On the Asymptotic Structure of a Navier-Stokes Flow Past a Rotating Body

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Consider a rigid body moving with a prescribed constant non-zero velocity and rotating with a prescribed constant non-zero angular velocity in a three-dimensional Navier-Stokes liquid. The asymptotic structure of a steady-state solution to the corresponding equations of motion is analyzed. In particular, an asymptotic expansion of the corresponding velocity field is obtained.

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1 Introduction

The aim of this this paper is to establish an asymptotic expansion of a solution to the steady-state three-dimensional Navier-Stokes equations written in a frame attached to a rigid body moving with non-zero translational velocity $\xi \in \mathbb{R}^3$ and non-zero angular velocity $\omega \in \mathbb{R}^3$. More specifically, we consider a body, with a connected boundary, moving in a Navier-Stokes liquid that fills the whole exterior of the body. If we denote by v the Eulerian velocity field of the liquid, and by p the corresponding pressure, the steady-state equations of motion written in a frame attached to the body read

$$\begin{cases} -\mu\Delta v + \nabla p + v\cdot\nabla v - \xi\cdot\nabla v - \omega\wedge x\cdot\nabla v + \omega\wedge v = f & \text{in } \Omega,\\ \operatorname{div} v = 0 & \text{in } \Omega,\\ v = v_* & \text{on } \partial\Omega,\\ \lim_{|x|\to\infty} v(x) = 0, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^3$ is an exterior domain, μ the (constant) kinematic viscosity coefficient, f an external force acting on the liquid, and v_* the velocity distribution on the liquid-structure boundary. We shall assume that ξ and ω are not orthogonal to each other. In this case, due to a simple transformation, see for example [16], we may take, without loss of generality, ξ and ω to be directed along the same axis, which we take to be e_3. Moreover, for simplicity we choose to consider only the so-called no-slip boundary condition, and do not take into account any external force in the liquid. In an appropriate non-dimensional form, the equations of motion then read

$$\begin{cases} -\Delta v + \nabla p - \mathcal{R}\partial_3 v + \mathcal{R} v\cdot\nabla v - \mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\big) = 0 & \text{in } \Omega,\\ \operatorname{div} v = 0 & \text{in } \Omega,\\ v = \mathrm{e}_3 + \mathcal{T}\,\mathrm{e}_3\wedge x & \text{on } \partial\Omega,\\ \lim_{|x|\to\infty} v(x) = 0, \end{cases} \tag{1.2}$$

where $\mathcal{R} > 0$ is a dimensionless constant, and $\mathcal{T} > 0$ the magnitude of the dimensionless angular velocity. Finally, we assume, again without loss of generality, that the origin of the frame of reference coincides with the body's center of mass. We then have $0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and $\int_{\mathbb{R}^3\setminus\overline{\Omega}} x\,\mathrm{d}x = 0$.

The above system is the classical steady-state Navier-Stokes problem with the additional term $\mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla v - \mathrm{e}_3\wedge v\big)$, which stems from the rotating frame of reference. Due to the unbounded coefficient $\mathrm{e}_3\wedge x$, this term can not be treated at a perturbation to the Oseen operator.

The main result of this paper is an asymptotic expansion as $|x| \to \infty$ of a so-called Leray solution v to (1.2), that is, of a solution with a bounded Dirichlet integral, also sometimes referred to as a D-solution. An asymptotic expansion of v is a decomposition

$$v(x) = \Gamma(x)\cdot\alpha + R(x) \tag{1.3}$$

where Γ and α is an explicitly known function and constant, respectively, and R some remainder term decaying faster than Γ as $|x| \to \infty$. In the case of a translating but non-rotating body ($\omega = 0$, $\xi \neq 0$), such an expansion was established for the first time by Finn, who showed in [9], see also [10], that any solution to (1.1) with v in the class $O\big(|x|^{-1/2+\varepsilon}\big)$ for all $\varepsilon > 0$ satisfies (1.3) with the Oseen fundamental solution in the role of Γ, the force $\mathcal{F}$ exerted by the liquid on the body as α, and $R(x) = O\big(|x|^{-3/2+\delta}\big)$ for all $\delta > 0$. Babenko later proved in [1] that the same holds true for Leray solutions; Finn had left this as an open question. The proof provided by Babenko, however, was not complete, and it was not until [11] that a full proof was available, see also [8]. In the case of a non-translating and non-rotating body ($\omega = 0$, $\xi = 0$), an asymptotic expansion was available only much later, and only for solutions corresponding to "small" data. This result is due to Korolev&Šverak [17], who showed that a Leray solution to (1.1) satisfies (1.3), but with the leading term $\Gamma(x)\cdot\alpha$ replaced by a so-called Landau solution depending only on $\mathcal{F}$, and $R(x) = O\big(|x|^{-2+\delta}\big)$. The result of Korolev&Šverak was extended to the rotating body case ($\omega \neq 0$, $\xi = 0$) in two papers by Farwig&Hishida [7] and Farwig&Galdi&Kyed [5], respectively. It is shown herein that the leading term in this case is again the Landau solution, but depending now only on the projection of $\mathcal{F}$ on the axis of rotation. In [7] the remainder term is estimated in a summability sense, whereas [5] establishes a point-wise estimate.

This leaves open only the case of a translating and rotating body ($\omega \neq 0$, $\xi \neq 0$), which is treated in this paper. As the main result, an asymptotic expansion in the sense of summability of a solution v will be established. The leading term in this expansion is identified as the Oseen fundamental solution multiplied by a constant vector. A computation of the constant will be carried out, and it will be shown that it equals the projection on the axis of rotation of the force exerted by the liquid on the body. The main theorem reads:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a C^2-smooth exterior domain, and $\mathcal{R}, \mathcal{T} > 0$. A solution*

$$(v,p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \cap W^{2,2}_{loc}(\overline{\Omega})^3 \times W^{1,2}_{loc}(\overline{\Omega}) \tag{1.4}$$

to (1.2) satisfies the asymptotic expansion ($j = 1, 2, 3$)

$$v(x) = \Gamma^{\mathcal{R}}_{\mathrm{O}}(x) \cdot \big(\mathcal{F} \cdot \mathrm{e}_3\big)\, \mathrm{e}_3 + R(x), \tag{1.5}$$

$$\partial_j v(x) = \partial_j \Gamma^{\mathcal{R}}_{\mathrm{O}}(x) \cdot \big(\mathcal{F} \cdot \mathrm{e}_3\big)\, \mathrm{e}_3 + S_j(x) \tag{1.6}$$

with

$$\forall q \in (4/3, \infty):\ \ R \in L^q(\Omega)^3, \tag{1.7}$$

$$\forall q \in (1, \infty):\ \ S_j \in L^q(\Omega)^3, \tag{1.8}$$

where

$$\mathcal{F} := \int_{\partial\Omega} \mathrm{T}(v,p) \cdot n \,\mathrm{d}S. \tag{1.9}$$

Here, $\Gamma^{\mathcal{R}}_{\mathrm{O}}$ denotes the fundamental solution to the Oseen equations, and $\mathrm{T}(v,p)$ the Cauchy stress tensor of the liquid (see below for the explicit definition).

