Leray's inequality in general multi-connected domains in $\mathbb{R}^n$

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Dedicated to Professor Izumi Takagi on the occasion of his 60th birthday.

Abstract

Consider the stationary Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ consists of $L+1$ smooth $n-1$ dimensional closed hypersurfaces $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$, where $\Gamma_1, \cdots, \Gamma_L$ lie inside of Γ_0 and outside of one another. The Leray inequality of the given boundary data β on $\partial\Omega$ plays an important role for the existence of solutions. It is known that if the flux $\gamma_i \equiv \int_{\Gamma_i} \beta \cdot \nu dS = 0$ on Γ_i(ν: the unit outer normal to Γ_i) is zero for each $i = 0, 1, \cdots, L$, then the Leray inequality holds. We prove that if there exists a sphere S in Ω separating $\partial\Omega$ in such a way that $\Gamma_1, \cdots, \Gamma_k$ $(1 \leqq k \leqq L)$ are contained inside of S and that the others $\Gamma_{k+1}, \cdots, \Gamma_L$ are outside of S, then the Leray inequality necessarily implies that $\gamma_1 + \cdots + \gamma_k = 0$. In particular, suppose that there are L spheres $S_1, \cdots, S_L$ in Ω such that Γ_i lies inside of S_i for all $i = 1, \cdots, L$. Then the Leray inequality holds if and only if $\gamma_0 = \gamma_1 = \cdots = \gamma_L = 0$.

1 Introduction.

We consider Leray's problem on the stationary Navier-Stokes equations with the *inhomogeneous* boundary data under the *general flux condition.* Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with smooth boundary $\partial\Omega$. Throughout this paper, we impose the following assumption on Ω.

Assumption. The boundary $\partial\Omega$ has $L+1$ connected components $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$ of $n-1$ dimensional C^∞-closed hypersurfaces such that $\Gamma_1, \cdots, \Gamma_L$ lie inside of Γ_0 and outside of one other;

$$\partial\Omega = \bigcup_{j=0}^{L} \Gamma_j.$$

In the most interesting case when $n = 3$, it is often called that Ω has the second Betti number L. In Ω we consider the boundary value problem for the stationary Navier-Stokes equations:

$$\text{(N-S)} \qquad \begin{cases} -\mu\Delta v + v\cdot\nabla v + \nabla p = 0 & \text{in } \Omega, \\ \text{div } v = 0 & \text{in } \Omega, \\ v = \beta & \text{on } \partial\Omega, \end{cases}$$

where $v = v(x) = (v_1(x), \cdots, v_n(x))$ and $p = p(x)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \cdots, x_n) \in \Omega$, while $\mu > 0$ is the given viscosity constant, and $\beta = \beta(x) = (\beta_1(x), \cdots, \beta_n(x))$ is the given boundary data on $\partial\Omega$. We use the standard notation as $\Delta v = \sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2}$, $\nabla p = \left(\frac{\partial p}{\partial x_1}, \cdots, \frac{\partial p}{\partial x_n}\right)$, div $v = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$, and $v\cdot\nabla v = \sum_{j=1}^{n} v_j \frac{\partial v}{\partial x_j}$. Since the solution v satisfies div $v = 0$ in Ω, the given boundary data β on $\partial\Omega$ is required to fulfill the following compatibility condition which we call the *general* flux condition:

$$\text{(G.F.)} \qquad \sum_{j=0}^{L} \int_{\Gamma_j} \beta\cdot\nu dS = 0,$$

where ν denotes the unit outer normal to $\partial\Omega$. Leray [11] proposed to solve the following problem.

Leray's problem. Let $n = 2, 3$. Suppose that $\beta \in H^{1/2}(\partial\Omega)$ satisfies the general flux condition (G.F.). Does there exist at least one weak solution $v \in H^1(\Omega)$ of (N-S) ?

Up to now, we are not yet successful to give a complete answer to this question. However, some partial answer has been proved by Leray [11], Fujita[3] and Ladyzehenskaya [10] under the *restricted* flux condition (R.F.) on β:

$$\text{(R.F.)} \qquad \gamma_i \equiv \int_{\Gamma_j} \beta\cdot\nu dS = 0 \quad \text{for all } j = 0, 1, \cdots, L.$$

Indeed, under the restricted flux condition (R.F.) on β, they showed that there exists at least one weak solution v of (N-S). Although more refined existence results were given by Galdi [6, Chapter VIII, Theorem 4.1] and the second and the third authors [9], it is still an open problem to prove an existence theorem under the general flux condition (G.F.).

