\frac{x}{\sqrt{t+1}}\right) + (-1)^{\max\{i-j,1+j-i\}}(1-\delta_{i,j})\sum_{n=0}^{\infty}\frac{(\Omega t)^{2n+1}}{(2n+1)!n!}\Psi_{i+j,n+1}\left(\frac{x}{\sqrt{t+1}}\right)$$

decays like $|x|^{-2}$, comparing (4.4), it remains to investigate the decay of the convolution $K_{i,j}(t) * v_j$. Since $\int_{\mathbb{R}^3} v_j(y)dy = 0$ the Taylor formula yields

$$|(K_{i,j}(t) * v_j)(x)| \leq \int_{|y|\leq\frac{|x|}{2}} |y||v_j(y)|dy \sup_{|z|\leq\frac{|x|}{2}} |\nabla K_{i,j}(x+z,t)| + \int_{|y|>\frac{|x|}{2}} |v_j(y)|dy|K_{i,j}(x,t)| + \int_{|y|>\frac{|x|}{2}} |v_j(y)||K_{i,j}(x-y,t)|dy.$$

Let $0 < \varepsilon < \frac{\mu-4}{3}$. The definition of v_j, see (5.2), implies

$$|v_j(y)| \leq |u_{0,1}(y)| + \mathcal{G}_1(y)\|u_{0,j}\|_1 \lesssim (1+|y|)^{-\mu}.$$

Applying the Lemmata 3.1 for any $\kappa > 0$ as well as the Hölder inequality yield

$$\begin{aligned}|(K_{i,j}(t) * v_j)(x)| &\lesssim \left(1 + (|\Omega|t)^5\right)\left(\sqrt{t} + |x|\right)^{-4} \int_{|y| \leq \frac{|x|}{2}} |y||v_j(y)|dy \\ &\quad + \left(1 + (|\Omega|t)^4\right)\left(\sqrt{t} + |x|\right)^{-3} \int_{|y| > \frac{|x|}{2}} |v_j(y)|dy \\ &\quad + \|K_{i,j}(t)\|_{1+\varepsilon}\left(\int_{|y| > \frac{|x|}{2}} |v_j(y)|^{\frac{1+\varepsilon}{\varepsilon}} dy\right)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\lesssim \left(1 + (|\Omega|t)^5 + \|K_{i,j}(t)\|_{1+\varepsilon}\right)\left(\min\{\sqrt{t}, 1\} + |x|\right)^{-4}.\end{aligned}$$

Note that due to the choice of ε there holds $(\mu - 4)\frac{1+\varepsilon}{\varepsilon} > 3$ which ensures the convergence of the integral $\int_{\mathbb{R}^3}(1 + |y|)^{(\mu-4)\frac{1+\varepsilon}{\varepsilon}} dy$.
Finally, we get for $i = 1, 2$ the asymptotic profile

$$\begin{aligned}u_i(x,t) &= |x|^{-3}\Big[V^{(1)}(x,t)\int_{\mathbb{R}^3} u_{0,i}(y)dy + (-1)^i V^{(2)}(x,t)\int_{\mathbb{R}^3} u_{0,3-i}(y)dy \\ &\quad + (-1)^i \frac{x_{3-i}x_3}{|x|^2} V^{(3)}(x,t)\int_{\mathbb{R}^3} u_{0,3}(y)dy\Big] + \mathcal{O}_t(|x|^{-4}), \\ u_3(x,t) &= |x|^{-3}\Big[V^{(1)}(x,t)\int_{\mathbb{R}^3} u_{0,3}(y)dy + \frac{x_2x_3}{|x|^2} V^{(3)}(x,t)\int_{\mathbb{R}^3} u_{0,1}(y)dy \\ &\quad - \frac{x_1x_3}{|x|^2} V^{(3)}(x,t)\int_{\mathbb{R}^3} u_{0,2}(y)dy\Big] + \mathcal{O}_t(|x|^{-4}),\end{aligned}$$