It is well-known that $\Gamma^{\mathcal{R}}_{\mathrm{O}}$ is *not* L^q-summable in a neighborhood of infinity for small q. More precisely, for $q \in [1, 2]$ one can show $\Gamma^{\mathcal{R}}_{\mathrm{O}} \notin L^q\big(\mathbb{R}^3 \setminus \mathrm{B}_r(0)\big)$ for any $r > 0$, see for example [12, Chapter VII.3]. Thus, in the sense of summability the remainder term R in the expansion (1.5) decays strictly faster as $|x| \to \infty$ than the leading term. Since for $q \in [1, 4/3]$ it is known that $\nabla\Gamma^{\mathcal{R}}_{\mathrm{O}} \notin L^q\big(\mathbb{R}^3 \setminus \mathrm{B}_r(0)\big)$ for any $r > 0$, see again [12, Chapter VII.3], the remainder S_j in (1.6) decays, again in the sense of summability, faster than $\nabla\Gamma^{\mathcal{R}}_{\mathrm{O}}$ as $|x| \to \infty$. Consequently, the decompositions (1.5) and (1.6) constitute valid asymptotic expansions at spatial infinity.

Note that $\mathcal{F}$, as defined in (1.9), equals the total force exerted by the liquid on the body. Since we in (1.2) consider the no-slip boundary condition, there is no contribution from momentum flux via the liquid-structure boundary to the total force.

As mentioned above, the no-slip boundary condition has been chosen for simplicity only. For the same reason, no external forces acting on the liquid are considered. However, with minor modifications to the proof of Theorem 1.1, arbitrary, but sufficiently smooth, boundary values can be included. Moreover, we can also introduce an external force of compact support, that is, a non-homogeneous right-hand side of compact support in $(1.2)_1$. Of course, with more general boundary values and external forces, the expression for $\mathcal{F}$ must be modified accordingly.

As a direct consequence of Theorem 1.1, we find that the kinetic energy of a Navier-Stokes flow past a rotating and translating body is infinite, unless the component in the direction of rotation of the force exerted by the liquid on the body vanishes. This follows from the fact that no entry of $\Gamma^{\mathcal{R}}_{\mathrm{O}}$ lies in $L^2(\Omega)$, which together with (1.5) and (1.7) implies that the velocity field of the flow is square summable if and only if $\mathcal{F}\cdot \mathrm{e}_3 = 0$. A similar result is known for a non-rotating body, and a rotating body in a linearized fluid, see [19], but this is the first time such a property is established for the fully non-linear Navier-Stokes flow past a rotating body.

Although a characterization of the remainder term in the expansion in terms of summability is favorable for deriving information on the energy of the flow, other applications require a point-wise decay estimate. To prove a point-wise estimate, one needs to take a slightly different approach than used here. This will be addressed in the forthcoming paper [18].

Parallel to the non-linear Navier-Stokes equations, asymptotic expansions of solutions to both the Stokes and Oseen linearizations have also been established. In the case of a non-rotating body, such results date back to the early works of Finn, see for example [3]. In the rotating body case, we refer to [6] and [19] for the non-translating and translating body case, respectively.

2 Notation and Preliminaries

Before proving the main theorem, we introduce some notation, recall well-known identities, and show two preliminary lemmas.

We denote by $L^q(\Omega)$, $1 \leq q \leq \infty$, the usual Lebesgue space with norm $\|\cdot\|_q$. For $m \in \mathbb{N}$, we use $W^{m,q}(\Omega)$ to denote the inhomogeneous Sobolev spaces with norm $\|\cdot\|_{m,q}$. We also introduce the homogeneous Sobolev space

$$D^{m,q}(\Omega) := \{v \in L^1_{loc}(\Omega) \mid |v|_{m,q} < \infty\},$$
$$|v|_{m,q} := \Big(\sum_{|\alpha|=m} \int_{\mathbb{R}^3} |\partial^\alpha v(x)|^q \,\mathrm{d}x\Big)^{\frac{1}{q}}.$$

Moreover, we introduce for $1 < q < 2$ the space

$$\mathrm{X}_q(\mathbb{R}^3) := \{(w, \mathfrak{q}) \in D^{2,q}(\mathbb{R}^3)^3 \times D^{1,q}(\mathbb{R}^3) \mid \|(w,\mathfrak{q})\|_{\mathrm{X}_q} < \infty\},$$
$$\|(w,\mathfrak{q})\|_{\mathrm{X}_q} := \|\nabla^2 w\|_q + \|\partial_3 w\|_q + \|\nabla w\|_{\frac{4q}{4-q}} + \|w\|_{\frac{2q}{2-q}} + \|\nabla \mathfrak{q}\|_q + \|\mathfrak{q}\|_{\frac{3q}{3-q}},$$

which one may identify as a canonical domain for the Oseen operator.

For functions $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ we let $\operatorname{div} u(x,t) := \operatorname{div}_x u(x,t)$, $\Delta u(x,t) := \Delta_x u(x,t)$ etc., that is, unless otherwise indicated, differential operators act in the spatial variable x only.

We put $\mathrm{B}_m := \{x \in \mathbb{R}^3 \mid |x| < m\}$ and $\mathrm{B}^m := \mathbb{R}^3 \setminus \mathrm{B}_m$.

We use the Landau symbol $O\big(R(x)\big)$ to characterize the class of functions u for which there is a constant $C > 0$ such that $|u(x)| \leq C|R(x)|$ for large $|x|$.

Constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear.