If the given boundary data β satisfies the general flux condition (G.F.), then there exists an extension b into Ω with $b|_{\partial\Omega} = \beta$ such that div $b = 0$. See e.g., Borchers-Sohr [2]. We call such b a solenoidal extension into Ω of β. Introducing a new unknown variable $u \equiv v - b$, we can reduce the original equations (N-S) to the following ones with the *homogeneous* boundary condition:

$$\text{(N-S')} \qquad \begin{cases} -\mu\Delta u + b\cdot\nabla u + u\cdot\nabla b + u\cdot\nabla u + \nabla p = \mu\Delta b - b\cdot\nabla b & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

To solve (N-S') we need to handle the linear convection term $b\cdot\nabla u + u\cdot\nabla b$ as a perturbation of $-\mu\Delta u$. More precisely, to prove the existence of the solution u of (N-S'), we rely on the following Leray inequality.

Definition 1 *Let Ω be as in the Assumption, and let $\beta \in H^{1/2}(\partial\Omega) \cap W^{1-\frac{2}{n},\frac{n}{2}}(\partial\Omega)$. Suppose that β fulfills (G.F.) . We say that β satisfies the Leray inequality in Ω if for every $\varepsilon > 0$ there exists $b_\varepsilon \in H^1(\Omega) \cap W^{1,\frac{n}{2}}(\Omega)$ with* div $b_\varepsilon = 0$ *in Ω and $b_\varepsilon = \beta$ on $\partial\Omega$ such that*

$$\text{(L.I.)} \qquad |(u \cdot \nabla b_\varepsilon, u)| \leqq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} \quad \text{for all } u \in H^1_{0,\sigma}(\Omega),$$

where $H^1_{0,\sigma}(\Omega) \equiv \{u \in H^1_0(\Omega); \operatorname{div} u = 0\}$ and $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)$.

For $u \in H^1_{0,\sigma}(\Omega)$ and $b_\varepsilon \in H^1(\Omega) \cap W^{1,\frac{n}{2}}(\Omega)$, the left hand side of (L.I.) is well-defined since we have by the Sobolev imbedding $H^1_0(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$ that

$$|(u \cdot \nabla b_\varepsilon, u)| \leqq \|u\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \|\nabla b_\varepsilon\|_{L^{\frac{n}{2}}(\Omega)} \leqq C\|\nabla u\|^2_{L^2(\Omega)} \|\nabla b_\varepsilon\|_{L^{\frac{n}{2}}(\Omega)}.$$

Notice that it holds a continuous imbedding $H^{1/2}(\partial\Omega) \subset W^{1-\frac{2}{n},\frac{n}{2}}(\partial\Omega)$ provided $n = 2, 3, 4$. So, the space $W^{1-\frac{2}{n},\frac{n}{2}}(\partial\Omega)$ for β is meaningful only for $n \geq 5$.

For a moment, let us assume that $n = 2$ or $n = 3$. Once (L.I.) is established, by the well-known identities $(b \cdot \nabla u, u) = (u \cdot \nabla u, u) = 0$ and $(\nabla p, u) = 0$ for $u \in H^1_{0,\sigma}(\Omega)$, we obtain from (L.I.) with $\varepsilon = \mu/2$ such an apriori estimate that

$$\|\nabla u\|_{L^2(\Omega)} \leqq 2\mu^{-1}\|\mu\Delta b - b \cdot \nabla b\|_{H^{-1}(\Omega)},$$

which yields the solution u of (N-S') with the aid of the Leray-Schauder fixed point theorem.

