Hadamard variational formula for the Green function of the boundary value problem on the Stokes equations.

Hideo KOZONO
Mathematical Institute
Tohoku University
Sendai 980-8578, Japan
e-mail:kozono@math.tohoku.ac.jp

Erika USHIKOSHI
Mathematical Institute
Tohoku University
Sendai 980-8578, Japan
e-mail:sa8m06@math.tohoku.ac.jp

Abstract

For every $\varepsilon > 0$, consider the Green matrix $G_\varepsilon(x,y)$ of the Stokes equations describing the motion of incompressible fluids in a bounded domain $\Omega_\varepsilon \subset \mathbb{R}^d$ which is a family of perturbation of domains from $\Omega \equiv \Omega_0$ with the smooth boundary $\partial\Omega$. Assuming the volume preserving property, i.e., $\mathrm{vol}.\Omega_\varepsilon = \mathrm{vol}.\Omega$ for all $\varepsilon > 0$, we give an explicit representation formula for $\delta G(x,y) \equiv \lim_{\varepsilon\to+0} \varepsilon^{-1}(G_\varepsilon(x,y) - G_0(x,y))$ in terms of the boundary integral on $\partial\Omega$ of $G_0(x,y)$. Our result may be regarded as a classical Hadamard variational formula for the Green functions of the elliptic boundary value problems.

AMS Mathematics Subject Classification(2000): 35Q30, 35J25, 35C20, 76D07
A running title: Hadamard formula for the Stokes equations

Introduction.

Let $\Omega \subset \mathbb{R}^d (d \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the stationary Stokes equations governing the motion of incompressible fluid.

$$\begin{cases} \Delta \boldsymbol{v} - \nabla q = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\boldsymbol{v} = \boldsymbol{v}(x) = (v^1(x), \cdots, v^d(x))$ and $q = q(x)$ denote the unknown velocity and the pressure at $x = (x^1, \cdots, x^d) \in \Omega$ respectively, while $\boldsymbol{f} = \boldsymbol{f}(x) = (f^1(x), \cdots, f^d(x))$ is the given external force. The purpose of this paper is to prove the Hadamard variational formula for the Green matrix associated with (0.1). Let $\{\Omega_\varepsilon\}_{\varepsilon\geq 0}$ be a family of domains with smooth boundary $\partial\Omega_\varepsilon$ satisfying $\Omega_0 = \Omega$. For small $\varepsilon \geq 0$, we regard Ω_ε as a perturbation of Ω. The Green matrix $G_\varepsilon(x,y) = \{G^i_{\varepsilon,n}(x,y)\}_{1\leq i,n\leq d}$, $R(x,y;\varepsilon) = \{R_{\varepsilon,n}(x,y)\}_{1\leq n\leq d}$ for the Stokes equations on Ω_ε is defined by

$$\begin{cases} \Delta_x \boldsymbol{G}_{\varepsilon,n}(x,y) - \nabla_x R_{\varepsilon,n}(x,y) = e_n \delta(x-y), & (x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon, \\ \operatorname{div} \boldsymbol{G}_{\varepsilon,n}(x,y) = 0, & (x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon, \\ \boldsymbol{G}_{\varepsilon,n}(x,y) = 0, & x \in \partial\Omega_\varepsilon, y \in \Omega_\varepsilon, \end{cases} \tag{0.2}$$

for $n = 1, \cdots, d$, where $\{e_1, \cdots, e_d\}$ denotes a canonical basis in $\mathbb{R}^d$. We abbreviate $G_0(x, y) = G(x, y)$, $R_0(x, y) = R(x, y)$. Our aim is to show a representation formula for $\delta G(x, y) := \lim_{\varepsilon\to 0} \varepsilon^{-1}(G_\varepsilon(x, y) - G(x, y))$. Concerning the usual Laplace operator $-\Delta$, such a formula was first obtained by Hadamard [8], and later on Garabedian-Shiffer [7], Garabedian [6] and Aomoto [2] treated more general case of perturbation Ω_ε of Ω, and gave several refined proofs. Indeed, they consider the perturbation Ω_ε of the domain Ω whose boundary $\partial\Omega_\varepsilon$ is expressed in such a way that $\partial\Omega_\varepsilon = \{y = x + \varepsilon\rho(x)\nu_x; x \in \partial\Omega\}$, where $\rho \in C^\infty(\partial\Omega)$ and ν_x is the unit outer normal to $\partial\Omega$. Then it holds that

$$\delta G(y, z) = \int_{\partial\Omega} \frac{\partial G(y, x)}{\partial\nu_x} \frac{\partial G(z, x)}{\partial\nu_x} \rho(x) d\sigma_x, \tag{0.3}$$

where $d\sigma_x$ denotes the surface element of $\partial\Omega$. Peetre [13] and Fujiwara-Ozawa [5] investigated a boundary value problem of general elliptic differential operators $A(x, D) = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$ of the $2m$-th order;

$$\begin{cases} A(x, D)u(x) = f(x) \quad \text{for } x \in \Omega_\varepsilon, \\ B_j(x, D)u(x) = 0, \quad j = 1, 2, \cdots, m \quad \text{for } x \in \partial\Omega_\varepsilon, \end{cases} \tag{0.4}$$

where $B_j(x, D)$ is the boundary differential operator of order m_j. Under the assumption that the system $\{A(x, D), B_j(x, D)\}_{j=1}^m$ satisfies the complementing condition in the sense of Agmon-Douglis-Nirenberg [1] and that the operator A defined by (0.4) with the domain $D(A) \equiv \{u \in H^{2m}(\Omega); B_j(x, D)u|_{\partial\Omega} = 0, j = 1, 2, \cdots, m\}$ is a bijective *self-adjoint operator* in $L^2(\Omega)$, Fujiwara-Ozawa [5] proved that

$$\delta G(y, z) = -\sum_{j=1}^m \int_{\partial\Omega} \frac{\partial}{\partial\nu_x} (B_j(x, D)G(x, y)) \overline{S_j(x, D)G(x, z)} \rho(x) d\sigma_x, \tag{0.5}$$

where $\{S_j(x, D)\}_{j=1}^m$ are boundary differential operators which are determined by Green's integral formula associated with the the system $\{A(x, D), B_j(x, D)\}_{j=1}^m$, that is

$$\begin{aligned} &\int_{\Omega_\varepsilon} A(x, D)u(x)\overline{v(x)}dx - \int_{\Omega_\varepsilon} u(x)\overline{A(x, D)v(x)}dx \\ &= \sum_{j=1}^m \int_{\partial\Omega_\varepsilon} S_j(x, D)u(x)\overline{B_j(x, D)v(x)}d\sigma_x - \sum_{j=1}^m \int_{\partial\Omega_\varepsilon} B_j(x, D)u(x)\overline{S_j(x, D)v(x)}d\sigma_x. \end{aligned} \tag{0.6}$$

The Hadamard formulae (0.3) and (0.5) are based on Green's integral formula (0.6). However, for the rigorous proof for an arbitrary displacement $\rho \in C^\infty(\partial\Omega)$, we need to handle $G(z, x)$ as the function of z defined on Ω_ε for each fixed $x \in \Omega$. To get around this difficulty, Fujiwara-Ozawa [5] made use of a Whitney extension $\tilde{G}(z, x)$ of $G(z, x)$ as a function of $z \in \mathbb{R}^d \setminus \{x\}$ for each fixed $x \in \Omega$ in such a way that

$$A(z, D)\tilde{G}(z, x) = \delta(z - x) + g(z, x), \tag{0.7}$$

where $g(\cdot, x) \in C^\infty(\mathbb{R}^d)$ satisfies $g(z, x) = 0$ for all $z \in \bar{\Omega}$. Then on account of a priori estimates for the elliptic equations due to Agmon-Douglis-Nirenberg [1], they succeed to treat $g(z, x)$ as the remainder term so that the desired Hadamard variational formula (0.5) can be established.

Unfortunately, such a method does not work in our case (0.2) because the Whitney extension $\tilde{\boldsymbol{G}}_n(z,x)$ of $\boldsymbol{G}_n(z,x)$ onto $z \in \mathbb{R}^d \setminus \{0\}$ may *not* preserve the divergence free property, i.e., div $\tilde{\boldsymbol{G}}_n(z,x) \neq 0$, $n = 1, \cdots, n$. Therefore, we transform the Stokes equations on Ω_ε to those on Ω by means of the diffeomorphsim $\Phi_\varepsilon : \bar{\Omega} \to \bar{\Omega}_\varepsilon$. Because of the divergence free property of $\{\boldsymbol{G}_n\}_{n=1,\cdots,d}$, it is necessary to assume that Φ_ε preserves the volume of Ω for all $\varepsilon \geq 0$. Then the transformed equations may be regarded as those for vector fields on the compact Riemannian manifold $(\bar{\Omega}, a_\varepsilon)$ with $\{a_\varepsilon\}_{\varepsilon \geq 0} \equiv \Phi_\varepsilon^* \delta$ a one-parameter family of Riemannian metrics on $\bar{\Omega}$, where $\delta = (\delta_{ij})_{1 \leq i,j \leq d}$ is the standard Euclidian metric on $\bar{\Omega}_\varepsilon$. Such a procedure was first introduced by Inoue-Wakimoto [9] who dealt with the moving boundary value problem for the non-stationary Navier-Stokes equations. Our method relies on the construction of the parametrix which approximates the Green matrix of the Stokes equations on $(\bar{\Omega}, a_\varepsilon)$, whose original idea is due to Garabedian [6]. For the usual Laplace equation with the homogeneous Dirichlet condition on $\partial\Omega$, we easily construct a parametrix $P(x,y)$ by multiplying the fundamental solution $\Gamma(x,y) = \omega_d^{-1}|x-y|^{2-d}$ by the cut-off function $\alpha(\cdot, y) \in C_0^\infty(\Omega)$ satisfying $\alpha(y,y) = 1$, that is $P(x,y) = \alpha(x,y)\Gamma(x,y)$. However, such a simple multiplication is unavailable to the Stokes equations because the divergence free condition is not preserved. To recover the divergence free property, we make use of the Bogovskiĭ formula so that the approximating argument is parallel to that of the Laplace equation. Similarly to (0.6), Green's integral formula associated with the Stokes equations on $(\bar{\Omega}, a_\varepsilon)$ plays an essential role in deriving our Hadamard variational formula. Instead of dealing with the remainder term $g(z,x)$ as in (0.7), we shall investigate the behavior as $\varepsilon \to 0$ of the compensating functions $\{\boldsymbol{q}_{\varepsilon,n}\}_{n=1,\cdots,d}$ which are defined by $(\Phi_\varepsilon^{-1})_* \boldsymbol{G}_{\varepsilon,n}(x,y) = \boldsymbol{u}_{\varepsilon,n}(x,y) + \boldsymbol{q}_{\varepsilon,n}(x,y)$ with $\boldsymbol{u}_{\varepsilon,n}(x,y)$ denoting the fundamental solutions of the Stokes equations on $(\bar{\Omega}, a_\varepsilon)$. For our proof, it is crucial to show that

$$\sup_{x \in \Omega} |D_x^\alpha \boldsymbol{q}_{\varepsilon,n}(x,y) - D_x^\alpha \boldsymbol{q}_{0,n}(x,y)| \to 0, \quad |\alpha| \leq 2, \quad n = 1, \cdots, d$$

for each $y \in \Omega$ as $\varepsilon \to 0$. To this end, we need to establish an ε-independent a priori estimate of Schauder's type for $\boldsymbol{q}_{\varepsilon,n}(\cdot, y)$ in the Hörder space $C^{2+\theta}(\Omega)$. Such an idea goes back to Ozawa [10], while Garabedian's [6] another approach is to expand $\boldsymbol{G}_{\varepsilon,n}(x,y)$ by the power series of ε through the integral equation of the Fredholm type.

This paper is organized as follows. In Section 1, we first impose the assumption on the perturbation $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of domains, and then state our main result. In Section 2, we introduce the fundamental tensor in $\mathbb{R}^d$ associated with (0.1) and investigate the properties of the singularity near $x = y$ of the Green matrix $\{G_{\varepsilon,n}^i(x,y)\}_{i,n=1,\cdots,d}$. Making use of the volume preserving diffeomorphism $\Phi_\varepsilon : \bar{\Omega} \to \bar{\Omega}_\varepsilon$, we transform our equations (0.2) to those in $(\bar{\Omega}, a_\varepsilon)$. Section 3 is devoted to constructing the parametrix $\{U_n^i\}_{i,n=1,\cdots,d}$ which approximates the Green matrix $\{G_n^i\}_{i,n=1,\cdots,d}$. The Bogovskiĭ formula plays an essential role in recovering the divergence free condition in Ω_ε. Since the Bogovskiĭ operator has a smoothing property, our construction of the parametrix preserves the same behavior near $x = y$ of singularity as that of the fundamental tensor $\boldsymbol{u}_{\varepsilon,n}(x,y)$, $n = 1, \cdots, d$. Finally in Section 4, we show continuous dependence as $\varepsilon \to 0$ of the compensating function $\{\boldsymbol{q}_{\varepsilon,n}\}_{n=1,\cdots,d}$, and then prove our main theorem.

Acknowledgement. This work was done while the second author visited Technical University of Darmstadt; she was partially supported by "Japanese-German Externship, International

Research Training Group 1529" associated with Technical University of Darmstadt and Waseda University, Tokyo. She is indebted to their kind hospitality. She was also partly supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

1 Result.

To state our result, we first introduce the assumption on the perturbation $\{\Omega_\varepsilon\}_{\varepsilon\ge 0}$ of domains from Ω.

Assumption. For every $\varepsilon \ge 0$, there is a diffeomorphism $\Phi_\varepsilon : \overline{\Omega} \to \overline{\Omega}_\varepsilon$ satisfying the following condition.

(A.1) $\Phi_\varepsilon = (\phi_\varepsilon^1, \phi_\varepsilon^2, \cdots, \phi_\varepsilon^d) \in C^\infty(\overline{\Omega})^d$;

(A.2) $\Phi_0(x) = x \quad$ for all $x \in \overline{\Omega}$;

(A.3) There exists $S = (S^1, S^2, \cdots, S^d) \in C^\infty(\overline{\Omega})^d$ such that $K(x;\varepsilon) := \Phi_\varepsilon(x) - x - S(x)\varepsilon$ satisfies

$$\sup_{x\in\overline{\Omega}} |K(x;\varepsilon)| + \sup_{x\in\overline{\Omega}} |\nabla K(x;\varepsilon)| = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0;$$

(A.4) It holds that

$$\det\left(\frac{\partial \phi_\varepsilon^i(x)}{\partial x^j}\right)_{i,j=1,\cdots,d} = 1 \qquad \text{for all } x \in \overline{\Omega} \text{ and all } \varepsilon \ge 0.$$

It should be noticed that by (A.4), Φ_ε defines a volume preserving diffeomorphism from $\overline{\Omega}$ onto $\overline{\Omega}_\varepsilon$ i.e., $\mathrm{vol}(\Omega_\epsilon) = \mathrm{vol}(\Omega)$ for all $\varepsilon \ge 0$. Moreover, the vector function $S \in C^\infty(\overline{\Omega})^d$ defined by (A.3) satisfies the divergence free property. Namely, it holds that

$$\operatorname{div} S(x) = 0 \quad \text{for all } x \in \overline{\Omega}. \tag{1.1}$$

For the proof, see Inoue-Wakimoto[9, Proposition 2.3].

Now we can state our result.

Theorem 1.1. *Let the Assumption hold. Let $\{G_n^m(x,y;\varepsilon)\}_{m,n=1,\cdots,d}$ be the Green matrix of the boundary value problem for (0.2). Let*

$$\delta G_n^m(y,z) := \lim_{\varepsilon\to 0} \frac{G_{\varepsilon,n}^m(y,z) - G_n^m(y,z)}{\varepsilon}, \quad m,n = 1,\cdots,d$$

for any y and z in Ω. Then we have the variational formula

$$\delta G_n^m(y,z) = \int_{\partial\Omega} \sum_{i=1}^d \left\{ \frac{\partial G_n^i}{\partial \nu_x}(x,y) \frac{\partial G_m^i}{\partial \nu_x}(x,z) - \left(R_n(x,y) \frac{\partial G_m^i}{\partial \nu_x}(x,z) + R_m(x,z) \frac{\partial G_n^i}{\partial \nu_x}(x,y) \right) \nu_x^i \right\} S(x)\cdot \nu_x \, d\sigma_x,$$

for $m,n = 1,\cdots,d$, where $\nu_x = (\nu_x^1,\cdots,\nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

Remarks. (1) Garabedian [6], Garabedian-Schiffer [7] and Aomoto [2] proved the Hadamard variational formula for the Laplace operator Δ in the domain Ω_ε with the Dirichlet condition on $\partial\Omega_\varepsilon = \{y = x + \varepsilon\rho(x)\nu_x; x \in \partial\Omega\}$, where $\rho \in C^\infty(\partial\Omega)$. Fujiwara-Ozawa [5] and Peetre [13] generalized it to the boundary value problem of single elliptic equations of the higher order which define the self-adjoint operators in $L^2(\Omega)$. Our theorem enables us to deal with the elliptic *system* which is not necessarily self-adjoint in $L^2(\Omega)$.

(2) In such a perturbation $\partial\Omega_\varepsilon = \{y = x + \varepsilon\rho(x)\nu_x; x \in \partial\Omega\}$ with $\rho \in C^\infty(\partial\Omega)$ as in the above (1), Theorem 1.1 has a simpler expression. Indeed, by taking a function $f \in C^\infty(\mathbb{R}^d)$ with $\nabla f \neq 0$ in some neighborhood of $\partial\Omega$, let us assume that $\Omega = \{x \in \mathbb{R}^d; f(x) < 0\}$ and that

$$\Omega_\varepsilon = \left\{ \tilde{x} = x + \varepsilon\rho(x)\frac{\nabla f(x)}{|\nabla f(x)|}; x \in \Omega \right\}, \quad \varepsilon \geq 0,$$

where ρ is extended as a smooth function on $\mathbb{R}^d$. It is easy to see that all hypotheses in the Assumption are fulfilled provided div $(\rho\nabla f) = 0$ in Ω with $S = \rho\dfrac{\nabla f}{|\nabla f|}$. Since $S(x)\cdot\nu_x = \rho(x)$ for all $x \in \partial\Omega$, it follows from Theorem 1.1 that

$$\begin{aligned} &\delta G_n^m(y,z) \\ &= \int_{\partial\Omega}\sum_{i=1}^d \left\{ \frac{\partial G_n^i}{\partial\nu_x}(x,y)\frac{\partial G_m^i}{\partial\nu_x}(x,z) - \left(R_n(x,y)\frac{\partial G_m^i}{\partial\nu_x}(x,z) + R_m(x,z)\frac{\partial G_n^i}{\partial\nu_x}(x,y)\right)\nu_x^i \right\}\rho(x)d\sigma_x \end{aligned}$$

for all $y, z \in \Omega$ and $m, n = 1, \cdots, d$.

(3) Because of the divergence free property div $\boldsymbol{v} = 0$ in (0.1), it is essential to assume that the diffeomorphism $\Phi_\varepsilon : \bar{\Omega} \to \bar{\Omega}_\varepsilon$ preserves the volume for all $\varepsilon \geq 0$, which is formulated by (A.4) in the Assumption. On the the other hand, it seems to be an interesting problem to consider the general perturbation Ω_ε which may not preserve the volume for $\varepsilon \geq 0$. For instance, the Hadamard variational formula as in Theorem 1.1 for perturbation of domains which change their topological types for $\varepsilon \geq 0$ is not obtained so far. There is another challenging problem such as the time-dependent case or the equations on the manifold. See e.g., Ozawa [11], [12]. We will discuss them in a forthcoming paper.

2 Preliminaries.

Although our main result Theorem 1.1 holds for all $d \geq 2$, for the sake of simplicity, we restrict ourselves to the case $d \geq 3$. Indeed, it is easy to see that our argument in the subsequent sections works even in the case $d = 2$.

2.1 Green matrix and fundamental identities.

In this subsection, we introduce the Green matrix of the Stokes equations. The Green matrix $\{G_n^i\}_{i,n=1,\cdots,d}$. for the velocity and the Green function $\{R_n\}_{n=1,\cdots,d}$ for the pressure can be represented by the fundamental tensor $\{u_n^i, p_n\}_{i,n=1,\cdots,d}$ of the Stokes equation (0.2) with the compensating function $\{q_n^i, q_n\}_{i,n=1,\cdots,d}$ as follows.

$$\text{(2.1)} \qquad \begin{cases} G_n^i(x,y) = u_n^i(x,y) - q_n^i(x,y), \\ R_n(x,y) = p_n(x,y) - q_n(x,y), \end{cases}$$

where

$$u_n^i(x,y) := -\frac{1}{2\omega_d(d-2)}\left[\frac{\delta^{in}}{|x-y|^{d-2}} + (d-2)\frac{(x^i-y^i)(x^n-y^n)}{|x-y|^d}\right], \tag{2.2}$$
$$p_n(x,y) := -\frac{1}{\omega_d}\frac{(x^n-y^n)}{|x-y|^d},$$

for $i,n=1,\cdots,d$, where ω_d denotes the volume of unit sphere in $\mathbb{R}^d$, i.e., $\omega_d := \frac{2\pi^{d/2}}{\Gamma(d/2)}$. The compensating function $\{q_n^i\}_{i,n=1,\cdots,d}$ and $\{q_n\}_{n=1,\cdots,d}$ are chosen so that (0.2) is satisfied, that is,

$$\begin{cases} \Delta q_n^i(x,y) - \nabla_{x^i} q_n(x,y) = 0, & x \in \Omega, \\ \displaystyle\sum_{i=1}^d \frac{\partial q_n^i(x,y)}{\partial x^i} = 0, & x \in \Omega, \\ q_n^i(x,y) = u_n^i(x.y), & x \in \partial\Omega, \qquad i,n=1,\cdots,d. \end{cases} \tag{2.3}$$

For any fixed $y \in \Omega$, $q_n^i(\cdot,y)$ and $q_n(\cdot,y)$ are analytic functions in Ω and continuous in $\overline{\Omega}$. For construction of the Hadamard variational formula, we need to investigate the behavior of the Green matrix defined by (2.1) around the singularity, so that we have various identities of both surface and volume integrals on $\partial\Omega$ and Ω. In this subsection, we collect some results on these fundamental identities.

Lemma 2.1. *Let $\{G_n^i\}_{i,n=1,\cdots,d}$ and $\{R_n\}_{n=1,\cdots,d}$ be the Green matrix for the Stokes equations as in (2.1).*

(1) *We define the stress tensor for the velocity $\boldsymbol{G}_n$ and the pressure R_n by*

$$T^{ij}(\boldsymbol{G}_n, R_n)(x,y) := -\delta^{ij}R_n(x,y) + \frac{\partial G_n^i(x,y)}{\partial x^j}, \qquad i,j,n=1,\cdots,d.$$

Then it holds that

$$\lim_{\rho\to 0}\int_{\partial B_\rho(y)} \sum_{i,j=1}^d T^{ij}(\boldsymbol{G}_n, R_n)(x,y)v^i(x)\nu_x^j\, d\sigma_x = v^n(y), \quad n=1,\cdots,d, \tag{2.4}$$

for all $y \in \Omega$ and all smooth vector functions $\boldsymbol{v} = (v^1,\cdots,v^d)$ near y, where $\partial B_\rho(y)$ denotes the surface centered at y with the radius ρ and ν_x is the unit outer normal vector to $\partial B(y)$ at x.

(2) *It holds that*

$$\lim_{\rho\to 0}\int_{\partial B_\rho(y)} \sum_{i,j,k=1}^d \left(\frac{\partial G_n^i(x,y)}{\partial x^k}\nu_x^j - \frac{\partial G_n^i(x,y)}{\partial x^j}\nu_x^k\right) h_{ijk}(x)\, d\sigma_x = 0, \quad n=1,\cdots,d \tag{2.5}$$

for all $y \in \Omega$ and all smooth tensors $\{h_{ijk}\}_{i,j,k=1,\cdots,d}$ near y.

