On Bianchi's identities in ECE theory

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(our comments in **blue**, quotations from Evans' texts in ***black italics***)

1. M.W. Evans' claims in his ECE theory

In their paper [1], The Bianchi Identity of differential geometry , the authors M.W. Evans and H. Eckardt assert:

A second Bianchi identity of Cartan geometry is given in ref {13} (= [3])

$$D \wedge R^a{}_b = 0 \qquad ([1], 11)$$

but this is a special case {13}, when ***the torsion is zero (the Einstein-Hilbert case)****. . . .*

and in M.W. Evans' paper [2]: The fundamental origin of curvature and torsion the author writes about the equation:

$$D \wedge R^a{}_b = 0 \qquad ([2], 10)$$

In the presence of the torsion form (i.e. $T^a \neq 0$) however the rigorously correct Bianchi identity is:

$$D \wedge T^a = R^a{}_b \wedge q^b \qquad ([2], 11)$$

and there is ***only one*** *Bianchi identity (see paper 88). The second one can be derived from Eq.(11).*

From the Abstract of the same paper we learn that

the conventional second Bianchi identity is ***true if and only if torsion is zero****.*

And on p. 8 of paper 88 the authors Evans and Eckardt introduce a "true second Bianchi identity" by:

[1] Please, do not confuse (as M.W. Evans does) the general covariant derivative operator ∇ (which acts on general tensor fields) with the exterior covariant derivative operator D (which acts on indexed form fields). See, e.g., [6] for details.

The true second Bianchi identity is obtained by taking the D∧ derivative of both sides of the true first Bianchi identity ([1], 1).

All these statements mean that M.W. Evans doubts the validity of the second Bianchi identity for the general case of a connection (covariant derivative[1]) with non-vanishing torsion. He is believing that the conventional 2nd Bianchi identity must be replaced with his "true second Bianchi identity", which is a trivial consequence of the 1st Bianchi identity. As we shall show below, the "conventional" 2nd Bianchi identity is a more far reaching statement.

The 2nd Bianchi identity

$$D \wedge R^a{}_b := d \wedge R^a{}_b + \omega^a{}_c \wedge R^c{}_b - \omega^c{}_b \wedge R^a{}_c = 0 \tag{1.1}$$

is well-known in literature [3, p.93 (3.141)], [4, p.489 (J.32)], [5, p.208 (C.1.69)], [6, p.123 (4,127 b)] . **It is a classical result, in exterior calculus going back to Élie Cartan (see e.g. [12], p.225, p.236) in the twenties of the last century.**

2. Evans' "true second Bianchi identity"

Before we present the simple (textbook) algebra proof of Eq.(1.1), which is valid for any general connection, we shall show that Eq.(1.1) does not follow from Eq. (paper 99, 11) as claimed by Evans (not even in the case of a connection with null torsion).

Indeed, taking the exterior covariant derivative of both sides of Eq. (paper 99, 11) we get

$$D \wedge (D \wedge T^a) = D \wedge (R^a{}_b \wedge q^b) . \tag{2.1}$$

Now recall some well-known rules of exterior calculus: For any r-form F and any s-form G, we have that

$$d \wedge d \wedge F = 0 , \tag{2.2}$$

$$F \wedge G = (-1)^{rs} G \wedge F, \; d \wedge (F \wedge G) = d \wedge F \wedge G + (-1)^r F \wedge d \wedge G . \tag{2.3}$$

Using these rules we get for the left hand side of Eq.(2.1)

$$\begin{aligned} D \wedge (D \wedge T^a) &= d \wedge (D \wedge T^a) + \omega^a{}_b \wedge (D \wedge T^b) \\ &= d \wedge (d \wedge T^a + \omega^a{}_b \wedge T^b) + \omega^a{}_b \wedge (d \wedge T^b + \omega^b{}_c \wedge T^c) \\ &= d \wedge \omega^a{}_b \wedge T^b - \omega^a{}_b \wedge d \wedge T^b + \omega^a{}_b \wedge d \wedge T^b + \omega^a{}_b \wedge \omega^b{}_c \wedge T^c \qquad (2.4) \\ &= (d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \wedge T^c \\ &= R^a{}_b \wedge T^b \end{aligned}$$

On the other hand, by using the rules (2.3-4) we obtain

$$D \wedge (R^a{}_b \wedge q^b) = d \wedge (R^a{}_b \wedge q^b) + \omega^a{}_c \wedge (R^c{}_b \wedge q^b)$$

$$= (d \wedge R^a{}_b + \omega^a{}_c \wedge R^c{}_b) \wedge q^b + R^a{}_b \wedge (d \wedge q^b) \tag{2.5}$$

$$= (D \wedge R^a{}_b) \wedge q^b + R^a{}_b \wedge T^b .$$

Hence, inserting the results (2.4) and (2.5) into Eq. (2.1) yields

$$R^a{}_b \wedge T^b = (D \wedge R^a{}_b) \wedge q^b + R^a{}_b \wedge T^b, \tag{2.6}$$

or

$$(D \wedge R^a{}_b) \wedge q^b = 0. \tag{2.7}$$

However, this result, a reformulation of Evans' “true 2nd Bianchi identity”, does not yet imply the more far reaching Eq. (2.1), the 2nd Bianchi identity, itself.