If the given boundary data β satisfies the restricted flux condition (R.F.), then we see that β fulfills (L.I.). Indeed, under the hypothesis of the restricted flux condition (R.F.) on β, the solenoidal extension b into Ω of β has a vector potential $w \in H^2(\Omega)$, which means that b can be expressed as $b = \operatorname{rot} w$. Taking a family $\{\theta_\varepsilon\}_{\varepsilon>0}$ of cut-off functions with $\theta_\varepsilon(x) \equiv 1$ for x near the boundary $\partial\Omega$ so that the support of θ_ε is confined in an arbitrarily narrow closed strip to $\partial\Omega$ as $\varepsilon \to +0$, and then redefining b_ε as $b_\varepsilon(x) \equiv \operatorname{rot}\,(\theta_\varepsilon(x)w(x))$, we see that β satisfies (L.I.). For instance, see Temam [14, Chapter II, Lemma 1.8] and Galdi [6, Chapter VIII, Lemma 4.2]. It should be noted that the boundary value of b_ε is invariant under the multiplication of w by θ_ε. In the previous work [8, Remark 1 (4)](see also [9]), we showed that the solenoidal extension b into Ω of β has a vector potential if and only if β satisfies the restricted flux condition (R.F.).

Now the natural question arises whether the general flux condition (G.F.) implies (L.I.). Unfortunately, Takeshita[13] gave a negative answer to this question. Indeed, he treated the annular domain $\Omega = \{x \in \mathbb{R}^n; R_1 < |x| < R_0\}$ with $\Gamma_0 = \{x \in \mathbb{R}^n; |x| = R_0\}$, $\Gamma_1 = \{x \in \mathbb{R}^n; |x| = R_1\}$, and proved that β satisfies (L.I.) in such an annulus Ω if and only if

$$\int_{\Gamma_0} \beta \cdot \nu dS = \int_{\Gamma_1} \beta \cdot \nu dS = 0.$$

Another refined proof in the 2D annular region was given by Galdi [6, page 23]. In the last part of Takeshita's paper [13, Theorm 2], he treated the domain Ω as in the Assumption and stated without any detail that if for each $i = 0, 1, \cdots, L$, Γ_i is diffeomorphically deformed to the sphere in $\bar{\Omega}$, then the Leray inequality (L.I.) is equivalent to (R.F.).

In this paper, we generalize Galdi-Takeshita's result with a simple proof. Although our result is not altogether new, we do not need to impose any *topological restriction* on the boundary, while Takeshita [13] requires that each Γ_i, $i = 0, 1, \cdots, L$, is diffeomorphic to the sphere. The main theorem now reads:

Theorem 1 *Let $n \geq 3$ and let Ω be as in the Assumption. Suppose that $\beta \in H^{1/2}(\partial\Omega) \cap W^{1-\frac{2}{n},\frac{n}{2}}(\partial\Omega)$ and that β satisfies (G.F.). Assume that there is a sphere S in Ω such that Γ_1, $\cdots$, and Γ_k lie inside of S and such that the others Γ_{k+1}, $\cdots$, Γ_L and Γ_0 lie outside of S. If β satisfies (L.I.) in Ω, then we have*

$$\gamma_1 + \cdots + \gamma_k = 0, \quad \gamma_{k+1} + \cdots + \gamma_L + \gamma_0 = 0. \tag{1.1}$$

As an immediate consequence of this theorem, we obtain the following necessary and sufficient condition on the Leray inequality.

Corollary 1 *Let $n \geq 3$ and let Ω be as in the Assumption. Suppose that $\beta \in H^{1/2}(\partial\Omega) \cap W^{1-\frac{2}{n},\frac{n}{2}}(\partial\Omega)$ and that β satisfies (G.F.). Assume that there exist L spheres $S_1, \cdots, S_L$ in Ω such that S_i contains only Γ_i in its inside and the rests $\partial\Omega \setminus \Gamma_i$ lie in the outside of S_i for all $i = 1, \cdots, L$. Then β satisfies (L.I.) in Ω if and only if (R.F.) holds.*

Remarks. 1. Corollary 1 may be regarded as a generalization of Takeshita [13, Theorm 2] since it is only assumed that each component Γ_i, $i = 1, \cdots, L$ is a smooth $n-1$ dimensional closed hypersurface in $\mathbb{R}^n$ with $n \geq 3$.

2. The assumption on regularity of the boundary $\partial\Omega$ can be relaxed so that the Stokes integral formula holds for vector fields on $\bar{\Omega}$. For instance, Theorem 1 holds for bounded locally Lipschitz domains Ω. More generally, we may treat the case when Ω is a bounded domain in $\mathbb{R}^n$ with locally finite perimeter as in Ziemer [15, Theorem 5.8.2].