Proof. (1) We substitute the Green matrix $\{G_n^i\}_{i,n=1,\cdots,d}$ and $\{R_n\}_{n=1,\cdots,d}$ to (2.4). Since the compensating functions $\{q_n^i\}_{i,n=1,\cdots,d}$ and $\{q_n\}_{n=1,\cdots,d}$ are smooth in Ω, it holds that

$$\lim_{\rho\to 0}\int_{\partial B_\rho(y)} \sum_{i,j=1}^d -\delta^{ij}q_n(x,y)v^i(x)\nu_x^j\, d\sigma_x = \lim_{\rho\to 0}\int_{\partial B_\rho(y)} \sum_{i,j=1}^d \frac{\partial q_n^i(x,y)}{\partial x^j}v^i(x)\nu_x^j\, d\sigma_x = 0 \tag{2.6}$$

for $n = 1, \cdots, d$. Since $\nu_x^j = (x^j - y^j)|x-y|^{-1}$, $1 \le j \le d$ for $x \in \partial B_\rho(y)$, it follows from (2.1) and (2.6) that

$$\begin{aligned}
&\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{i,j=1}^{d} T^{ij}(\boldsymbol{G}_n, R_n)(x,y)v^i(x)\nu_x^j\, d\sigma_x \\
&= \lim_{\rho\to 0}\frac{1}{2\omega_d}\int_{\partial B_\rho(y)}\sum_{i=1}^{d}\left(\frac{\delta^{in}}{|x-y|^{d-1}} + d\frac{(x^i-y^i)(x^n-y^n)}{|x-y|^{d+1}}\right)v^i(x)\, d\sigma_x \\
&:= \lim_{\rho\to 0}(I_1(\rho) + I_2(\rho)),
\end{aligned}$$

where

$$\begin{aligned}
I_1(\rho) &= \frac{1}{2\omega_d}\int_{\partial B_\rho(y)}\sum_{i=1}^{d}\left(\frac{\delta^{in}}{|x-y|^{d-1}} + d\frac{(x^i-y^i)(x^n-y^n)}{|x-y|^{d+1}}\right)(v^i(x) - v^i(y))\, d\sigma_x, \\
I_2(\rho) &= \frac{1}{2\omega_d}\int_{\partial B_\rho(y)}\sum_{i=1}^{d}\left(\frac{\delta^{in}}{|x-y|^{d-1}} + d\frac{(x^i-y^i)(x^n-y^n)}{|x-y|^{d+1}}\right)v^i(y)\, d\sigma_x.
\end{aligned}$$

We first treat $I_1(\rho)$. Indeed, it holds that

$$\begin{aligned}
|I_1(\rho)| &= \frac{1}{2\omega_d}\left|\int_{\partial B_\rho(y)}\sum_{i=1}^{d}\left(\frac{\delta^{in}}{|x-y|^{d-1}} + d\frac{(x^i-y^i)(x^n-y^n)}{|x-y|^{d+1}}\right)(v^i(x) - v^i(y))\, d\sigma_x\right| \\
&\le \frac{1}{2\omega_d}\int_{\partial B_\rho(y)}\sum_{i=1}^{d}\left|\frac{\delta^{in}}{|x-y|^{d-1}} + d\frac{(x^i-y^i)(x^n-y^n)}{|x-y|^{d+1}}\right|\cdot\left|\left(\frac{d}{d\theta}\int_0^1 v^i(\theta x + (1-\theta)y)\, d\theta\right)\right| d\sigma_x \\
&\le \frac{1}{2\omega_d}\int_{\partial B_\rho(y)}\sum_{i=1}^{d}\left|\frac{\delta^{in}}{|x-y|^{d-1}} + d\frac{(x^i-y^i)(x^n-y^n)}{|x-y|^{d+1}}\right| |x-y|\int_0^1 |\nabla v^i(\theta x + (1-\theta)y)|d\theta\, d\sigma_x. \\
&\le C\max_{i=1,\cdots,d}\sup_{x\in\Omega}|(\nabla v^i)(x)|\rho,
\end{aligned}$$

with a constant C independent of ρ, which yields

$$\lim_{\rho\to 0} I_1(\rho) = 0. \tag{2.7}$$

Concerning the estimate of $I_2(\rho)$, we change variables $x \to z := \frac{x-y}{\rho}$ to obtain

$$\begin{aligned}
I_2(\rho) &= \frac{1}{2\omega_d}\int_{\partial B_\rho(y)}\sum_{i=1}^{d}\left(\frac{\delta^{in}}{|x-y|^{d-1}} + d\frac{(x^i-y^i)(x^n-y^n)}{|x-y|^{d+1}}\right)v^i(y)\, d\sigma_x. \\
&= \frac{1}{2}v^n(y) + \frac{d}{2\omega_d}\sum_{i=1}^{d} v^i(y)\int_{\partial B_1(0)} z^i z^n\, d\sigma_z \\
&= v^n(y), \qquad n = 1, \cdots, d
\end{aligned} \tag{2.8}$$

for all sufficiently small $\rho > 0$. Here we note that

$$\int_{\partial B_1(0)} z^i z^n\, d\sigma_z = \delta^{in}\frac{\omega_d}{d}, \qquad i, n = 1, \cdots, d. \tag{2.9}$$

By (2.7) and (2.8), we have (2.4).

(2) It follows from (2.6) that

$$\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{i,j,k=1}^{d}\left(\frac{\partial G_n^i(x,y)}{\partial x^k}\nu_x^j-\frac{\partial G_n^i(x,y)}{\partial x^j}\nu_x^k\right)h_{ijk}(x)\,d\sigma_x := \lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{i,j,k=1}^{d}F_n^{ijk}(x,y)h_{ijk}(x)\,d\sigma_x$$
$$= \lim_{\rho\to 0}(J_1(\rho)+J_2(\rho)),$$

where

$$J_1(\rho) := \int_{\partial B_\rho(y)}\sum_{i,j,k=1}^{d}F_n^{ijk}(x,y)(h_{ijk}(x)-h_{ijk}(y))\,d\sigma_x,$$
$$J_2(\rho) := \int_{\partial B_\rho(y)}\sum_{i,j,k=1}^{d}F_n^{ijk}(x,y)h_{ijk}(y)\,d\sigma_x$$

with

$$F_n^{ijk}(x,y) := \left(\frac{\partial u_n^i(x,y)}{\partial x^k}\nu_x^j-\frac{\partial u_n^i(x,y)}{\partial x^j}\nu_x^k\right), \qquad i,j,k,n=1,\cdots,d.$$

Moreover, since the unit outer normal vector $\nu_x=(\nu_x^1,\cdots,\nu_x^d)$ to $\partial B_\rho(y)$ can be represented explicitly by $\nu_x^j=(x^j-y^j)|x-y|^{-1}$, $1\le j\le d$, we have that

$$\begin{aligned}F_n^{ijk}(x,y) = -\frac{1}{2\omega_d}\Bigg\{&\left(\delta_n^k\frac{(x^i-y^i)(x^j-y^j)}{|x-y|^{d+1}}+\delta^{ik}\frac{(x^j-y^j)(x^n-y^n)}{|x-y|^{d+1}}\right)\\ &-\left(\delta_n^j\frac{(x^i-y^i)(x^k-y^k)}{|x-y|^{d+1}}+\delta^{ij}\frac{(x^k-y^k)(x^n-y^n)}{|x-y|^{d+1}}\right)\Bigg\}.\end{aligned} \tag{2.10}$$

Since $|F_n^{ijk}(x,y)|\le C|x-y|^{1-d}$ for all $i,j,k,n=1,\cdots,d$ and all $x,y\in\Omega$ with $x\ne y$, in the same way as (2.7), we have

$$|J_1(\rho)|\le C\max_{\{i,j,k\}\in\{1,\cdots,d\}}\sup_{x\in\Omega}|(\nabla h_{ijk})(x)|\rho\to 0 \qquad \text{as } \rho\to 0, \tag{2.11}$$

where $C=C(y,\Omega)$ is a constant independent of ρ. Concerning the estimate of $J_2(\rho)$, we change the variables in (2.10) by $x\to z:=\frac{x-y}{\rho}$, and it follows from (2.9) that

$$\begin{aligned}J_2(\rho) &= -\frac{1}{2\omega_d}\sum_{i,j,k=1}^{d}h_{ijk}(y)\int_{\partial B_1(0)}(\delta^{kn}z^iz^j+\delta^{ik}z^jz^n-\delta^{jn}z^iz^k-\delta^{ij}z^kz^n)\,d\sigma_z\\ &= -\frac{1}{2\omega_d}\Bigg(\sum_{i,j=1}^{d}h_{ijn}(y)\int_{\partial B_1(0)}z^iz^j\,d\sigma_z+\sum_{i,j=1}^{d}h_{iji}(y)\int_{\partial B_1(0)}z^jz^n\,d\sigma_z\\ &\qquad -\sum_{i,k=1}^{d}h_{ink}(y)\int_{\partial B_1(0)}z^iz^k\,d\sigma_z-\sum_{i,k=1}^{d}h_{iik}(y)\int_{\partial B_1(0)}z^kz^n\,d\sigma_z\Bigg)\\ &= -\frac{1}{2d}\left(\sum_{i,j=1}^{d}h_{ijn}(y)\delta^{ij}+\sum_{i,j=1}^{d}h_{iji}(y)\delta^{jn}-\sum_{i,k=1}^{d}h_{ink}(y)\delta^{ik}-\sum_{i,k=1}^{d}h_{iik}(y)\delta^{kn}\right)\\ &= 0.\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we have the desired identity (2.5). □

The following lemma will be used to calculate the volume integral in Ω associated with the Green matrix.

Lemma 2.2. *Let $\{G^i_n\}_{i,n=1,\cdots,d}$ be the Green matrix for the Stokes equations defined by (2.1). Then we obtain the following identity.*

$$\int_\Omega \sum_{i,j=1}^d \frac{\partial v^i(x,z)}{\partial x^j}\frac{\partial G^i_n(x,y)}{\partial x^j}\,dx = v^n(y,z), \qquad n=1,\cdots,d \tag{2.13}$$

for all $y,z\in\Omega$ with $y\neq z$ and all smooth vector functions $\boldsymbol{v}(\cdot,z)=(v^1(\cdot,z),\cdots,v^d(\cdot,z))\in C^2(\Omega\setminus\{z\})$ with $\operatorname{div}\boldsymbol{v}(x,z)=0$ in $\Omega\setminus\{z\}$ and $\boldsymbol{v}(x,z)=0$ for $x\in\partial\Omega$ satisfying $|\boldsymbol{v}(x,z)|\le C|x-z|^{2-d}$, where $C=C(\Omega,z)$ is a constant which may depend on $z\in\Omega$, but not on $x\in\Omega\setminus\{z\}$.

Proof. Since $\boldsymbol{v}(\cdot,z)=\{v^i(\cdot,z)\}_{i=1,\cdots,d}$ is a solenoidal vector field in $\Omega\setminus\{z\}$, it holds that

$$\sum_{i,j=1}^d -\delta^{ij}\frac{\partial v^i(x,z)}{\partial x^j}R_n(x,y)=0 \quad \text{for} \quad x\in\Omega\setminus\{y\}\cup\{z\}, \quad n=1,\cdots,d, \tag{2.14}$$

where $\{R_n(\cdot,y)\}_{n=1,\cdots,d}$ is the Green function defined by (2.1). Adding (2.14) to the left hand side of (2.13), we have by (0.2) and (2.4) that

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^d \frac{\partial v^i(x,z)}{\partial x^j}\frac{\partial G^i_n(x,y)}{\partial x^j}\,dx \\
&= \lim_{\rho\to0}\int_{\Omega\setminus\{B_\rho(y)\cup B_\rho(z)\}}\sum_{i,j=1}^d T^{ij}(\boldsymbol{G}_n,R_n)(x,y)\frac{\partial v^i(x,z)}{\partial x^j}\,dx \\
&= \lim_{\rho\to0}\int_{\partial\{\Omega\setminus\{B_\rho(y)\cup B_\rho(z)\}\}}\sum_{i,j=1}^d T^{ij}(\boldsymbol{G}_n,R_n)(x,y)v^i(x,z)\nu^j_x\,d\sigma_x \\
&\qquad - \lim_{\rho\to0}\int_{\Omega\setminus\{B_\rho(y)\cup B_\rho(z)\}}\sum_{i,j=1}^d \frac{\partial}{\partial x^j}T^{ij}(\boldsymbol{G}_n,R_n)(x,y)v^i(x,z)\,dx \\
&= v^n(y,z), \qquad n=1,\cdots,d,
\end{aligned} \tag{2.15}$$

which yields the desired identity (2.13). □

2.2 Reduction of the problem via volume preserving diffeomorphism.

In this subsection, we reduce our problem in Ω_ε to that in Ω by means of the diffeomorphism $\Phi_\varepsilon : x\in\overline{\Omega}\to\tilde{x}=\Phi_\varepsilon(x)\in\overline{\Omega}_\varepsilon$ in the Assumption. More precisely, we regard $(\overline{\Omega}_\varepsilon,\delta)$ as a Riemannian manifold $(\overline{\Omega},a_\varepsilon)$ with a one parameter family $\{a_\varepsilon\}_{\varepsilon\ge0}$ of metrics, where $\delta=(\delta_{ij})_{i,j=1,\cdots,d}$ is the standard Euclidean metric in $\mathbb{R}^d$, and where $a_\varepsilon=\{a_{\varepsilon,ij}\}_{1\le i,j\le d}$ has an expression

$$a_{\varepsilon,ij} := \sum_{k=1}^d \frac{\partial\tilde{x}^k}{\partial x^i}\frac{\partial\tilde{x}^k}{\partial x^j}, \qquad i,j=1,\cdots,d. \tag{2.16}$$

Then the Green matrix $\{G^i_{\varepsilon,n}\}_{i,n=1,\cdots,d}$ should be treated as d-vector field on $(\Omega_\varepsilon, \delta)$ which is transformed to $\boldsymbol{g}_{\varepsilon,n} = (\Phi_\varepsilon^{-1})_* \boldsymbol{G}_{\varepsilon,n}$, $n = 1, \cdots, d$ on $(\overline{\Omega}, a_\varepsilon)$ with the expression

$$g^i_{\varepsilon,n}(x,y) = \sum_{j=1}^d \frac{\partial x^i}{\partial \tilde{x}^j} G^j_{\varepsilon.n}(\tilde{x}, \tilde{y}), \qquad i,n = 1, \cdots, d. \tag{2.17}$$

Such an argument was first established by Inoue-Wakimoto [9]. Moreover, we introduce the Green function $\{r_{\varepsilon,n}(\cdot, y)\}_{n=1,\cdots,d}$ for the pressure which is transformed on $(\overline{\Omega}, a_\varepsilon)$ with the expression

$$r_{\varepsilon,n}(x,y) = R_{\varepsilon,n}(\tilde{x}, \tilde{y}), \qquad n = 1, \cdots, d. \tag{2.18}$$

Since the Green matrix $\{G^i_{\varepsilon,n}(\tilde{x}, \tilde{y})\}_{i,n=1,\cdots,d}$ is a solution of the Stokes equations (0.1) on Ω_ε, the Green matrix $\{G^i_{\varepsilon,n}(\tilde{x}, \tilde{y})\}_{i,n=1,\cdots,d}$ attains the minimum of the Dirichlet integral in $\Omega_\varepsilon \setminus K_{\rho,\varepsilon}(\tilde{y})$ with $K_{\rho,\varepsilon}(\tilde{y}) := \Phi_\varepsilon(B_\rho(y))$. More precisely, the Green matrix $\{G^i_{\varepsilon,n}(\tilde{x}, \tilde{y})\}_{i,n=1,\cdots,d}$ satisfies

$$\int_{\Omega_\varepsilon \setminus K_{\rho,\varepsilon}(\tilde{y})} \sum_{i,j=1}^d \left(\frac{\partial G^i_{\varepsilon,n}}{\partial \tilde{x}^j}(\tilde{x}, \tilde{y}) \right)^2 d\tilde{x} = \min_{H^1_\sigma(\Omega_\varepsilon \setminus K_{\rho,\varepsilon}(\tilde{y}))} E(f), \qquad n = 1, \cdots, d, \tag{2.19}$$

where $E(f) := \int_{\Omega_\varepsilon \setminus K_{\rho,\varepsilon}(\tilde{y})} \sum_{i,j=1}^d \left(\frac{\partial f^i}{\partial \tilde{x}^j} \right)^2 d\tilde{x}$ and $H^1_\sigma(\Omega) := \{\boldsymbol{v} \in H^1(\Omega)^d \ ; \ \operatorname{div} \boldsymbol{v} = 0\}$. Therefore the vector functions $\{g^i_{\varepsilon,n}\}_{i,n=1,\cdots,d}$ on Ω defined by (2.17) satisfy

$$\int_{\Omega \setminus B_\rho(y)} \sum_{i,j,k,l=1}^d \left\{ \frac{\partial}{\partial x^k} \left(\frac{\partial \tilde{x}^i}{\partial x^l} g^l_{\varepsilon,n}(x,y) \right) \frac{\partial x^k}{\partial \tilde{x}^j} \right\}^2 dx = \min_{H^1_\sigma(\Omega \setminus B_\rho(y))} E(f), \quad n = 1, \cdots, d. \tag{2.20}$$

By a standard procedure, we obtain the following transformed differential equation for the vector functions $\{g^i_{\varepsilon,n}\}_{i,n=1,\cdots,d}$ and the Green function $\{r_{\varepsilon,n}\}_{n=1,\cdots,d}$.

$$\mathcal{L}_\varepsilon(\boldsymbol{g}_{\varepsilon,n}, r_{\varepsilon,n})(x,y) = 0, \qquad x \in \Omega \setminus B_\rho(y), \tag{2.21}$$

where $\mathcal{L}_\varepsilon(\boldsymbol{v}, \pi) = (\mathcal{L}^1_\varepsilon(\boldsymbol{v}, \pi), \cdots, \mathcal{L}^d_\varepsilon(\boldsymbol{v}, \pi))$ has an expression as

$$\mathcal{L}^r_\varepsilon(\boldsymbol{v}, \pi)(x) := \sum_{i,k,l,p,s=1}^d \frac{\partial}{\partial x^k} \left(a^{ks}_\varepsilon \frac{\partial}{\partial x^s} \left(\frac{\partial \tilde{x}^i}{\partial x^l} v^l(x) \right) \right) \frac{\partial \tilde{x}^i}{\partial x^p} a^{pr}_\varepsilon - \sum_{i=1}^d a^{ri}_\varepsilon \frac{\partial \pi}{\partial x^i}(x), \quad r = 1, \cdots, d \tag{2.22}$$

with the variable coefficient

$$a^{ks}_\varepsilon := \sum_{j=1}^d \frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^j}, \quad k,s = 1, \cdots, d. \tag{2.23}$$

The differential equation (2.21) is the Euler-Lagrange equation with respect to the variational problem of the Dirichlet integral (2.20) and $\mathcal{L}_\varepsilon$ is the Stokes operator on the the Riemannian manifold $(\overline{\Omega}, a_\varepsilon)$. It should be noted that $(a^{ks}_\varepsilon) = ((a_{\varepsilon,ks})^{-1})$ and that, implied by (A.4),

$\det(a_\varepsilon^{ks}) = \det(a_{\varepsilon,ks}) = 1$, where $\{a_{\varepsilon,ks}\}_{k,s=1,\cdots,d}$ is the Riemannian metric defined by (2.16). Hence the divergence operator is invariant under the diffeomorphism Φ_ε in the sense that

$$\sum_{j=1}^{d} \frac{\partial u^j}{\partial \tilde{x}^j}(\tilde{x}) = 0 \quad \text{in} \quad \Omega_\varepsilon,$$

which is equivalent to

$$\sum_{i=1}^{d} \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{d} \frac{\partial x^i}{\partial \tilde{x}^j} u^j(\tilde{x}) \right) = 0 \quad \text{in} \quad \Omega.$$

In the next step, let us recall the following Green integral formula for the Stokes operator $\mathcal{L}_\varepsilon$ defined by (2.22);

$$\begin{aligned} &\int_\Omega \sum_{q,r=1}^{d} a_{\varepsilon,qr} \left\{ \mathcal{L}_\varepsilon^r(\boldsymbol{v},\pi)(x) w^q(x) - \mathcal{L}_\varepsilon^r(\boldsymbol{w},\tilde{\pi})(x) v^q(x) \right\} dx \\ &= \int_{\partial\Omega} \sum_{k,q=1}^{d} \left\{ T_\varepsilon^{kq}(\boldsymbol{v},\pi)(x) w^q(x) - T_\varepsilon^{kq}(\boldsymbol{w},\tilde{\pi})(x) v^q(x) \right\} \nu_x^k \, d\sigma_x, \end{aligned} \tag{2.24}$$

where $\{a_{\varepsilon,kq}\}_{1\le k,q\le d}$ is the same as in (2.16) and where $\{T_\varepsilon^{kq}\}_{k,q=1,\cdots,d}$ is defined by

$$T_\varepsilon^{kq}(\boldsymbol{v},\pi)(x) := -\delta^{kq}\pi(x) + \sum_{i,l,s=1}^{d} a_\varepsilon^{ks} \frac{\partial}{\partial x_s} \left(\frac{\partial \tilde{x}^i}{\partial x^l} v^l(x) \right) \frac{\partial \tilde{x}^i}{\partial x^q}, \quad k,q =, \cdots, d \tag{2.25}$$

for vector functions $\boldsymbol{v}, \boldsymbol{w} \in C^2(\overline{\Omega})^d$ with $\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{w} = 0$ in Ω and scalar functions $\pi, \tilde{\pi} \in C^1(\overline{\Omega})$. We need to investigate the behavior as $\varepsilon \to 0$ of the integral identity (2.24). For that purpose, based on the Assumption, we expand various functions on $(\overline{\Omega}, a_\varepsilon)$ with respect for ε.

Proposition 2.1. *Let Φ_ε be as in the Assumption. Suppose that $\{a_{\varepsilon,ij}\}_{i,j=1,\cdots,d}$ and $\{a_\varepsilon^{ks}\}_{k,s=1,\cdots,d}$ are the same as in (2.16) and (2.23), respectively. Then it holds that*

$$\begin{aligned} &\frac{\partial \tilde{x}^i}{\partial x^j} = \delta^i_j + \varepsilon \frac{\partial S^i}{\partial x^j} + O(\varepsilon^2), \quad \frac{\partial x^i}{\partial \tilde{x}^j} = \delta^i_j - \varepsilon \frac{\partial S^j}{\partial x^i} + O(\varepsilon^2), \\ &a_{\varepsilon,ij} = \delta_{ij} + \varepsilon \left(\frac{\partial S^i}{\partial x^j} + \frac{\partial S^j}{\partial x^i} \right) + O(\varepsilon^2), \quad a_\varepsilon^{ij} = \delta^{ij} - \varepsilon \left(\frac{\partial S^i}{\partial x^j} + \frac{\partial S^j}{\partial x^i} \right) + O(\varepsilon^2), \\ &\frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} = \varepsilon \frac{\partial^2 S^i}{\partial x^j \partial x^k} + O(\varepsilon^2), \\ &\frac{1}{|\tilde{x}-\tilde{y}|^N} = \frac{1}{|x-y|^N} - \varepsilon N \frac{(S(x)-S(y))\cdot(x-y)}{|x-y|^{N+2}} + O(\varepsilon^2) \end{aligned}$$

for $i,j,k = 1,\cdots,d$ and for $N \in \mathbb{N}$ as $\varepsilon \to 0$, where $\tilde{x} = \Phi_\varepsilon(x)$, $\tilde{y} = \Phi_\varepsilon(y)$ for $x,y \in \Omega$.

The proof is an immediate consequence of (A.3). So, we may omit the detail.

The following lemma may be regarded as a generalization of the Gauß formula with respect to the modified stress tensor $\{T_\varepsilon^{kr}\}_{k,r=1,\cdots,d}$ on the surface integral.

Lemma 2.3. *Let $\{g^l_{\varepsilon,n}\}_{n,l=1,\cdots,d}$ be the matrix as in (2.17) and let $\{r_{\varepsilon,n}\}_{n=1,\cdots,d}$ be the Green function defined by (2.18). Then it holds that*

(2.26)
$$\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{k,q=1}^{d} T^{kq}_\varepsilon(\boldsymbol{g}_{\varepsilon,n}, r_{\varepsilon,n})(x,y)v^q(x)\nu^k_x\, d\sigma_x = \sum_{q=1}^{d}\frac{\partial\tilde{x}^n}{\partial x^q}(y)v^q(y) + O(\varepsilon^2), \quad n=1,\cdots,d$$

for all $y\in\Omega$ and all smooth vector functions $v=(v^1,\cdots,v^d)$ near y as $\varepsilon\to 0$. Here $\{T^{kq}_\varepsilon(\cdot,\cdot)\}_{k,q=1,\cdots,d}$ is the operator defined by (2.25).

