Uniform asymptotic regularity for Mann iterates

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Abstract

In [16] we obtained an effective quantitative analysis of a theorem due to Borwein, Reich and Shafrir on the asymptotic behavior of general Krasnoselski-Mann iterations for nonexpansive self-mappings of convex sets C in arbitrary normed spaces. We used this result to obtain a new strong uniform version of Ishikawa's theorem for bounded C. In this paper we give a qualitative improvement of our result in the unbounded case and prove the uniformity result for the bounded case under the weaker assumption that C contains a point x whose Krasnoselski-Mann iteration (x_k) is bounded.
We also consider more general iterations for which asymptotic regularity is known only for uniformly convex spaces (Groetsch). We give uniform effective bounds for (an extension of) Groetsch's theorem which generalize previous results by Kirk/Martinez-Yanez and the author.

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1 Krasnoselski-Mann iterations

This paper is concerned with quantitative estimates on the rate of asymptotic regularity for so-called Krasnoselski-Mann iterations of nonexpansive mappings.

Definition 1.1 *Let* $(X, \|\cdot\|)$ *be a normed linear space and* $C \subseteq X$ *be a subset of* X*. A mapping* $f : C \to C$ *is called nonexpansive if*

$$\forall x, y \in C,\ \ \|f(x) - f(y)\| \leq \|x - y\|.$$

In the following, $(X, \|\cdot\|)$ will be an arbitrary normed linear space, $C \subseteq X$ a non-empty convex subset of X and $f : C \to C$ a nonexpansive mapping.
We consider the so-called Krasnoselski-Mann iteration starting from $x \in C$

$$x_0 := x, \quad x_{k+1} := (1 - \lambda_k)x_k + \lambda_k f(x_k), \tag{1}$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of real numbers in $[0, 1]$ (for more information on the relevance of this kind of generalized Krasnoselski ([20]) iterations (see e.g [21],[24],[1],[6]).

Under quite general circumstances the sequence $(\|x_k - f(x_k)\|)$ is known to converge towards $r_C(f) := \inf\limits_{x\in C} \|x - f(x)\|$. In many cases $r_C(f) = 0$ so that from sufficiently large k on x_k is an arbitrarily good approximate fixed point. If this is the case for all starting points x of the iteration, f is called 'asymptotically regular'. We will consider effective uniform bounds on the rate of convergence towards $r_C(f)$ both in the general case as well as in the case where $r_C(f) = 0$.

One simple fact we will use is the following:

Lemma 1.2 *If* C *is bounded, then* $r_C(f) = 0$.

Proof: We use the following well-known construction (see e.g. [9](prop.1.4)): $f_t(x) := (1 - t)f(x) + tc$ for some $c \in C$ and $t \in (0, 1]$. $f_t : C \to C$ is a contraction and therefore Banach's fixed point theorem applies. Since we only need approximate fixed points it is not necessary to assume that X is complete or that C is closed. For full details see [17]. $\Box$

For the rest of this section we assume (following [1]) that $(\lambda_k)_{k\in\mathbb{N}}$ is divergent in sum, which can – as we will use later – be expressed (since $\lambda_k \geq 0$) as

$$\text{for every } n, i \in \mathbb{N} \text{ there exists } k \in \mathbb{N} \text{ with } \sum_{j=i}^{i+k} \lambda_j \geq n, \tag{2}$$

and that

$$\limsup_{k\to\infty} \lambda_k < 1. \tag{3}$$

Theorem 1.3 ([1]) *Suppose that* $(\lambda_k)_{k\in\mathbb{N}}$ *satisfies the conditions* (2) *and* (3) *and that* (x_k) *is defined as in* (1)*. Then*

$$\|x_k - f(x_k)\| \stackrel{k\to\infty}{\to} r_C(f).$$

Together with the previous lemma theorem 2.1 implies the following important result due to Ishikawa [12] (for constant $\lambda_k := \lambda$ it was independently obtained also in [5]):

Corollary 1.4 ([12],[8],[1]) *Under the assumptions of theorem 1.3 plus the additional assumption that* C *is bounded the following holds:*

$$\forall x \in C,\ \|x_k - f(x_k)\| \stackrel{k\to\infty}{\to} 0.$$

Using an inequality due to [15] the following lemma was proved in [1] (see also [7]):

Lemma 1.5 *Let* (x_k) *be given by* (1)*, then, for all* $k \geq 1$*,*

$$\|x - x_k\| \geq \sum_{i=0}^{k-1} \lambda_i r_C(f).$$

Remark 1.6 In [1] it is assumed that X is complete and C is closed in order to have Banach's fixed point theorem available. However, the proof can be rewritten with approximate fixed points instead whose existence follows without these assumptions. Alternatively, one can infer the lemma by applying the one proved in [1] to the completion of X.

