Spatial random permutations with cycle weights

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We investigate a model of spatial permutations which is motivated by its connection to the theory of Bose-Einstein condensation [5, 7, 6, 1]. We consider pairs $(\boldsymbol{x}, \pi)$ with $\boldsymbol{x} \in \Lambda^N$ (Λ is a cubic box in $\mathbb{R}^d$) and $\pi \in \mathcal{S}_N$ (the group of permutations of N elements). N is the number of "particles" of the system. The weight of $(\boldsymbol{x}, \pi)$ is given by the "Gibbs factor" $\mathrm{e}^{-H(\boldsymbol{x},\pi)}$ with Hamiltonian of the form

$$H(\boldsymbol{x}, \pi) = \sum_{i=1}^{N} \xi(x_i - x_{\pi(i)}) + \sum_{\ell \geq 1} \alpha_\ell r_\ell(\pi). \tag{1}$$

We always assume that ξ is a function $\mathbb{R}^d \to [0, \infty]$, with $\int \mathrm{e}^{-\xi} = 1$. The cycle parameters $\alpha_1, \alpha_2, \ldots$ are some fixed numbers, and $r_\ell(\pi)$ is the number of cycles of length ℓ in the permutation π. The most relevant choice for the function ξ is $\xi(x) = \gamma|x|^2$, $\gamma > 0$, which is related to the quantum Bose gas. We mainly consider the case where the weights α_ℓ decay at infinity faster than $1/\log \ell$, and where $\mathrm{e}^{-\xi}$ has positive Fourier transform. Intuitively, the Gibbs factor restricts the permutations so each jump is local, i.e. the distances $|x_i - x_{\pi(i)}|$ remain finite even for large systems.

Our main result on this model states that macroscopic cycles occur in the thermodynamic limit $N, |\Lambda| \to \infty$ when the density $\rho = N/|\Lambda|$ is larger than the *critical density* $\rho_c \leq \infty$. We also give an explicit formula for ρ_c, cf. (4). When $\alpha_\ell = 0$ for all ℓ, we obtain the model of spatial random permutations that corresponds to the ideal Bose gas; in this case ρ_c is the well-known critical density for Bose-Einstein condensation for non-interacting particles.

Setting and main result. The state space is $\Omega_{\Lambda,N} = \Lambda^N \times \mathcal{S}_N$, with the Borel σ-algebra on Λ^N, and the discrete σ-algebra on $\mathcal{S}_N$. Write $\boldsymbol{x} = (x_1, \ldots, x_N)$. Our Hamiltonian is given by a slight modification of (1): define ξ_Λ through $\mathrm{e}^{-\xi_\Lambda(x)} = \sum_{y \in \mathbb{Z}^d} \mathrm{e}^{-\xi(x - Ly)}$, and put

$$H_\Lambda(\boldsymbol{x}, \pi) = \sum_{i=1}^{N} \xi_\Lambda(x_i - x_{\pi(i)}) + \sum_{\ell \geq 1} \alpha_\ell r_\ell(\pi). \tag{2}$$

The important point is that $\mathrm{e}^{-\xi_\Lambda}$ has a Λ-independent Fourier transform $\mathrm{e}^{-\varepsilon(k)}$ in finite volume. With the additional assumption that $\mathrm{e}^{-\varepsilon(k)}$ is positive, this enables us to relate our model with a probability model on Fourier modes. Note that $\int_\Lambda \mathrm{e}^{-\xi_\Lambda} = 1$, and that $H_\Lambda(\boldsymbol{x}, \pi) = H(\boldsymbol{x}, \pi)$ for large enough Λ if $\mathrm{e}^{-\xi}$ has compact support.

We introduce a probability measure on $\Omega_{\Lambda,N}$ such that a random variable $\theta : \Omega_{\Lambda,N} \to \mathbb{R}$ has expectation

$$E_{\Lambda,N}(\theta) = \frac{1}{Y(\Lambda, N) N!} \int_{\Lambda^N} \mathrm{d}\boldsymbol{x} \sum_{\pi \in \mathcal{S}_N} \theta(\boldsymbol{x}, \pi) \, \mathrm{e}^{-H_\Lambda(\boldsymbol{x},\pi)}. \tag{3}$$

$Y(\Lambda, N) = \frac{1}{N!} \int_{\Lambda^N} \mathrm{d}\boldsymbol{x} \sum_{\pi \in \mathcal{S}_N} \mathrm{e}^{-H_\Lambda(\boldsymbol{x},\pi)}$ is the partition function. Put

$$\rho_{\mathrm{c}} = \sum_{n \geq 1} \mathrm{e}^{-\alpha_n} \int_{\mathbb{R}^d} \mathrm{e}^{-n\varepsilon(k)} \, \mathrm{d}k. \tag{4}$$