Proof. It follows from (2.6) that

$$\begin{aligned}
(2.27)\qquad &\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{k,q=1}^{d} T^{kq}_\varepsilon(\boldsymbol{g}_{\varepsilon,n}, r_{\varepsilon,n})(x,y)v^q(x)\nu^k_x\, d\sigma_x\\
&= \lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{k,q=1}^{d} T^{kq}_\varepsilon\left(\sum_{j=1}^{d}\frac{\partial \boldsymbol{x}}{\partial x^j}G^j_{\varepsilon.n}(\tilde{x},\tilde{y}), R_{\varepsilon,n}(\tilde{x},\tilde{y})\right)v^q(x)\nu^k_x\, d\sigma_x\\
&= \lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{k,q=1}^{d} T^{kq}_\varepsilon\left(\sum_{j=1}^{d}\frac{\partial \boldsymbol{x}}{\partial x^j}u^j_n(\tilde{x},\tilde{y}), p_n(\tilde{x},\tilde{y})\right)v^q(x)\nu^k_x\, d\sigma_x\\
&= \lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{k,q=1}^{d}\Big\{-\delta^{kq}p_n(\tilde{x},\tilde{y}) + \sum_{i,s=1}^{d}a^{ks}_\varepsilon\frac{\partial}{\partial x_s}\left(u^i_n(\tilde{x},\tilde{y})\right)\frac{\partial\tilde{x}^i}{\partial x^q}\Big\}v^q(x)\nu^k_x\, d\sigma_x.
\end{aligned}$$

Here we note that

$$(2.28)\qquad \sum_{l=1}^{d}\frac{\partial\tilde{x}^i}{\partial x^l}\frac{\partial x^l}{\partial\tilde{x}^j} = \delta^i_j, \qquad i,j=1\cdots,d.$$

Therefore, by (2.2) it suffices to prove the following identity.

$$(2.29)\qquad \lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{q=1}^{d}F^q_n(x,y)v^q(x)\, d\sigma_x = \sum_{q=1}^{d}\frac{\partial\tilde{x}^n}{\partial x^q}(y)v^q(y) + O(\varepsilon^2), \qquad n=1,\cdots,d,$$

as $\varepsilon\to 0$, where

$$\begin{aligned}
&F^q_n(x,y)\\
&:= -\frac{1}{2\omega_d(d-2)}\sum_{k=1}^{d}\Big\{2(d-2)\delta^{kq}\frac{(\tilde{x}^n-\tilde{y}^n)}{|\tilde{x}-\tilde{y}|^d}\\
&\qquad + \sum_{i,s=1}^{d}a^{ks}_\varepsilon\frac{\partial}{\partial x^s}\left(\frac{\delta^{in}}{|\tilde{x}-\tilde{y}|^{d-2}} + (d-2)\frac{(\tilde{x}^i-\tilde{y}^i)(\tilde{x}^n-\tilde{y}^n)}{|\tilde{x}-\tilde{y}|^d}\right)\frac{\partial\tilde{x}^i}{\partial x^q}\Big\}\nu^k_x.
\end{aligned}$$

Since the unit outer normal vector $\nu_x = (\nu_x^1, \cdots, \nu_x^d)$ to $\partial B_\rho(y)$ can be represented explicitly by $\nu_x^k = (x^k - y^k)|x-y|^{-1}$, $1 \le k \le d$, by (2.23) we have a more precise expression of $F_n^q(x,y)$ as

$$F_n^q(x,y) := -\frac{1}{2\omega_d}\left(f_n^{(1)q}(x,y) + f_n^{(2)q}(x,y) + f_n^{(3)q}(x,y) + f_n^{(4)q}(x,y)\right), \tag{2.30}$$

where

$$\begin{aligned}
f_n^{(1)q}(x,y) &= \sum_{k=1}^d 2\delta^{kq}\frac{(\tilde{x}^n - \tilde{y}^n)}{|\tilde{x}-\tilde{y}|^d}\frac{(x^k - y^k)}{|x-y|},\\
f_n^{(2)q}(x,y) &= \sum_{i,k,l=1}^d -\frac{\partial x^k}{\partial \tilde{x}^l}\frac{\partial \tilde{x}^i}{\partial x^q}\delta^{in}\frac{(\tilde{x}^l - \tilde{y}^l)}{|\tilde{x}-\tilde{y}|^d}\frac{(x^k - y^k)}{|x-y|},\\
f_n^{(3)q}(x,y) &= \sum_{i,k,l=1}^d \frac{\partial x^k}{\partial \tilde{x}^l}\frac{\partial \tilde{x}^i}{\partial x^q}\left\{\frac{(\tilde{x}^i - \tilde{y}^i)}{|\tilde{x}-\tilde{y}|^d}\delta^{ln} + \frac{(\tilde{x}^n - \tilde{y}^n)}{|\tilde{x}-\tilde{y}|^d}\delta^{il}\right\}\frac{(x^k - y^k)}{|x-y|},\\
f_n^{(4)q}(x,y) &= \sum_{i,k,l=1}^d -d\frac{\partial x^k}{\partial \tilde{x}^l}\frac{\partial \tilde{x}^i}{\partial x^q}\frac{(\tilde{x}^i - \tilde{y}^i)(\tilde{x}^l - \tilde{y}^l)(\tilde{x}^n - \tilde{y}^n)}{|\tilde{x}-\tilde{y}|^{d+2}}\frac{(x^k - y^k)}{|x-y|}
\end{aligned}$$

for $n, q = 1, \cdots, d$. In the next step, we deal with $f_n^{(1)q}$, $f_n^{(2)q}$, $f_n^{(3)q}$ and $f_n^{(4)q}$. Since $\tilde{x} = \Phi_\varepsilon(x)$ and $\tilde{y} = \Phi_\varepsilon(y)$, we have by the Taylor expansion around $y \in \Omega$ that

$$\begin{aligned}
(\tilde{x}^i - \tilde{y}^i) &= \sum_{k=1}^d \frac{\partial \tilde{x}^i}{\partial x^k}(y)(x^k - y^k) + O(\rho^2), \quad i = 1, \cdots, d,\\
\frac{1}{|\tilde{x}-\tilde{y}|^N} &= \left(\sum_{i=1}^d (\tilde{x}^i - \tilde{y}^i)^2\right)^{-\frac{N}{2}} = \left(\sum_{i,j,k=1}^d \frac{\partial \tilde{x}^i}{\partial x^j}(y)\frac{\partial \tilde{x}^i}{\partial x^k}(y)(x^j - y^j)(x^k - y^k) + O(\rho^3)\right)^{-\frac{N}{2}}\\
&= \left(\sum_{j,k=1}^d a_{\varepsilon,jk}(y)(x^j - y^j)(x^k - y^k)\right)^{-\frac{N}{2}} + O(\rho^{2-N}),
\end{aligned}$$

as $\rho := |x-y| \to 0$, which yields that

$$\begin{aligned}
f_n^{(1)q}(x,y) &= -2\sum_{i=1}^d \frac{\partial \tilde{x}^n}{\partial x^i}(y)\frac{(x^i - y^i)(x^q - y^q)}{\left[\sum_{j,l=1}^d a_{\varepsilon,jl}(y)(x^j - y^j)(x^l - y^l)\right]^{\frac{d}{2}}}\frac{1}{|x-y|} + O(\rho^{3-d}),\\
f_n^{(2)q}(x,y) &= -\frac{\partial \tilde{x}^n}{\partial x^q}(y)\frac{|x-y|}{\left[\sum_{j,l=1}^d a_{\varepsilon,jl}(y)(x^j - y^j)(x^l - y^l)\right]^{\frac{d}{2}}} + O(\rho^{3-d}),\\
f_n^{(3)q}(x,y) &= \sum_{i,k=1}^d \frac{\partial x^k}{\partial \tilde{x}^n}(y)a_{\varepsilon,qi}(y)\frac{(x^i - y^i)(x^k - y^k)}{\left[\sum_{j,l=1}^d a_{\varepsilon,jl}(y)(x^j - y^j)(x^l - y^l)\right]^{\frac{d}{2}}}\frac{1}{|x-y|}\\
&\quad + \frac{\partial \tilde{x}^n}{\partial x^k}(y)\frac{(x^q - y^q)(x^k - y^k)}{\left[\sum_{j,l=1}^d a_{\varepsilon,jl}(y)(x^j - y^j)(x^l - y^l)\right]^{\frac{d}{2}}}\frac{1}{|x-y|} + O(\rho^{3-d}),
\end{aligned} \tag{2.31}$$

$$f_n^{(4)q}(x,y) = -d\sum_{i,k=1}^{d}\frac{\partial\tilde{x}^n}{\partial x^i}(y)a_{\varepsilon,kq}(y)\frac{(x^i-y^i)(x^k-y^k)|x-y|}{\left[\sum_{j,l=1}^{d}a_{\varepsilon,jl}(y)(x^j-y^j)(x^l-y^l)\right]^{\frac{d+2}{2}}} + O(\rho^{3-d})$$

for $n,q=1,\cdots,d$ as $\rho := |x-y| \to 0$. Changing variables $x \to \frac{x-y}{\rho}+y$, we have by (2.30) and (2.31) that

$$\begin{aligned}
&\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{q=1}^{d}F_n^q(x,y)v^q(x)\,d\sigma_x \\
&= \frac{1}{2\omega_d}\sum_{i,q=1}^{d}\frac{\partial\tilde{x}^n}{\partial x^i}(y)v^q(y)X^{iq} + \frac{1}{2\omega_d}\sum_{q=1}^{d}\frac{\partial\tilde{x}^n}{\partial x^q}(y)v^q(y)Y \\
&\quad - \frac{1}{2\omega_d}\sum_{i,s,q=1}^{d}\frac{\partial x^s}{\partial\tilde{x}^n}(y)a_{\varepsilon,qi}(y)v^q(y)X^{is} + \frac{d}{2\omega_d}\sum_{i,k,q=1}^{d}\frac{\partial\tilde{x}^n}{\partial x^i}(y)a_{\varepsilon,kq}(y)v^q(y)Z^{ik}
\end{aligned} \tag{2.32}$$

for $n=1,\cdots,d$, where

$$\begin{aligned}
X^{iq} &:= \int_{\partial B_1(y)}\frac{(x^i-y^i)(x^q-y^q)}{\left[\sum_{j,l=1}^{d}a_{\varepsilon,jl}(y)(x^j-y^j)(x^l-y^l)\right]^{\frac{d}{2}}}\,d\sigma_x, \\
Y &:= \int_{\partial B_1(y)}\frac{1}{\left[\sum_{j,l=1}^{d}a_{\varepsilon,jl}(y)(x^j-y^j)(x^l-y^l)\right]^{\frac{d}{2}}}\,d\sigma_x, \\
Z^{iq} &:= \int_{\partial B_1(y)}\frac{(x^i-y^i)(x^q-y^q)}{\left[\sum_{j,l=1}^{d}a_{\varepsilon,jl}(y)(x^j-y^j)(x^l-y^l)\right]^{\frac{d+2}{2}}}\,d\sigma_x
\end{aligned} \tag{2.33}$$

for $i,q=1,\cdots,d$. By (1.1) and Proposition 2.1, we expand each term of (2.33) with respect to ε, and obtain that

$$\begin{aligned}
Y &= \int_{B_1(y)}d\sigma_x - \varepsilon d\sum_{i=1}^{d}\frac{\partial S^i}{\partial x^i}(y)\int_{B_1(y)}(x^i-y^i)^2d\sigma_x + O(\varepsilon^2) \\
&= \omega_d - \varepsilon\omega_d \operatorname{div} S(y) + O(\varepsilon^2) \\
&= \omega_d + O(\varepsilon^2), \qquad \text{as } \varepsilon\to 0.
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
X^{iq} &= \int_{\partial B_1(y)}(x^i-y^i)(x^q-y^q)d\sigma_x \\
&\quad - d\varepsilon\sum_{j,l=1}^{d}\frac{\partial S^j}{\partial x^l}(y)\int_{\partial B_1(y)}(x^j-y^j)(x^l-y^l)(x^i-y^i)(x^q-y^q)d\sigma_x + O(\varepsilon^2) \\
&= \begin{cases}\dfrac{-2\sqrt{\pi}^d}{(d+2)\Gamma(\frac{d}{2})}\left(\dfrac{\partial S^i}{\partial x^q}+\dfrac{\partial S^q}{\partial x^i}\right)\varepsilon + O(\varepsilon^2), & (1\le i\ne q\le d), \\ \dfrac{2\sqrt{\pi}^d}{\Gamma(\frac{d}{2})}\left(\dfrac{1}{d} - \dfrac{2\varepsilon}{(d+2)}\dfrac{\partial S^i}{\partial x^i}(y)\right) + O(\varepsilon^2), & (1\le i = q\le d),\end{cases} \qquad \text{as } \varepsilon\to 0.
\end{aligned} \tag{2.35}$$

$$Z^{iq} = \int\limits_{\partial B_1(y)} (x^i - y^i)(x^q - y^q) d\sigma_x$$

(2.36)

$$- (d+2)\varepsilon \sum_{j,l=1}^{d} \frac{\partial S^j}{\partial x^l}(y) \int\limits_{\partial B_1(y)} (x^j - y^j)(x^l - y^l)(x^i - y^i)(x^q - y^q) d\sigma_x + O(\varepsilon^2)$$

$$= \begin{cases} \dfrac{-2\sqrt{\pi}^d}{d\Gamma(\frac{d}{2})} \left(\dfrac{\partial S^i}{\partial x^q} + \dfrac{\partial S^q}{\partial x^i} \right) \varepsilon + O(\varepsilon^2), & (1 \le i \ne q \le d), \\ \dfrac{2\sqrt{\pi}^d}{d\Gamma(\frac{d}{2})} \left(1 - 2\varepsilon \dfrac{\partial S^i}{\partial x^i} \right) + O(\varepsilon^2), & (1 \le i = q \le d), \quad \text{as } \varepsilon \to 0. \end{cases}$$

where Γ is the gamma function. Hence it follows from (2.32)–(2.36) and Proposition 2.1 that

$$\lim_{\rho \to 0} \int\limits_{\partial B_\rho(y)} \sum_{q=1}^{d} F_n^q(x,y) v^q(x)\, d\sigma_x = v^n(y) + \varepsilon \sum_{q=1}^{d} \frac{\partial S^n}{\partial x^q}(y) v^q(y) + O(\varepsilon^2)$$

$$= \sum_{q=1}^{d} \frac{\partial \tilde{x}^n}{\partial x^q}(y) v^q(y) + O(\varepsilon^2), \qquad n = 1, \cdots, d,$$

as $\varepsilon \to 0$, which implies (2.29). Thus we complete the proof of Lemma 2.3. □

3 Construction of the parametrix.

Since the Green matrix $\{G_n^i(x,y)\}_{i,n=1,\cdots,d}$ has the singularity at $x = y$, which is governed by the fundamental solution $\{u_n^i(x,y)\}_{i,n=1,\cdots,d}$ in (2.2), we need to construct a parametrix $\{U_n^i(x,y)\}_{i,n=1,\cdots,d}$ as an approximation of $\{G_n^i(x,y)\}_{i,n=1,\cdots,d}$. For that purpose, we prepare several fundamental lemmas.

3.1 Bogovskiĭ formula.

The following proposition gives a solution of the inhomogeneous divergence equation with the Dirichlet boundary condition. In particular, when the domain is star-like with respect to some ball, we have an explicit representation in terms of the integral kernel.

Proposition 3.1. *Let $D \subset \mathbb{R}^d$, $d \ge 2$, be a star-like bounded domain with respect to the ball $B = B_R(x_0) \subset D$. Consider the inhomogeneous divergence equation with the Dirichlet condition on the Lipschitz boundary ∂D of D;*

(3.1)
$$\begin{cases} \operatorname{div} \boldsymbol{v}(x) = f(x) & \text{in } D, \\ \boldsymbol{v}(x) = 0 & \text{on } \partial D. \end{cases}$$

For any $f \in L^q(D)$ with $1 < q < \infty$, there exists at least one solution $\boldsymbol{v} \in W_0^{1,q}(D)^d$ of (3.1). Suppose that $\omega \in C_0^\infty(D)$ has such properties as supp$(\omega) \subset B$ and $\int_B \omega(x)\, dx = 1$. Then the solution $\boldsymbol{v}$ of (3.1) can be represented as

(3.2)
$$\boldsymbol{v}(x) = \int_\Omega \boldsymbol{N}(x,z) f(z)\, dz,$$

where

$$\boldsymbol{N}(x,z) := \frac{x-z}{|x-z|^d}\int_{|x-z|}^{\infty}\omega\left(z+\xi\frac{x-z}{|x-z|}\right)\xi^{d-1}\,d\xi. \tag{3.3}$$

For the proof, see Bogovskiĭ [3]. The proposition 3.1 is generalized to the divergence equation in an arbitrary bounded domain in $\mathbb{R}^d$.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^d$, $d \ge 2$, be a general bounded Lipschitz domain, and let $1 < q < \infty$, $k \in \mathbb{N}$. Then for every $g \in W_0^{k,q}(\Omega)$ with $\int_\Omega g(x)\,dx = 0$, there exists at least one $\boldsymbol{v} \in W_0^{k+1,q}(\Omega)^d$ satisfying*

$$\operatorname{div}\boldsymbol{v} = g$$

with the estimate

$$\|\boldsymbol{v}\|_{W_0^{k+1,q}(\Omega)} \le C\|g\|_{W_0^{k,q}(\Omega)},$$

where $C = C(k,q,\Omega) > 0$ is a constant independent of g.

For the proof, see Borchers-Sohr [4] for instance.

In the next step, we introduce the key lemma to construct the parametrix. Proposition 3.1 assures the existence of the auxiliary function $\boldsymbol{Q}_{\varepsilon,n}(\cdot,y) = \{Q^l_{\varepsilon,n}(\cdot,y)\}_{n,l=1,\cdots,d}$ for any $y \in \Omega$ which preserves the behavior of the singularity of $\{\boldsymbol{u}_{\varepsilon,n}(x,y)\}_{n=1,\cdots,d}$ at $x = y$.

Lemma 3.1. *For each fixed $y \in \Omega$, we define a cut-off function $\alpha(\cdot,y) \in C_0^\infty(\Omega)$ satisfying*

$$\alpha(x,y) = \begin{cases} 1 & \text{for all } x \in B_{\frac{d_y}{4}}(y), \\ 0 & \text{for all } x \in \Omega \setminus B_{\frac{d_y}{2}}(y), \end{cases} \tag{3.4}$$

where $d_y := dist(y,\partial\Omega)$. Suppose that $\boldsymbol{u}_n(\cdot,\tilde{y}) = \{u^i_n(\cdot,\tilde{y})\}_{i,n=1,\cdots,d}$ is the fundamental tensor of the Stokes equations in $\mathbb{R}^d$, i.e. ,

$$u^i_n(\tilde{x},\tilde{y}) = -\frac{1}{2\omega_d(d-2)}\left\{\frac{\delta^{in}}{|\tilde{x}-\tilde{y}|^{d-2}} + (d-2)\frac{(\tilde{x}^i-\tilde{y}^i)(\tilde{x}^n-\tilde{y}^n)}{|\tilde{x}-\tilde{y}|^d}\right\}, \qquad i,n = 1,\cdots,d.$$

For every $0 \le \varepsilon < 1$ we define $\boldsymbol{u}_{\varepsilon,n}(\cdot,y) = \{u^i_{\varepsilon,n}(\cdot,y)\}_{i,n=1,\cdots,d}$ by

$$u^i_{\varepsilon,n}(x,y) := \sum_{l=1}^{d}\frac{\partial x^i}{\partial \tilde{x}^l}u^l_n(\tilde{x},\tilde{y}), \quad i,n = 1,\cdots,d. \tag{3.5}$$

Then there exists $\boldsymbol{Q}_{\varepsilon,n}(x,y) = \{Q^l_{\varepsilon,n}(x,y)\}_{n,l=1,\cdots,d} \in W^{1,r}(\Omega)^d$ for all $r \in (1,\frac{d}{d-2})$ such that

$$\begin{cases} \operatorname{div}\boldsymbol{Q}_{\varepsilon,n}(x,y) = \displaystyle\sum_{i,j,k=1}^{d} a_{\varepsilon,ij}\left(a^{ik}_\varepsilon\frac{\partial\alpha}{\partial x^k}(x,y)u^j_{\varepsilon,n}(x,y)\right) & \text{in } \Omega, \\ \boldsymbol{Q}_{\varepsilon,n}(x,y) = 0 & \text{on } \partial\Omega, \qquad n = 1,\cdots,d, \end{cases} \tag{3.6}$$

where $\{a_{\varepsilon,ij}\}_{i,j=1,\cdots,d}$ *and* $\{a_\varepsilon^{ik}\}_{i,k=1,\cdots,d}$ *are the same as in (2.16) and (2.23), respectively. Moreover, such a* $\boldsymbol{Q}_{\varepsilon,n}$ *satisfies the following additional properties;*

$$|\boldsymbol{Q}_{\varepsilon,n}(x,y)| \le \frac{C_y}{|x-y|^{d-3}}, \tag{3.7}$$

$$|(\boldsymbol{Q}_{\varepsilon,n} - \boldsymbol{Q}_{0,n})(x,y)| \le \frac{\varepsilon C_y}{|x-y|^{d-3}}, \tag{3.8}$$

$$|(\nabla_x Q^i_{\varepsilon,n})(x,y)| \le \frac{C_y}{|x-y|^{d-2}}, \tag{3.9}$$

$$|\nabla_x (Q^i_{\varepsilon,n} - Q^i_{0,n})(x,y)| \le \frac{\varepsilon C_y}{|x-y|^{d-2}}, \tag{3.10}$$

$$|\nabla_x^2 (Q^i_{\varepsilon,n})(x,y)| \le \frac{C_y}{|x-y|^{d-1}}, \tag{3.11}$$

$$|\nabla_x^2 (Q^i_{\varepsilon,n} - Q^i_{0,n})(x,y)| \le \frac{\varepsilon C_y}{|x-y|^{d-1}}, \tag{3.12}$$

for all $x \in \Omega$ *and all* $0 \le \varepsilon < 1$, $i, n = 1, \cdots, d$, *where* C_y *is a constant which may depend on* y, *but not on* x, ε.

Remark 3.1. We regard Ω as the Riemannian manifold with the metric $a_\varepsilon = \{a_{\varepsilon.ij}\}_{i,j=1,\cdots,d}$. Hence the operator div in (3.6) should be understood as the divergence operator on $(\overline{\Omega}, a_\varepsilon)$. However, on account of (A.4), we may treat it as the usual divergence operator in the standard Euclidean space $(\overline{\Omega}, \delta)$.

Proof. First of all, we prove the existence of the solution of (3.6). Since

$$\sum_{i=1}^d a_{\varepsilon,ij} a_\varepsilon^{ik} = \delta_j^k, \qquad k, j = 1, \cdots, d,$$

by integration by parts, it holds that

$$\begin{aligned} &\int_\Omega \sum_{i,j,k=1}^d a_{\varepsilon,ij} (a_\varepsilon^{ik} \frac{\partial \alpha}{\partial x^k} u^j_{\varepsilon,n})(x,y)\, dx = \int_\Omega \sum_{j=1}^d (\frac{\partial \alpha}{\partial x^j} u^j_{\varepsilon,n})(x,y)\, dx \\ = &\lim_{\rho\to 0} \Big\{ -\int_{\Omega\setminus B_\rho(y)} \alpha(x,y) \mathrm{div}\, \boldsymbol{u}_{\varepsilon,n}(x,y)\, dx + \int_{\partial\{\Omega\setminus B_\rho(y)\}} \sum_{j=1}^d \alpha(x,y) u^j_{\varepsilon,n}(x,y) \nu_x^j \, d\sigma_x \Big\} \\ = &\ 0. \end{aligned}$$

Here, we have used

$$|\boldsymbol{u}_{\varepsilon,n}(x,y)| = O(|x-y|^{2-d}), \quad n = 1, \cdots, d, \quad \text{as } x \to y.$$

Since $\boldsymbol{u}_{\varepsilon,n}(\cdot, y) \in L^r(\Omega)^d$ for all $r \in (1, \frac{d}{d-2})$, it follows from Proposition 3.2 that there exists a solution $\boldsymbol{Q}_{\varepsilon,n}(\cdot, y) \in W_0^{1,r}(\Omega)^d$ of (3.6) for all $r \in (1, \frac{d}{d-2})$.