For a fluid velocity field $v : \mathbb{R}^3 \to \mathbb{R}^3$ and pressure $p : \mathbb{R}^3 \to \mathbb{R}$, we let

$$\mathrm{T}(v,p) := \nabla v + \nabla v^\top - pI$$

denote the Cauchy stress tensor of the (Newtonian) fluid corresponding to the non-dimensional form (1.2) of the Navier-Stokes equations. We let

$$\Gamma_{\mathrm{O}}^{\mathcal{R}} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^{3\times 3}, \quad \big[\Gamma_{\mathrm{O}}^{\mathcal{R}}(x)\big]_{ij} := (\delta_{ij}\Delta - \partial_i\partial_j)\Phi^{\mathcal{R}}(x),$$
$$\Phi^{\mathcal{R}}(x) := \frac{1}{4\pi\mathcal{R}} \int_0^{\mathcal{R}(|x|+x_3)/2} \frac{1-\mathrm{e}^{-\tau}}{\tau}\,\mathrm{d}\tau$$

denote the three-dimensional Oseen fundamental solution tensor, see [12, Chapter VII.3] for a closed-form expression. Finally, we denote by

$$\Gamma_{\mathrm{L}} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad \Gamma_{\mathrm{L}}(x) := \frac{1}{4\pi}\frac{1}{|x|}$$

the fundamental solution to the Laplace equation.

The summability properties of $\Gamma_{\mathrm{O}}^{\mathcal{R}}$ will play a fundamental role in form of the following lemma:

Lemma 2.1. *Let $H \in C^\infty(\mathbb{R}^3)^{3\times 3}$ satisfy*

$$\forall q \in (1,\infty): \ H \in W^{2,q}(\mathbb{R}^3)^{3\times 3}. \tag{2.1}$$

Then[1] $(i = 1,2,3)$

$$U_i(x) := \big[\Gamma_{\mathrm{O}}^{\mathcal{R}} * \operatorname{div} H\big]_i(x) := \int_{\mathbb{R}^3} \big[\Gamma_{\mathrm{O}}^{\mathcal{R}}(y)\big]_{ij}\,\partial_k H_{jk}(x-y)\,\mathrm{d}y \tag{2.2}$$

is well-defined with

$$\forall q \in (4/3,\infty): \ U \in L^q(\mathbb{R}^3)^3, \tag{2.3}$$
$$\forall q \in (1,\infty): \ \nabla U \in L^q(\mathbb{R}^3)^{3\times 3}. \tag{2.4}$$

Proof. It is well-known, see for example [12, Chapter VII.3], that $\Gamma_{\mathrm{O}}^{\mathcal{R}}$ enjoys the summability properties

$$\forall q \in (2,\infty): \qquad \Gamma_{\mathrm{O}}^{\mathcal{R}} \in L^q(\mathbb{R}^3 \setminus \mathrm{B}_r)^{3\times 3} \text{ for any } r > 0, \tag{2.5}$$
$$\forall q \in [1,3): \qquad \Gamma_{\mathrm{O}}^{\mathcal{R}} \in L^q_{loc}(\mathbb{R}^3)^{3\times 3}, \tag{2.6}$$
$$\forall q \in (4/3,\infty): \qquad \nabla\Gamma_{\mathrm{O}}^{\mathcal{R}} \in L^q(\mathbb{R}^3 \setminus \mathrm{B}_r)^{3\times 3\times 3} \text{ for any } r > 0, \tag{2.7}$$
$$\forall q \in [1,3/2): \qquad \nabla\Gamma_{\mathrm{O}}^{\mathcal{R}} \in L^q_{loc}(\mathbb{R}^3)^{3\times 3\times 3}, \tag{2.8}$$
$$\forall q \in (1,\infty): \qquad \nabla^2\Gamma_{\mathrm{O}}^{\mathcal{R}} \in L^q(\mathbb{R}^3 \setminus \mathrm{B}_r)^{3\times 3\times 3\times 3} \text{ for any } r > 0. \tag{2.9}$$

[1] Following the summation convention, we implicitly sum over repeated indices.

By (2.5), (2.6), and Young's inequality, it is clear that the convolution in (2.2) is well-defined. Taking into account (2.7) and (2.8), we further see that $U = \nabla\Gamma^{\mathcal{R}}_{\mathrm{O}} * H \in L^q(\mathbb{R}^3)^3$ for all $q \in (4/3, \infty)$. Thus we deduce (2.3). To prove (2.4), we split

$$\begin{aligned}\partial_l U_i(x) &= \int_{\mathbb{R}^3} \partial_l \big[\Gamma^{\mathcal{R}}_{\mathrm{O}}(y)\big]_{ij}\, \partial_k H_{jk}(x-y)\,\mathrm{d}y \\ &= \int_{\mathrm{B}_1} \partial_l \big[\Gamma^{\mathcal{R}}_{\mathrm{O}}(y)\big]_{ij}\, \partial_k H_{jk}(x-y)\,\mathrm{d}y \\ &\quad + \int_{\partial\,\mathrm{B}_1} \partial_l \big[\Gamma^{\mathcal{R}}_{\mathrm{O}}(y)\big]_{ij}\, H_{jk}(x-y)\, n_k(y)\,\mathrm{d}S(y) \\ &\quad - \int_{\mathbb{R}^3\setminus\overline{B_1}} \partial_k\partial_l \big[\Gamma^{\mathcal{R}}_{\mathrm{O}}(y)\big]_{ij}\, H_{jk}(x-y)\,\mathrm{d}y =: I(x) + J(x) + K(x).\end{aligned}$$

We again employ Young's inequality and deduce from (2.8) that $I \in L^q(\mathbb{R}^3)^3$ for all $q \in (1, \infty)$. Minkowski's inequality yields for any $q \in (1, \infty)$

$$\|J\|_q \le \int_{\partial\,\mathrm{B}_1} \bigg(\int_{\mathbb{R}^3} |\Gamma^{\mathcal{R}}_{\mathrm{O}}(y)|^q |H(x-y)|^q \,\mathrm{d}x \bigg)^{\frac{1}{q}} \mathrm{d}S(y) = \|\Gamma^{\mathcal{R}}_{\mathrm{O}}\|_{L^1(\partial\,\mathrm{B}_1)} \|H\|_q < \infty.$$

Finally, by (2.9) and Young's inequality, we have $K \in L^q(\mathbb{R}^3)^3$ for all $q \in (1, \infty)$. We conclude (2.4). □

Next, we consider the linearization of (1.2) and establish a very strong L^q-estimate for solutions corresponding to a special class of data. For this purpose, let $\mathrm{E}_3 \in \mathrm{skew}_{3\times 3}(\mathbb{R})$ denote the skew-symmetric adjoint of e_3, and put

$$Q(t) := \exp(\mathcal{T}\mathrm{E}_3 t) = \begin{pmatrix} \cos(\mathcal{T}t) & -\sin(\mathcal{T}t) & 0 \\ \sin(\mathcal{T}t) & \cos(\mathcal{T}t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.10}$$