3. A similar argument to make use of the sphere covering each component of the boundary was established by Kobayashi [7] in the two-dimensional multi-connected domains. Indeed, he proved the corresponding result to Corollary 1 in the plane. However, it seems difficult to apply his method directly to our higher-dimensional case.

4. Under some hypothesis on symmetry of the multi-connected domain Ω in $\mathbb{R}^2$, Amick [1] and Fujita [4] proved (L.I.) for all solenoidal vector fields u with symmetry. See also Morimoto [12].

5. For solvability of (N-S') itself in the case $n = 2, 3$, the Leray inequality (L.I.) can be relaxed to the following weaker condition (E.C.).

$$-(u \cdot \nabla b_\varepsilon, u) \leqq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} \quad \text{for all } u \in H^1_{0,\sigma}(\Omega). \tag{E.C.}$$

Galdi[6, page 21] called it *extension condition* on β, and proved that in the 2D annular region (E.C.) necessarily implies (R.F.) for $j = 0, 1$ under the inflow condition that $\gamma_1 \leqq 0$. It is also possible to prove in Theorem 1 that (E.C.) yields the same conclusion as (1.1) provided $\gamma_1 + \cdots + \gamma_k \leqq 0$.

In Theorem 1, it is sufficient to cover $\Gamma_1, \cdots, \Gamma_k$ by the sphere S in Ω so that the rests of $\Gamma_{k+1}, \cdots, \Gamma_L$ and Γ_0 lie outside of S. Taking a slightly larger sphere S' in Ω containing S with its same origin so that $\Gamma_{k+1}, \cdots, \Gamma_L$ and Γ_0 also lie outside of S', we may reduce the problem to that in the annular domain D between S and S'. Indeed, if the given boundary data β satisfies (G.F.) and (L.I.) with some solenoidal vector field $b_\varepsilon \in H^1(\Omega) \cap W^{1,\frac{n}{2}}(\Omega)$, then it holds

$$\int_S b_\varepsilon \cdot \nu dS = \sum_{i=1}^{k} \gamma_i \equiv \gamma, \quad \int_{S'} b_\varepsilon \cdot \nu dS = -\gamma.$$

In the similar manner to Takeshita [13], by introducing the mean $M(b_\varepsilon)$ of b_ε with respect to the normalized Haar measure on the $SO(n)$-action, we see that each flux on S and S' of $M(b_\varepsilon)$ remains invariant, and that the inequality

$$\left| \int_D u \cdot \nabla M(b_\varepsilon) \cdot u dx \right| \leqq \varepsilon \int_D |\nabla u|^2 dx \tag{1.2}$$

holds for all $u \in C_0^\infty(D)$ with div $u = 0$. An appropriate choice of u in (1.2) enables us to obtain $\gamma = 0$.

2 Proof of Theorem 1.

Suppose that the boundary data $\beta \in H^{1/2}(\partial\Omega) \cap W^{1-\frac{2}{n},\frac{n}{2}}(\partial\Omega)$ satisfies the Leray inequality in Ω in the sense of Definition 1 Then, for every $\varepsilon > 0$ there exists $b_\varepsilon \in H^1(\Omega) \cap W^{1,\frac{n}{2}}(\Omega)$ with div $b_\varepsilon = 0$ in Ω and $b_\varepsilon = \beta$ such that (L.I.) holds. By the hypothesis on $\partial\Omega$, without loss of generality, we may take $0 < R < R'$ such that both spheres $S_R \equiv \{x \in \mathbb{R}^n; |x| = R\}$ and $S_{R'} \equiv \{x \in \mathbb{R}^n; |x| = R'\}$ are contained in Ω, $\Gamma_1, \cdots, \Gamma_k$ lie inside of S_R and such that $\Gamma_{k+1}, \cdots, \Gamma_L$ and Γ_0 lie outside of $S_{R'}$. Since $\sum_{i=0}^{L} \gamma_i = 0$, implied by (G.F.), and since div $b_\varepsilon = 0$ in Ω with $b_\varepsilon = \beta$ on $\partial\Omega$, it holds

$$\int_{S_R} b_\varepsilon \cdot \nu dS = \gamma \equiv \sum_{i=1}^{k} \gamma_i, \quad \int_{S_{R'}} b_\varepsilon \cdot \nu dS = -\gamma. \tag{2.1}$$

Now we reduce our problem to that in the concentric spherical domain $D \equiv \{x \in \mathbb{R}^n; R < |x| < R'\}$ and follow the argument given by Takeshita [13].

