Counter examples to M.W. Evans' Hodge dual of the first Bianchi identity

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(our comments in **blue**, quotations from Evans' texts in ***black italics***)

One of the basic equations of differential geometry is the first Bianchi identity

$$d \wedge T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge q^b . \qquad ([1],20)$$

In their web note [1] the authors M.W. Evans and H. Eckardt attempt to dualize that identity by claiming the Hodge dual of Eq.([1],20) to be

$$d \wedge T^{a\sim} + \omega^a{}_b \wedge T^{b\sim} = R^a{}_b{}^{\sim} \wedge q^b \qquad ([1],21)$$

as an up to now unknown further *basic equation of differential geometry.*

To prove that new equation ([1],21) is transformed to the tensorial equation

$$D_\mu T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu} \qquad ([1],1)$$

which those authors consider to be equivalent[1] to the dualized 1st Bianchi identity ([1],21). Therefore the tensorial equation ([1],1) should be valid for arbitrary sample manifolds of Riemannian differential geometry. We'll check this assertion ([1],1) by an elementary example below. The reader will find a much deeper treatment of the topic by W.A. Rodrigues Jr. in [4].

The authors of [1] have tested their claim themselves. They write in the Introduction of [1]:

. . . In this chapter, various classes of solutions of the Einstein field equation are tested numerically against equation (1), by directly evaluating the curvature tensor. It is found that the Einstein field equation fails the test of Eq. (1) because the Einstein field equation is based on a geometry that neglects torsion . . .

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[1] Supposed that Eq.([1],21) were a valid equation which in fact is not. The correct equation for $DT^{a\sim}$ may be found in [4]). Eq. ([1],21) does not imply Eq.([1],1) though claimed in [1].

This means that the authors do not grasp that their claim (if true) must be satisfied also for Christoffel connections, i.e. for torsion-free cases, and the failing of their numerical tests is just another counter example to their claim ([1],1). See Evans' and Eckardt's paper [2]: Sect.1.1.4 gives (symmetric) Christoffel connection, while due to Sect.1.1.12 they have $R^{o}{}_{\mu}{}^{\mu o} \neq 0$, again contradicting their own claim Eq. ([1],1) .

1. The unit-2-sphere S^2 in R^3

Using some basic information from S.M. Carroll [3] we have the metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 = dx^{1\,2} + \sin^2 x^1\, dx^{2\,2} \tag{1.1}$$

with the metric tensors

$$(g_{\mu\nu}) = \mathrm{diag}(\,1, \sin^2 x^1) , \qquad (g^{\mu\nu}) = \mathrm{diag}(\,1, {}^1/_{\sin^2 x^1}) . \tag{1.2}$$

using the numbering of indices

$$1 \sim \theta,\ 2 \sim \varphi \tag{1.3}$$

There are only a few non-vanishing Christoffel coefficients

$$\Gamma^1{}_{22} = -\sin x^1 \cos x^1 , \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot x^1 , \tag{1.4}$$

while all other Christoffels $\Gamma^\kappa{}_{\mu\nu}$ vanish.

The torsion T^κ is given by

$$T^\kappa{}_{\mu\nu} = \Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu} = 0 , \tag{1.5}$$

i.e. vanishing due to the symmetry of the Christoffels in their lower indices μ,ν.

2. The Riemann tensor of S^2

The Riemann tensor is given by

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa{}_{\lambda\nu} - \partial_\nu \Gamma^\kappa{}_{\lambda\mu} + \Gamma^\kappa{}_{\mu\rho}\Gamma^\rho{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\rho}\Gamma^\rho{}_{\mu\lambda} \tag{2.1}$$

being antisymmetric in μ,ν as is well-known. Therefore we have especially

$$R^\kappa{}_{\lambda\mu\nu} = 0 \quad \text{if} \quad \mu = \nu. \tag{2.2}$$

3. The check

Due to the vanishing of torsion (1.5) the equation (2) to be checked reduces to

$$0 = R^{\kappa}{}_{\mu}{}^{\mu\nu} = R^{\kappa}{}_{\mu\alpha\beta}\, g^{\mu\alpha}\, g^{\nu\beta} \,. \qquad (3.1)$$

Therefore, due to the diagonal form (1.2) of ($g^{\mu\rho}$), we have to check:

$$(R^{\kappa}{}_{11\beta}\, g^{11} + R^{\kappa}{}_{22\beta}\, g^{22})\, g^{\nu\beta}$$

$$= (R^{\kappa}{}_{111}\, g^{11} + R^{\kappa}{}_{221}\, g^{22})\, g^{\nu 1} + (R^{\kappa}{}_{112}\, g^{11} + R^{\kappa}{}_{222}\, g^{22})\, g^{\nu 2} \,. \qquad (3.2)$$

This is:

for ν=1: $R^{\kappa}{}_{\mu}{}^{\mu 1} = (R^{\kappa}{}_{111}\, g^{11} + R^{\kappa}{}_{221}\, g^{22})\, g^{11} + 0 = (R^{\kappa}{}_{111}\, g^{11} + R^{\kappa}{}_{221}\, g^{22})\, g^{11} = R^{\kappa}{}_{221}\, g^{22}\, g^{11}$,

for ν=2: $R^{\kappa}{}_{\mu}{}^{\mu 2} = 0 + (R^{\kappa}{}_{112}\, g^{11} + R^{\kappa}{}_{222}\, g^{22})\, g^{22} = (R^{\kappa}{}_{112}\, g^{11} + R^{\kappa}{}_{222}\, g^{22})\, g^{22} = R^{\kappa}{}_{112}\, g^{11}\, g^{22}$,

i.e. the test reduces to:

$$R^{\kappa}{}_{221} = 0 \,? \qquad \text{and} \qquad R^{\kappa}{}_{112} = 0 \,? \qquad (3.3)$$

We consider the special case $\kappa = 1$ to obtain:

$$R^{1}{}_{221} = 0 \,? \qquad \text{and} \qquad R^{1}{}_{112} = 0 \,? \qquad (3.4)$$

The check '$R^{1}{}_{221} = 0$?' means in detail

$$R^{1}{}_{221} = \underbrace{\partial_2 \Gamma^{1}{}_{21}}_{=0} - \partial_1 \Gamma^{1}{}_{22} + (\underbrace{\Gamma^{1}{}_{21}\Gamma^{1}{}_{12}}_{=0} + \Gamma^{1}{}_{22}\Gamma^{2}{}_{12}) - (\underbrace{\Gamma^{1}{}_{11}\Gamma^{1}{}_{22}}_{=0} + \underbrace{\Gamma^{1}{}_{12}\Gamma^{2}{}_{22}}_{=0}) = 0 \,? \qquad (3.5)$$

thus

$$\mathbf{R^{1}{}_{221} = -\,\partial_1 \Gamma^{1}{}_{22} + \Gamma^{1}{}_{22}\Gamma^{2}{}_{12} = -\,\sin^2 x^1 \neq 0} \,. \qquad \mathbf{(3.6)}$$

Therefore we have obtained a **negative check result**: The test equation ([1],1) is not fulfilled for the unit-2-sphere S^2 which means:

Evans' duality hypothesis Eq. ([1],1) is invalid in general.

References

[1] M.W. Evans, H. Eckardt, *Violation of the Dual Bianchi Identity by Solutions of the Einstein Field Equation*
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[3] S.M. Carroll, *Lecture Notes on General Relativity*, p.60 f.,
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[4] W.A. Rodrigues Jr., *Differential Forms on Riemannian (Lorentzian) and Riemann-Cartan Structures and Some Applications to Physics*
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