We then have the following lemma:

Lemma 2.2. *Let $\mathcal{R}, \mathcal{T} > 0$ and $1 < q < \infty$. For any $f \in L^q(\mathbb{R}^3)^3 \cap C^\infty(\mathbb{R}^3)^3$ satisfying*

$$\int_0^{2\pi/\mathcal{T}} Q(t) f\big(Q(t)^\top x\big)\,\mathrm{d}t = 0 \tag{2.11}$$

there exists a solution $(w, \mathfrak{q}) \in W^{2,q}(\mathbb{R}^3)^3 \times D^{1,q}(\mathbb{R}^3)$ to

$$\begin{cases} -\Delta w + \nabla\mathfrak{q} - \mathcal{R}\partial_3 w - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla w - \mathrm{e}_3 \wedge w\big) = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{2.12}$$

that satisfies

$$\|w\|_{2,q} + \|\nabla\mathfrak{q}\|_q \le C_1 \|f\|_q, \tag{2.13}$$

where $C_1 = C_1(\mathcal{R}, \mathcal{T})$. Moreover, if, for some $1 < r, s < \infty$, $(\tilde{w}, \tilde{\mathfrak{q}}) \in L^r(\mathbb{R}^3)^3 \cap W^{2,s}_{loc}(\mathbb{R}^3)^3 \times W^{1,s}_{loc}(\mathbb{R}^3)$ is another solution, then necessarily $w = \tilde{w}$ and $\mathfrak{q} = \tilde{\mathfrak{q}} + c$ for some constant $c \in \mathbb{R}$.

Proof. The uniqueness statement of the lemma follows directly from [15, Lemma 4.1]. We therefore only need to show the existence of a solution $(w, \mathfrak{q})$ to (2.12) that satisfies (2.13).

Consider first $1 < q < 2$. By [4, Theorem 1.1 and Corollary 1.2], see also [14, Theorem 1.1], there exists for any $f \in L^q(\mathbb{R}^3)^3$ a solution $(w, \mathfrak{q}) \in \mathrm{X}_q(\mathbb{R}^3)$ to (2.12) that satisfies

$$\|\nabla^2 w\|_q + \|\nabla \mathfrak{q}\|_q \leq c_1 \|f\|_q. \tag{2.14}$$

If f is smooth, standard regularity theory for elliptic systems implies that also w and $\mathfrak{q}$ are smooth. We shall now show that when $f \in L^q(\mathbb{R}^3)^3 \cap C^\infty(\mathbb{R}^3)^3$ further satisfies (2.11), additional summability of w can be established. For this purpose, put

$$\begin{aligned}
&u : \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right] \to \mathbb{R}^3, \quad u(x,t) := Q(t) w\big(Q(t)^\top x\big),\\
&\mathfrak{p} : \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right] \to \mathbb{R}, \quad \mathfrak{p}(x,t) := \mathfrak{q}\big(Q(t)^\top x\big),\\
&F : \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right] \to \mathbb{R}^3, \quad F(x,t) := Q(t) f\big(Q(t)^\top x\big).
\end{aligned}$$

As one may easily verify, $(u, \mathfrak{p})$ satisfies

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - \mathcal{R}\partial_3 u = F & \text{in } \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right], \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right]. \end{cases} \tag{2.15}$$

Note that u, $\mathfrak{p}$, and F are smooth and $\frac{2\pi}{\mathcal{T}}$-periodic in t. We can therefore expand these fields in their Fourier-series with respect to t. More precisely, we have

$$\begin{aligned}
&u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)\, \mathrm{e}^{i\mathcal{T}kt}, \quad \mathfrak{p}(x,t) = \sum_{k\in\mathbb{Z}} \mathfrak{p}_k(x)\, \mathrm{e}^{i\mathcal{T}kt},\\
&F(x,t) = \sum_{k\in\mathbb{Z}} F_k(x)\, \mathrm{e}^{i\mathcal{T}kt},
\end{aligned} \tag{2.16}$$

with

$$\begin{aligned}
&u_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} u(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t, \quad \mathfrak{p}_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \mathfrak{p}(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t,\\
&F_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} F(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t.
\end{aligned}$$

Inserting the Fourier series from (2.16) into (2.15), we find that each Fourier coefficient satisfies

$$\begin{cases} i\mathcal{T}k u_k - \Delta u_k + \nabla \mathfrak{p}_k - \mathcal{R}\partial_3 u_k = F_k & \text{in } \mathbb{R}^3, \\ \operatorname{div} u_k = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.17}$$

Clearly, $(u_k, \mathfrak{p}_k)$ enjoys the same summability properties as $(w, \mathfrak{q})$, that is, we have $(u_k, \mathfrak{p}_k) \in \mathrm{X}_q(\mathbb{R}^3)$. We now use that f satisfies (2.11), which implies that $F_0 = 0$. Consequently, $(u_0, \mathfrak{p}_0)$ is a solution to the homogeneous whole space Oseen problem. It follows that $u_0 = 0$. Now consider $k \neq 0$. Using Minkowski's integral inequality, we obtain

$$\|\nabla^2 u_k\|_q \leq \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \bigg(\int_{\mathbb{R}^3} |\nabla^2 u(x,t)|^q \,\mathrm{d}x \bigg)^{1/q} \mathrm{d}t \leq \|\nabla^2 w\|_q = c_1 \|f\|_q,$$

and similarly $\|\nabla \mathfrak{p}_k\|_q \leq c_1 \|f\|_q$. Consequently, we can deduce directly from (2.17) that

$$\begin{aligned} |\mathcal{T}k| \|u_k\|_q &\leq \|\Delta u_k\|_q + \|\nabla \mathfrak{p}_k\|_q + \mathcal{R}\|\partial_3 u_k\|_q + \|F_k\|_q \\ &\leq c_2 \|f\|_q + \mathcal{R}\|\partial_3 u_k\|_q. \end{aligned} \tag{2.18}$$