Let us take the mean $M(b_\varepsilon)$ of b_ε with respect to the normalized Haar measure dg on $SO(n)$-action. That is,

$$\begin{aligned} M(b_\varepsilon) &= \int_{SO(n)} T_g b_\varepsilon dg, \\ T_g b_\varepsilon(x) &= g b_\varepsilon(g^{-1}x), \quad x \in D, g \in SO(n). \end{aligned}$$

By (2.1) it holds

$$\begin{cases} \text{div } M(b_\varepsilon) = 0 \quad \text{in } D, \\ \displaystyle\int_{S_R} M(b_\varepsilon) \cdot \nu dS = \gamma, \quad \int_{S_{R'}} M(b_\varepsilon) \cdot \nu dS = -\gamma. \end{cases} \tag{2.2}$$

Furthermore, by (L.I.) we have

$$\left|\int_D v\cdot\nabla M(b_\varepsilon)\cdot v dx\right| \leqq \varepsilon\int_D |\nabla v|^2 dx \quad \text{for all } v\in C^\infty_{0,\sigma}(D), \tag{2.3}$$

where $C^\infty_{0,\sigma}(D)$ is the set of all solenoidal vector fields with compact support in D. Indeed, since $\det g = 1$, by changing the variable $x\in D\to y=g^{-1}x\in D$, we have

$$\int_D v\cdot\nabla(T_g b_\varepsilon)\cdot v dx = \int_D T_g^{-1}v\cdot\nabla b_\varepsilon\cdot T_g^{-1} v dy$$

for all $g\in SO(n)$, which yields with the aid of the Fubini theorem that

$$\begin{aligned} \left|\int_D v\cdot\nabla(M b_\varepsilon)\cdot v dx\right| &= \left|\int_{SO(n)}\left(\int_D T_g^{-1}v\cdot\nabla b_\varepsilon\cdot T_g^{-1}v dy\right)dg\right| \\ &\leqq \int_{SO(n)}\left|\int_D T_g^{-1}v\cdot\nabla b_\varepsilon\cdot T_g^{-1}v dy\right| dg. \end{aligned} \tag{2.4}$$

Since $T_g^{-1}v\in C^\infty_{0,\sigma}(D)$ and since $|\nabla T_g^{-1}v(y)|^2 = |\nabla v(gy)|^2$ for all $y\in D$, we have by (L.I.) and again by changing variable $y\in D\to x=gy\in D$ with $\det g^{-1}=1$ that

$$\begin{aligned} \left|\int_D T_g^{-1}v\cdot\nabla b_\varepsilon\cdot T_g^{-1}v dy\right| &= \left|\int_\Omega T_g^{-1}v\cdot\nabla b_\varepsilon\cdot T_g^{-1}v dy\right| \\ &\leqq \varepsilon\int_\Omega |\nabla T_g^{-1}v|^2 dy \\ &= \varepsilon\int_D |\nabla T_g^{-1}v|^2 dy \\ &= \varepsilon\int_D |\nabla v|^2 dx \end{aligned} \tag{2.5}$$

for all $g\in SO(n)$. It follows from (2.4) and (2.5) that

$$\left|\int_D v\cdot\nabla(M b_\varepsilon)\cdot v dx\right| \leqq \varepsilon\int_{SO(n)}\left(\int_D |\nabla v|^2 dx\right) dg = \varepsilon\int_D |\nabla v|^2 dx,$$

which implies (2.3).

In the next step, we test (2.3) by an appropriate $v\in C^\infty_{0,\sigma}(D)$. First, it follows from (2.2) that $M(b_\varepsilon)$ has the representation as

$$M(b_\varepsilon) = \frac{\gamma}{\omega_n r^n}x, \quad x\in D, \tag{2.6}$$

where $r=|x|$ and $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}$ is the surface area of S^{n-1}. Now, we choose a test vector function v of (2.3) as

$$v(x) = (-\rho(r)x_2, \rho(r)x_1, 0, \cdots, 0), \quad x=(x_1,\cdots,x_n)\in D$$

with $\rho \in C_0^\infty((R,R'))$. It is easy to see that $v \in C_{0,\sigma}^\infty(D)$ with the property that $v(x)\cdot x = 0$ for all $x \in D$. Since