In the next step, we prove estimates from (3.7) to (3.12). We choose a special solution $\boldsymbol{Q}_{\varepsilon,n}(\cdot, y) \in W_0^{1,r}(\Omega)^d$ of (3.6) which is represented by the kernel function as in (3.2) with the analogue of Borchers-Sohr [4]. For that purpose, let us take bounded domains U_i and functions $\psi_i \in C_0^\infty(U_i), i = 1, \cdots, N$ so that $\Omega \subset \cup_{i=1}^N \Omega_i$, $\Omega_i := \Omega \cap U_i$ are star like, and so that $\sum_{i=1}^N \psi_i(x) = 1$ for all $x \in \Omega$. Then for every $f \in L^1(\Omega)$ with $\int_\Omega f(x)dx = 0$, we can decompose f as $f = \Gamma_1(f) + \Gamma_2(f) + \cdots + \Gamma_N(f)$ in such a way that

$$\int_{\Omega_i} (\Gamma_i(f))(x)dx = 0, \qquad \mathit{supp}\, \Gamma_i(f) \subset \Omega_i, \qquad i = 1, \cdots, N.$$

Although N may be a large natural number, we can reduce such a decompositions of Ω and f to that in the case $N = 2$. This is due to an inductive argument as in Borchers-Sohr [4, Theorem 2.4]. See also Sohr [14, II. 2.3]. Hence it suffices to deal with the following simpler case like

$$\begin{cases} \Gamma_1(f) := \psi_1 f - (\int_{\Omega_1} \psi_1 f dx)\eta, \\ \Gamma_2(f) := f - \Gamma_1(f), \end{cases} \tag{3.13}$$

where supp $\psi_i \subset \Omega_i$, $i = 1, 2$, $\psi_1(x) + \psi_2(x) \equiv 1$ for $x \in \Omega$ and $\eta \in C_0^\infty(\Omega_1 \cap \Omega_2)$ with $\int_{\Omega_1 \cap \Omega_2} \eta(x)dx = 1$. We define $\{\boldsymbol{Q}_{\varepsilon,n}\}_{n=1,\cdots,d}$ by

$$\boldsymbol{Q}_{\varepsilon,n}(x, y) = \sum_{i=1}^{2} \int_{\Omega_i} \Gamma_i \left(\sum_{l,j,k=1}^{d} a_{\varepsilon,lj} \left(a_\varepsilon^{lk} \frac{\partial \alpha}{\partial x^k} u_{\varepsilon,n}^j \right) \right) (y, z) \boldsymbol{N}_i(x, z)\, dz \tag{3.14}$$

for $n = 1, \cdots, d$, where $\boldsymbol{N}_i(x, z)$ is defined as in (3.3) with D replaced by Ω_i for $i = 1, 2$. By Propositions 3.1 and 3.2, we see that this $\{\boldsymbol{Q}_{\varepsilon,n}\}_{n=1,\cdots,d}$ solves (3.6).

On the other hand, it follows from (3.3) that

$$|\boldsymbol{N}_i(x, z)| \le \frac{C}{|x - z|^{d-1}} \quad \text{for all } x, z \in \Omega \text{ with } x \neq z \text{ and all } i = 1, 2, \tag{3.15}$$

where $C = C(\Omega)$ is a constant independent of $x, z \in \Omega$. Moreover, by the definition of $\{\Gamma_i\}_{i=1,2.}$ in (3.13), for any fixed $y \in \Omega$, we obtain that

$$\begin{aligned} &\left| \Gamma_1 \left(\sum_{l,j,k=1}^{d} a_{\varepsilon,lj} \left(a_\varepsilon^{lk} \frac{\partial \alpha}{\partial x^k} u_{\varepsilon,n}^j \right) \right) (y, z) \right| \\ &= \left| \Gamma_1 \left(\sum_{j=1}^{d} \frac{\partial \alpha}{\partial x^j} u_{\varepsilon,n}^j \right) (y, z) \right| \\ &= \left| \psi_1(z) \left(\sum_{j=1}^{d} \frac{\partial \alpha}{\partial x^j} u_{\varepsilon,n}^j \right) (y, z) - \left(\int_{\Omega_1} \psi_1(x) \left(\sum_{j=1}^{d} \frac{\partial \alpha}{\partial x^j} u_{\varepsilon,n}^j \right) (y, x) dx \right) \eta(z) \right| \\ &\le C \frac{1}{|y - z|^{d-2}} + \left| \left(\int_{\Omega_1} \psi_1(x) \left(\sum_{j=1}^{d} \frac{\partial \alpha}{\partial x^j} u_{\varepsilon,n}^j \right) (y, x) dx \right) \eta(z) \right| \\ &\le C \frac{1}{|y - z|^{d-2}} + C \int_{\Omega_1} \left| \sum_{j=1}^{d} \frac{\partial \alpha}{\partial x^j} u_{\varepsilon,n}^j \right| (y, x) dx \\ &\le C \frac{1}{|y - z|^{d-2}} + C \int_{\Omega} |\boldsymbol{u}_{\varepsilon,n}(y, x)|\, dx, \qquad n = 1, \cdots, d \end{aligned} \tag{3.16}$$

with $C = C(y, \Omega)$ depending on $y \in \Omega$ but not on $z \in \Omega$. On the other hand, by (A.3) and Proposition 2.1, it holds that

$$
\begin{aligned}
& u^j_{\varepsilon,n}(x,y) \\
&= \sum_{l=1}^d \frac{\partial x^j}{\partial \tilde{x}^l} u^l_n(\tilde{x},\tilde{y}) \\
&= \sum_{l=1}^d \left(\delta^j_l - \varepsilon \frac{\partial S^j}{\partial x^l}(x) \right) u^l_n(\tilde{x},\tilde{y}) + O(\varepsilon^2) \\
&= -\frac{1}{2\omega_d(d-2)} \sum_{l=1}^d \left(\delta^j_l - \varepsilon \frac{\partial S^j}{\partial x^l}(x) \right) \left\{ \frac{\delta^{ln}}{|\tilde{x}-\tilde{y}|^{d-2}} + (d-2) \frac{(\tilde{x}^l - \tilde{y}^l)(\tilde{x}^n - \tilde{y}^n)}{|\tilde{x}-\tilde{y}|^d} \right\} + O(\varepsilon^2) \\
&= -\frac{1}{2\omega_d(d-2)} \sum_{l=1}^d \left(\delta^j_l - \varepsilon \frac{\partial S^j}{\partial x^l}(x) \right) \\
&\times \Bigg[\left(\frac{\delta^{ln}}{|x-y|^{d-2}} + (d-2) \frac{(x^l-y^l)(x^n-y^n)}{|x-y|^d} \right) \\
&\qquad - \varepsilon(d-2) \Bigg\{ \frac{\delta^{ln} \sum_{k=1}^d (S^k(x)-S^k(y))(x^k-y^k)}{|x-y|^d} \\
&\qquad\qquad - \frac{(S^l(x)-S^l(y))(x^n-y^n) + (S^n(x)-S^n(y))(x^l-y^l)}{|x-y|^d} \\
&\qquad\qquad + d \frac{\sum_{k=1}^d (S^k(x)-S^k(y))(x^k-y^k)(x^l-y^l)(x^n-y^n)}{|x-y|^{d+2}} \Bigg\} \Bigg] + O(\varepsilon^2) \\
&= u^j_n(x,y) \\
&- \frac{\varepsilon}{2\omega_d(d-2)} \Bigg[\sum_{l=1}^d \frac{\partial S^j}{\partial x^l}(x) \left(\frac{\delta^{ln}}{|x-y|^{d-2}} + (d-2) \frac{(x^l-y^l)(x^n-y^n)}{|x-y|^d} \right) \\
&\qquad - (d-2) \Bigg\{ \frac{\delta^{jn} \sum_{k=1}^d (S^k(x)-S^k(y))(x^k-y^k)}{|x-y|^d} \\
&\qquad\qquad - \frac{(S^j(x)-S^j(y))(x^n-y^n) + (S^n(x)-S^n(y))(x^j-y^j)}{|x-y|^d} \\
&\qquad\qquad + d \frac{\sum_{k=1}^d (S^k(x)-S^k(y))(x^k-y^k)(x^j-y^j)(x^n-y^n)}{|x-y|^{d+2}} \Bigg\} \Bigg] + O(\varepsilon^2)
\end{aligned}
\tag{3.17}
$$

for $j, n = 1, \cdots, d$, as $\varepsilon \to 0$. Hence by (3.16), we have

$$
\begin{aligned}
& \left| \Gamma_1 \left(\sum_{l,j,k=1}^d a_{\varepsilon,lj} \left(a^{lk}_\varepsilon \frac{\partial \alpha}{\partial x^k} u^j_{\varepsilon,n} \right) \right)(y,z) \right| \\
& \leq C \frac{1}{|y-z|^{d-2}} + C \left(\int_\Omega \frac{1}{|x-y|^{d-2}} dx + \varepsilon \int_\Omega \frac{1}{|x-y|^{d-2}} dx \right) + O(\varepsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
&< C\frac{1}{|y-z|^{d-2}} + 2C\int_\Omega \frac{1}{|x-y|^{d-2}}dx \\
&< \frac{C}{|y-z|^{d-2}}, \quad n=1,\cdots,d \qquad \text{for all } z\in\Omega_1 \text{ with } z\neq y,
\end{aligned}
$$

where $C = C(y,\Omega)$ is a constant depending on $y \in \Omega$ but not on $z \in \Omega$. Similarly, for $i = 2$, we have

$$
\left|\Gamma_2\left(\sum_{l,j,k=1}^d a_{\varepsilon,lj}\left(a_\varepsilon^{lk}\frac{\partial\alpha}{\partial x^k}u^j_{\varepsilon,n}\right)\right)(y,z)\right| < \frac{C}{|y-z|^{d-2}}, \quad n=1,\cdots,d \qquad \text{for all } z\in\Omega_2 \text{ with } z\neq y,
$$

where $C = C(y,\Omega)$ is a constant depending on $y \in \Omega$ but not on $z \in \Omega$. Hence it follows from (3.14) that

$$
\begin{aligned}
|\boldsymbol{Q}_{\varepsilon,n}(x,y)| &= \left|\sum_{i=1}^2\int_{\Omega_i}\Gamma_i\left(\sum_{l,j,k=1}^d a_{\varepsilon,lj}\left(a_\varepsilon^{lk}\frac{\partial\alpha}{\partial x^k}u^j_{\varepsilon,n}\right)\right)(y,z)\boldsymbol{N}_i(x,z)\,dz\right| \\
&\le \sum_{i=1}^2\int_{\Omega_i}\left|\Gamma_i\left(\sum_{l,j,k=1}^d a_{\varepsilon,lj}\left(a_\varepsilon^{lk}\frac{\partial\alpha}{\partial x^k}u^j_{\varepsilon,n}\right)\right)(y,z)\right||\boldsymbol{N}_i(x,z)|\,dz \\
&< C\int_\Omega \frac{1}{|y-z|^{d-2}}\frac{1}{|x-z|^{d-1}}\,dz \\
&\le \frac{C}{|x-y|^{d-3}}, \qquad n=1,\cdots,d \qquad \text{for all } x\in\Omega \text{ with } x\neq y,
\end{aligned}
\tag{3.18}
$$

where $C = C(y,\Omega)$. This implies (3.7).

Next, we prove (3.8). In the same manner as in (3.7), (3.15), we have by (3.13) and (3.17) that

$$
\begin{aligned}
&\left|\Gamma_i\left(\sum_{l,j,k=1}^d a_{\varepsilon,lj}\left(a_\varepsilon^{lk}\frac{\partial\alpha}{\partial x^k}u^j_{\varepsilon,n}\right)-(\nabla\alpha\cdot u_n)\right)(y,z)\right| \\
&= \left|\Gamma_i\left(\sum_{j,k=1}^d \delta^k_j\frac{\partial\alpha}{\partial x^k}u^j_{\varepsilon,n}-(\nabla\alpha\cdot u_n)\right)(y,z)\right| \\
&= \left|\Gamma_i\left(\sum_{j=1}^d \frac{\partial\alpha}{\partial x^j}\left(u^j_{\varepsilon,n}-u^j_n\right)(y,z)\right)\right| \\
&= \left|\psi_i(z)\left(\sum_{j=1}^d \frac{\partial\alpha}{\partial x^j}\left(u^j_{\varepsilon,n}-u^j_n\right)(y,z)\right)-\left(\int_{\Omega_i}\psi_i(x)\left(\sum_{j=1}^d \frac{\partial\alpha}{\partial x^j}\left(u^j_{\varepsilon,n}-u^j_n\right)(y,x)\right)dx\right)\eta(z)\right| \\
&\le C\left|\left(u^j_{\varepsilon,n}-u^j_n\right)(y,z)\right| + C\int_\Omega\left|\left(u^j_{\varepsilon,n}-u^j_n\right)(y,x)\right|dx \\
&\le \frac{C\varepsilon}{|y-z|^{d-2}}, \quad n=1,\cdots,d \qquad \text{for all } z\in\Omega_i \text{ with } y\neq z \text{ and } i=1,2
\end{aligned}
$$

with $C = C(y,\Omega)$. Therefore, similarly to (3.16), we have

$$
\begin{aligned}
&|(\boldsymbol{Q}_{\varepsilon,n} - \boldsymbol{Q}_{0,n})(x,y)| \\
&= \left| \sum_{i=1}^{2} \int_{\Omega_i} \Gamma_i \left(\sum_{l,j,k=1}^{d} a_{\varepsilon,lj} \left(a_\varepsilon^{lk} \frac{\partial \alpha}{\partial x^k} u_{\varepsilon,n}^j \right) - \nabla\alpha \cdot u_n \right) (y,z) \boldsymbol{N}_i(x,z)\, dz \right| \\
&\le C\varepsilon \int_\Omega \frac{1}{|y-z|^{d-2}} \frac{1}{|x-z|^{d-1}} \\
&\le \frac{C\varepsilon}{|x-y|^{d-3}}, \quad n = 1, \cdots, d \quad \text{for all } x \in \Omega \text{ with } x \neq y,
\end{aligned}
$$

where $C = C(y,\Omega)$, which implies (3.8).

Since the rest of proofs for (3.9)–(3.12) can be handled similarly as above, we may omit it. □

Lemma 3.1 shows that the order of the singularity of $\boldsymbol{Q}_{\varepsilon,n}(x,y) = \{Q^l_{\varepsilon,n}(x,y)\}_{l,n=1,\cdots,d}$ near $x = y$ is weaker by one degree than that of the fundamental solutions as in (2.2). Making use of the $\boldsymbol{Q}_{\varepsilon,n}(x,y) = \{Q^l_{\varepsilon,n}(x,y)\}_{l,n=1,\cdots,d}$, we shall construct a parametrix in the following way.

Let us define $p_{\varepsilon,n}(\cdot,y) := \{p_{\varepsilon,n}(\cdot,y)\}_{n=1,\cdots,d}$ by

$$
p_{\varepsilon,n}(x,y) := p_n(\Phi_\varepsilon(x), \Phi_\varepsilon(y)), \tag{3.19}
$$

where p_n is the fundamental solutions of the Stokes equations (2.2) for the pressure, that is,

$$
p_n(\tilde{x},\tilde{y}) := -\frac{1}{\omega_d} \frac{(\tilde{x}^n - \tilde{y}^n)}{|\tilde{x}-\tilde{y}|^d}, \quad n = 1, \cdots, d.
$$

Now we define the parametrix $\{U^l_{\varepsilon,n}(\cdot,y)\}_{l,n=1,\cdots,d}$ for the velocity and $\{P_{\varepsilon,n}(\cdot,y)\}_{n=1,\cdots,d}$ for the pressure by

$$
\begin{aligned}
\boldsymbol{U}_{\varepsilon,n}(x,y) &:= \alpha(x,y)\boldsymbol{u}_{\varepsilon,n}(x,y) - \boldsymbol{Q}_{\varepsilon,n}(x,y), \\
P_{\varepsilon,n}(x,y) &:= \alpha(x,y) p_{\varepsilon,n}(x,y), \qquad n = 1, \cdots, d,
\end{aligned} \tag{3.20}
$$

where $\alpha(\cdot,y)$ and $\boldsymbol{u}_{\varepsilon,n}(\cdot,y) = \{u^i_{\varepsilon,n}(\cdot,y)\}_{i,n=1,\cdots,d}$ are the same as in (3.4) and (3.5), respectively. We may regard $\boldsymbol{Q}_{\varepsilon,n}(\cdot,y) := \{Q^i_{\varepsilon,n}(\cdot,y)\}_{i,n=1,\cdots,d}$ as a compensation for recovery of the divergence free condition of the parametrix $\{U^l_{\varepsilon,n}(\cdot,y)\}_{l,n=1,\cdots,d}$ in Ω.

3.2 Properties of the parametrix.

In this subsection, we shall show several properties of the parametrix $\{\boldsymbol{U}_{\varepsilon,n}, P_{\varepsilon,n}\}_{n=1,\cdots,d}$ defined by (3.20).

Lemma 3.2. *For any $0 \le \varepsilon < 1$ and for each fixed $y \in \Omega$, the parametrix $\boldsymbol{U}_{\varepsilon,n} = \{\boldsymbol{U}_{\varepsilon,n}(\cdot,y)\}_{n=1,\cdots,d}$ and $P_{\varepsilon,n} = \{P_{\varepsilon,n}(\cdot,y)\}_{n=1,\cdots,d}$ defined by (3.20) satisfy the following properties.*

(1) It holds that

$$
\begin{cases}
\operatorname{div} \boldsymbol{U}_{\varepsilon,n}(\cdot,y) = 0 & \text{in } \Omega, \\
\boldsymbol{U}_{\varepsilon,n}(\cdot,y) = 0 & \text{on } \partial\Omega, \qquad n = 1, \cdots, d.
\end{cases} \tag{3.21}
$$

(2) The parametrix $\{U_{0,n}^l(\cdot,y)\}_{n,l=1,\cdots,d}$ and $\{P_{0,n}(\cdot,y)\}_{n=1,\cdots,d}$ approximate the Green matrix $\{G_n^l(\cdot,y)\}_{n,l=1,\cdots,d}$ and $\{R_n(\cdot,y)\}_{n=1,\cdots,d}$ in the sense that

$$\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{l=1}^d \nabla_x\left(G_n^l(x,y)-U_{0,n}^l(x,y)\right)\cdot\nu_x v^l(x)\,d\sigma_x=0, \tag{3.22}$$

$$\lim_{\rho\to 0}\int_{\partial B_\rho(y)}(R_n(x,y)-P_{0,n}(x,y))(v(x)\cdot\nu_x)\,d\sigma_x=0 \tag{3.23}$$

for $n=1,\cdots,d$, where $\boldsymbol{v}=\{v^l\}_{l=1,\cdots,d.}$ is a smooth vector function around $y\in\Omega$, ν_x denotes the unit outer normal vector to $\partial B_\rho(y)$ and $d\sigma_x$ denotes the surface element of $\partial B_\rho(y)$.

(3) It holds that

$$|\boldsymbol{U}_{\varepsilon,n}(x,y)-\boldsymbol{U}_{0,n}(x,y)|\le\frac{C\varepsilon}{|x-y|^{d-2}}, \tag{3.24}$$

$$|\boldsymbol{P}_{\varepsilon,n}(x,y)-\boldsymbol{P}_{0,n}(x,y)|\le\frac{C\varepsilon}{|x-y|^{d-1}},\quad n=1,\cdots,d \tag{3.25}$$

for all $x\in\Omega$ with $x\neq y$, where $C=C(y,d)$ is a constant which may depend on y, but not on $x\in\Omega$.

(4) For the operator $\mathcal{L}_\varepsilon$ defined by (2.22), we obtain the following estimate.

$$|\mathcal{L}_\varepsilon(\boldsymbol{G}_n-\boldsymbol{U}_{0,n}+\boldsymbol{U}_{\varepsilon,n},R_n-P_{0,n}+P_{\varepsilon,n})(x,y)|\le\frac{C\varepsilon}{|x-y|^{d-1}},\quad n=1,\cdots,d \tag{3.26}$$

for $x\in\Omega$ with $x\neq y$, where $C=C(y,d)$ is a constant which may depend on y, but not on $x\in\Omega$.

Proof. (1) we see that (3.21) is an immediate consequence of (3.4), (3.6) and (3.20).

(2) Let us show (3.22) and (3.23). First we note that

$$U_{0,n}^l(x,y)=\alpha(x,y)u_n^l(x,y)-Q_{0,n}^l(x,y),\qquad n,l=1,\cdots,d, \tag{3.27}$$

where $\boldsymbol{u}_n(\cdot,y):=\{u_n^l(\cdot,y)\}_{n,l=1,\cdots,d}$ is the fundamental solution of the Stokes equations introduced by (2.2). Since $\boldsymbol{u}_n(x,y)=O(|x-y|^{d-2})$ for $n=1,\cdots,d$, as $x\to y$, we obtain from (2.6) and (3.9) that

$$\begin{aligned}
&\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{l=1}^d\nabla_x\left(G_n^l(x,y)-U_{0,n}^l(x,y)\right)\cdot\nu_x v^l(x)\,d\sigma_x\\
&=\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{k.l=1}^d\left\{\frac{\partial}{\partial x^k}(1-\alpha(x,y))u_n^l(x,y)+(1-\alpha(x,y))\frac{\partial}{\partial x^k}u_n^l(x,y)+\frac{\partial Q_{0,n}^l}{\partial x^k}(x,y)\right\}v^l(x)\nu_x^k\,d\sigma_x\\
&=\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{k.l=1}^d(1-\alpha(x,y))\frac{\partial}{\partial x^k}u_n^l(x,y)v^l(x)\nu_x^k\,d\sigma_x\\
&=\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{k.l=1}^d(\alpha(y,y)-\alpha(x,y))\frac{\partial}{\partial x^k}u_n^l(x,y)v^l(x)\nu_x^k\,d\sigma_x\\
&:=\lim_{\rho\to 0}I_n(\rho),\qquad n=1,\cdots,d.
\end{aligned}$$

Since $\nabla_x u_n^l(x,y) = O(|x-y|^{d-1})$ for $l, n = 1, \cdots, d$, as $x \to y$, in the same way as (2.11), we have

$$|I_n(\rho)| \leq C \sup_{x\in\Omega} |(\nabla_x \alpha)(x,y)|\rho \to 0, \quad n = 1, \cdots, d, \quad \text{as} \quad \rho \to 0$$

with a constant $C = C(y, \Omega)$ independent of ρ, which implies (3.22). It is easy to see that (3.23) can be handled similarly to (3.22).

(3) We next show (3.24) and (3.25). By (3.20), (3.17) and (3.8), it holds that

$$\begin{aligned} |U_{\varepsilon,n}^l(x,y) - U_{0,n}^l(x,y)| &\leq \sup_{x\in\Omega} |\alpha(x,y)||u_{\varepsilon,n}^l(x,y) - u_n^l(x,y)| + |Q_{\varepsilon,n}^l(x,y) - Q_{0,n}^l(x,y)| \\ &\leq C\varepsilon\{|x-y|^{2-d} + |x-y|^{3-d}\}, \qquad n = 1, \cdots, d \end{aligned}$$

with a constant $C = C(y, \Omega)$, which yields that

$$|\boldsymbol{U}_{\varepsilon,n}(x,y) - \boldsymbol{U}_{0,n}(x,y)| \leq \frac{C\varepsilon}{|x-y|^{d-2}}, \qquad n = 1, \cdots, d,$$

where $C = C(y, \Omega)$ is a constant. This implies (3.24). The proof of (3.25) is parallel to that of (3.24).

(4) Since the Green matrix $\boldsymbol{G}_n(x,y) = \{G_n^l(x,y)\}_{n,l=1,\cdots,d}$ and the Green function $R_n(x,y) = \{R_n(x,y)\}_{n=1,\cdots,d}$ satisfy the Stokes equations (0.2), we obtain that

$$\begin{aligned} &|\mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)| \\ &= |\mathcal{L}_\varepsilon^r(\boldsymbol{U}_{\varepsilon,n}, P_{\varepsilon,n})(x,y) - \mathcal{L}_0^r(\boldsymbol{U}_{0,n}, P_{0,n})(x,y) \\ &\qquad + \mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n}, R_n - P_{0,n})(x,y) - \mathcal{L}_0^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n}, R_n - P_{0,n})(x,y)| \\ &\leq |\mathcal{L}_\varepsilon^r(\boldsymbol{U}_{\varepsilon,n}, P_{\varepsilon,n})(x,y) - \mathcal{L}_0^r(\boldsymbol{U}_{0,n}, P_{0,n})(x,y)| \\ &\qquad + |\mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n}, R_n - P_{0,n})(x,y) - \mathcal{L}_0^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n}, R_n - P_{0,n})(x,y)| \end{aligned}$$

for $n, r = 1, \cdots, d$. Therefore, for the proof of (3.26), it suffices to show that

$$|\mathcal{L}_\varepsilon^r(\boldsymbol{U}_{\varepsilon,n}, P_{\varepsilon,n})(x,y) - \mathcal{L}_0^r(\boldsymbol{U}_{0,n}, P_{0,n})(x,y)| \leq \frac{C\varepsilon}{|x-y|^{d-1}}, \tag{3.28}$$

$$|\mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n}, R_n - P_{0,n})(x,y) - \mathcal{L}_0^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n}, R_n - P_{0,n})(x,y)| \leq \frac{C\varepsilon}{|x-y|^{d-1}} \tag{3.29}$$

for $n, r = 1, \cdots, d$, where $C = C(y, d)$.