In the following (x^*_k) always refers to the Krasnoselski-Mann iteration starting from $x^* \in C$.

Corollary 1.7 ([1]) [1] *If* C *contains a point* $x^* \in C$ *such that* (x^*_k) *is bounded, then* $r_C(f) = 0$.

As observed in [1], theorem 1.3 combined with the previous lemma allows to derive the conclusion of corollary 1.4 under the weaker assumption that C contains an element whose Krasnoselski-Mann iteration is bounded:

Theorem 1.8 ([1]) *Under the assumptions of theorem 1.3 we have: if* C *contains an* x^* *such that* (x^*_k) *is bounded, then*

$$\forall x \in C,\ \|x_k - f(x_k)\| \stackrel{k\to\infty}{\to} 0.$$

[1]The corollary follows also from [12], see remark 1.9 below.

Remark 1.9 The case where $x = x^*$ in theorem 1.8 is already proved in [12].

In the next section we use a result from [16] to prove a uniform bound on the convergence in theorem 1.8 thereby generalizing a corresponding result for corollary 1.4 from [16]. We also give a qualitative improvement of the quantitative version of theorem 2.1 obtained in [16]. In the final section we prove a new bound on Groetsch's theorem on the asymptotic regularity in the case of uniformly convex spaces where the conditions (2),(3) on the sequence (λ_k) are replaced by the weaker condition

$$\sum_{k=0}^{\infty} \lambda_k(1-\lambda_k) = +\infty. \tag{4}$$

2 Uniform bounds on asymptotic regularity

In [16], we obtained the following quantitative version of theorem 1.3:

Theorem 2.1 ([16]) *Let $(X, \|\cdot\|)$ be a normed linear space, $C \subseteq X$ a non-empty convex subset and $f : C \to C$ a nonexpansive mapping. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $[0,1)$ which satisfies (2),(3) and $K \in \mathbb{N}$ such that*

$$\lambda_k \le 1 - \frac{1}{K} \quad \textit{for all } k \in \mathbb{N}.$$

Let $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be such that for all $i, n \in \mathbb{N}$

$$\alpha(i,n) \le \alpha(i+1,n) \quad \textit{and} \quad n \le \sum_{k=i}^{i+\alpha(i,n)-1} \lambda_k.$$

$(x_k)_{k\in\mathbb{N}}$ is defined as in (1). Then the following holds

$$\forall x, x^* \in C, \forall \varepsilon > 0, \forall k \ge h(\varepsilon, x, x^*, f, K, \alpha),\ \|x_k - f(x_k)\| < \|x^* - f(x^*)\| + \varepsilon,$$

where[2]

$$\begin{aligned}
&h(\varepsilon, x, x^*, f, K, \alpha) := \widehat{\alpha}(\lceil 2\|x - f(x)\| \cdot \exp(K(M+1))\rceil - 1, M),\\
&\textit{with } M := \left\lceil \tfrac{1+2\|x-x^*\|}{\varepsilon} \right\rceil \textit{ and}\\
&\widehat{\alpha}(0, M) := \tilde{\alpha}(0, M),\ \widehat{\alpha}(m+1, M) := \tilde{\alpha}(\widehat{\alpha}(m, M), M) \textit{ with}\\
&\tilde{\alpha}(m, M) := m + \alpha(m, M) \quad (m \in \mathbb{N})
\end{aligned}$$

[2] Here we stipulate that $\lceil 0 \rceil := 1$ so that the first argument of α always is a non-negative integer.

(Instead of M we may use any upper bound $\mathbb{N} \ni \tilde{M} \geq \frac{1+2\|x-x^*\|}{\varepsilon}$. Likewise, $\|x - f(x)\|$ may be replaced by any upper bound.)