A simple interpolation argument yields

$$\|\partial_3 u_k\|_q \leq c_3 \big(\varepsilon \|u_k\|_q + \varepsilon^{-1} \|\nabla^2 u_k\|_q \big) \tag{2.19}$$

for all $\varepsilon > 0$. We choose $\varepsilon = |\mathcal{T}k| / (2\mathcal{R} c_3)$ in (2.19), and apply the resulting estimate in (2.18) to obtain

$$\|u_k\|_q \leq c_4 \frac{1}{|\mathcal{T}k|} \bigg(1 + \frac{\mathcal{R}^2}{|\mathcal{T}k|} \bigg) \|f\|_q \quad (k \neq 0), \tag{2.20}$$

with c_4 independent of k. We can now estimate $\|w\|_q$. First, observe that

$$\|w\|_q^q = \bigg(\frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \bigg[\int_{\mathbb{R}^3} |u(x,t)|^q \,\mathrm{d}x \bigg]^{\frac{1}{q-1}} \mathrm{d}t \bigg)^{q-1}.$$

Since $1 < q < 2$, Minkowski's integral inequality yields

$$\|w\|_q^q \leq \int_{\mathbb{R}^3} \bigg(\frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} |u(x,t)|^{\frac{q}{q-1}} \,\mathrm{d}t \bigg)^{q-1} \mathrm{d}x.$$

Employing the Hausdorff-Young inequality for Fourier series, see for example [2, Proposition 4.2.7], we then find that

$$\|w\|_q^q \leq \int_{\mathbb{R}^3} \sum_{k \in \mathbb{Z}} |u_k(x)|^q = \sum_{k \in \mathbb{Z}} \|u_k\|_q^q.$$

We now recall (2.20) and the fact that $u_0 = 0$, and finally obtain

$$\|w\|_q \leq c_4 \bigg(\sum_{k \neq 0} \frac{1}{|\mathcal{T}k|^q} \bigg(1 + \frac{\mathcal{R}^2}{\mathcal{T}} \bigg)^q \|f\|_q^q \bigg)^{1/q} \leq c_5 \|f\|_q, \tag{2.21}$$

with $c_5 = c_5(\mathcal{R}, \mathcal{T})$. By (2.14) and (2.21), we conclude (2.13) in the case $1 < q < 2$.

In the case $q=2$, the existence of a solution $(w,\mathfrak{q})\in D^{2,2}(\mathbb{R}^3)^3\cap D^{1,2}(\mathbb{R}^3)^3\cap L^6(\mathbb{R}^3)^3\times D^{1,2}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)$ was shown in [13, Lemma 4.14] and [16, Theorem 2]. With this solution, we repeat the arguments above and obtain (2.13) also in the case $q=2$.

Consider now $2<q<\infty$. In this case, we can not utilize the inequalities of Hausdorff-Young and Minkowski as above. Instead, we shall use a duality argument. Assume for the moment that $f\in C_0^\infty(\mathbb{R}^3)^3$ and satisfies (2.11). The existence of a solution $(w,\mathfrak{q})\in D^{2,q}(\mathbb{R}^3)^3\times D^{1,q}(\mathbb{R}^3)$ satisfying (2.14) follows by [4, Theorem 1.1]. Since $f\in C_0^\infty(\mathbb{R}^3)^3$, [4, Corollary 1.2] even yields $(w,\mathfrak{q})\in \mathrm{X}_r(\mathbb{R}^3)$ for all $1<r<2$. Moreover, by standard regularity theory for elliptic systems, w and $\mathfrak{q}$ are smooth. Now let $\varphi\in C_0^\infty(\mathbb{R}^3)^3$ and put

$$\tilde{\varphi}(x):=\varphi-\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} Q(t)\varphi\big(Q(t)^\top x\big)\,\mathrm{d}t.$$

Then $\tilde{\varphi}$ satisfies (2.11). Observe that $q'\in(1,2)$, where q' denotes the Hölder conjugate of q. Consequently, by arguments as above, there exists a solution $(\psi,\eta)\in W^{2,q'}(\mathbb{R}^3)^3\times D^{1,q'}(\mathbb{R}^3)$ to the adjoint problem

$$\begin{cases} -\Delta\psi-\nabla\eta+\mathcal{R}\partial_3\psi+\mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla\psi-\mathrm{e}_3\wedge\psi\big)=\tilde{\varphi} & \text{in } \mathbb{R}^3,\\ \operatorname{div}\psi=0 & \text{in } \mathbb{R}^3\end{cases}$$

satisfying

$$\|\psi\|_{2,q'}+\|\nabla\eta\|_{q'}\leq c_6\|\tilde{\varphi}\|_{q'}\leq c_7\|\varphi\|_{q'}.$$

In view of the good summability properties of both $(w,\mathfrak{q})$ and (ψ,η), we compute

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3} w\cdot\tilde{\varphi}\,\mathrm{d}x\Big|\\
&=\Big|\lim_{R\to\infty}\int_{\mathrm{B}_R} w\cdot\big[-\Delta\psi-\nabla\eta+\mathcal{R}\partial_3\psi+\mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla\psi-\mathrm{e}_3\wedge\psi\big)\big]\,\mathrm{d}x\Big|\\
&=\Big|\lim_{R\to\infty}\int_{\mathrm{B}_R}\big[-\Delta w+\nabla\mathfrak{q}-\mathcal{R}\partial_3 w-\mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla w-\mathrm{e}_3\wedge w\big)\big]\cdot\psi\,\mathrm{d}x\Big|\\
&=\Big|\int_{\mathbb{R}^3} f\cdot\psi\,\mathrm{d}x\Big|\leq\|f\|_q\|\psi\|_{q'}\leq c_7\|f\|_q\|\varphi\|_{q'}.
\end{aligned}\tag{2.22}$$

Note that in order derive the second equality above, it is used that

$$\begin{aligned}
\int_{\mathrm{B}_R} w\cdot\big(\mathrm{e}_3\wedge x\cdot\nabla\psi\big)\,\mathrm{d}x&=\int_{\partial\mathrm{B}_R} w\cdot\psi\big[(\mathrm{e}_3\wedge x)\cdot n\big]\,\mathrm{d}S-\int_{\mathrm{B}_R}\psi\cdot\big(\mathrm{e}_3\wedge x\cdot\nabla w\big)\,\mathrm{d}x\\
&=-\int_{\mathrm{B}_R}\psi\cdot\big(\mathrm{e}_3\wedge x\cdot\nabla w\big)\,\mathrm{d}x,
\end{aligned}$$

where the boundary integral vanishes due to $n=x/|x|$ on $\partial\mathrm{B}_R$. We can now reintroduce u, $\mathfrak{p}$, F and the Fourier coefficients u_k, $\mathfrak{p}_k$, F_k from the first part of the proof. Recall that (2.11) implies $F_0=0$ and thus $u_0=0$. Consequently,

$$\int_{\mathbb{R}^3} w(x)\cdot\Big[\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} Q(t)\varphi\big(Q(t)^\top x\big)\,\mathrm{d}t\Big]\,\mathrm{d}x=\int_{\mathbb{R}^3} u_0(x)\cdot\varphi(x)\,\mathrm{d}x=0.\tag{2.23}$$