First we treat (3.28). Since $\boldsymbol{u}_{0,n} = \boldsymbol{u}_n$, it follows from (3.20) that

$$\begin{aligned} &\mathcal{L}_\varepsilon^r(\boldsymbol{U}_{\varepsilon,n}, P_{\varepsilon,n})(x,y) - \mathcal{L}_0^r(\boldsymbol{U}_{0,n}, P_{0,n})(x,y) \\ &= \mathcal{L}_\varepsilon^r(\alpha \boldsymbol{u}_{\varepsilon,n}, \alpha p_{\varepsilon,n})(x,y) - \mathcal{L}_0^r(\alpha \boldsymbol{u}_{0,n}, \alpha p_{0,n})(x,y) - \mathcal{L}_\varepsilon^r(\boldsymbol{Q}_{\varepsilon,n}, 0)(x,y) + \mathcal{L}_0^r(\boldsymbol{Q}_{0,n}, 0)(x,y) \end{aligned}$$

(3.30)

$$\begin{aligned} &= \sum_{i,j,k,l,p,s=1}^{d} \frac{\partial}{\partial x^k}\left\{a_\varepsilon^{ks}\frac{\partial}{\partial x^s}\left(\frac{\partial \tilde{x}^i}{\partial x^l}\alpha(x,y)\frac{\partial x^l}{\partial \tilde{x}^j}u_n^j(\tilde{x},\tilde{y})\right)\right\}\frac{\partial \tilde{x}^i}{\partial x^p}a_\varepsilon^{pr} - \sum_{i=1}^{d} a_\varepsilon^{ri}\frac{\partial}{\partial x^i}\{\alpha(x,y)p_n(\tilde{x},\tilde{y})\} \\ &\quad - \Big[\sum_{i,j,k,l,p,s=1}^{d} \frac{\partial}{\partial x^k}\left\{\delta^{ks}\frac{\partial}{\partial x^s}\left(\delta^{il}\alpha(x,y)\delta^{lj}u_n^j(x,y)\right)\right\}\delta^{ip}\delta^{pr} - \sum_{i=1}^{d}\delta^{ri}\frac{\partial}{\partial x^i}\{\alpha(x,y)p_n(x,y)\}\Big] \\ &\quad - \mathcal{L}_\varepsilon^r(\boldsymbol{Q}_{\varepsilon,n}, 0)(x,y) + \mathcal{L}_0^r(\boldsymbol{Q}_{0,n}, 0)(x,y) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,k,p,s=1}^{d} \frac{\partial}{\partial x^k}\left\{a_\varepsilon^{ks}\frac{\partial}{\partial x^s}\left(\alpha(x,y)u_n^i(\tilde{x},\tilde{y})\right)\right\}\frac{\partial \tilde{x}^i}{\partial x^p}a_\varepsilon^{pr} - \sum_{i=1}^{d} a_\varepsilon^{ri}\frac{\partial}{\partial x^i}\{\alpha(x,y)p_n(x,y)\} \\
&\quad - \left[\sum_{i,k,p,s=1}^{d} \frac{\partial}{\partial x^k}\left\{\delta^{ks}\frac{\partial}{\partial x^s}\left(\alpha(x,y)u_n^i(x,y)\right)\right\}\delta^{ip}\delta^{pr} - \sum_{i=1}^{d}\delta^{ri}\frac{\partial}{\partial x^i}\{\alpha(x,y)p_n(x,y)\}\right] \\
&\quad - \mathcal{L}_\varepsilon^r(\boldsymbol{Q}_{\varepsilon,n},0)(x,y) + \mathcal{L}_0^r(\boldsymbol{Q}_{0,n},0)(x,y) \\
&:= I_{1,n}^r + I_{2,n}^r + I_{3,n}^r + I_{4,n}^r + I_{5,n}^r + I_{6,n}^r, \qquad n,r = 1,\cdots,d,
\end{aligned}
$$

where

$$
\begin{aligned}
I_{1,n}^r &:= \alpha(x,y)\left(\sum_{i,j,k,l,p,q,s=1}^{d} a_\varepsilon^{ks}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial \tilde{x}^q}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^s}\frac{\partial^2 u_n^i}{\partial \tilde{x}^q \partial \tilde{x}^j}(\tilde{x},\tilde{y}) - \sum_{i,j=1}^{d} a_\varepsilon^{ri}\frac{\partial \tilde{x}_j}{\partial x^i}\frac{\partial p_n}{\partial \tilde{x}^j}(\tilde{x},\tilde{y})\right) \\
&\quad - \alpha(x,y)\left(\sum_{i,k,p,s=1}^{d}\delta^{ks}\delta^{ip}\delta^{pr}\frac{\partial^2 u_n^i}{\partial x^k \partial x^s}(x,y) - \sum_{i=1}^{d}\delta^{ri}\frac{\partial p_n}{\partial x^i}(x,y)\right), \\
I_{2,n}^r &:= \alpha(x,y)\sum_{i,j,k,p,s=1}^{d}\left(\frac{\partial a_\varepsilon^{ks}}{\partial x^k}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial \tilde{x}^j}{\partial x^s}\frac{\partial u_n^i}{\partial \tilde{x}^j}(\tilde{x},\tilde{y}) + a_\varepsilon^{ks}a_\varepsilon^{pr}\frac{\partial^2 \tilde{x}^j}{\partial x^k \partial x^s}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial u_n^i}{\partial \tilde{x}^j}(\tilde{x},\tilde{y})\right) \\
&\quad + \sum_{i,k,p,s=1}^{d}\frac{\partial \alpha}{\partial x^s}(x,y)\frac{\partial a_\varepsilon^{ks}}{\partial x^k}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}u_n^i(\tilde{x},\tilde{y}), \\
I_{3,n}^r &:= -\sum_{i=1}^{d} a_\varepsilon^{ri}\frac{\partial \alpha}{\partial x^i}(x,y)p_n(\tilde{x},\tilde{y}) + \sum_{i=1}^{d}\delta^{ri}\frac{\partial \alpha}{\partial x^i}(x,y)p_n(x,y), \\
I_{4,n}^r &:= \sum_{i,j,k,p,s=1}^{d}\left(\frac{\partial \alpha}{\partial x^k}(x,y)\frac{\partial \tilde{x}^j}{\partial x^s} + \frac{\partial \alpha}{\partial x^s}(x,y)\frac{\partial \tilde{x}^j}{\partial x^k}\right)a_\varepsilon^{ks}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial u_n^i}{\partial \tilde{x}^j}(\tilde{x},\tilde{y}) \\
&\quad - \sum_{i,k,p,s=1}^{d}\delta^{ks}\delta^{ip}\delta^{pr}\left(\frac{\partial \alpha}{\partial x^k}(x,y)\frac{\partial u_n^i}{\partial x^s}(x,y) + \frac{\partial \alpha}{\partial x^s}(x,y)\frac{\partial u_n^i}{\partial x^k}(x,y)\right), \\
I_{5,n}^r &:= \sum_{i,k,p,s=1}^{d}\frac{\partial^2 \alpha}{\partial x^k \partial x^s}(x,y)a_\varepsilon^{ks}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}u_n^i(\tilde{x},\tilde{y}) - \sum_{i,k,p,s=1}^{d}\delta^{ks}\delta^{ip}\delta^{pr}\frac{\partial^2 \alpha}{\partial x^k \partial x^s}(x,y)u_n^i(x,y), \\
I_{6,n}^r &= -\{\mathcal{L}_\varepsilon^r(\boldsymbol{Q}_{\varepsilon,n},0)(x,y) - \mathcal{L}_0^r(\boldsymbol{Q}_{0,n},0)(x,y)\}, \qquad n,r = 1,\cdots,d.
\end{aligned}
$$

Concerning the identity of $I_{1,n}^r$, it follows from (2.16) and (2.28) that

$$
\begin{aligned}
&\alpha(x,y)\left(\sum_{i,j,k,p,q,s=1}^{d} a_\varepsilon^{ks}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial \tilde{x}^q}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^s}\frac{\partial^2 u_n^i}{\partial \tilde{x}^q \partial \tilde{x}^j}(\tilde{x},\tilde{y}) - \sum_{i,j=1}^{d} a_\varepsilon^{ri}\frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j}p_n(\tilde{x},\tilde{y})\right) \\
&= \alpha(x,y)\sum_{i=1}^{d}\frac{\partial x^r}{\partial \tilde{x}^i}\left(\Delta_{\tilde{x}}u_n^i(\tilde{x},\tilde{y}) - \frac{\partial p_n(\tilde{x},\tilde{y})}{\partial \tilde{x}^i}\right) \\
&= 0, \qquad n,r = 1,\cdots,d,
\end{aligned} \tag{3.31}
$$

Similarly, since $\{\boldsymbol{u}_n(x,y), p_n(x,y)\}_{n=1,\cdots,d}$ solves the Stokes equations in Ω for $x \neq y$, it holds that $\mathcal{L}_0(\boldsymbol{u}_n, p_n)(x,y) = 0$ for $x, y \in \Omega$ with $x \neq y$, which yields

$$
\begin{aligned}
(3.32) \qquad & \alpha(x,y)\left(\sum_{i,k,p,s=1}^{d} \delta^{ks}\delta^{ip}\delta^{pr}\frac{\partial^2 u_n^i}{\partial x^k \partial x^s}(x,y) - \sum_{i=1}^{d}\delta^{ri}\frac{\partial p_n}{\partial x^i}(x,y)\right) \\
&= \alpha(x,y)\left(\Delta u_n^r(x,y) - \frac{\partial p}{\partial x^r}(x,y)\right) \\
&= 0, \qquad n, r = 1, \cdots, d, \quad \text{for} \ \ x \in \Omega \text{ with } x \neq y.
\end{aligned}
$$

This implies $I_{1,n}^r = 0$ for $n, r = 1, \cdots, d$ and for $x, y \in \Omega$ with $x \neq y$. By Proposition 2.1, we have

$$
\begin{aligned}
|I_{2,n}^r| = & \left|\alpha(x,y)\sum_{i,j,k,p,s=1}^{d}\left(\frac{\partial a_\varepsilon^{ks}}{\partial x^k}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial \tilde{x}^j}{\partial x^s}\frac{\partial u_n^i}{\partial \tilde{x}^j}(\tilde{x},\tilde{y}) + a_\varepsilon^{ks}a_\varepsilon^{pr}\frac{\partial^2 \tilde{x}^j}{\partial x^k \partial x^s}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial u_n^i}{\partial \tilde{x}^j}(\tilde{x},\tilde{y})\right)\right| \\
& + \left|\sum_{i,k,p,s=1}^{d}\frac{\partial \alpha}{\partial x^s}(x,y)\frac{\partial a_\varepsilon^{ks}}{\partial x^k}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}u_n^i(\tilde{x},\tilde{y})\right| \leq \frac{C\varepsilon}{|x-y|^{d-1}}, \\
& \qquad n, r = 1, \cdots, d, \quad \text{for} \ \ x \in \Omega \text{ with } x \neq y,
\end{aligned}
$$

$$
\begin{aligned}
|I_{3,n}^r| &= \left|\sum_{i=1}^{d}\left(a_\varepsilon^{ri}\frac{\partial \alpha}{\partial x^i}(x,y)p_n(\tilde{x},\tilde{y}) - \delta^{ri}\frac{\partial \alpha}{\partial x^i}(x,y)p_n(x,y)\right)\right| \\
&= \left|\frac{\partial \alpha}{\partial x^i}\sum_{i=1}^{d}\left\{\left(\delta^{ri} + \varepsilon\left(\frac{\partial S^i}{\partial x^r} + \frac{\partial S^r}{\partial x^i}\right) + O(\varepsilon^2)\right)\right.\right. \\
&\qquad \left.\left. \times\left(p_n(x,y) + \varepsilon\nabla_x p_n(x,y)\{(\tilde{x}-x) + (y-\tilde{y})\} + O(\varepsilon^2)\right) - \delta^{ir}p_n(x,y)\right\}\right| \\
&= \left|\frac{\partial \alpha}{\partial x^i}\sum_{i=1}^{d}\varepsilon\left(\frac{\partial S^i}{\partial x^r} + \frac{\partial S^r}{\partial x^i}\right)p_n(x,y) + O(\varepsilon^2)\right| \\
&\leq C\frac{\varepsilon}{|x-y|^{d-1}}, \qquad n, r = 1, \cdots, d, \quad \text{for} \ \ x \in \Omega \text{ with } x \neq y,
\end{aligned}
$$

$$
\begin{aligned}
|I_{4,n}^r| = & \Bigg|\sum_{i,j,k,p,s=1}^{d}\left(\frac{\partial \alpha}{\partial x^k}(x,y)\frac{\partial \tilde{x}^j}{\partial x^s} + \frac{\partial \alpha}{\partial x^s}(x,y)\frac{\partial \tilde{x}^j}{\partial x^k}\right)a_\varepsilon^{ks}a_\varepsilon^{pr}\frac{\partial \tilde{x}^i}{\partial x^p}\frac{\partial u_n^i}{\partial \tilde{x}^j}(\tilde{x},\tilde{y}) \\
& - \sum_{i,k,p,s=1}^{d}\delta^{ks}\delta^{ip}\delta^{pr}\left(\frac{\partial \alpha}{\partial x^k}(x,y)\frac{\partial u_n^i}{\partial x^s}(x,y) + \frac{\partial \alpha}{\partial x^s}(x,y)\frac{\partial u_n^i}{\partial x^k}(x,y)\right)\Bigg| \\
= & \Bigg|\sum_{i,j,k,p,s=1}^{d}\left\{\frac{\partial \alpha}{\partial x^k}(x,y)\left(\delta_s^j + \varepsilon\frac{\partial S^j}{\partial x^s} + O(\varepsilon^2)\right) + \frac{\partial \alpha}{\partial x^s}(x,y)\left(\delta_k^j + \varepsilon\frac{\partial S^j}{\partial x^k} + O(\varepsilon^2)\right)\right\} \\
& \times\left(\delta^{ks} + \varepsilon\left(\frac{\partial S^k}{\partial x^s} + \frac{\partial S^s}{\partial x^k}\right) + O(\varepsilon^2)\right)\left(\delta^{pr} + \varepsilon\left(\frac{\partial S^p}{\partial x^r} + \frac{\partial S^r}{\partial x^p}\right) + O(\varepsilon^2)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \left(\delta^i_p + \varepsilon\frac{\partial S^i}{\partial x^p} + O(\varepsilon^2)\right)\frac{\partial u^i_n}{\partial \tilde{x}^j}(\tilde{x},\tilde{y}) \\
&\quad - \sum_{i,k,p,s=1}^{d} \delta^{ks}\delta^{ip}\delta^{pr}\left(\frac{\partial \alpha}{\partial x^k}(x,y)\frac{\partial u^i_n}{\partial x^s}(x,y) + \frac{\partial \alpha}{\partial x^s}(x,y)\frac{\partial u^i_n}{\partial x^k}(x,y)\right)\Bigg| \\
&\le C\frac{\varepsilon}{|x-y|^{d-1}}, \qquad n,r=1,\cdots,d, \quad \text{for } \ x\in\Omega \text{ with } x\neq y,
\end{aligned}
$$

$$
\begin{aligned}
|I^r_{5,n}| &= \left|\sum_{i,k,p,s=1}^{d}\left(\frac{\partial^2\alpha}{\partial x^k\partial x^s}(x,y)a^{ks}_\varepsilon a^{pr}_\varepsilon\frac{\partial \tilde{x}^i}{\partial x^p}u^i_n(\tilde{x},\tilde{y}) - \delta^{ks}\delta^{ip}\delta^{pr}\frac{\partial^2\alpha}{\partial x^k\partial x^s}(x,y)u^i_n(x,y)\right)\right| \\
&= \left|\sum_{i,k,p,s=1}^{d}\frac{\partial^2\alpha}{\partial x^k\partial x^s}(x,y)\left(a^{ks}_\varepsilon a^{pr}_\varepsilon\frac{\partial \tilde{x}^i}{\partial x^p}u^i_n(\tilde{x},\tilde{y}) - \delta^{ks}\delta^{ip}\delta^{pr}u^i_n(x,y)\right)\right| \\
&= \Bigg|\sum_{i,k,p,s=1}^{d}\frac{\partial^2\alpha}{\partial x^k\partial x^s} \\
&\qquad \times \Bigg[\Bigg\{\left(\delta^{ks} + \varepsilon\left(\frac{\partial S^k}{\partial x^s} + \frac{\partial S^s}{\partial x^k}\right) + O(\varepsilon^2)\right)\left(\delta^{pr} + \varepsilon\left(\frac{\partial S^p}{\partial x^r} + \frac{\partial S^r}{\partial x^p}\right) + O(\varepsilon^2)\right) \\
&\qquad\quad \times \left(\delta^{ip} + \varepsilon\frac{\partial S^i}{\partial x^p} + O(\varepsilon^2)\right)\left(u^i_n(x,y) + \varepsilon\nabla_x u^i_n(x,y)\{(\tilde{x}-x)+(y-\tilde{y})\} + O(\varepsilon^2)\right)\Bigg\} \\
&\qquad\qquad - \delta^{ks}\delta^{ip}\delta^{pr}u^i_n(x,y)\Bigg]\Bigg| \\
&\le \frac{C\varepsilon}{|x-y|^{d-1}}, \qquad n,r=1,\cdots,d, \quad \text{for } \ x\in\Omega \text{ with } x\neq y,
\end{aligned}
$$

$$
\begin{aligned}
|I^r_{6,n}| &= |\mathcal{L}^r_\varepsilon(\boldsymbol{Q}_{\varepsilon,n},0)(x,y) - \mathcal{L}^r_0(\boldsymbol{Q}_{0,n},0)(x,y)| \\
&= \Bigg|\sum_{i,k,l,p,s=1}^{d}\frac{\partial}{\partial x^k}\left\{a^{ks}_\varepsilon\frac{\partial}{\partial x^s}\left(\frac{\partial \tilde{x}^i}{\partial x^l}Q^l_{\varepsilon,n}(x,y)\right)\right\}\frac{\partial \tilde{x}^i}{\partial x^p}a^{pr}_\varepsilon \\
&\qquad\qquad - \sum_{i,k,l,p,s=1}^{d}\frac{\partial}{\partial x^k}\left\{\delta^{ks}\frac{\partial}{\partial x^s}\left(\delta^{il}Q^l_{0,n}(x,y)\right)\right\}\delta^{ip}\delta^{pr}\Bigg| \\
&\le J^r_{1,n} + J^r_{2,n}, \qquad n,r=1,\cdots,d,
\end{aligned}
$$

where

$$
\begin{aligned}
J^r_{1,n} &:= \left|\sum_{k,l,p,s=1}^{d}a^{ks}_\varepsilon a_{\varepsilon,pl}a^{pr}_\varepsilon\frac{\partial^2 Q^l_{\varepsilon,n}}{\partial x^k\partial x^s} - \sum_{i,k,l,p,s=1}^{d}\delta^{ks}\delta^{il}\delta^{ip}\delta^{pr}\frac{\partial^2 Q^l_{0,n}}{\partial x^k\partial x^s}\right|, \\
J^r_{2,n} &:= \Bigg|\sum_{i,k,l,p,s=1}^{d}\left\{\frac{\partial a^{ks}_\varepsilon}{\partial x^k}\frac{\partial \tilde{x}^i}{\partial x^l}\frac{\partial Q^l_{\varepsilon,n}}{\partial x^s} + a^{ks}_\varepsilon\left(\frac{\partial^2\tilde{x}^i}{\partial x^s\partial x^l}\frac{\partial Q^l_{\varepsilon,n}}{\partial x^k} + \frac{\partial^2\tilde{x}^i}{\partial x^s\partial x^l}\frac{\partial Q^l_{\varepsilon,n}}{\partial x^k}\right)\right\}\frac{\partial \tilde{x}^i}{\partial x^p}a^{pr}_\varepsilon \\
&\qquad + \sum_{i,k,l,p,s=1}^{d}\left\{a^{ks}_\varepsilon\frac{\partial^3\tilde{x}^i}{\partial x^s\partial x^k\partial x^l}Q^l_{\varepsilon,n}\frac{\partial \tilde{x}^i}{\partial x^p}a^{pr}_\varepsilon + \frac{\partial a^{ks}_\varepsilon}{\partial x^k}\frac{\partial^2\tilde{x}^i}{\partial x^s\partial x^l}Q^l_{\varepsilon,n}\frac{\partial \tilde{x}^i}{\partial x^p}a^{pr}_\varepsilon\right\}\Bigg|, \quad n,r=1,\cdots,d.
\end{aligned}
$$

By (3.7), (3.9), (3.11), (3.12) and Proposition 2.1, we have that

$$
\begin{aligned}
J^r_{1,n} = \Bigg| \sum_{k,l,p,s=1}^{d} & \left(\delta^{ks} - \varepsilon\left(\frac{\partial S^k}{\partial x^s} + \frac{\partial S^s}{\partial x^k}\right) + O(\varepsilon^2)\right)\left(\delta_{pl} + \varepsilon\left(\frac{\partial S^p}{\partial x^l} + \frac{\partial S^l}{\partial x^p}\right) + O(\varepsilon^2)\right) \\
& \times \left(\delta^{pr} + \varepsilon\left(\frac{\partial S^p}{\partial x^r} + \frac{\partial S^r}{\partial x^p}\right) + O(\varepsilon^2)\right)\frac{\partial^2 Q^l_{\varepsilon,n}}{\partial x^k \partial x^s} - \sum_{i,k,l,p,s=1}^{d} \delta^{ks}\delta^{il}\delta^{ip}\delta^{pr}\frac{\partial^2 Q^l_{0,n}}{\partial x^k \partial x^s}\Bigg| \\
\le \frac{C\varepsilon}{|x-y|^{d-1}}, & \qquad n,r = 1,\cdots,d, \qquad \text{for all } x \in \Omega \text{ with } x \ne y.
\end{aligned}
$$

Similarly, we have

$$
J^r_{2,n} \le \frac{C\varepsilon}{|x-y|^{d-2}}, \qquad n,r = 1,\cdots,d, \qquad \text{for all } x \in \Omega \text{ with } x \ne y,
$$

where $C = C(y,\Omega)$. These two estimates of $J^r_{1,n}$ and $J^r_{2,n}$ for $n,r = 1,\cdots,d$ yield that

$$
|I^r_{6,n}| \le \frac{C\varepsilon}{|x-y|^{d-1}}, \qquad n,r = 1,\cdots,d, \qquad \text{for all } x \in \Omega \text{ with } x \ne y
$$

with $C = C(y,\Omega)$. From above estimates, we see that

$$
|I^r_{j,n}| \le \frac{C\varepsilon}{|x-y|^{d-1}}, \qquad j = 1,\cdots,6, \qquad n,r = 1,\cdots,d, \qquad \text{for all } x \in \Omega \text{ with } x \ne y, \tag{3.33}
$$

where $C = C(y,\Omega)$ is independent of $x \in \Omega$. By (3.30) and (3.33), we have (3.28).