Remark 2.2 *Note that a mapping α satisfying the conditions of theorem 2.1 can easily be computed from a mapping $\beta : \mathbb{N} \to \mathbb{N}$ satisfying the weaker requirement*

$$n \leq \sum_{k=0}^{\beta(n)} \lambda_k, \quad \textit{for all } n. \tag{5}$$

Just define $\beta'(i,n) := \beta(n+i)-i+1$ and $\beta^+(i,n) := \max_{j \leq i}(\beta'(j,n))$. Then β^+ satisfies the conditions imposed on α so that theorem 2.1 holds with $h(\varepsilon, x, x^*, f, K, \beta^+)$, where β satisfies (5)*.*

Corollary 2.3 ([16])
Under the same assumptions as in theorem 2.1 plus the assumption that C has a positive[3] finite diameter $d(C)$ the following holds:

$$\forall x \in C,\ \forall \varepsilon > 0,\ \forall n \geq h(\varepsilon, d(C), K, \alpha), \quad \|x_n - f(x_n)\| \leq \varepsilon,$$

where

$$h(\varepsilon, d(C), K, \alpha) := \widehat{\alpha}(\lceil 2d(C) \cdot \exp(K(M+1)) \rceil - 1, M), \ \textit{with } M := \left\lceil \frac{1+2d(C)}{\varepsilon} \right\rceil$$

and $\widehat{\alpha}$ as in the previous theorem.
The bound $h(\varepsilon, d(C), K, \alpha)$ can be replaced also by $h(\frac{\varepsilon}{d(C)}, 1, K, \alpha)$.
Instead of $d(C)$ we can use any upper bound $d \geq d(C)$.

Remark 2.4 Perhaps the most interesting aspect of corollary 2.3 is that the bound $h(\varepsilon, d(C), K, \alpha)$ is independent from x, f and depends only weakly on C, λ_k via d resp. α, K. This generalizes uniformity results of [5] and [8] which established independence from x and f. Only for constant λ, independence from C (except via d) had been established before in [2] where for this special case an optimal quadratic bound was obtained. Our result implies a uniform exponential bound (only depending on ε, d, K) for the much more general case of sequences $(\lambda_k) \subset [\frac{1}{K}, 1 - \frac{1}{K}]$, where $2 \leq K \in \mathbb{N}$. Already for this case (which still is more restrictive than the general result obtained in corollary 2.3) no effective bound at all was known before (for more

[3]For $d(C) = 0$ things are trivial.

information on this see [16]). In contrast to [5] and [8], our proof of corollary 2.3 doesn't use any functional analytic embeddings but a logical transformation of the non-constructive proof of (non-uniform) convergence from [1]: in a series of papers (see [17, 19] for an extensive list of references) we have proved general logical meta-theorems which provide algorithms to extract effective bounds from certain types of proofs even if the latter appear to be hopelessly non-constructive. Moreover, the complexity of these bounds can be a-priori estimated in terms of logical properties of the given proof. The extraction is based on a logical transformation of the original proof which results in a new proof which again can be written in ordinary mathematical terms without reference to the logical methods which were used to find it. The proof of theorem 1.3 given in [1] as well as the proofs of corollary 1.4 in [12] and [8] are all of the type covered by these meta-theorems. They are non-constructive as convergence is shown by reference to the fact that bounded monotone sequences in $\mathbb{R}$ converge which is well-known to fail in computable analysis. Our proof of theorem 2.1 was found applying this method to the proof in [1]. That approach has the further benefit that only the specific mathematical ingredients of the original proof are used in the transformed one. That is why the uniformity results in [16] and the present paper do not require any further functional theoretic arguments. Moreover, the effective results easily generalize to other settings to which the proof idea of the original proof applies (see [18]). For general information on this logic-based approach and a detailed discussion of it applicability in fixed point theory as well as other areas, see [19] and the references given there.

We now prove the following strengthened version of corollary 2.3:

Theorem 2.5 *Under the assumptions of theorem 2.1 the following holds. Let $d > 0, x, x^* \in C$ be such that $\|x^*_k\| \le d$ for all $k \in \mathbb{N}$ and $\|x - x^*\| \le d$. Then for all $\varepsilon > 0$ and all $k \in \mathbb{N}$*

$$k \ge h(\varepsilon, d, K, \alpha) \to \|x_k - f(x_k)\| \le \varepsilon,$$

where

$$h(\varepsilon, d, K, \alpha) := \widehat{\alpha}(\lceil 12d \cdot \exp(K(M+1)) \rceil - 1, M),$$
$$\text{with } M := \left\lceil \frac{1+6d}{\varepsilon} \right\rceil \text{ and } \widehat{\alpha} \text{ as in theorem 2.1.}$$

The bound $h(\varepsilon, d, K, \alpha)$ can be replaced also by $h(\frac{\varepsilon}{d}, 1, K, \alpha)$.