Combining (2.22) and (2.23), we find

$$\Big|\int_{\mathbb{R}^3} w\cdot\varphi\,\mathrm{d}x\Big| \leq c_7\|f\|_q\|\varphi\|_{q'}.$$

Since $\varphi \in C_0^\infty(\mathbb{R}^3)^3$ was arbitrary, we conclude $\|w\|_q \leq c_7\|f\|_q$, which, combined with the fact that $(w,\mathfrak{q})$ satisfies (2.14), implies (2.13). By a standard density argument, we finally extend this assertion to all $f \in L^q(\mathbb{R}^3)^3 \cap C^\infty(\mathbb{R}^3)^3$ that satisfies (2.11). □

Remark 2.3. The assertions in Lemma 2.2 remain true also for non-smooth $f \in L^q(\mathbb{R}^3)^3$. In this case, the integral in (2.11) should be understood as a Bochner integral in the space $L^q(\mathbb{R}^3)^3$. Such an interpretation is valid since the mapping $t \to Q(t)f\big(Q(t)^\top x\big)$ belongs to the space $C\big([0,2\pi/\mathcal{T}];L^q(\mathbb{R}^3)^3\big)$. The estimate (2.13) can then be established for a general $f \in L^q(\mathbb{R}^3)^3$ by a density argument. In this paper, however, we only need Lemma 2.2 in a context of smooth data f.

3 Proof of Main Theorem

We are now in a position to prove the main theorem.

Proof of Theorem 1.1. In the first step of the proof, we will reduce (1.2) to a whole space problem. For this purpose, choose $\rho > 0$ so large that $\mathbb{R}^3 \setminus \Omega \subset \mathrm{B}_\rho$. Let $\psi_\rho \in C^\infty(\mathbb{R}^3)$ be a "cut-off" function with $\psi_\rho = 0$ in B_ρ and $\psi_\rho = 1$ in $\mathbb{R}^3 \setminus \mathrm{B}_{2\rho}$. Since (v,p) solves (1.2), standard regularity theory for elliptic systems implies that $(v,p) \in C^\infty(\Omega\setminus \mathrm{B}_\rho)^3 \times C^\infty(\Omega \setminus \mathrm{B}_\rho)$. We can therefore define

$$\begin{aligned} &w:\mathbb{R}^3\to\mathbb{R}^3, \quad w(x) := \psi_\rho(x)v(x) - \mathfrak{B}[\nabla\psi_\rho\cdot v](x),\\ &\mathfrak{q}:\mathbb{R}^3\to\mathbb{R}, \quad \mathfrak{q}(x) := \psi_\rho(x)p(x), \end{aligned} \tag{3.1}$$

where $\mathfrak{B}$ denotes the so-called "Bogovskiĭ operator", that is, an operator

$$\mathfrak{B}: C_0^\infty(\mathrm{B}_{2\rho}) \to C_0^\infty(\mathrm{B}_{2\rho})^3$$

with the property that $\operatorname{div}\mathfrak{B}(f) = f$ whenever $\int_{\mathrm{B}_{2\rho}} f(x)\,\mathrm{d}x = 0$. We refer to [12, Theorem III.3.2] for details on this operator. Observe that

$$\begin{aligned} \int_{\mathrm{B}_{2\rho}} \nabla\psi_\rho\cdot v\,\mathrm{d}x &= \int_{\partial\mathrm{B}_{2\rho}} v\cdot n\,\mathrm{d}x = \int_{\partial\Omega} v\cdot n\,\mathrm{d}S\\ &= \int_{\partial\Omega} (\mathrm{e}_3 + \mathcal{T}\,\mathrm{e}_3\wedge x)\cdot n\,\mathrm{d}S\\ &= \int_{\mathbb{R}^3\setminus\overline{\Omega}} \operatorname{div}(\mathrm{e}_3 + \mathcal{T}\,\mathrm{e}_3\wedge x)\,\mathrm{d}x = 0, \end{aligned} \tag{3.2}$$

whence $(w,\mathfrak{q})$ is a smooth solution in the class (1.4) to the whole space problem

$$\begin{cases} -\Delta w + \nabla\mathfrak{q} - \mathcal{R}\partial_3 w - \mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla w - \mathrm{e}_3\wedge w\big) = g - \mathcal{R}w\cdot\nabla w & \text{in } \mathbb{R}^3,\\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

with $g \in C_0^\infty(\mathbb{R}^3)^3$ and $\operatorname{supp} g \subset \mathrm{B}_{2\rho}$. In addition, from [15, Theorem 4.4] we obtain

$$\forall r \in (1,2): \ (w, \mathfrak{q}) \in \mathrm{X}_r(\mathbb{R}^3). \tag{3.3}$$

In the next step, we proceed as in the proof of Lemma 2.2 and transform the whole space problem above into an equivalent time dependent Oseen problem. As in the proof of Lemma 2.2, we put

$$\begin{aligned} &u : \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right] \to \mathbb{R}^3, \quad u(x,t) := Q(t) w\big(Q(t)^\top x\big), \\ &\mathfrak{p} : \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right] \to \mathbb{R}, \quad \mathfrak{p}(x,t) := \mathfrak{q}\big(Q(t)^\top x\big), \\ &G : \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right] \to \mathbb{R}^3, \quad G(x,t) := Q(t) g\big(Q(t)^\top x\big). \end{aligned} \tag{3.4}$$

Then $(u, \mathfrak{p})$ satisfies

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} - \mathcal{R}\partial_3 u = G - \mathcal{R} u \cdot \nabla u & \text{in } \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right], \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times \left[0, \frac{2\pi}{\mathcal{T}}\right]. \end{cases} \tag{3.5}$$

Since u, $\mathfrak{p}$, and G are smooth and $\frac{2\pi}{\mathcal{T}}$-periodic in t, we can expand these fields in their Fourier-series with respect to t:

$$\begin{aligned} &u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)\, \mathrm{e}^{i\mathcal{T}kt}, \quad \mathfrak{p}(x,t) = \sum_{k\in\mathbb{Z}} \mathfrak{p}_k(x)\, \mathrm{e}^{i\mathcal{T}kt}, \\ &G(x,t) = \sum_{k\in\mathbb{Z}} G_k(x)\, \mathrm{e}^{i\mathcal{T}kt}, \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} &u_k(x) := \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} u(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t, \quad \mathfrak{p}_k(x) := \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} \mathfrak{p}(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t, \\ &G_k(x) := \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} G(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t. \end{aligned}$$