In the next step, let us show (3.29). Since $\boldsymbol{u}_{0,n} = \boldsymbol{u}_n$, it follows from (2.1) and (3.20) that

$$
\begin{aligned}
& \mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n}, R_n - P_{0,n})(x,y) - \mathcal{L}^r_0(\boldsymbol{G}_n - \boldsymbol{U}_{0,n}, R_n - P_{0,n})(x,y) \\
& = \mathcal{L}^r_\varepsilon((1-\alpha)\boldsymbol{u}_n, (1-\alpha)p_n)(x,y) - \mathcal{L}^r_0((1-\alpha)\boldsymbol{u}_n, (1-\alpha)p_n)(x,y) \\
& \qquad + \mathcal{L}^r_\varepsilon(\boldsymbol{Q}_{0,n}, 0)(x,y) - \mathcal{L}^r_0(\boldsymbol{Q}_{0,n}, 0)(x,y) \\
& = \sum_{i,k,l,p,s=1}^{d} \frac{\partial}{\partial x^k}\left\{a^{ks}_\varepsilon \frac{\partial}{\partial x^s}\left(\frac{\partial \tilde{x}^i}{\partial x^l}(1-\alpha(x,y))u^l_n(x,y)\right)\right\}\frac{\partial \tilde{x}^i}{\partial x^p}a^{pr}_\varepsilon - \sum_{i=1}^{d} a^{ri}\frac{\partial}{\partial x^i}\{(1-\alpha(x,y))p_n(x,y)\} \\
& \qquad - \left[\sum_{i,k,l,p,s=1}^{d} \frac{\partial}{\partial x^k}\left\{\delta^{ks}\frac{\partial}{\partial x^s}\left(\delta_{il}(1-\alpha(x,y))u^l_n(x,y)\right)\right\}\delta^{ip}\delta^{pr} - \sum_{i=1}^{d}\delta^{ri}\frac{\partial}{\partial x^i}\{(1-\alpha(x,y))p_n(x,y)\}\right] \\
& \qquad\qquad + \mathcal{L}^r_\varepsilon(\boldsymbol{Q}_{0,n}, 0)(x,y) - \mathcal{L}^r_0(\boldsymbol{Q}_{0,n}, 0)(x,y) \\
& := E^r_{1,n} + E^r_{2,n} + E^r_{3,n} + E^r_{4,n} + E^r_{5,n} + E^r_{6,n}, \qquad n,r = 1,\cdots,d,
\end{aligned}
$$

where

$$
\begin{aligned}
E^r_{1,n} := (1-\alpha(x,y)) & \left(\sum_{i,k,l,p,s=1}^{d} a^{ks}_\varepsilon \frac{\partial \tilde{x}^i}{\partial x^l}\frac{\partial \tilde{x}^i}{\partial x^p}a^{pr}_\varepsilon \frac{\partial^2 u^l_n}{\partial x^k \partial x^s} - \sum_{i=1}^{d} a^{ri}_\varepsilon \frac{\partial p_n}{\partial x^i}\right) \\
& - (1-\alpha(x,y))\left(\sum_{i,k,p,s=1}^{d} \delta^{ks}\delta_{ip}\delta_{pr}\frac{\partial^2 u^i_n}{\partial x^k \partial x^s} - \sum_{i=1}^{d}\delta^{ri}\frac{\partial p_n}{\partial x^i}\right),
\end{aligned}
$$

$$
\begin{aligned}
E^r_{2,n} &:= (1-\alpha(x,y)) \\
&\quad \times \sum_{i,k,l,p,s=1}^{d} \left\{ \frac{\partial a_\varepsilon^{ks}}{\partial x^k} \frac{\partial \tilde{x}^i}{\partial x^l} \frac{\partial \tilde{x}^i}{\partial x^p} a_\varepsilon^{pr} \frac{\partial u_n^l}{\partial x^s} + a_\varepsilon^{ks} \left(\frac{\partial^2 \tilde{x}^i}{\partial x^s \partial x^l} \frac{\partial u_n^l}{\partial x^k} + \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^l} \frac{\partial u_n^l}{\partial x^s} \right) \right\} \frac{\partial \tilde{x}^i}{\partial x^p} a_\varepsilon^{pr} \\
&\quad + \sum_{i,k,l,p,s=1}^{d} \left\{ \frac{\partial a_\varepsilon^{ks}}{\partial x^k} \frac{\partial^2 \tilde{x}^i}{\partial x^s \partial x^l} (1-\alpha(x,y)) \right. \\
&\qquad\qquad + a_\varepsilon^{ks} \left(\frac{\partial^3 \tilde{x}^i}{\partial x^k \partial x^s \partial x^l} (1-\alpha(x,y)) + \frac{\partial \tilde{x}^i}{\partial x^l} \frac{\partial^2 (1-\alpha(x,y))}{\partial x^k \partial x^s} \right. \\
&\qquad\qquad\quad \left. \left. + \frac{\partial^2 \tilde{x}^i}{\partial x^s \partial x^l} \frac{\partial (1-\alpha(x,y))}{\partial x^k} + \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^l} \frac{\partial (1-\alpha(x,y))}{\partial x^s} \right) \right\} \frac{\partial \tilde{x}^i}{\partial x^p} a_\varepsilon^{pr} u_n^l, \\
E^r_{3,n} &:= \sum_{i=1}^{d} -a_\varepsilon^{ri} \frac{\partial (1-\alpha(x,y))}{\partial x^i} p_n + \sum_{i=1}^{d} \delta^{ri} \frac{\partial (1-\alpha(x,y))}{\partial x^i} p_n, \\
E^r_{4,n} &:= \sum_{k,l,p,s=1}^{d} a_\varepsilon^{ks} a_{\varepsilon,lp} a_\varepsilon^{pr} \left(\frac{\partial (1-\alpha(x,y))}{\partial x^k} \frac{\partial u_n^l}{\partial x^s} + \frac{\partial (1-\alpha(x,y))}{\partial x^s} \frac{\partial u_n^l}{\partial x^k} \right) \\
&\quad - \sum_{i,k,p,s=1}^{d} \delta^{ks} \delta_{ip} \delta_{pr} \left(\frac{\partial (1-\alpha(x,y))}{\partial x^k} \frac{\partial u_n^i}{\partial x^s} + \frac{\partial (1-\alpha(x,y))}{\partial x^s} \frac{\partial u_n^i}{\partial x^k} \right), \\
E^r_{5,n} &:= \sum_{i,k,l,p,s=1}^{d} a_\varepsilon^{ks} \frac{\partial \tilde{x}^i}{\partial x^l} \frac{\partial^2 (1-\alpha(x,y))}{\partial x^k \partial x^s} \frac{\partial \tilde{x}^i}{\partial x^p} a_\varepsilon^{pr} u_n^l - \sum_{k,p,q,s=1}^{d} \delta^{ks} \delta_{ip} \delta_{pr} \frac{\partial^2 (1-\alpha(x,y))}{\partial x^k \partial x^s} u_n^i, \\
E^r_{6,n} &:= \mathcal{L}^r_\varepsilon(\boldsymbol{Q}_{0,n}, 0)(x,y) - \mathcal{L}^r_0(\boldsymbol{Q}_{0,n}, 0)(x,y), \qquad n, r = 1, \cdots, d.
\end{aligned}
$$

First we consider $E^r_{1,n}$. Since the cut-off function $\alpha(x,y) = 1$ for $x \in B_{\frac{d_y}{4}}(y)$ as in (3.4), we have $|1-\alpha(x,y)| \le C|x-y|$ for all $x \in B_{\frac{d_y}{4}}(y)$, which yields that

$$
\left| (1-\alpha(x,y)) \left(\sum_{k,l,p,q,s=1}^{d} a_\varepsilon^{ks} \frac{\partial \tilde{x}^i}{\partial x^l} \frac{\partial \tilde{x}^i}{\partial x^p} a_\varepsilon^{pr} \frac{\partial^2 u_n^l}{\partial x^k \partial x^s} - \sum_{i=1}^{d} a_\varepsilon^{ri} \frac{\partial p_n}{\partial x^i} \right) \right| \le \frac{C\varepsilon}{|x-y|^{d-1}},
$$

for $n, r = 1, \cdots, d$ and all $x \in \Omega$ with $x \ne y$, where $C = C(y, \Omega)$.

In the same manner as for $I^r_{1,n}$, $I^r_{2,n}$, $I^r_{3,n}$, $I^r_{4,n}$, $I^r_{5,n}$ and $I^r_{6,n}$, we can handle $E^r_{j,n}$ for $j = 1, \cdots, 6$ and $n, r = 1, \cdots, d$ so that (3.29) is established. This completes the proof of Lemma 3.2. □

4 Proof of the theorem.

4.1 Expansion of G_ε.

Our aim of this subsection is to investigate a continuous ε-dependence of G_ε, which yields necessarily an explicit representation of G_ε by means of the volume integral form containing the parametrix as in (3.20).

Lemma 4.1. *Let* $\{G_n^l\}_{n,l=1,\cdots,d}$ *and* $\{R_n\}_{n=1,\cdots,d}$ *be the Green matrix and the Green function defined by (2.1). Let* $\boldsymbol{U}_{\varepsilon,n} = \{U_{\varepsilon,n}^l\}_{n,l=1,\cdots,d}$ *and* $\{P_{\varepsilon,n}\}_{n=1,\cdots,d}$ *be the parametrix defined by (3.20). Then we have*

$$
\begin{aligned}
(4.1) \qquad & G_{\varepsilon,n}^m(\tilde{y},\tilde{z}) - G_n^m(y,z) \\
& = \{-U_{0,m}^n(z,y) + U_{\varepsilon,m}^n(z,y)\} + \varepsilon \sum_{j=1}^{d} \frac{\partial S^m}{\partial x^r}(z) G_n^r(z,y) \\
& + \int_\Omega \sum_{i,q,r=1}^{d} a_{\varepsilon,qr} \mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y) \frac{\partial x^q}{\partial \tilde{x}^i} G_m^i(x,z)\, dx + o(\varepsilon)
\end{aligned}
$$

for $m,n = 1,\cdots,d$ *and for all* $y,z \in \Omega$*, as* $\varepsilon \to 0$.

For the proof of Lemma 4.1, by the Green integral formula as in (2.24), we first express G_ε by an integral equation of the Fredholm type.

Proposition 4.1. *Let* $\{G_n^l\}_{l,n=1,\cdots,d}$ *and* $\{R_n\}_{n=1,\cdots,d}$ *be the Green matrix and function as in (2.1). Let* $\{U_{\varepsilon,n}^l\}_{n,l=1,\cdots,d}$ *and* $\{P_{\varepsilon,n}\}_{n=1,\cdots,d}$ *be the parametrix as in (3.20). Then we have*

$$
\begin{aligned}
& G_{\varepsilon,m}^n(\tilde{y},\tilde{z}) \\
(4.2) \qquad & = \{G_n^m(z,y) - U_{0,n}^m(z,y) + U_{\varepsilon,n}^m(z,y)\} + \varepsilon \sum_{r=1}^{d} \frac{\partial S^m}{\partial x^r}(z) G_n^r(z,y) \\
& + \int_\Omega \sum_{q,r=1}^{d} a_{\varepsilon,qr} \mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y) g_{\varepsilon,m}^q(x,z)\, dx + O(\varepsilon^2)
\end{aligned}
$$

for $m,n = 1,\cdots,d$ *and for all* $y,z \in \Omega$ *as* $\varepsilon \to 0$*, where* $\mathcal{L}_\varepsilon$ *is the operator defined by (2.22),* $\{g_{\varepsilon,m}^q\}_{m,q=1,\cdots,d}$ *is the function defined by (2.17) and where* $\{S^i\}_{i=1,\cdots,d}$ *is the vector function as in (A.3).*

Proof. We apply the Green integral formula (2.24) in $\Omega \setminus B_\rho(y) \cup B_\rho(z)$ to the functions

$$
\begin{cases}
v^i(x) = G_n^i(x,y) - U_{0,n}^i(x,y) + U_{\varepsilon,n}^i(x,y), \\
\pi(x) = R_n(x,y) - P_{0,n}(x,y) + P_{\varepsilon,n}(x,y),
\end{cases}
$$

$$
\begin{cases}
w^i(x) = g_{\varepsilon,m}^i(x,z), \\
\tilde{\pi}(x) = r_{\varepsilon,m}(x,z), \qquad\qquad i,m,n = 1,\cdots,d.
\end{cases}
$$

Since $G_n^i(\cdot,y)$, $U_{0,n}^i(\cdot,y)$, $U_{\varepsilon,n}^i(\cdot,y)$ and $g_{\varepsilon,m}^i(\cdot,z)$ vanish on $\partial\Omega$ for all $y,z \in \Omega$, $i,m,n = 1,\cdots,d$, we have by (2.21) and (2.24) that

$$
\begin{aligned}
(4.3) \qquad & \int_{\Omega \setminus B_\rho(y) \cup B_\rho(z)} \sum_{q,r=1}^{d} a_{\varepsilon,qr} \Big\{ \mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y) g_{\varepsilon,m}^q(x,z) \\
& \qquad - \mathcal{L}_\varepsilon^r(\boldsymbol{g}_{\varepsilon,m}, r_{\varepsilon,m})(x,z)(G_n^q - U_{0,n}^q + U_{\varepsilon,n}^q)(x,y) \Big\}\, dx
\end{aligned}
$$

$$
= \int\limits_{\partial B_\rho(y)\cup\partial B_\rho(z)} \Big\{ \sum_{k,q=1}^{d} T_\varepsilon^{kq}(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y) g_{\varepsilon,m}^q(x,z)\nu_x^k
- T_\varepsilon^{kq}(\boldsymbol{g}_{\varepsilon,m}, r_{\varepsilon,m})(x,z)(G_n^q - U_{0,n}^q + U_{\varepsilon,n}^q)(x,y)\nu_x^k \Big\} \, d\sigma_x
$$

for $m, n = 1, \cdots, d$ and for all sufficiently small $\rho > 0$. By (3.9), (3.22) and (3.23), we have similarly to (2.27) that

$$
\begin{aligned}
&\lim_{\rho\to 0} \int\limits_{\partial B_\rho(y)\cup\partial B_\rho(z)} \sum_{k,q=1}^{d} T_\varepsilon^{kq}(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y) g_{\varepsilon,m}^q(x,z)\nu_x^k \, d\sigma_x \\
&= \lim_{\rho\to 0} \int\limits_{\partial B_\rho(y)} \sum_{k,q=1}^{d} T_\varepsilon^{kq}(\boldsymbol{U}_{\varepsilon,n}, P_{\varepsilon,n})(x,y) g_{\varepsilon,m}^q(x,z)\nu_x^k \, d\sigma_x \\
&= \lim_{\rho\to 0} \int\limits_{\partial B_\rho(y)} \sum_{k,q=1}^{d} T_\varepsilon^{kq}(\alpha \boldsymbol{u}_{\varepsilon,n}, \alpha p_{\varepsilon,n})(x,y) g_{\varepsilon,m}^q(x,z)\nu_x^k \, d\sigma_x \\
&= \lim_{\rho\to 0} \int\limits_{\partial B_\rho(y)} \sum_{k,q=1}^{d} T_\varepsilon^{kq} \left(\alpha \sum_{j=1}^{d} \frac{\partial \boldsymbol{x}}{\partial \tilde{x}^j} u_n^j(\tilde{x},\tilde{y}), \alpha p_n(\tilde{x},\tilde{y}) \right) g_{\varepsilon,m}^q(x,z)\nu_x^k \, d\sigma_x \\
&= \sum_{q=1}^{d} \frac{\partial \tilde{x}^n}{\partial x^q} g_{\varepsilon,m}^q(y,z) + O(\varepsilon^2), \qquad m,n = 1,\cdots,d, \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{4.4}
$$

Note that $\alpha(y,y) = 1$ and that

$$
\begin{aligned}
\int_{\partial B_\rho(y)} |\nabla_x \alpha(x,y)||u_n(\tilde{x},\tilde{y})| d\sigma_x &\le C(y) \int_{\partial B_\rho(y)} \frac{d\sigma_x}{|\tilde{x}-\tilde{y}|^{d-2}} \\
&\le C(y) \int_{\partial B_\rho(y)} \frac{d\sigma_x}{|x-y|^{d-2}} + O(\varepsilon^2) \\
&\to O(\varepsilon^2), \qquad n = 1,\cdots,d, \quad \text{as } \rho \to 0.
\end{aligned} \tag{4.5}
$$

By Lemma 2.3 with y replaced by z, we have

$$
\begin{aligned}
&\lim_{\rho\to 0} \int\limits_{\partial B_\rho(y)} \sum_{k,q=1}^{d} T_\varepsilon^{kq}(\boldsymbol{g}_{\varepsilon,m}, r_{\varepsilon,m})(x,z)(G_n^q - U_{0,n}^q + U_{\varepsilon,n}^q)(x,y)\nu_x^k d\sigma_x \\
&= \sum_{q=1}^{d} \frac{\partial \tilde{x}^m}{\partial x^q}(z)(G_n^q - U_{0,n}^q + U_{\varepsilon,n}^q)(z,y) + O(\varepsilon^2), \qquad m,n = 1,\cdots,d, \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{4.6}
$$

Letting $\rho \to 0$ in (4.3), we obtain from (4.4), (4.5) and (4.6) with the aid of Proposition 2.1 and (3.24) that

$$
\int\limits_{\Omega} \sum_{q,r=1}^{d} a_{\varepsilon,qr} \mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y) g_{\varepsilon,m}^q(x,z) \, d\sigma_x
$$

$$= \sum_{q=1}^{d} \left\{ \frac{\partial \tilde{x}^n}{\partial x^q}(y) g^q_{\varepsilon,m}(y,z) - \frac{\partial \tilde{x}^m}{\partial x^q}(z)(G^q_n - U^q_{0,n} + U^q_{\varepsilon,n})(z,y) \right\}$$

$$= \sum_{j,q=1}^{d} \frac{\partial \tilde{x}^n}{\partial x^q} \frac{\partial x^q}{\partial \tilde{x}^j} G^j_{\varepsilon,m}(\tilde{y},\tilde{z}) - \sum_{q=1}^{d} \left(\delta^{mq} + \varepsilon \frac{\partial S^m}{\partial x^q} + O(\varepsilon^2) \right) (G^q_n - U^q_{0,n} + U^q_{\varepsilon,n})(z,y)$$

$$= G^n_{\varepsilon,m}(\tilde{y},\tilde{z}) - \left\{ G^m_n(z,y) - U^m_{0,n}(z,y) + U^m_{\varepsilon,n}(z,y) + \varepsilon \sum_{q=1}^{d} \frac{\partial S^m}{\partial x^q}(z) G^q_n(z,y) \right\} + O(\varepsilon^2),$$

for $m, n = 1, \cdots, d$, as $\varepsilon \to 0$, which implies (4.2). □

Since $\{g^r_{\varepsilon,m}\}_{m,r=1,\cdots,d}$ is expressed by G_ε as in (2.17), we may regard (4.2) as an integral equation for G_ε of the Fredholm type. Garabedian [6] and Garabedian-Schiffer [7] solved this integral equation for G_ε in terms of the Fredholm alternative theorem. On the other hand, our method relies on treating G_ε directly by means of the fundamental solution $\boldsymbol{u}_\varepsilon$ and the compensation term $\boldsymbol{q}_{\varepsilon,n}$. Such an idea was introduced by Ozawa [10]. To this end, we need to investigate the behavior of $\boldsymbol{q}_{\varepsilon,n}$ as $\varepsilon \to 0$. We first establish an a priori estimate for the operator $\mathcal{L}_\varepsilon$.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with the $C^{2+\theta}$-boundary $\partial\Omega$, $0 < \theta < 1$. There exist $\varepsilon_0(\Omega) > 0$ and $C(\Omega) > 0$ such that if $\varepsilon \le \varepsilon_0$, then it holds that*

$$\|\boldsymbol{v}\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla \pi\|_{C^\theta(\overline{\Omega})} \le C \left(\|\mathcal{L}_\varepsilon(\boldsymbol{v},\pi)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}\|_{C^{2+\theta}(\partial\Omega)} \right) \tag{4.7}$$

for all $\boldsymbol{v} \in C^{2+\theta}(\overline{\Omega})^d$ and $\pi \in C^{1+\theta}(\overline{\Omega})$.

Proof. We shall show by a contradiction argument. Suppose that (4.7) is not true. Then for any $m = 1, 2, \cdots$, there exist $\boldsymbol{v}_m \in C^{2+\theta}(\overline{\Omega})^d$ and $\pi_m \in C^{1+\theta}(\overline{\Omega})$ with $\|\boldsymbol{v}_m\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla \pi_m\|_{C^\theta(\overline{\Omega})} \equiv 1$ such that

$$\frac{1}{m} > \left(\|\mathcal{L}_{\frac{1}{m}}(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}_m\|_{C^{2+\theta}(\partial\Omega)} \right), \quad m = 1, 2, \cdots. \tag{4.8}$$

On the other hand, by (A.3), (4.8) and an a priori estimate for the Stokes equations(see e.g., (Solonnikov [15, Theorem 3.1]), it holds that

$$\begin{aligned} &\|\boldsymbol{v}_m\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla \pi_m\|_{C^\theta(\overline{\Omega})} \\ &\le M \left(\|\mathcal{L}(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}_m\|_{C^{2+\theta}(\partial\Omega)} \right) \\ &\le M \left(\|(\mathcal{L}_0 - \mathcal{L}_{\frac{1}{m}})(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})} + \|\mathcal{L}_{\frac{1}{m}}(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}_m\|_{C^{2+\theta}(\partial\Omega)} \right) \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\le M \left\{ \frac{1}{m}(\|\boldsymbol{v}_m\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla \pi_m\|_{C^\theta(\overline{\Omega})}) + (\|\mathcal{L}_{\frac{1}{m}}(\boldsymbol{v}_m,\pi_m)\|_{C^\theta(\overline{\Omega})} + \|\boldsymbol{v}_m\|_{C^{2+\theta}(\partial\Omega)}) \right\} \\ &\le 2M \frac{1}{m} \to 0, \quad \text{as} \quad m \to \infty, \end{aligned}$$

where M is a constant independent of m. Since $\|\boldsymbol{v}_m\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla \pi_m\|_{C^\theta(\overline{\Omega})} \equiv 1$ for $m = 1, 2, \cdots$, this causes a contradiction. □

We define the compensation term of the Green function $\{\boldsymbol{q}_{\varepsilon,n}, q_n\}_{n=1,\cdots,d}$ by

$$
\begin{aligned}
\boldsymbol{q}_{\varepsilon,n}(x,y) &:= \boldsymbol{g}_{\varepsilon,n}(x,y) - \boldsymbol{u}_{\varepsilon,n}(x,y) \\
q_{\varepsilon,n}(x,y) &:= r_{\varepsilon,n}(x,y) - p_{\varepsilon,n}(x,y), \quad n = 1,\cdots,d,
\end{aligned} \tag{4.10}
$$

where $\{\boldsymbol{g}_{\varepsilon,n}\}_{n=1,\cdots,d}$, $\{\boldsymbol{u}_{\varepsilon,n}\}_{1,\cdots,d}$, $\{r_{\varepsilon,n}(x,y)\}_{1,\cdots,d}$ and $\{p_{\varepsilon,n}\}_{1,\cdots,d}$ are introduced by (2.17), (3.5), (2.1) and (3.19), respectively.

By Proposition 4.2, we have a continuous ε-dependence of the compensation term $\{\boldsymbol{q}_{\varepsilon,n}\}_{n=1,\cdots,d}$ defined in (4.10) in the topology of $C^{2+\theta}(\overline{\Omega})$. Indeed, it holds;

Proposition 4.3. *For every $y \in \Omega$, we have*

$$
\|\boldsymbol{q}_{\varepsilon,n}(\cdot,y) - \boldsymbol{q}_{0,n}(\cdot,y)\|_{C^{2+\theta}(\overline{\Omega})} \to 0, \qquad n = 1,\cdots,d \quad \text{as } \ \varepsilon \to 0, \tag{4.11}
$$

where $\{\boldsymbol{q}_{\varepsilon,n}\}_{n=1,\cdots,d}$ is the compensation term defined by (4.10).

Proof. By Proposition 4.2, we have

$$
\begin{aligned}
&\|\boldsymbol{q}_{\varepsilon,n}(\cdot,y)\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla q_{\varepsilon,n}(\cdot,y)\|_{C^{\theta}(\overline{\Omega})} \\
&\qquad \le C\left(\|\mathcal{L}_\varepsilon(\boldsymbol{q}_{\varepsilon,n}, q_{\varepsilon,n})(\cdot,y)\|_{C^{\theta}(\overline{\Omega})} + \|\boldsymbol{q}_{\varepsilon,n}(\cdot,y)\|_{C^{2+\theta}(\partial\Omega)}\right).
\end{aligned} \tag{4.12}
$$

for all $\varepsilon \le \varepsilon_0$. On the other hand, for each fixed $y \in \Omega$, there exists an $\varepsilon_0'(y) > 0$ such that

$$
\|\boldsymbol{q}_{\varepsilon,n}(\cdot,y)\|_{C^{2+\theta}(\partial\Omega)} = \|\boldsymbol{u}_{\varepsilon,n}(\cdot,y)\|_{C^{2+\theta}(\partial\Omega)} < 2\|\boldsymbol{u}_{0,n}(\cdot,y)\|_{C^{2+\theta}(\partial\Omega)} \tag{4.13}
$$

for all $0 < \varepsilon \le \varepsilon_0'(y)$. Since $\mathcal{L}_\varepsilon(\boldsymbol{q}_{\varepsilon,n}, q_{\varepsilon,n})(\cdot,y) = 0$, implied by (4.10), it follows from (4.12) and (4.13) that

$$
\|\boldsymbol{q}_{\varepsilon,n}(\cdot,y)\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla q_{\varepsilon,n}(\cdot,y)\|_{C^{\theta}(\overline{\Omega})} \le 2C\|\boldsymbol{u}_{0,n}(\cdot,y)\|_{C^{2+\theta}(\partial\Omega)} := C_y \tag{4.14}
$$

for all $0 < \varepsilon \le \min\{\varepsilon_0, \varepsilon_0'(y)\}$, where C_y is the constant depending only on $y \in \Omega$.