Proof: Let $x^*, x \in C$ and $d > 0$ be as in the theorem. Then

$$\|x^* - x^*_k\| \le 2d, \text{ for all } k \tag{6}$$

and therefore

$$\|x - x_k^*\| \le 3d, \text{ for all } k. \tag{7}$$

Using the nonexpansivity of f we get

$$\|f(x^*) - f(x_k^*)\| \le 2d, \text{ for all } k, \text{ and } \|f(x^*) - f(x)\| \le d. \tag{8}$$

By theorem 1.8 we obtain that for any $\delta > 0$ there exists a k s.t.

$$\|x_k^* - f(x_k^*)\| \le \delta. \tag{9}$$

Thus

$$\begin{cases} \|x - f(x)\| \le \|x - x^*\| + \|x^* - x_k^*\| + \|x_k^* - f(x_k^*)\| \\ \qquad +\|f(x_k^*) - f(x^*)\| + \|f(x^*) - f(x)\| \le 6d + \delta. \end{cases} \tag{10}$$

So be letting δ tend to 0 we conclude

$$\|x - f(x)\| \le 6d. \tag{11}$$

By (9), let k_δ again be such that $\|x_{k_\delta}^* - f(x_{k_\delta}^*)\| \le \delta$.
Let $h(\varepsilon, d, K, \alpha)$ be defined as in the theorem.
Now we apply theorem 2.1 to x and $x_{k_\delta}^*$ and use that because of (7) and (11) we can take $3d$ resp. $6d$ as upper bound for $\|x - x_{k_\delta}^*\|$ resp. for $\|x - f(x)\|$. This yields

$$\forall k \ge h(\varepsilon, d, K, \alpha), \quad \|x_k - f(x_k)\| \le \|x_{k_\delta}^* - f(x_{k_\delta}^*)\| + \varepsilon \le \delta + \varepsilon \tag{12}$$

By letting δ tend to 0, (12) implies the theorem. □

Remark 2.6 *Using a simple renorming argument the dependency of the bound from ε and d can be improved to the dependency on ε/d only: define $\|x\|^* := \|x\|/d$. Then the assumptions of the theorem are satisfied for $(X, \|\cdot\|^*)$ with $d = 1$. So by the result we just proved we get that*

$$k \ge h(\varepsilon, 1, K, \alpha) \to \|x_k - f(x_k)\|^* \le \varepsilon$$

and hence

$$k \ge h(\varepsilon/d, 1, K, \alpha) \to \|x_k - f(x_k)\| \le \varepsilon.$$

As briefly discussed in remark 2.4, our results from [16] were obtained by applying general results from logic about the extractability of effective data from ineffective proofs to the proof of theorem 2.1 as given in [1]. To understand the reason for the dependence of the bound in theorem 2.1 (compared to the one in corollary 2.3) from the additional input x^*, let us consider the formal logical structure of the statement of theorem 2.1: when formalized appropriately it translates into

$$\forall \varepsilon > 0 \exists n \in \mathbb{N} \forall m \geq n \forall x^* \in C(\|x_m - f(x_m)\| < \|x^* - f(x^*)\| + \varepsilon) \tag{13}$$

where – since $(\|x_k - f(x_k)\|)_k$ is non-increasing (see lemma 3.1 below) – the quantifier '$\forall m \geq n$' is superfluous, i.e. (13) is equivalent to

$$\forall \varepsilon > 0 \exists n \in \mathbb{N} \forall x^* \in C(\|x_n - f(x_n)\| < \|x^* - f(x^*)\| + \varepsilon). \tag{14}$$

An effective bound on n in (14) would (relatively to the computability of $(\lambda_k), f, x, \|\cdot\|$) imply the computability of r_C which is unlikely to hold for general C. In order to make the aforementioned logical meta-theorem applicable one has to reverse the quantifier alternation $\exists n \forall x^*$ into a $\forall\exists$-alternation. The easiest way to do this is just by replacing it by '$\forall x^* \exists n$'. This is what we did in [16] thereby making x^* an input for the bound on n:

$$\forall \varepsilon > 0 \forall x^* \in C \exists n \in \mathbb{N}(\|x_n - f(x_n)\| < \|x^* - f(x^*)\| + \varepsilon). \tag{15}$$