ρ_{c} is the critical density of the system. Precisely, define the finite volume free energy for density ρ through $q_\Lambda(\rho) = -\frac{1}{|\Lambda|} \log Y(\Lambda, |\Lambda|\rho)$. Then [2] there exists a convex function $q(\rho)$ such whenever $\rho_n \to \rho \geq 0$, we have $\lim_{n\to\infty} q_{\Lambda_n}(\rho_n) = q(\rho)$. q is an analytic function of ρ except at the critical density ρ_{c}. The non-analyticity of $q(\rho)$ at ρ_{c} is caused by the appearance of macroscopic cycles: let $\ell_i(\pi) = 1, 2, \dots$ denote the length of the cycle of π that contains the index i. Let $V = L^d$, and $\boldsymbol{\varrho}_{m,n}(\pi) = \frac{1}{|\Lambda|} \#\{i = 1, 2, \dots : m \leq \ell_i(\pi) \leq n\}$.

Theorem: *Assume* $\sum_{\ell \geq 1} \frac{|\alpha_\ell|}{\ell} < \infty$*,* $\mathrm{e}^{-\xi}$ *has Fourier transform* $\mathrm{e}^{-\varepsilon(k)} \geq 0$*, and* $\rho_{\mathrm{c}} < \infty$*. For any function* η *with* $\eta(V) \to \infty$ *and* $\eta(V)/V \to 0$ *as* $V \to \infty$*, and all* $s \geq 0$*, we have*

$$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{1,\eta(V)}) = \begin{cases} \rho & \text{if } \rho \leq \rho_{\mathrm{c}}; \\ \rho_{\mathrm{c}} & \text{if } \rho \geq \rho_{\mathrm{c}}; \end{cases} \qquad \textit{(microscopic cycles)}$$

$$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{\eta(V),V/\eta(V)}) = 0; \qquad \textit{(mesoscopic cycles)}$$

$$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{V/\eta(V),sV}) = \begin{cases} 0 & \text{if } \rho \leq \rho_{\mathrm{c}}; \\ s & \text{if } 0 \leq s \leq \rho - \rho_{\mathrm{c}}, \\ \rho - \rho_{\mathrm{c}} & \text{if } 0 \leq \rho - \rho_{\mathrm{c}} \leq s. \end{cases} \qquad \textit{(macroscopic cycles)}$$

When $\alpha_\ell = 0$ for all ℓ, we obtain the model of spatial random permutations that corresponds to the ideal Bose gas; in this case ρ_{c} is the well-known critical density for Bose-Einstein condensation for non-interacting particles. There the occurrence of macroscopic cycles has been understood in [7, 8]. The present setting with general functions ξ was considered in [1].

Main ideas of the proof. We express (3) as a model of random permutations on Fourier modes. Let $\Lambda^* = \frac{1}{L}\mathbb{Z}^d$. For $\boldsymbol{k} \in \Lambda^*$ We define

$$p_{\Lambda,N}(\boldsymbol{k}, \pi) = \frac{1}{\widehat{Y}(\Lambda,N)N!} \, \mathrm{e}^{-\widehat{H}(\boldsymbol{k},\pi)} \prod_{i=1}^{N} \delta_{k_i, k_{\pi(i)}}, \tag{5}$$

with $\widehat{H}(\boldsymbol{k}, \pi) = \sum_{i=1}^N \varepsilon(k_i) + \sum_{\ell \geq 1} \alpha_\ell r_\ell(\pi)$. This model offers an alternative representation to the model of spatial permutations, as far as the permutations are concerned: for any permutation π,

$$\int_{\Lambda^N} \mathrm{e}^{-H_\Lambda(\boldsymbol{x},\pi)} \, \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k} \in (\Lambda^*)^N} \mathrm{e}^{-\widehat{H}(\boldsymbol{k},\pi)} \prod_{i=1}^{N} \delta_{k_i, k_{\pi(i)}}.$$