3. The 2nd Bianchi identity

We repeat next the simple (textbook) algebra proof of the 2nd Bianchi identity Eq.(1.1), which is valid for any general connection:

Proof

The curvature 2-forms are by definition[2]

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b . \tag{3.1}$$

Now due to the rules (2.3-4) we can write

$$d\wedge R^a{}_b = d\wedge(d\wedge\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b) = 0 + d\wedge(\omega^a{}_c\wedge\omega^c{}_b) = \omega^c{}_b\wedge d\wedge\omega^a{}_c - \omega^a{}_c\wedge d\wedge\omega^c{}_b . \tag{3.2}$$

On the other hand, by using the provided rules we can evaluate

$$\begin{aligned} \omega^a{}_c\wedge R^c{}_b &= \omega^a{}_c\wedge(d\wedge\omega^c{}_b + \omega^c{}_d\wedge\omega^d{}_b) = \omega^a{}_c\wedge d\wedge\omega^c{}_b + \omega^a{}_c\wedge\omega^c{}_d\wedge\omega^d{}_b \\ \omega^c{}_b\wedge R^a{}_c &= \omega^c{}_b\wedge(d\wedge\omega^a{}_c + \omega^a{}_d\wedge\omega^d{}_c) = \omega^c{}_b\wedge d\wedge\omega^a{}_c + \omega^c{}_b\wedge\omega^a{}_d\wedge\omega^d{}_c \end{aligned} \tag{3.3}$$

and due to $\omega^a{}_c\wedge\omega^c{}_d\wedge\omega^d{}_b = \omega^c{}_b\wedge\omega^a{}_d\wedge\omega^d{}_c$ we get

[2] Note that the connection 1-forms $\omega^a{}_b$ do not behave as tensors under changes of the tetrad frame. Hence a presumed exterior covariant derivative of $\omega^a{}_b$. built in analogy to Eq. (3.5) fails to transform correctly, i.e. “covariantly”. Nevertheless, the combination $d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ behaves as a tensor: It is identical with $R^a{}_b$.

$$\begin{aligned} & d \wedge R^a{}_b \qquad\qquad + \quad \omega^a{}_c \wedge R^c{}_b - \omega^c{}_b \wedge R^a{}_c \\ &= d\wedge\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d \wedge \omega^c{}_b \quad + \quad \omega^a{}_c \wedge d\wedge\omega^c{}_b - \omega^c{}_b \wedge d \wedge \omega^a{}_c \\ &= d\wedge\omega^a{}_c \wedge \omega^c{}_b \qquad + \quad 0 \qquad - \omega^c{}_b \wedge d \wedge \omega^a{}_c = 0\,, \end{aligned} \tag{3.4}$$

or

$$\mathbf{D \wedge R^a{}_b = d \wedge R^a{}_b + \boldsymbol{\omega}^a{}_c \wedge R^c{}_b - \boldsymbol{\omega}^c{}_b \wedge R^a{}_c = 0\,.} \tag{3.5}$$

Remark: More comments on the other serious confusions and errors that M.W. Evans made in his use of differential geometry (in particular, Cartan calculus of differential forms) will be presented in this series of short and pedagogical notes. See also [7].

References

[1] M.W. Evans and H. Eckardt, paper 88: The Bianchi Identity of differential geometry

[2]: M.W. Evans, paper 99: The fundamental origin of curvature and torsion

[3] S.M. Carroll, *Lecture Notes on General Relativity*,
http://xxx.lanl.gov/PS_cache/gr-qc/pdf/9712/9712019v1.pdf

[4] S.M. Carroll, *Spacetime and Geometry*, Addison Wesley 2004, ISBN 0-8053-8732-3

[5] F.W. Hehl and Y.N. Obukhov, *Foundations of Classical Electrodynamics - Charge, Flux, and Metric*, Birkhäuser 2003, ISBN 0-8176-4222-6

[6] W.A. Rodrigues Jr. and E.C. de Oliveira, *The Many Faces of Maxwell, Dirac and Einstein Equations*,
Lecture Notes in Physics 722, Springer 2007, ISBN 978-3-540-71292-3

[7] G. W. Bruhn, A. Jadczyk and W.A. Rodrigues Jr., *Exterior Covariant Derivatives of (p+q)–Indexed Forms and the Correction of Some Errors Concerning the Bianchi Identities and their Coordinate Representations in the ECE Series of Papers and also in other Publications*. Note#

[8] W.A. Rodrigues Jr., Differential Forms on Riemannian (Lorentzian) and Riemann-Cartan Structures and Some Applications to Physics, *Ann. Fond. L. de Broglie* **32,4**, 425-478 (2007). http://arxiv.org/PS_cache/arxiv/pdf/0712/0712.3067v3.pdf

[9] I.M. Been and R.W. Tucker, An Introduction to Spinors and Geometry and Applications in Physics, Adam Hilger, Bristol, 1987.

[10] Y. Choquet-Bruhat, C. DeWitt-Morette with M. Dillard-Bleick, *Analysis, Manifolds and Physics* (revised edition), North-Holland Publ. Co., Amsterdam, 1982, ISBN0-444-86017-7

[11] M.W. Evans, paper 15: The spinning and curving of spacetime:the electromagnetic and gravitational fields in the Evans unified field theory

[12] S-S. Chern and C. Chevalley, *ÉLIE CARTAN AND HIS MATHEMATICAL WORK* Bull. Amer. Math. Soc. Volume 58, Number 2 (1952), 217-250.