Although (15) actually is equivalent to (14) (and hence to (13)), and so still a faithful formalization of theorem 2.1, there is no effective way to get from a bound on n in (15) one on n in (14).
A more subtle variant is to replace (14) by

$$\forall \varepsilon > 0 \forall (y_k)_{k \in \mathbb{N}} \subset C \exists k \in \mathbb{N}(\|x_k - f(x_k)\| < \|y_k - f(y_k)\| + \varepsilon), \tag{16}$$

where $(y_k)_{k \in \mathbb{N}}$ is an **arbitrary** sequence in C whereas (x_k) still denotes the Krasnoselski-Mann iteration starting from $x \in C$. Obviously, any bound for (16) yields also one for (15) just by applying it to the constant sequence defined by $y_k := x^*$.
The next theorem shows that (as guaranteed by our general logical results,[19]) an effective bound for n in (16) can indeed be obtained. It provides an upper bound for an k at which the sequence (x_k) 'catches up' (with an error of at most ε) with the arbitrarily given sequence $(y_k)_{k \in \mathbb{N}}$ w.r.t. its approximate fixed point behaviour:

Theorem 2.7 *Under the same assumptions as in theorem 2.1 the following holds: For any $x \in C$, $(y_k)_{k \in \mathbb{N}} \subset C$, $\varepsilon > 0$ there exists a $k \leq j(\varepsilon, x, (y_k)_{k \in \mathbb{N}}, f, K, \alpha)$ s.t.*

$$\|x_k - f(x_k)\| < \|y_k - f(y_k)\| + \varepsilon,$$

where (omitting the arguments f, K, α for better readability)

$$j(\varepsilon, x, (y_n)_{n\in\mathbb{N}}) := \max_{i \le k(\varepsilon, x, (y_n)_{n\in\mathbb{N}})} h(\varepsilon/2, x, y_i)$$

with

$$k(\varepsilon, x, (y_n)_{n\in\mathbb{N}}) := \max_{j<N} g^j(0),\ \ g(n) := h(\varepsilon/2, x, y_n), N := \left\lceil \frac{2\|y_0 - f(y_0)\|}{\varepsilon} \right\rceil.$$

Here h is the bound from theorem 2.1 and $g^n(0)$ is defined recursively:
$g^0(0) := 0,\ g^{n+1}(0) := g(g^n(0))$.
Instead of N, we can take any integer upper bound for $2\|y_0 - f(y_0)\|/\varepsilon$.

Proof: By theorem 2.1 we have

$$\forall k \in \mathbb{N},\ \ \|x_{g(k)} - f(x_{g(k)})\| < \|y_k - f(y_k)\| + \frac{\varepsilon}{2}, \tag{17}$$

where

$$g(k) := h(\frac{\varepsilon}{2}, x, y_k, f, K, \alpha).$$

We now construct (uniformly in $\varepsilon, x, (y_n)_{n\in\mathbb{N}}, f, K, \alpha$) a $k \in \mathbb{N}$ such that

$$\|y_i - f(y_i)\| \le \|y_{g(i)} - f(y_{g(i)})\| + \frac{\varepsilon}{2},\ \text{for some } i \le k. \tag{18}$$

(17) and (18) imply

$$\|x_{g(i)} - f(x_{g(i)})\| < \|y_{g(i)} - f(y_{g(i)})\| + \varepsilon,\ \text{for some } i \le k \tag{19}$$

so that the theorem is satisfied with

$$j(\varepsilon, x, (y_n)_{n\in\mathbb{N}}, f, K, \alpha) := \max_{i\le k} g(i).$$

Define

$$k := \max_{j<N} g^j(0),\ \text{where } \mathbb{N} \ni N \ge \left\lceil \frac{2\|y_0 - f(y_0)\|}{\varepsilon} \right\rceil. \tag{20}$$

Claim:

$$\|y_{(g^j(0))} - f(y_{(g^j(0))})\| \le \|y_{(g^{j+1}(0))} - f(y_{(g^{j+1}(0))})\| + \frac{\varepsilon}{2}),\ \text{for some } j < N.$$