which is nothing but the componentwise presentation of the assertion. □

The following Lemma will be helpfull to develop an asymptotic profile far from the rotating axis, i.e. more precisely in an area such that $|x_3|^{1+\varepsilon} \leq |x|$ for an arbitrary $\varepsilon > 0$. In contrast to the previous result the profile of Theorem 4.2 have thus a very simple spatial configuration. For this goal let us denote by $V_0^{(j)}(t)$, $j = 1, 2$, the series

$$\begin{aligned}V_0^{(1)}(t) &:= \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!(n-1)!}(\Omega t)^{2n}, \\ V_0^{(2)}(t) &:= \frac{1}{2\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2}(\Omega t)^{2n+1}.\end{aligned}$$

These series are nothing but the summands of the corresponding series $V^{(j)}(t)$, see (3.8), with respect to $l = 0$. In the case $j = 3$ the series $3 \cdot V_0^{(2)}(t)$ even equals the summands of the series $V^{(3)}(t)$ with respect to $l = 0$.

Lemma 5.2: *Let $\varepsilon > 0$ and $\delta := \frac{2\varepsilon}{1+\varepsilon}$. Then there holds for all $x \in \{y \in \mathbb{R}^3 \mid |y_3|^{1+\varepsilon} \leq |y|\}$:*

$$|V^{(1)}(x,t) - V_0^{(1)}(t)| + |V^{(2)}(x,t) - V_0^{(2)}(t)| + |V^{(3)}(x,t) - 3V_0^{(2)}(t)| = \mathcal{O}_t(|x|^{-\delta}).$$

Proof: Due to the assumption $|x_3|^{1+\varepsilon} \leq |x|$ we obtain

$$\begin{aligned} |V^{(1)}(x,t) - V_0^{(1)}(t)| &= \left| \pi^{-\frac{3}{2}} \sum_{l=1}^{\infty} \sum_{n=l}^{\infty} \frac{(-1)^{n+l}\Gamma(l+\frac{3}{2})(\Omega t)^{2n}}{4^{n-l}(n-1)!(n-l)!(2l)!} \left(\frac{x_3}{|x|}\right)^{2l} \right| \\ &\leq \pi^{-\frac{3}{2}} \sum_{l=1}^{\infty} \frac{4^l (l+1)!}{(2l)!} |x|^{-\delta l} \sum_{n=l}^{\infty} \frac{(\Omega t)^{2n}}{4^n (n-1)!} \\ &< \pi^{-\frac{3}{2}} e^{(\Omega t)^2} \left[\cosh(2|\Omega| t |x|^{-\frac{\delta}{2}}) - 1 \right] = \mathcal{O}_t(|x|^{-\delta}), \end{aligned}$$

since $4^n > n(n+1)$. We applied Taylor's theorem to get the inequality

$$\sum_{n=l}^{\infty} \frac{y^{2n}}{n!} \leq \frac{\exp(y^2)}{l!} y^{2l}$$

and hence the last estimate above. Finally, we obtain the following limiting behaviour in the same way:

$$\begin{aligned} &|V^{(2)}(x,t) - V_0^{(2)}(t)| + |V^{(3)}(x,t) - 3V_0^{(2)}(t)| \\ &\qquad \lesssim \left(|\Omega| t + (|\Omega| t)^{-1} \right) e^{(\Omega t)^2} \left[\cosh(2|\Omega| t |x|^{-\frac{\delta}{2}}) - 1 \right] = \mathcal{O}_t(|x|^{-\delta}). \end{aligned}$$

□

Eventually we recognise the Bessel functions I_ν which we already defined in (2.1) and easily obtain the equations

$$V_0^{(1)}(t) = -\frac{\Omega t}{4\pi} J_1(\Omega t) \qquad \text{and} \qquad V_0^{(2)}(t) = \frac{\Omega t}{2\pi} J_0(\Omega t).$$

These terms are non-negative spatial constants but analytic functions in time and can be estimated by

$$|J_\nu(\Omega t)| \lesssim \frac{\cos(|\Omega| t - \frac{\pi}{4}(2\nu + 1))}{\sqrt{|\Omega| t}}$$

for all $|\Omega t| \gg 1$, see [18, chapter 3]. With these series we are able to state the main result of this work. Combining Lemma 5.1 with Lemma 5.2 we get the assertion about the spatial asymptotic behaviour of the Navier-Stokes equations in a rotating frame without external force away from the rotating axis. This finishes the proof of Theorem 2.2.