Note that $G_k \in C_0^\infty(\mathbb{R}^3)^3$ and that $(u_k, \mathfrak{p}_k)$ enjoys the same summability properties as $(w, \mathfrak{q})$, that is, $(u_k, \mathfrak{p}_k) \in \mathrm{X}_r(\mathbb{R}^3)$ for all $r \in (1,2)$. Inserting the Fourier series from (3.6) into (3.5), we find that each Fourier coefficient satisfies

$$\begin{cases} i\mathcal{T}k u_k - \Delta u_k + \nabla \mathfrak{p}_k - \mathcal{R}\partial_3 u_k = G_k - \operatorname{div} H_k & \text{in } \mathbb{R}^3, \\ \operatorname{div} u_k = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.7}$$

where

$$H_k(x) := \mathcal{R}\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} u(x,t) \otimes u(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t. \tag{3.8}$$

Observe that H_k has the same summability properties as $w \otimes w$. Recalling (3.3) and the fact that w is smooth, we thus deduce that $H_k \in C^\infty(\mathbb{R}^3)^{3\times 3}$ and satisfies (2.1).

We now focus on the Fourier coefficient u_0. By (3.7), $(u_0, \mathfrak{p}_0)$ satisfies the classical whole space Oseen problem with non-homogeneous data $G_0 - \operatorname{div} H_0$. Consequently,

$$u_0(x) = \Gamma_{\mathrm{O}}^{\mathcal{R}} * \big[G_0 - \operatorname{div} H_0\big](x). \tag{3.9}$$

We shall briefly prove this assertion. As in the proof of Lemma 2.1, it follows from the summability properties (2.5) and (2.6) in combination with Young's inequality that the convolution above is well-defined as an element in $L^r(\mathbb{R}^3)^3$ for all $r \in (2, \infty)$. Consider now $l \in (1, 2)$. Let $\{h_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^3)^3$ be a sequence with $\lim_{n\to\infty} h_n = G_0 - \operatorname{div} H_0$ in $L^s(\mathbb{R}^3)$ for all $s \in (1, \infty)$. Then, by well-known theory for the Oseen problem, $(\Gamma_{\mathrm{O}}^{\mathcal{R}} * h_n, \nabla\Gamma_{\mathrm{L}} * h_n) \in \mathrm{X}_l(\mathbb{R}^3)$ and satisfies the whole space Oseen problem with respect to data h_n, see for example [12, Theorem VII.4.1]. Moreover, $\{(\Gamma_{\mathrm{O}}^{\mathcal{R}} * h_n, \nabla\Gamma_{\mathrm{L}} * h_n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathrm{X}_l(\mathbb{R}^3)$, and thus converges to some element $(\tilde{u}_0, \tilde{\mathfrak{p}}_0) \in \mathrm{X}_l(\mathbb{R}^3)$. Clearly, $(\tilde{u}_0, \tilde{\mathfrak{p}}_0)$ satisfies the whole space Oseen problem with respect to data $G_0 - \operatorname{div} H_0$. Hence, by classical uniqueness results for the Oseen problem, $(\tilde{u}_0, \tilde{\mathfrak{p}}_0) = (u_0, \mathfrak{p}_0)$. On the other hand we have, by Young's inequality,

$$\|\Gamma_{\mathrm{O}}^{\mathcal{R}} * h_n - \Gamma_{\mathrm{O}}^{\mathcal{R}} * \big[G_0 - \operatorname{div} H_0\big]\|_{\frac{2l}{2-l}} \leq \|h_n - \big[G_0 - \operatorname{div} H_0\big]\|_s \|\Gamma_{\mathrm{O}}^{\mathcal{R}}\|_r$$

with $r \in (2, 3)$ and $s \in (1, 2)$. Letting $n \to \infty$, we conclude (3.9). We now employ Lemma 2.1 and find, by (2.3), that

$$\forall q \in (4/3, \infty): \ \Gamma_{\mathrm{O}}^{\mathcal{R}} * \operatorname{div} H_0 \in L^q(\mathbb{R}^3)^3. \tag{3.10}$$

Since $G_0 \in C_0^\infty(\mathbb{R}^3)^3$, it is well known, see for example [12, Chapter VII.3], that

$$\Gamma_{\mathrm{O}}^{\mathcal{R}} * G_0(x) = \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot \left(\int_{\mathbb{R}^3} G_0(y)\,\mathrm{d}y \right) + O\big(\nabla\Gamma_{\mathrm{O}}^{\mathcal{R}}(x)\big) \quad \text{as} \quad |x| \to \infty,$$

from which we infer, by the summability property (2.7) of $\nabla\Gamma_{\mathrm{O}}^{\mathcal{R}}$, that

$$\forall q \in (4/3, \infty): \ \Gamma_{\mathrm{O}}^{\mathcal{R}} * G_0 - \Gamma_{\mathrm{O}}^{\mathcal{R}} \cdot \left(\int_{\mathbb{R}^3} G_0(y)\,\mathrm{d}y \right) \in L^q(\mathbb{R}^3)^3. \tag{3.11}$$

We now wish to evaluate the integral in the identity above. We start by computing

$$\begin{aligned}
\int_{\mathbb{R}^3} g\,\mathrm{d}x &= \int_{\mathrm{B}_{2\rho}} \operatorname{div}\big[- \mathrm{T}(w, \mathfrak{q}) - \mathcal{R} w \otimes \mathrm{e}_3 - \mathcal{T} w \otimes (\mathrm{e}_3 \wedge x) + \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes w \\
&\qquad\qquad + \mathcal{R} w \otimes w\big]\,\mathrm{d}x \\
&= \int_{\partial \mathrm{B}_{2\rho}} \big[- \mathrm{T}(v, p) - \mathcal{R} v \otimes \mathrm{e}_3 - \mathcal{T} v \otimes (\mathrm{e}_3 \wedge x) + \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes v \\
&\qquad\qquad + \mathcal{R} v \otimes v\big] \cdot n\,\mathrm{d}S \\
&= - \int_{\partial\Omega} \big[- \mathrm{T}(v, p) - \mathcal{R} v \otimes \mathrm{e}_3 - \mathcal{T} v \otimes (\mathrm{e}_3 \wedge x) + \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes v \\
&\qquad\qquad + \mathcal{R} v \otimes v\big] \cdot n\,\mathrm{d}S.
\end{aligned}$$