Proof of claim: Suppose not, then for all $j < N$

$$\|y_{(g^{j+1}(0))} - f(y_{(g^{j+1}(0))})\| < \|y_{(g^j(0))} - f(y_{(g^j(0))})\| - \frac{\varepsilon}{2}$$

and therefore

$$\|y_{(g^N(0))} - f(y_{(g^N(0))})\| < \|y_0 - f(y_0)\| - N \cdot \frac{\varepsilon}{2} \leq 0$$

which is a contradiction.
By the claim, (18) is satisfied with k as defined in (20). $\Box$

Remark 2.8 *Again, the most interesting aspect of the rather complicated bound in theorem 2.7 is its limited dependence on the various parameters: j is independent of C and depends on $x, (y_k)_{k\in\mathbb{N}}, f$ only via upper bounds $d \geq \|x - f(x)\|$ and $M(k) \geq \|x - y_k\|$ (for all k). This follows from the fact that because of*

$$\begin{aligned} &\|y_0 - f(y_0)\| \leq \|y_0 - x\| + \|x - f(x)\| + \|f(x) - f(y_0)\| \\ &\leq 2\|y_0 - x\| + \|x - f(x)\| \leq 2M(0) + d \end{aligned}$$

one gets a bound on $\|y_0 - f(y_0)\|$ in terms of $M(0)$ and d as well. Moreover, the bound depends on (λ_k) only via the rather general inputs α, K.

3 The uniformly convex case

The assumptions (2),(3) on the sequence (λ_k) in $[0,1]$ made in Ishikawa's paper are still the most general ones for which asymptotic regularity has been proved for arbitrary normed spaces. In [2] it is conjectured that Ishikawa's theorem holds true if (2),(3) are replaced by the weaker condition (4) which is symmetric in $\lambda_k, 1-\lambda_k$. For the case of uniformly convex normed spaces, this has been proved by Groetsch [10] (see also [23]).[4] In this section we give a uniform quantitative bound on (a generalization of) Groetsch's theorem.
The following easy lemma holds in arbitrary normed linear spaces $(X, \|\cdot\|)$:

Lemma 3.1 *Let $C \subset X$ be convex, $(\lambda_k) \subset [0,1]$ and $f : C \to C$ nonexpansive. Then $\|x_{k+1} - f(x_{k+1})\| \leq \|x_k - f(x_k)\|$ for all k.*

[4]For recent applications of Groetsch's theorem to elliptic Cauchy problems see [6].

Definition 3.2 ([4]) *A normed linear space* $(X, \|\cdot\|)$ *is uniformly convex if for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\|x\|, \|y\| \le 1\ ,\ \ \|x-y\| \ge \varepsilon \ \textit{implies}\ \|\frac{1}{2}(x+y)\| \le 1-\delta,\ \textit{for all}\ x, y \in X.$$

A mapping $\eta : (0,2] \to (0,1]$ *providing such a* $\delta := \eta(\varepsilon) > 0$ *for given* $\varepsilon > 0$ *is called a modulus of uniform convexity.*

Lemma 3.3 ([10]) *Let* $(X, \|\cdot\|)$ *be uniformly convex with modulus* η.
If $\|x\|, \|y\| \le 1$ *and* $\|x-y\| \ge \varepsilon > 0$, *then*

$$\|\lambda x + (1-\lambda)y\| \le 1 - 2\lambda(1-\lambda)\eta(\varepsilon), \quad 0 \le \lambda \le 1.$$

Groetsch's theorem ([10]) states that in uniformly convex spaces

$$\|x_k - f(x_k)\| \stackrel{k\to\infty}{\to} 0$$

holds if (λ_k) satisfies the condition (4) and f has a fixed point in C. We now give a quantitative version of a strengthening of Groetsch's theorem which only assumes the existence of approximate fixed points in some neighborhood of x:

Theorem 3.4
Let $(X, \|\cdot\|)$ *be a uniformly convex normed linear space with modulus of uniform convexity* η, $d > 0$, $C \subseteq X$ *a (non-empty) convex subset,* $f : C \to C$ *nonexpansive and* $(\lambda_k) \subset [0,1]$ *and* $\gamma : \mathbb{N} \to \mathbb{N}$ *such that*

$$\forall n \in \mathbb{N}, \quad \sum_{k=0}^{\gamma(n)} \lambda_k(1-\lambda_k) \ge n.$$