Furthermore, by a direct calculation, we see that the pair $\{\boldsymbol{q}_{\varepsilon,n} - \boldsymbol{q}_{0,n}, q_{\varepsilon,n} - q_{0,n}\}$ is subject to the following identities.

$$
\begin{cases}
\mathcal{L}(\boldsymbol{q}_{\varepsilon,n} - \boldsymbol{q}_{0,n}, q_{\varepsilon,n} - q_{0,n})(x,y) = (\mathcal{L} - \mathcal{L}_\varepsilon)(\boldsymbol{q}_{\varepsilon,n}, q_{\varepsilon,n})(x,y), & x \in \Omega, \\
\operatorname{div}(\boldsymbol{q}_{\varepsilon,n} - \boldsymbol{q}_{0,n})(x,y) = 0, & x \in \Omega, \\
(\boldsymbol{q}_{\varepsilon,n} - \boldsymbol{q}_{0,n})(x,y) = (-\boldsymbol{u}_{\varepsilon,n} + \boldsymbol{u}_{0,n})(x,y), & x \in \partial\Omega
\end{cases} \tag{4.15}
$$

for $n = 1,\cdots,d$. Hence by (4.14), (A.3) and an a priori estimate for the Stokes equations(see e.g., Solonnikov [15, Theorem 3.1]), we have that

$$
\begin{aligned}
&\|(\boldsymbol{q}_{\varepsilon,n} - \boldsymbol{q}_{0,n})(\cdot,y)\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla(q_{\varepsilon,n} - q_{0,n})(\cdot,y)\|_{C^{\theta}(\overline{\Omega})} \\
&\le M\left(\|(\mathcal{L} - \mathcal{L}_\varepsilon)(\boldsymbol{q}_{\varepsilon,n}, q_{\varepsilon,n})(\cdot,y)\|_{C^{\theta}(\overline{\Omega})} + \|(-\boldsymbol{u}_{\varepsilon,n} + \boldsymbol{u}_{0,n})(\cdot,y)\|_{C^{2+\theta}(\partial\Omega)}\right) \\
&\le M\bigl\{\varepsilon(\|\boldsymbol{q}_{\varepsilon,n}(\cdot,y)\|_{C^{2+\theta}(\overline{\Omega})} + \|\nabla q_{\varepsilon,n}(\cdot,y)\|_{C^{\theta}(\overline{\Omega})}) + \|(-\boldsymbol{u}_{\varepsilon,n} + \boldsymbol{u}_{0,n})(\cdot,y)\|_{C^{2+\theta}(\partial\Omega)}\bigr\} \\
&\le M(\varepsilon C_y + \|(-\boldsymbol{u}_{\varepsilon,n} + \boldsymbol{u}_{0,n})(\cdot,y)\|_{C^{2+\theta}(\partial\Omega)})
\end{aligned}
$$

for all $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_0'(y)\}$. Since M is the constant independent of ε as in (4.9), from (3.17) and this estimate, we obtain the desired result (4.11). □

Proposition 4.3 enables us to expand G_ε in terms of G_0 with respect to $\varepsilon \ll 1$. Now by (4.2) and (4.11), we are in a position to prove Lemma 4.1.

Proof of Lemma 4.1. We deal with the volume integral of the right hand side of (4.2). It holds by (2.17) that

$$
\begin{aligned}
(4.16)\qquad & \int_\Omega \sum_{q,r=1}^{d} a_{\varepsilon,qr}\mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y) g^q_{\varepsilon,m}(x,z)\,dx \\
&= \int_\Omega \sum_{i,q,r=1}^{d} a_{\varepsilon,qr}\mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial x^q}{\partial \tilde{x}^i} G^i_{\varepsilon,m}(\tilde{x},\tilde{z})\,dx \\
&= \int_\Omega \sum_{i,q,r=1}^{d} a_{\varepsilon,qr}\mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial x^q}{\partial \tilde{x}^i} G^i_{m}(x,z)\,dx + I,
\end{aligned}
$$

where

$$
I := \int_\Omega \sum_{i,q,r=1}^{d} a_{\varepsilon,qr}\mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial x^q}{\partial \tilde{x}^i}(G^i_{\varepsilon,m}(\tilde{x},\tilde{z}) - G^i_m(x,z))\,dx
$$

for $m, n = 1, \cdots, d$. By (2.1), (3.5), (4.10) and Proposition 2.1, we have

$$
\begin{aligned}
&\sum_{i=1}^{d} \frac{\partial x^q}{\partial \tilde{x}^i}(G^i_{\varepsilon,m}(\tilde{x},\tilde{z}) - G^i_m(x,z)) \\
&= \sum_{i=1}^{d}\left(\frac{\partial x^q}{\partial \tilde{x}^i}G^i_{\varepsilon,m}(\tilde{x},\tilde{z}) - \frac{\partial x^q}{\partial \tilde{x}^i}G^i_m(x,z)\right) \\
&= \sum_{i=1}^{d}\left\{\frac{\partial x^q}{\partial \tilde{x}^i}G^i_{\varepsilon,m}(\tilde{x},\tilde{z}) - \left(\delta^{qi} - \varepsilon\frac{\partial S^q}{\partial x^i}\right)G^i_m(x,z)\right\} + O(\varepsilon^2) \\
&= \left\{(u^q_{\varepsilon,m}(x,z) - q^q_{\varepsilon,m}(x,z)) - (u^q_m(x,z) - q^q_m(x,z))\right\} + \varepsilon\sum_{i=1}^{d}\frac{\partial S^q}{\partial x^i}G^i_m(x,z) + O(\varepsilon^2) \\
&= \left\{(u^q_{\varepsilon,m}(x,z) - u^q_m(x,z)) - (q^q_{\varepsilon,m}(x,z) - q^q_m(x,z))\right\} + \varepsilon\sum_{i=1}^{d}\frac{\partial S^q}{\partial x^i}G^i_m(x,z) + O(\varepsilon^2)
\end{aligned}
$$

for $m, q = 1, \cdots, d$, as $\varepsilon \to 0$. Therefore, I can be divided to the following three terms,

$$
I = I_1 + I_2 + I_3 + O(\varepsilon^2), \qquad \text{as} \ \ \varepsilon \to 0,
$$

where

$$
I_1 = \int_\Omega \sum_{q,r=1}^{d} a_{\varepsilon,qr}\mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)(u^q_{\varepsilon,m}(x,z) - u^q_m(x,z))\,dx,
$$

$$
I_2 = -\int_\Omega \sum_{q,r=1}^{d} a_{\varepsilon,qr}\mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)(q^q_{\varepsilon,m}(x,z) - q^q_m(x,z))\,dx,
$$

$$I_3 = \varepsilon \int_\Omega \sum_{i,q,r=1=1}^{d} a_{\varepsilon,qr} \mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y) \frac{\partial S^q}{\partial x^i}(x) G^i_m(x,z)\, dx.$$

First we consider I_1. Since we have by (3.17)

$$u^q_{\varepsilon,m}(x,z) = u^q_m(x,z) + O(\varepsilon|x-z|^{2-d}), \qquad m,q = 1,\cdots,d, \quad \text{as } \varepsilon \to 0, \quad x \to z,$$

it follows from (3.26) that

$$\begin{aligned} I_1 &\le \int_\Omega \frac{\varepsilon C}{|x-y|^{d-1}} |\boldsymbol{u}_{\varepsilon,m}(x,z) - \boldsymbol{u}_m(x,z)|\, dx \\ &\le \varepsilon^2 C \int_\Omega \frac{1}{|x-y|^{d-1}} \frac{1}{|x-z|^{d-2}}\, dx \\ &\le \varepsilon^2 C |y-z|^{d-3}, \end{aligned} \tag{4.17}$$

where $C = C(y,\Omega)$. Concerning I_2, by Proposition 4.3, we have similarly that

$$\begin{aligned} I_2 &\le \varepsilon C \|q^q_{\varepsilon,m}(\cdot,z) - q^q_m(\cdot,z)\|_{C^0(\Omega)} \int_\Omega \frac{1}{|x-y|^{d-1}}\, dx \\ &\le \varepsilon C \|\boldsymbol{q}_{\varepsilon,m}(\cdot,z) - \boldsymbol{q}_m(\cdot,z)\|_{C^0(\Omega)} \to 0, \qquad \text{as } \varepsilon \to 0, \end{aligned} \tag{4.18}$$

where $C = C(y,\Omega)$. Finally in the same manner as in I_1, we obtain

$$I_3 \le \varepsilon^2 C |y-z|^{d-3}, \tag{4.19}$$

where $C = C(y,\Omega)$. Hence by (4.17), (4.18) and (4.19), it holds that $I = o(\varepsilon)$ as $\varepsilon \to 0$, from which and (4.2) with (4.16), we obtain the desired identity (4.1). This proves Lemma 4.1. □

4.2 How to handle the parametrix.

In the previous subsection, we obtain the representation for G_ε by the integral equation with the parametrix. In this subsection, we shall establish an expression for G_ε without the parametrix.

Lemma 4.2. *Let $\{G^l_n\}_{n,l=1,\cdots,d}$ and $\{R_n\}_{n=1,\cdots,d}$ be the Green matrix and the Green function defined by (2.1). Then we have*

(4.20)

$$\begin{aligned} &G^n_{\varepsilon,m}(\tilde{y},\tilde{z}) - G^n_m(y,z) \\ &= \varepsilon \sum_{k=1}^{d} \frac{\partial S^m}{\partial x^k}(z) G^k_n(z,y) \\ &- \varepsilon \int_\Omega \Bigg\{ \sum_{i,k,s=1}^{d} - \left(\frac{\partial S^k(x)}{\partial x^s} + \frac{\partial S^s(x)}{\partial x^k} \right) \frac{\partial G^i_n(x,y)}{\partial x^k} \frac{\partial G^i_m(x,z)}{\partial x^s} + \sum_{i,k,l=1}^{d} \frac{\partial S^i(x)}{\partial x^l} \frac{\partial G^l_n(x,y)}{\partial x^k} \frac{\partial G^i_m(x,z)}{\partial x^k} \\ &\qquad + \sum_{j,k=1}^{d} \frac{\partial S^k(x)}{\partial x^j} R_n(x,y) \frac{\partial G^j_m(x,z)}{\partial x^k} + \sum_{i,k,l=1}^{d} \frac{\partial^2 S^i(x)}{\partial x^k \partial x^l} G^l_n(x,y) \frac{\partial G^i_m(x,z)}{\partial x^k} \Bigg\}\, dx + O(\varepsilon^2) \end{aligned}$$

for $m,n = 1,\cdots,d$ and for all $y,z \in \Omega$ as $\varepsilon \to 0$, where $\{S^i\}_{i=1,\cdots,d}$ is the vector function introduced by (A.3).

For the proof of Lemma 4.2, we need to investigate the volume integral of the right hand side of (4.1) as in Lemma 4.1.

Proposition 4.4. *Let $\{G_n^l\}_{n,l=1,\cdots,d}$ and $\{R_n\}_{n=1,\cdots,d}$ be the Green matrix and the Green function defined by (2.1). Let $\{U_{\varepsilon,n}^l\}_{n,l=1,\cdots,d}$ and $\{P_{\varepsilon,n}\}_{n=1,\cdots,d}$ be the parametrix defined by (3.20). Then we have*

$$\begin{aligned}
(4.21)\qquad &\int_\Omega \sum_{j,q,r=1}^d a_{\varepsilon,qr}\mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial x^q}{\partial \tilde{x}^j}G_m^j(x,z)\,dx \\
&= U_{0,n}^m(z,y) - U_{\varepsilon,n}^m(z,y) \\
&- \varepsilon \int_\Omega \Bigg\{ \sum_{i,k,s=1}^d -\left(\frac{\partial S^k(x)}{\partial x^s} + \frac{\partial S^s(x)}{\partial x^k}\right)\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s} \\
&\qquad + \sum_{i,k,l=1}^d \frac{\partial S^i(x)}{\partial x^l}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k} + \sum_{j,k=1}^d \frac{\partial S^k(x)}{\partial x^j}R_n(x,y)\frac{\partial G_m^j(x,z)}{\partial x^k} \\
&\qquad + \sum_{i,k,l=1}^d \frac{\partial^2 S^i(x)}{\partial x^k \partial x^l}G_n^l(x,y)\frac{\partial G_m^i(x,z)}{\partial x^k}\Bigg\}\,dx + O(\varepsilon^2), \quad m,n = 1,\cdots,d
\end{aligned}$$

for all $y,z \in \Omega$ as $\varepsilon \to 0$, where $\mathcal{L}_\varepsilon$ is the operator defined by (2.22) and $\{S^i\}_{i=1,\cdots,d}$ is the vector function introduced by (A.3).

Proof of Lemma 4.2. It is easy to see that Lemma 4.1 and Proposition 4.4 yield Lemma 4.2. □

Now, it remains to prove Proposition 4.4.

Proof of Proposition 4.4. Integrating by parts, we have by (3.22), (3.23) and Lemma 2.3 that

(4.22)

$$\begin{aligned}
&\int_\Omega \sum_{j,p,r=1}^d a_{\varepsilon,qr}\mathcal{L}_\varepsilon^r(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial x^q}{\partial \tilde{x}^j}G_m^j(x,z)\,dx \\
&= \lim_{\rho\to 0}\int_{\partial B_\rho(y)} \sum_{j,k,p=1}^d T_\varepsilon^{kq}(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial x^q}{\partial \tilde{x}^j}G_m^j(x,z)\nu_x^k\,d\sigma_x \\
&\quad - \lim_{\rho\to 0}\int_{\Omega\setminus B_\rho(y)} \Bigg[\sum_{j,k,p=1}^d -\delta^{kq}(R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial}{\partial x^k}\left\{\frac{\partial x^q}{\partial \tilde{x}^j}G_m^j(x,z)\right\} \\
&\qquad + \sum_{i,j,k,l,q,s=1}^d a_\varepsilon^{ks}\frac{\partial}{\partial x^s}\left(\frac{\partial \tilde{x}^i}{\partial x^l}(G_n^l - U_{0,n}^l + U_{\varepsilon.n}^l)(x,y)\right)\frac{\partial}{\partial x^k}\left\{\frac{\partial \tilde{x}^i}{\partial x^q}\frac{\partial x^q}{\partial \tilde{x}^j}G_m^j(x,z)\right\}\Bigg]\,dx \\
&= \sum_{i,q=1}^d \frac{\partial \tilde{x}^n}{\partial x^q}(y)\frac{\partial x^q}{\partial \tilde{x}^i}(y)G_m^i(y,z)
\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \Big[\sum_{j,k,q=1}^{d} -\delta^{kq}(R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial}{\partial x^k}\Big\{ \frac{\partial x^q}{\partial \tilde{x}^j} G^j_m(x,z) \Big\} \\
&\qquad + \sum_{i,k,l,s=1}^{d} a^{ks}_\varepsilon \frac{\partial}{\partial x^s}\left(\frac{\partial \tilde{x}^i}{\partial x^l}(G^l_n - U^l_{0.n} + U^l_{\varepsilon.n})(x,y)\right) \frac{\partial G^i_m(x,z)}{\partial x^k} \Big] \, dx \\
&= G^n_m(y,z) \\
&\quad -\int_\Omega \Big[\sum_{j,k,q=1}^{d} -\delta^{kq}(R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial}{\partial x^k}\Big\{ \frac{\partial x^q}{\partial \tilde{x}^j} G^j_m(x,z) \Big\} \\
&\qquad + \sum_{i,k,l,s=1}^{d} a^{ks}_\varepsilon \frac{\partial}{\partial x^s}\left(\frac{\partial \tilde{x}^i}{\partial x^l}(G^l_n - U^l_{0.n} + U^l_{\varepsilon.n})(x,y)\right) \frac{\partial G^i_m(x,z)}{\partial x^k} \Big] \, dx, \quad m,n = 1,\cdots,d.
\end{aligned}
$$

Applying Proposition 2.1 and Lemma 3.2 together with the fact that $\operatorname{div} \boldsymbol{G}_m = 0$, and then using Lemma 2.2 to the second and third terms of the right hand side, we have

(4.23)

$$
\begin{aligned}
&\int_\Omega \sum_{j,q,r=1}^{d} a_{\varepsilon,qr}\mathcal{L}^r_\varepsilon(\boldsymbol{G}_n - \boldsymbol{U}_{0,n} + \boldsymbol{U}_{\varepsilon,n}, R_n - P_{0,n} + P_{\varepsilon,n})(x,y)\frac{\partial x^q}{\partial \tilde{x}^j} G^j_m(x,z)\, dx \\
&= G^n_m(y,z) \\
&\quad - \int_\Omega \sum_{i,k=1}^{d} \frac{\partial (G^i_n - U^i_{0,n} + U^i_{\varepsilon,n})(x,y)}{\partial x^k}\frac{\partial G^i_m(x,z)}{\partial x^k}\, dx \\
&\quad - \varepsilon \int_\Omega \Big\{ \sum_{i,k,l=1}^{d} \frac{\partial^2 S^i(x)}{\partial x^k \partial x^l} G^l_n(x,y)\frac{\partial G^i_m(x,z)}{\partial x^k} + \sum_{i,k,l=1}^{d} \frac{\partial S^i(x)}{\partial x^l}\frac{\partial G^l_n(x,y)}{\partial x^k}\frac{\partial G^i_m(x,z)}{\partial x^k} \\
&\qquad + \sum_{j,k=1}^{d} \frac{\partial S^k(x)}{\partial x^j} R_n(x,y) \frac{\partial G^j_m(x,z)}{\partial x^k} - \sum_{i,k,s=1}^{d}\left(\frac{\partial S^k(x)}{\partial x^s} + \frac{\partial S^s(x)}{\partial x^k}\right)\frac{\partial G^i_n(x,y)}{\partial x^k}\frac{\partial G^i_m(x,z)}{\partial x^s} \Big\}\, dx \\
&= U^m_{0,n}(z,y) - U^m_{\varepsilon,n}(z,y) \\
&\quad - \varepsilon \int_\Omega \Big\{ -\sum_{i,k,s=1}^{d}\left(\frac{\partial S^k(x)}{\partial x^s} + \frac{\partial S^s(x)}{\partial x^k}\right)\frac{\partial G^i_n(x,y)}{\partial x^k}\frac{\partial G^i_m(x,z)}{\partial x^s} + \sum_{i,k,l=1}^{d} \frac{\partial S^i(x)}{\partial x^l}\frac{\partial G^l_n(x,y)}{\partial x^k}\frac{\partial G^i_m(x,z)}{\partial x^k} \\
&\qquad + \sum_{j,k=1}^{d} \frac{\partial S^k(x)}{\partial x^j} R_n(x,y) \frac{\partial G^j_m(x,z)}{\partial x^k} + \sum_{i,k,l=1}^{d} \frac{\partial^2 S^i(x)}{\partial x^k \partial x^l} G^l_n(x,y)\frac{\partial G^i_m(x,z)}{\partial x^k} \Big\}\, dx + O(\varepsilon^2)
\end{aligned}
$$

for $m,n = 1,\cdots,d$ as $\varepsilon \to 0$, which implies (4.21). □

4.3 Expression by the surface integral.

By Lemma 4.2, we have succeeded to express G_ε by means of the volume integral over Ω. In this subsection, we shall establish a representation in terms of the *surface* integral on $\partial\Omega$.

Lemma 4.3. *Let $\{G_n^l\}_{n,l=1,\cdots,d}$ and $\{R_n\}_{n=1,\cdots,d}$ be the Green matrix and the Green function defined by (2.1). Then we have*

$$
\begin{aligned}
&G_{\varepsilon,m}^n(\tilde{y},\tilde{z}) - G_m^n(y,z) \\
&= \varepsilon\sum_{s=1}^d \left(\frac{\partial G_m^n(y,z)}{\partial y^s}S^s(y) + \frac{\partial G_m^n(y,z)}{\partial z^s}S^s(z)\right) \\
&+ \varepsilon\int_{\partial\Omega}\sum_{i=1}^d \Big\{\frac{\partial G_m^i(x,z)}{\partial\nu_x}\frac{\partial G_n^i(x,y)}{\partial\nu_x} \\
&\qquad - \left(R_m(x,z)\frac{\partial G_n^i(x,y)}{\partial\nu_x} + \frac{\partial G_m^i(x,z)}{\partial\nu_x}R_n(x,y)\right)\nu_x^i\Big\}S(x)\cdot\nu_x\,d\sigma_x + o(\varepsilon), \quad m,n=1,\cdots,d
\end{aligned}
$$

for all $y,z\in\Omega$ as $\varepsilon\to 0$, where $\{S^i\}_{i=1,\cdots,d}$ is the vector function introduced by (A.3), $\nu_x = (\nu_x^1,\cdots,\nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x\in\partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

For the proof of Lemma 4.3, we should investigate the volume integral as in Lemma 4.2. By a direct calculation, we have the following representation for the volume integral in (4.20).

Proposition 4.5. *The volume integral over Ω in (4.20) can be expressed by the following surface integral on $\partial\Omega$. Namely, we have the identity*

(4.24)

$$
\begin{aligned}
&\int_\Omega\Big\{-\sum_{i,k,s=1}^d\left(\frac{\partial S^k(x)}{\partial x^s}+\frac{\partial S^s(x)}{\partial x^k}\right)\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s} + \sum_{i,k,l=1}^d\frac{\partial S^i(x)}{\partial x^l}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k} \\
&\qquad + \sum_{j,k=1}^d\frac{\partial S^k(x)}{\partial x^j}R_n(x,y)\frac{\partial G_m^j(x,z)}{\partial x^k} + \sum_{i,k,l=1}^d\frac{\partial^2 S^i(x)}{\partial x^k\partial x^l}G_n^l(x,y)\frac{\partial G_m^i(x,z)}{\partial x^k}\Big\}\,dx \\
&= \sum_{k=1}^d\frac{\partial S^m}{\partial x^k}(z)G_n^k(z,y) - \sum_{s=1}^d\left(\frac{\partial G_n^m(z,y)}{\partial z^s}S^s(y) + \frac{\partial G_m^n(y,z)}{\partial y^s}S^s(y)\right) \\
&+ \int_{\partial\Omega}\Big\{\sum_{i,k,s=1}^d\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^s(x)\nu_x^s - \sum_{i,k,s=1}^d T^{ik}(\boldsymbol{G}_n,R_n)(x,y)\frac{\partial G_m^i(x,z)}{\partial x^s}S^s(x)\nu_x^k \\
&\qquad - \sum_{i,k,l=1}^d T^{lk}(\boldsymbol{G_m},R_m)(x,z)\frac{\partial G_n^l(x,y)}{\partial x^i}S^i(x)\nu_x^k\Big\}\,d\sigma_x, \quad m,n=1,\cdots,d
\end{aligned}
$$

for all $y,z\in\Omega$, where $\{S^i\}_{i=1,\cdots,d}$ is the vector function introduced by (A.3), $\nu_x=(\nu_x^1,\cdots,\nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x\in\partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

Proof of Lemma 4.3. Since $\{G_n^i\}_{i,n=1,\cdots,d}=0$ on $\partial\Omega$, we have

$$\frac{\partial G_n^i(x,y)}{\partial x^k} = \frac{\partial G_n^i(x,y)}{\partial\nu_x}\nu_x^k, \quad i,k,n=1,\cdots,d, \tag{4.25}$$

for $x \in \partial\Omega$. Then it is easy to see that Lemma 4.2 and Proposition 4.5 yield Lemma 4.3. $\square$

Now, it remains to prove Proposition 4.5.