In particular $\widehat{Y}(\Lambda, N) = Y(\Lambda, N)$, and $E_{\Lambda,N}(\Lambda^N \times \{\pi\}) = p_{\Lambda,N}(\Omega^*_{\Lambda,N}, \pi)$ for all π. In order to separate the spatial component of the model from the permutations, we introduce a model of non-spatial permutations. For $\pi \in \mathcal{S}_n$, we put

$$p_n(\pi) = \frac{1}{h_n n!} \exp\Big\{ -\sum_{\ell \geq 1} \alpha_\ell r_\ell(\pi) \Big\} \tag{6}$$

with normalization $h_n = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \mathrm{e}^{-\sum_\ell \alpha_\ell r_\ell(\pi)}$. $r_\ell(\pi)$ denotes the number of cycles of length ℓ in the permutation π. About this model, very detailed results can be obtained [2, 3]. In particular, put $N_{a,b} = \#\{i = 1, 2, \cdots : a \leq \ell_i(\pi) \leq b\}$. If $\sum_{\ell \geq 1} \frac{|\alpha_\ell|}{\ell} < \infty$, then

$$\lim_{n \to \infty} \tfrac{1}{n} E_n(N_{1,sn}) = s. \tag{7}$$

Next we introduce occupation numbers. Let $\mathcal{N}_\Lambda$ be the set of sequences $\boldsymbol{n} = (n_k)$ of integers indexed by $k \in \Lambda^*$, and $\mathcal{N}_{\Lambda,N} = \{\boldsymbol{n} \in \mathcal{N}_\Lambda : \sum_{k \in \Lambda^*} n_k = N\}$. To each $\boldsymbol{k} \in (\Lambda^*)^N$ corresponds an element $\boldsymbol{n} \in \mathcal{N}_{\Lambda,N}$, with n_k counting the number of indices i such that $k_i = k$. Thus we can view $\boldsymbol{n}$ as a subset of $(\Lambda^*)^N$. The probability (5) yields a probability on occupation numbers: summing over permutations and over compatible vectors $\boldsymbol{k}$, we have $p_{\Lambda,N}(\boldsymbol{n}) = \frac{1}{Y(\Lambda,N)} \prod_{k \in \Lambda^*} \mathrm{e}^{-n_k \varepsilon(k)} h_{n_k}$. Separation of the spatial and non-spatial aspects of the measure 3 is achieved by the identity

$$E_{\Lambda,N}(\boldsymbol{\varrho}_{a,b}) = \frac{1}{V} \sum_{\boldsymbol{n} \in \mathcal{N}_{\Lambda,N}} p_{\Lambda,N}(\boldsymbol{n}) \sum_{k \in \Lambda^*} E_{n_k}(N_{ab}). \tag{8}$$

In the light of (7) and (8), we can now focus on the quantity $p_{\Lambda,N}(\boldsymbol{n})$. By (7), macroscopic cycles appear if and only if at least one mode is macroscopically occupied, i.e. iff $p_{\Lambda,N}(n_k \geq sN) > 0$ uniformly in $N \in \mathbb{N}$ and Λ such that $N = \rho\Lambda$. It turns out that macroscopic occupation can occur only for $k = 0$ and that it occurs if and only if $\rho \geq \rho_\mathrm{c}$. The main step in proving this is a result that gives detailed information about the limiting distribution of the random variable n_0/V: putting $\rho_0 = \max(0, \rho - \rho_\mathrm{c})$, we have, for all $\lambda \geq 0$,

$$\lim_{V \to \infty} E_{\Lambda,\rho V}(\,\mathrm{e}^{\lambda n_0/V}\,) = \mathrm{e}^{\lambda \rho_0}\,.$$

The proof [2] is based on ideas of Buffet and Pulé [4] for the ideal Bose gas.

References

[1] V. Betz, D. Ueltschi, *Spatial random permutations and infinite cycles*, Commun. Math. Phys. 285, 469–501 (2009)

[2] V. Betz, D. Ueltschi, *Spatial random permutations with small cycle weights*, preprint (2008).

[3] V. Betz, D. Ueltschi, Y. Velenik, *Random permutations with cycle weights*, in preparation

[4] E. Buffet, J. V. Pulé, *Fluctuation properties of the imperfect Bose gas*, J. Math. Phys. 24, 1608–1616 (1983)

[5] R. P. Feynman, *Atomic theory of the λ transition in Helium*, Phys. Rev. 91, 1291–1301 (1953)

[6] D. Gandolfo, J. Ruiz, D. Ueltschi, *On a model of random cycles*, J. Stat. Phys. 129, 663–676 (2007)

[7] A. Sütő, *Percolation transition in the Bose gas*, J. Phys. A 26, 4689–4710 (1993)

[8] A. Sütő, *Percolation transition in the Bose gas II*, J. Phys. A 35, 6995–7002 (2002)