Then for all $x \in C$ *which satisfy that for all* $\varepsilon > 0$ *there is a* $y \in C$ *with*

$$\|x-y\| \le d \ \textit{and}\ \|y - f(y)\| < \varepsilon,$$

one has

$$\forall \varepsilon > 0,\ \forall k \ge h(\varepsilon, d, \gamma), \quad \|x_k - f(x_k)\| \le \varepsilon,$$

where $h(\varepsilon, d, \gamma) := \gamma\left(\left\lceil \frac{3(d+1)}{2\varepsilon\cdot\eta(\frac{\varepsilon}{d+1})}\right\rceil\right)$ *for* $\varepsilon < 2d$ *and* $h(\varepsilon, d) := 0$ *otherwise.*
Moreover, if $\eta(\varepsilon)$ *can be written as* $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ *with*

$$\varepsilon_1 \ge \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \ge \tilde{\eta}(\varepsilon_2),\ \textit{for all}\ \varepsilon_1, \varepsilon_2 \in (0,2], \tag{21}$$

then the bound $h(\varepsilon, d, \gamma)$ *can be replaced (for* $\varepsilon < 2d$*) by*

$$\tilde{h}(\varepsilon, d, \gamma) := \gamma\left(\left\lceil \frac{d+1}{2\varepsilon \cdot \tilde{\eta}(\frac{\varepsilon}{d+1})}\right\rceil\right).$$

Proof: The case $\varepsilon \geq 2d$ is trivial as the assumption on x implies that $\|x - f(x)\| \leq 2d$. So we may assume that $\varepsilon < 2d$.
Let $\delta > 0$ be such that $\delta < \min(1/(2h(\varepsilon, d, \gamma) + 2), \varepsilon/3)$ and let $y \in C$ be point satisfying

$$\|y - f(y)\| < \delta \text{ and } \|x - y\| \leq d. \tag{22}$$

Define

$$n_\varepsilon := \gamma\left(\left\lceil \frac{3(d+1)}{2\varepsilon \cdot \eta(\varepsilon/(d+1))} \right\rceil\right).$$

Since for all k (using that f is nonexpansive)

$$\begin{aligned}
&\|x_{k+1} - y\| = \\
&\|(1-\lambda_k)x_k + \lambda_k f(x_k)) - y\| = \|(1-\lambda_k)(x_k - y) + \lambda_k(f(x_k) - y)\| \leq \\
&\|(1-\lambda_k)(x_k - y)\| + \|\lambda_k(f(x_k) - f(y))\| + \lambda_k\|f(y) - y\| \leq \|x_k - y\| + \delta
\end{aligned}$$

we have for all $k \leq n_\varepsilon$

$$\|x_k - y\| \leq \|x - y\| + k\delta \leq d + \frac{1}{2}. \tag{23}$$

Assume that $k \leq n_\varepsilon$ and

$$\|x_k - y\| \geq \frac{\varepsilon}{3} \text{ and} \tag{24}$$

$$\|x_k - f(x_k)\| = \|(x_k - y) - (f(x_k) - y)\| > \varepsilon. \tag{25}$$

Then

$$\left\| \frac{x_k - y}{\|x_k - y\| + \delta} - \frac{f(x_k) - y}{\|x_k - y\| + \delta} \right\| > \frac{\varepsilon}{\|x_k - y\| + \delta} \overset{(23)}{\geq} \frac{\varepsilon}{d+1}. \tag{26}$$

Because of

$$\|f(x_k) - y\| \overset{(22)}{\leq} \|f(x_k) - f(y)\| + \delta \leq \|x_k - y\| + \delta, \tag{27}$$

we have

$$\left\| \frac{x_k - y}{\|x_k - y\| + \delta} \right\|, \left\| \frac{f(x_k) - y}{\|x_k - y\| + \delta} \right\| \leq 1 \tag{28}$$

and therefore by lemma 3.3

$$\left\{\begin{array}{l} \left\|(1-\lambda_k)\left(\frac{x_k-y}{\|x_k-y\|+\delta}\right)+\lambda_k\left(\frac{f(x_k)-y}{\|x_k-y\|+\delta}\right)\right\| \\ \leq 1-2\lambda_k(1-\lambda_k)\eta(\varepsilon/(d+1)). \end{array}\right. \tag{29}$$