$$\frac{\partial}{\partial x_j} M(b_\varepsilon)_k = \frac{\gamma}{\omega_n r^n}\left(\delta_{jk} - n\frac{x_j}{r}\frac{x_k}{r}\right), \quad j,k = 1,\cdots,n$$

and since $v(x)\cdot x = 0$ for all $x \in D$, it holds that

$$\begin{aligned} v\cdot\nabla M(b_\varepsilon)\cdot v = \sum_{j,k=1}^n v_j \frac{\partial}{\partial x_j} M(b_\varepsilon)_k v_k &= \frac{\gamma}{\omega_n r^n}\left(|v|^2 - n\left(\frac{v\cdot x}{r}\right)^2\right) \\ &= \frac{\gamma}{\omega_n r^n}|v|^2 \end{aligned} \tag{2.7}$$

in D. Hence it follows from (2.3) and (2.7) that

$$\frac{|\gamma|}{\omega_n}\int_D \frac{|v|^2}{r^n}dx \leqq \varepsilon \int_D |\nabla v|^2 dx \tag{2.8}$$

for all $\varepsilon > 0$. Since the left and side of (2.8) is independent of ε and since $\nabla v \neq 0$, by letting $\varepsilon \to 0$ we conclude from (2.8) that

$$\gamma = 0.$$

This proves Theorem 1.

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References

[1] Amick, C.J., *Existence of solutions to the nonhomogeneous steady Navier-Stokes equations.* Indiana Univ. Math. J., **33**, 817–830 (1984).

[2] Borchers, W., Sohr, H., *On the equations* rot $v = g$ *and* div $u = f$ *with zero boundary conditions.* Hokkaido Math. J., **19**, 67–87 (1990).

[3] Fujita, H., *On the existence and regularity of the steady-state solutions of the Navier-Stokes equations.* J. Fac. Sci. Univ. Tokyo Sec.I. A.**9**, 59–102 (1960).

[4] Fujita, H., *On stationary solutions to Navier-Stokes equation in symmetric plane domains under general outflow condition.* Navier-Stokes equations: theory and numerical methods, Varenna 1997 Pitmann Res. Notes Math. Ser. **388** Longman-Harlow, 16–30, 1998.

[5] Fujita, H., Morimoto, H., *A remark on the existence of the Navier-Stokes flow with non-vanishing outflow condition.* Proceeding of the 4th MSJ-IRI on Nonlinear Waves, Agemi et. al ed., 53–61 (1997).

[6] Galdi, G.P., *An Introduction to the Mathematical Theory of the Navier-Stokes Equations.* vol. II, Springer-Verlag New York–Berlin–Heidelberg-Tokyo, 1994.

[7] Kobayashi., T., *Takeshita's examples for Leray's inequality.* Preprint

[8] Kozono,H., Yanagisawa,T., *L^r-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains.* Indiana Univ. Math. J. **58**, 1853-1920 (2009).

[9] Kozono,H., Yanagisawa,T., *Leray's problem on the stationary Navier-Stokes equations with inhomogeneous boundary data.* Math. Z. **262**, 27–39 (2009).

[10] Ladyzehnskaya, O.A., *The mathematical theory of viscous incompressible flow.* Gordon and Breach. London, 1969.

[11] Leray, J., *Etude de diverses équations intégrals non linés et de quelques problėms que pose lH́ydrodynamique.* J. Math. Pures Appl. **12**, 1–82 (1933).

[12] Morimoto, H., *General outflow condition for Navier-Stokes flow.* Recent topics in mathematical theory of viscous incompressible fluids, Tsukuba 1996, edited by H.Kozono & Y.Shibata, Lec. Notes Num. Appl.**16**, Kinokuniya, Tokyo. 209–224, 1998.

[13] Takeshita, A., *A Remark on Leray's inequality.* Pacific J. Math. **157**, 151–158 (1993).

[14] Temam, R., *Navier-Stokes equations. Theory and Numerical Analysis.* North-Holland Pub. Co., Amsterdam-New York–Oxford 1979.

[15] Ziemer, W.P., *Weakly differentiable functions.* Springer-Verlag New York–Berlin–Heidelberg-London-Paris-Tokyo-Hong Kong, 1989.