Proof of Proposition 4.5. It follows from (1.1) that

$$\sum_{k,l,q=1}^{d} -\frac{\partial^2 S^i(x)}{\partial x^k \partial x^l}\delta^{ki} R_m(x,z) G_n^l(x,y) = 0, \quad m,n = 1,\cdots,d, \tag{4.26}$$

for all $x \in \Omega$. Adding the left hand side of (4.26) to that of (4.24), we have by integration by parts that

(4.27)

$$\begin{aligned}
&\int_\Omega \Bigg\{ -\sum_{i,k,s=1}^{d}\left(\frac{\partial S^k(x)}{\partial x^s}+\frac{\partial S^s(x)}{\partial x^k}\right)\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s} + \sum_{i,k,l=1}^{d}\frac{\partial S^i(x)}{\partial x^l}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k} \\
&\qquad + \sum_{k,j=1}^{d}\frac{\partial S^k(x)}{\partial x^j}R_n(x,y)\frac{\partial G_m^j(x,z)}{\partial x^k} + \sum_{i,k,l=1}^{d}\frac{\partial^2 S^i(x)}{\partial x^k\partial x^l}G_n^l(x,y)\frac{\partial G_m^i(x,z)}{\partial x^k}\Bigg\}\,dx \\
&= \int_\Omega \Bigg\{ -\sum_{i,k,s=1}^{d}\left(\frac{\partial S^k(x)}{\partial x^s}+\frac{\partial S^s(x)}{\partial x^k}\right)\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s} \\
&\qquad + \sum_{i,k,l=1}^{d}\frac{\partial S^i(x)}{\partial x^l}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k} + \sum_{j,k=1}^{d}\frac{\partial S^k(x)}{\partial x^j}R_n(x,y)\frac{\partial G_m^j(x,z)}{\partial x^k} \\
&\qquad + \sum_{i,k,l=1}^{d}\frac{\partial^2 S^i(x)}{\partial x^k\partial x^l}G_n^l(x,y)\frac{\partial G_m^i(x,z)}{\partial x^k} - \sum_{i,k,l=1}^{d}\frac{\partial^2 S^i(x)}{\partial x^k\partial x^l}\delta^{ki}R_m(x,z)G_n^l(x,y)\Bigg\}\,dx \\
&= \lim_{\rho\to 0}\int_{\partial\{\Omega\backslash B_\rho(y)\cup B_\rho(z)\}} \Bigg\{ -\sum_{i,k,s=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s}S^k(x)\nu_x^s - \sum_{i,k,s=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s}S^s(x)\nu_x^k \\
&\qquad + \sum_{i,k,l=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^i(x)\nu_x^l + \sum_{j,k=1}^{d}R_n(x,y)\frac{\partial G_m^j(x,z)}{\partial x^k}S^k(x)\nu_x^j \\
&\qquad + \sum_{i,k,l=1}^{d}T^{ik}(\boldsymbol{G}_m,R_m)(x,z)G_n^l(x,y)\frac{\partial S^i(x)}{\partial x^l}\nu_x^k\Bigg\}\,d\sigma_x \\
&\quad - \lim_{\rho\to 0}\int_{\Omega\backslash B_\rho(y)\cup B_\rho(z)} \Bigg\{ -\sum_{k=1}^{d}\left(\sum_{i,s=1}^{d}\frac{\partial^2 G_n^i(x,y)}{\partial x^s\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s} + \sum_{i=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\Delta G_m^i(x,z)\right)S^k(x) \\
&\qquad - \sum_{s=1}^{d}\left(\sum_{i=1}\mathcal{L}^i(\boldsymbol{G}_n,R_n)(x,y)\frac{\partial G_m^i(x,z)}{\partial x^s} + \sum_{i,k=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial^2 G_m^i(x,z)}{\partial x^k\partial x^s}\right)S^s(x)
\end{aligned}$$

$$+\sum_{i,k,l=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial^2 G_m^i(x,z)}{\partial x^l\partial x^k}S^i(x)$$

$$+\sum_{i,l=1}^{d}\left(\sum_{k=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}T^{ik}(\boldsymbol{G}_m,R_m)(x,z)+\mathcal{L}^i(\boldsymbol{G}_m,R_m)(x,z)G_n^l(x,y)\right)\frac{\partial S^i(x)}{\partial x^l}\Bigg\}\,dx$$

$$\equiv I_s+(I_v^1+I_v^2),$$

where

$$I_s:=\lim_{\rho\to 0}\int_{\partial\{\Omega\setminus B_\rho(y)\cup B_\rho(z)\}}\Bigg\{-\sum_{i,k,s=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s}S^k(x)\nu_x^s-\sum_{i,k,s=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s}S^s(x)\nu_x^k$$

$$+\sum_{i,k,l=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^i(x)\nu_x^l+\sum_{j,k=1}^{d}R_n(x,y)\frac{\partial G_m^j(x,z)}{\partial x^k}S^k(x)\nu_x^j$$

$$+\sum_{i,k,s=1}^{d}T^{ik}(\boldsymbol{G}_m,R_m)(x,z)G_n^l(x,y)\frac{\partial S^i(x)}{\partial x^l}\nu_x^k\Bigg\}\,d\sigma_x,$$

$$I_v^1:=-\lim_{\rho\to 0}\int_{\Omega\setminus B_\rho(y)\cup B_\rho(z)}\Bigg\{-\left(\sum_{i,k,s=1}^{d}\frac{\partial^2 G_n^i(x,y)}{\partial x^s\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s}S^k(x)+\sum_{i,k=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\Delta G_m^i(x,z)S^k(x)\right)$$

$$+\sum_{i,k,l=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial^2 G_m^i(x,z)}{\partial x^l\partial x^k}S^i(x)\Bigg\}\,dx,$$

$$I_v^2:=-\lim_{\rho\to 0}\int_{\Omega\setminus B_\rho(y)\cup B_\rho(z)}\Bigg\{-\sum_{i,k,s=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial^2 G_m^i(x,z)}{\partial x^k\partial x^s}S^s(x)$$

$$+\sum_{i,k,l=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}T^{ik}(\boldsymbol{G}_m,R_m)(x,z)\frac{\partial S^i(x)}{\partial x^l}\Bigg\}\,dx,\quad m,n=1,\cdots,d.$$

In the above calculation, we have used the fact that $\operatorname{div}\boldsymbol{G}_n=0$ in Ω and $\mathcal{L}^i(\boldsymbol{G}_n,R_n)(x,y)=\mathcal{L}^i(\boldsymbol{G}_m,R_m)(x,z)=0$ in $\Omega\setminus B_\rho(y)\cap B_\rho(z)$, $i,m,n=1,\cdots,d$. Since $\operatorname{div}S=\operatorname{div}\boldsymbol{G_n}=0$ in Ω, we have by integration by parts that

$$I_v^2=-\lim_{\rho\to 0}\int_{\partial\{\Omega\setminus B_\rho(y)\cup B_\rho(z)\}}\Bigg\{-\sum_{i,k,s=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^s(x)\nu_x^s$$

$$+\sum_{i,k,l=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}T^{ik}(\boldsymbol{G}_m,R_m)(x,z)S^i(x)\nu_x^l\Bigg\}\,d\sigma_x$$

$$+\lim_{\rho\to 0}\int_{\Omega\setminus B_\rho(y)\cup B_\rho(z)}\Bigg\{-\sum_{i,k,s=1}^{d}\frac{\partial^2 G_n^i(x,y)}{\partial x^s\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^s(x)$$

$$+\sum_{i,k,l=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}\left(\frac{\partial^2 G_m^i(x,z)}{\partial x^l\partial x^k}-\delta^{ik}\frac{\partial R_m(x,z)}{\partial x^l}\right)S^i(x)\Big\}\,dx$$

for $m,n=1,\cdots,d$. Since we note that $\{\boldsymbol{G}_n,R_n\}_{n=1,\cdots,d}$ is a solution of the Stokes equations, it follows from the above calculation that

(4.28)

$$\begin{aligned}
I_v^1+I_v^2=&-\lim_{\rho\to 0}\int_{\partial\{\Omega\setminus B_\rho(y)\cup B_\rho(z)\}}\Big\{-\sum_{i,k,s=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^s(x)\nu_x^s\\
&\qquad+\sum_{i,k,l=1}^{d}\frac{\partial G_n^l(x,y)}{\partial x^k}T^{ik}(\boldsymbol{G}_m,R_m)(x,z)S^i(x)\nu_x^l\Big\}\,d\sigma_x\\
&-\lim_{\rho\to 0}\int_{\Omega\setminus B_\rho(y)\cup B_\rho(z)}\sum_{i,k=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\mathcal{L}^i(\boldsymbol{G}_m,R_m)(x,z)S^k(x)\,dx\\
=&-\lim_{\rho\to 0}\int_{\partial\{\Omega\setminus B_\rho(y)\cup B_\rho(z)\}}\sum_{i,k,s=1}^{d}\Big\{-\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^s(x)\nu_x^s\\
&\qquad+\frac{\partial G_n^l(x,y)}{\partial x^k}T^{ik}(\boldsymbol{G}_m,R_m)(x,z)S^i(x)\nu_x^l\Big\}\,d\sigma_x
\end{aligned}$$

for $m,n=1,\cdots,d$. Applying (4.28) to (4.27), we have that

(4.29)

$$\begin{aligned}
&\int_\Omega\Big\{-\sum_{i,k,s=1}^{d}\left(\frac{\partial S^k(x)}{\partial x^s}+\frac{\partial S^s(x)}{\partial x^k}\right)\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^s}\\
&\quad+\sum_{i,k,l=1}^{d}\frac{\partial S^i(x)}{\partial x^l}\frac{\partial G_n^l(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}+\sum_{k,j=1}^{d}\frac{\partial S^k(x)}{\partial x^j}R_n(x,y)\frac{\partial G_m^j(x,z)}{\partial x^k}\\
&\quad+\sum_{i,k,l=1}^{d}\frac{\partial^2 S^i(x)}{\partial x^k\partial x^l}G_n^l(x,y)\frac{\partial G_m^i(x,z)}{\partial x^k}-\sum_{i,k,l=1}^{d}\frac{\partial^2 S^i(x)}{\partial x^k\partial x^l}\delta^{ik}R_m(x,z)G_n^l(x,y)\Big\}\,dx\\
&=\lim_{\rho\to 0}\int_{\partial\{\Omega\setminus B_\rho(y)\cup B_\rho(z)\}}\Big\{\sum_{i,k,l=1}^{d}\frac{\partial S^i(x)}{\partial x^l}G_n^l(x,y)T^{ik}(\boldsymbol{G}_m,R_m)(x,z)\nu_x^k+\sum_{i,k,s=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^s(x)\nu_x^s\\
&\quad-\sum_{i,k,s=1}^{d}T^{ik}(\boldsymbol{G}_n,R_n)(x,y)\frac{\partial G_m^i(x,z)}{\partial x^s}S^s(x)\nu_x^k-\sum_{i,k,l=1}^{d}T^{lk}(\boldsymbol{G}_m,R_m)(x,z)\frac{\partial G_n^l(x,y)}{\partial x^i}S^i(x)\nu_x^k\Big\}\,d\sigma_x
\end{aligned}$$

for $m,n=1,\cdots,d$. We next investigate the limit as $\rho\to 0$ at the points y and z of the above surface integral. Since $\operatorname{div}\boldsymbol{G}_n=0$ in Ω, we have an identity

$$-\sum_{i,k,l=1}^{d}\delta^{ik}R_m(x,z)\frac{\partial G_n^i(x,y)}{\partial x^k}S^l(x)\nu_x^l=0,\quad m,n=1,\cdots,d, \tag{4.30}$$

for all $x \in \Omega$. Adding the the left hand side of (4.30) to the right hand side of (4.29) , we have by (2.4) and (2.5) that

(4.31)

$$\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\Big\{\sum_{i,k,l=1}^{d}\frac{\partial S^i(x)}{\partial x^l}G_n^l(x,y)T^{ik}(\boldsymbol{G}_m,R_m)(x,z)\nu_x^k+\sum_{i,k,l=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^l(x)\nu_x^l$$
$$-\sum_{i,k,l=1}^{d}\delta^{ik}R_m(x,z)\frac{\partial G_n^i(x,y)}{\partial x^k}S^l(x)\nu_x^l-\sum_{i,k,s=1}^{d}T^{ik}(\boldsymbol{G}_n,R_n)(x,y)\frac{\partial G_m^i(x,z)}{\partial x^s}S^s(x)\nu_x^k$$
$$-\sum_{i,k,l=1}^{d}T^{lk}(\boldsymbol{G}_m,R_m)(x,z)\frac{\partial G_n^l(x,y)}{\partial x^i}S^i(x)\nu_x^k\Big\}\,d\sigma_x$$
$$=-\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{i,k,s=1}^{d}T^{ik}(\boldsymbol{G}_n,R_n)(x,y)\frac{\partial G_m^i(x,z)}{\partial x^s}S^s(x)\nu_x^k\,d\sigma_x$$
$$-\lim_{\rho\to 0}\int_{\partial B_\rho(y)}\sum_{i,k,l=1}^{d}T^{lk}(\boldsymbol{G}_m,R_m)(x,z)\left(\frac{\partial G_n^l(x,y)}{\partial x^k}\nu_x^i-\frac{\partial G_n^l(x,y)}{\partial x^i}\nu_x^k\right)S^i(x)\,d\sigma_x$$
$$=-\sum_{s=1}^{d}\frac{\partial G_m^n(y,z)}{\partial y^s}S^s(y),\qquad m,n=1,\cdots,d.$$

We can also handle $I_s+(I_v^1+I_v^2)$ around the singularity z in the same manner as the case of y and obtain

(4.32)

$$\lim_{\rho\to 0}\int_{\partial B_\rho(z)}\Big\{\sum_{i,k,l=1}^{d}\frac{\partial S^i(x)}{\partial x^l}G_n^l(x,y)T^{ik}(\boldsymbol{G}_m,R_m)(x,z)\nu_k+\sum_{i,k,l=1}^{d}\frac{\partial G_n^i(x,y)}{\partial x^k}\frac{\partial G_m^i(x,z)}{\partial x^k}S^l(x)\nu_x^l$$
$$-\sum_{i,k,s=1}^{d}T^{ik}(\boldsymbol{G}_n,R_n)(x,y)\frac{\partial G_m^i(x,z)}{\partial x^s}S^s(x)\nu_x^k-\sum_{i,k,l=1}^{d}T^{lk}(\boldsymbol{G}_m,R_m)(x,z)\frac{\partial G_n^l(x,y)}{\partial x^i}S^i(x)\nu_x^k\Big\}\,d\sigma_x$$
$$=\sum_{k=1}^{d}\frac{\partial S^m}{\partial x^k}(z)G_n^k(z,y)-\sum_{s=1}^{d}\frac{\partial G_n^m(z,y)}{\partial z^s}S^s(y),\qquad m,n=1,\cdots,d.$$

Since $\{G_n^i\}_{i,n=1,\cdots,d}=0$ on $\partial\Omega$, by (4.31) and (4.32), we have the desired identity (4.24). This proves Proposition 4.5. □

4.4 Proof of Theorem 1.1.

The proof of Theorem 1.1 is based on Lemma 4.3. We first need to express $\{G_{\varepsilon,m}^n(\tilde{y},\tilde{z})\}_{n,m=1,\cdots,d}$ by means of $\{G_{\varepsilon,m}^n(y,z)\}_{n,m=1,\cdots,d}$ for $\tilde{y}=\Phi_\varepsilon(y)$, $\tilde{z}=\Phi_\varepsilon(z)$ with $(y,z)\in\Omega\times\Omega$. By the Taylor expansion of $\{G_{\varepsilon,m}^n(\tilde{y},\tilde{z})\}_{n,m=1,\cdots,d}$ around $(y,z)\in\Omega\times\Omega$, we have

(4.33)

$$G_{\varepsilon,m}^n(\tilde{y},\tilde{z})-G_{\varepsilon,m}^n(y,z)$$
$$=\nabla_{\tilde{y}}G_{\varepsilon,m}^n(y,z)\cdot(\tilde{y}-y)+\nabla_{\tilde{z}}G_{\varepsilon,m}^n(y,z)\cdot(\tilde{z}-z)+\nabla_{\tilde{z}}\nabla_{\tilde{y}}G_{\varepsilon,m}^n(y,z+\theta_1(\tilde{z}-z))(\tilde{y}-y)\cdot(\tilde{z}-z)$$

$$+\frac{1}{2}\nabla_{\tilde{y}}^2 G_{\varepsilon,m}^n(y+\theta_2(\tilde{y}-y),\tilde{z})(\tilde{y}-y)\cdot(\tilde{y}-y)+\frac{1}{2}\nabla_{\tilde{z}}^2 G_{\varepsilon,m}^n(y,z+\theta_3(\tilde{z}-z))(\tilde{z}-z)\cdot(\tilde{z}-z)$$

for some $0<\theta_1,\theta_2,\theta_3<1$.

By (A.3) and (4.14), it holds that

$$\sup_{\varepsilon>0}|\nabla_{\tilde{z}}\nabla_{\tilde{y}}G_{\varepsilon,m}^n(y,z+\theta_1(\tilde{z}-z))|\le C \tag{4.34}$$

with some constant C which may depend on $y,z\in\Omega$. Hence we have again by (A.3) that

$$\nabla_{\tilde{z}}\nabla_{\tilde{y}}G_{\varepsilon,m}^n(y,z+\theta_1(\tilde{z}-z))(\tilde{y}-y)\cdot(\tilde{z}-z)=O(\varepsilon^2),\quad n,m=1,\cdots,d,$$

as $\varepsilon\to 0$. The last two remainders of the right hand side of (4.33) can be handled in same way and we have

$$G_{\varepsilon,m}^n(\tilde{y},\tilde{z})-G_{\varepsilon,m}^n(y,z)=\nabla_{\tilde{y}}G_{\varepsilon,m}^n(y,z)\cdot(\tilde{y}-y)+\nabla_{\tilde{z}}G_{\varepsilon,m}^n(y,z)\cdot(\tilde{z}-z)+O(\varepsilon^2),\quad m,n=1,\cdots,d, \tag{4.35}$$

for all $y,z\in\Omega$ with $\tilde{y}=\Phi_\varepsilon(y),\ \tilde{z}=\Phi_\varepsilon(z)$ as $\varepsilon\to 0$. In the above argument, it should be noted that $G_{\varepsilon,m}^n=G_{\varepsilon,m}^n(\tilde{x},\tilde{y})$ is regarded as a function on $\Omega_\varepsilon\times\Omega_\varepsilon$ with variables $(\tilde{x},\tilde{y})$. By (A.3), (4.14), (4.35) and Lemma 4.3, it holds that

$$\begin{aligned}
&G_{\varepsilon,m}^n(y,z)-G_m^n(y,z)\\
&=\varepsilon\sum_{s=1}^d\left\{\left(\frac{\partial G_m^n}{\partial y^s}(y,z)-\frac{\partial G_{\varepsilon,m}^n}{\partial\tilde{y}^s}(y,z)\right)S^s(y)+\left(\frac{\partial G_m^n}{\partial z^s}(y,z)-\frac{\partial G_{\varepsilon,m}^n}{\partial\tilde{z}^s}(y,z)\right)S^s(z)\right\}\\
&+\varepsilon\int_{\partial\Omega}\sum_{i=1}^d\left\{\frac{\partial G_m^i(x,z)}{\partial\nu_x}\frac{\partial G_n^i(x,y)}{\partial\nu_x}\right.\\
&\qquad\left.-\left(R_m(x,z)\frac{\partial G_n^i(x,y)}{\partial\nu_x}+\frac{\partial G_m^i(x,z)}{\partial\nu_x}R_n(x,y)\right)\nu_x^i\right\}S(x)\cdot\nu_x\,d\sigma_x+o(\varepsilon),
\end{aligned} \tag{4.36}$$

for $m,n=1,\cdots,d$. Since $\sup_{\varepsilon>0}\left|\left(\frac{\partial\phi_\varepsilon^i}{\partial x^j}(x)\right)\right|<\infty$ for $i,j=1,\cdots,d$ and for all $x\in\Omega$, it follows from (2.17), (4.10) and Proposition 4.3 that

$$\begin{aligned}
&\lim_{\varepsilon\to0}|\nabla_{\tilde{y}}G_{\varepsilon,m}^n(y,z)-\nabla_y G_m^n(y,z)|=0,\\
&\lim_{\varepsilon\to0}|\nabla_{\tilde{z}}G_{\varepsilon,m}^n(y,z)-\nabla_z G_m^n(y,z)|=0,\quad m,n=1,\cdots,d
\end{aligned} \tag{4.37}$$

for all $y,z\in\Omega$. Now from (4.36) and (4.37), we obtain

$$\begin{aligned}
&\delta G_n^m(y,z)\\
&=\lim_{\varepsilon\to0}\sum_{s=1}^d\left\{\left(\frac{\partial G_m^n}{\partial y^s}(y,z)-\frac{\partial G_{\varepsilon,m}^n}{\partial\tilde{y}^s}(y,z)\right)S^s(y)+\left(\frac{\partial G_m^n}{\partial z^s}(y,z)-\frac{\partial G_{\varepsilon,m}^n}{\partial\tilde{z}^s}(y,z)\right)S^s(z)\right\}\\
&+\int_{\partial\Omega}\sum_{i=1}^d\left\{\frac{\partial G_m^i(x,z)}{\partial\nu_x}\frac{\partial G_n^i(x,y)}{\partial\nu_x}\right.\\
&\qquad\left.-\left(R_m(x,z)\frac{\partial G_n^i(x,y)}{\partial\nu_x}+\frac{\partial G_m^i(x,z)}{\partial\nu_x}R_n(x,y)\right)\nu_x^i\right\}S(x)\cdot\nu_x\,d\sigma_x
\end{aligned}$$

$$= \int_{\partial\Omega} \sum_{i=1}^{d} \left\{ \frac{\partial G_m^i(x,z)}{\partial \nu_x} \frac{\partial G_n^i(x,y)}{\partial \nu_x} \right.$$

$$\left. - \left(R_m(x,z) \frac{\partial G_n^i(x,y)}{\partial \nu_x} + \frac{\partial G_m^i(x,z)}{\partial \nu_x} R_n(x,y) \right) \nu_x^i \right\} S(x) \cdot \nu_x \, d\sigma_x, \quad m,n = 1, \cdots, d,$$

which implies Theorem 1.1. □

References

[1] Agmon, S., Douglis, A., Nirenberg, L., *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623-727.

[2] Aomoto, K., *Formule variationnelle d'Hadamard et modèle euclidien des variétés différentiables plongées*. J. Funct. Anal. **34** no. 3 (1979), 493-523.

[3] Bogovskiĭ, M. E., *Solutions of some problems of vector analysis, associated with the operators* div *and* grad, Trudy Sem. S. L. Soboleva, No. 1, Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1980.

[4] Borchers, W., Sohr, H., *On the equations* rot *v=g and* div *u=f with zero boundary conditions.* Hokkaido Math. J. **19** no. 1 (1990), 67-87.

[5] Fujiwara, D., Ozawa, S., *The Hadamard variational formula for the Green functions of some normal elliptic boundary value problems.* Proc. Japan Acad. Ser. A Math. Sci. **54** no. 8 (1978), 215-220.

[6] Garabedian, P. R., *Partial Differential Equations.* John Wiley & Sons Inc., New York-London-Sydney, 1964.

[7] Garabedian, P. R., Schiffer, M., *Convexity of domain functionals.* J.Analyse Math., **2** (1953), 281-368.

[8] Hadamard, J., *Mémoire sur le probleme d'analyse relatif à l'equilibre des plaques élastiques encastrées.* Oeuvres, C. N. R. S., **2**, Anatole France, 1968, 515-631.

[9] Inoue, A., Wakimoto, M., *On existence of solutions of the Navier-Stokes equation in a time dependent domain.* J. Fac. Sci. Univ. Tokyo Sect. IA Math., **24** no. 2 (1977), 303-319.

[10] Ozawa, S., *Research for the Hadamard variational formula (in Japanese).* Master Thesis, Faculty of Science University of Tokyo (1979).

[11] Ozawa, S., *Perturbation of domains and Green Kernels of heat equations.* Proc. Japan Acad. Ser. A Math. Sci., **54** no. 10 (1978), 322-325.

[12] Ozawa, S., *Surgery of domain and the Green function of the Laplacian.* Proc. Japan Acad. Ser. A Math. Sci., **56** no. 10 (1980), 459-461.

[13] Peetre, J., *On Hadamard's variational formula.* J. Differential Equations., **36** no. 3 (1980), 335–346.

[14] Sohr, H., *The Navier-Stokes equations : An elementary functional analytic approach.* Birkhäuser Verlag, Basel-Boston-Berlin, 2001.

[15] Solonnikov, V.A., *General boundary value problems for Douglis-Nirenberg elliptic systems II.* Proc. Steklov Ints. Math. **92** (1966), 269–339.