References

[1] Babin, A., Mahalov, A., Nicolaenko, B., *Global Regularity of the 3D Rotating Navier-Stokes Equations for Resonant Domains*, Asympt. Anal. **15**, 103-150 (1997).

[2] Bae, H.-O., Brandolese, L., *On the Effect of External Forces on Incompressible Fluid Motions at Large Distances*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **55**, 225-238 (2009).

[3] Brandolese, L., Vigneron, F., *New Asymptotic Profiles of the Nonstationary Solutions of the Navier-Stokes System*, J. Math. Pures Appl. **88**, 64-86 (2007).

[4] Chen, Q., Miao, C., Zhang, Z., *Global Well-Posedness for the 3D Rotating Navier-Stokes Equations with Highly Oscillating Initial Data*, arXiv:0919.3064v2 [math.AP] (2009).

[5] Farwig, R., Hishida, T., *Asymptotic Profiles of Steady Stokes and Navier-Stokes Flows around a Rotating Obstacle*, Ann. Univ. Ferrara **55**, 263-277 (2009).

[6] Farwig, R., Hishida, T., *Asymptotic Profile of Steady Stokes Flow around a Rotating Obstacle*, Manuscripta Math. **135**, 315-338 (2011).

[7] Farwig, R., Galdi, G. P., Kyed, M., *Asymptotic Structure of a Leray Solution to the Navier-Stokes Flow Around a Rotating Body*, Pacific Math J. **253**, 367-382 (2011).

[8] Farwig, R., Schulz, R., Yamazaki, M., *Concentration-Diffusion Phenomena of Heat Convection in an Incompressible Fluid*, TU Darmstadt, Dept. Math., Preprint no. 2649 (2012).

[9] Giga, Y., Inui, K., Mahalov, A., Matsui, S., *Uniform Local Solvability for the Navier-Stokes Equations with the Coriolis Force*, Methods Appl. Anal. **12**, 381-393 (2005).

[10] Giga, Y., Inui, K., Mahalov, A., Matsui, S., *Navier-Stokes Equations in a Rotating Frame with Initial Data Nondecreasing at Infinity*, Hokkaido Mathematical Journal **35**, No. 2, 321-364 (2006).

[11] Giga, Y., Inui, K., Mahalov, Saal, J., *Uniform Global Solvability of the Rotating Navier-Stokes Equations for Nondecaying Initial Data*, Indiana Univ. Math. J. **57**, No. 6, 2775-2791 (2008).

[12] Hieber, M., Shibata, Y., *The Fujita-Kato Approach to the Navier-Stokes equations in the Rotatinal Framework*, Math. Z., 265:481-491 (2010).

[13] Hishida, T., *An Existence Theorem for the Navier-Stokes Flow in the Exterior of a Rotating Obstacle*, Arch. Ration. Mech. Anal. **150**, 307-348 (1999).

[14] Konieczny, P., Yoneda, T., *On Dispersive Effect of the Coriolis Force for the Stationary NavierStokes Equations*, J. Differential Equations **250**, 3859-3873 (2011).

[15] Kyed, M., *Asymptotic Profile of a Linearized Navier-Stokes Flow Past a Rotating Body*, Preprint 2638 TU Darmstadt, (2011).

[16] Lemarié-Rieusset, P. G., *Recent Developments In the Navier-Stokes Problem*, Chapman & Hall, CRC Press Boca Raton (2002).

[17] Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences **44**, Springer (1983).

[18] Watson, G.N., *A Treatise on the Thoery of Bessel Functions*, Cambridge University Press, (1980)

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