Inserting the boundary values $(1.2)_3$ for v on $\partial\Omega$, an elementary calculation similar to (3.2) shows that all but the first term in the last integral above vanish. Thus

$$\int_{\mathbb{R}^3} g \,\mathrm{d}x = \int_{\partial\Omega} \mathrm{T}(v,p)\cdot n \,\mathrm{d}S = \mathcal{F}.$$

We then find, by the definition of G_0, that

$$\begin{aligned}\int_{\mathbb{R}^3} G_0(y)\,\mathrm{d}y &= \int_{\mathbb{R}^3} \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} Q(t)g(Q(t)^\top y)\,\mathrm{d}t\mathrm{d}y \\ &= \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} Q(t)\Big(\int_{\mathbb{R}^3} g(y)\,\mathrm{d}y\Big)\mathrm{d}t = \big(\mathcal{F}\cdot \mathrm{e}_3\,\big)\,\mathrm{e}_3\,.\end{aligned} \tag{3.12}$$

Combining now (3.9), (3.10), (3.11), and (3.12), we conclude

$$\forall q\in(4/3,\infty):\ \ u_0 - \Gamma_{\mathrm{O}}^{\mathcal{R}}\cdot(\mathcal{F}\cdot\mathrm{e}_3)\,\mathrm{e}_3 \in L^q(\mathbb{R}^3)^3. \tag{3.13}$$

We now return to the expansion (1.5) of v. By standard regularity theory for elliptic systems, v is continuous up to the boundary of Ω. It is therefore enough to show (1.5) for large $|x|$. For this purpose, we split v into two parts. More precisely, we put

$$\mathfrak{z}(x) := \sum_{k\neq 0} u_k(x), \qquad \pi(x) := \sum_{k\neq 0} \mathfrak{p}_k(x),$$

and observe that for $|x|>2\rho$ holds

$$v(x) = w(x) = u(x,0) = \sum_{k\in\mathbb{Z}} u_k(x) = u_0(x) + \mathfrak{z}(x). \tag{3.14}$$

Thus, recalling (3.13), we see that (1.5) is established once we show that $\mathfrak{z}\in L^q(\mathbb{R}^3)^3$ for all $q\in(1,\infty)$. Since both $(u_0,\mathfrak{p}_0)$ and $(w,\mathfrak{q})$ belong to $\mathrm{X}_r(\mathbb{R}^3)$ for all $r\in(1,2)$, so does $(\mathfrak{z},\pi)$. Moreover, $(\mathfrak{z},\pi)$ satisfies

$$\begin{cases} -\Delta\mathfrak{z} + \nabla\pi - \mathcal{R}\partial_3\mathfrak{z} - \mathcal{T}\big(\mathrm{e}_3\wedge x\cdot\nabla\mathfrak{z} - \mathrm{e}_3\wedge\mathfrak{z}\big) & \\ \qquad\qquad\qquad\qquad = \big(g - \mathcal{R}w\cdot\nabla w\big) - \big(G_0 - \operatorname{div} H_0\big) & \text{in } \mathbb{R}^3,\\ \operatorname{div}\mathfrak{z} = 0 & \text{in } \mathbb{R}^3\end{cases}$$

As one may easily verify, $\big(g-\mathcal{R}w\cdot\nabla w\big) - \big(G_0 - \operatorname{div} H_0\big)$ satisfies condition (2.11). Hence, for any $q\in(1,\infty)$ Lemma 2.2 yields

$$\begin{aligned}\|\mathfrak{z}\|_{2,q} &\leq c_1\big\|\big(g-\mathcal{R}w\cdot\nabla w\big) - \big(G_0-\operatorname{div} H_0\big)\big\|_q \\ &\leq c_2(\|g\|_q + \|w\cdot\nabla w\|_q).\end{aligned} \tag{3.15}$$

Due to (3.3), the right-hand side above is finite for all $q\in(1,\infty)$. We thus conclude $\mathfrak{z}\in L^q(\mathbb{R}^3)^3$ for all $q\in(1,\infty)$, and thereby (1.5).

It remains to show (1.6). By standard regularity theory, also $\partial_j v$ is continuous up to the boundary. Again, it is therefore enough to establish (1.6) for large $|x|$. It is well-known, see again [12, Chapter VII.3], that

$$\partial_j \big[\Gamma^{\mathcal{R}}_{\mathrm{O}} * G_0\big](x) = \partial_j \Gamma^{\mathcal{R}}_{\mathrm{O}}(x) \cdot \bigg(\int_{\mathbb{R}^3} G_0(y)\,\mathrm{d}y\bigg) + O\big(\nabla^2 \Gamma^{\mathcal{R}}_{\mathrm{O}}(x)\big) \quad \text{as} \quad |x| \to \infty,$$

which, combined with the summability property (2.9) of $\nabla^2 \Gamma^{\mathcal{R}}_{\mathrm{O}}$, implies

$$\forall q \in (1,\infty): \ \partial_j \big[\Gamma^{\mathcal{R}}_{\mathrm{O}} * G_0\big] - \partial_j \Gamma^{\mathcal{R}}_{\mathrm{O}} \cdot \bigg(\int_{\mathbb{R}^3} G_0(y)\,\mathrm{d}y\bigg) \in L^q(\mathbb{R}^3)^3. \tag{3.16}$$

From Lemma 2.1 we obtain

$$\forall q \in (1,\infty): \ \partial_j \big[\Gamma^{\mathcal{R}}_{\mathrm{O}} * \operatorname{div} H_0\big] \in L^q(\mathbb{R}^3)^3. \tag{3.17}$$

Combining (3.9), (3.14), (3.16), and (3.17), we conclude that

$$\partial_j v(x) = \partial_j \Gamma^{\mathcal{R}}_{\mathrm{O}}(x) \cdot (\mathcal{F} \cdot \mathrm{e}_3)\,\mathrm{e}_3 + S(x) + \partial_j \mathfrak{z}(x),$$

with $S \in L^q(\mathbb{R}^3)^3$ for all $q \in (1,\infty)$. By (3.15), $\partial_j \mathfrak{z} \in L^q(\mathbb{R}^3)^3$ for all $q \in (1,\infty)$. Thus, (1.6) follows. □

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