Hence

$$\left\{\begin{array}{l} \|x_{k+1}-y\| = \\ \|(1-\lambda_k)x_k+\lambda_k f(x_k))-y\| = \|(1-\lambda_k)(x_k-y)+\lambda_k(f(x_k)-y)\| \leq \\ \|x_k-y\|+\delta-(\|x_k-y\|+\delta)2\lambda_k(1-\lambda_k)\cdot\eta(\varepsilon/(d+1)) \stackrel{(24)}{\leq} \\ \|x_k-y\|+\delta-\frac{2\varepsilon}{3}\lambda_k(1-\lambda_k)\cdot\eta(\varepsilon/(d+1)). \end{array}\right. \tag{30}$$

If (24),(25) both hold for all $k \leq n_\varepsilon$, then (30) yields

$$\left\{\begin{array}{l} \|x_{n_\varepsilon+1}-y\| \leq \|x_0-y\|-\frac{2\varepsilon}{3}\cdot\eta(\varepsilon/(d+1))\cdot\sum\limits_{k=0}^{n_\varepsilon}\lambda_k(1-\lambda_k)+(n_\varepsilon+1)\cdot\delta \\ \leq \|x_0-y\|-(d+1)+\frac{1}{2} < \|x_0-y\|-d \stackrel{(22)}{\leq} 0, \end{array}\right. \tag{31}$$

which is a contradiction.
Hence there exists a $k \leq n_\varepsilon$ such that

$$\|x_k-y\| \leq \frac{\varepsilon}{3} \text{ or } \|x_k-f(x_k)\| \leq \varepsilon. \tag{32}$$

By the choice of δ, (22) and the nonexpansivity of f, the first disjunct implies $\|f(x_k)-x_k\| \leq \varepsilon$ too and so by lemma 3.1

$$\forall k \geq n_\varepsilon, \quad \|x_k-f(x_k)\| \leq \varepsilon. \tag{33}$$

The last claim in the theorem follows by choosing $y \in C$ as a δ-fixed point of f with $\delta < \min(1/(2\tilde{h}(\varepsilon,d,\gamma)+2), \varepsilon/3)$, replacing n_ε by $\tilde{n}_\varepsilon := \tilde{h}(\varepsilon,d,\gamma)$ and the following modifications of (29),(30) to

$$\left\{\begin{array}{l} \left\|(1-\lambda_k)\left(\frac{x_k-y}{\|x_k-y\|+\delta}\right)+\lambda_k\left(\frac{f(x_k)-y}{\|x_k-y\|+\delta}\right)\right\| \leq \\ 1-2\lambda_k(1-\lambda_k)\eta(\varepsilon/(\|x_k-y\|+\delta)). \end{array}\right. \tag{34}$$

$$
\begin{cases}
\|x_{k+1} - y\| = \\
\|(1-\lambda_k)x_k + \lambda_k f(x_k)) - y\| = \|(1-\lambda_k)(x_k - y) + \lambda_k(f(x_k) - y)\| \leq \\
\|x_k - y\| + \delta - (\|x_k - y\| + \delta)2\lambda_k(1-\lambda_k) \cdot \eta(\varepsilon/(\|x_k - y\| + \delta)) \leq \\
\|x_k - y\| + \delta - 2\varepsilon\lambda_k(1-\lambda_k) \cdot \tilde{\eta}(\varepsilon/(\|x_k - y\| + \delta)) \overset{(21)}{\leq} \\
\|x_k - y\| + \delta - 2\varepsilon\lambda_k(1-\lambda_k) \cdot \tilde{\eta}(\varepsilon/(d+1))
\end{cases} \tag{35}
$$

(note that we can apply η to $\varepsilon/(\|x_k - y\| + \delta)$ since (25) and

$$
\|f(x_k) - y\| \overset{(23)}{\leq} \|f(x_k) - f(y)\| + \delta \leq \|x_k - y\| + \delta
$$

imply

$$
\varepsilon \leq \|x_k - y\| + \|f(x_k) - y\| \leq 2(\|x_k - y\| + \delta)
$$

and therefore

$$
\varepsilon/(\|x_k - y\| + \delta) \in (0, 2]).
$$

$\dashv$

Corollary 3.5 *If C has finite diameter d_C, theorem 3.4 holds with d_C instead of d for all $x \in C$.*

Proof: Follows from theorem 3.4 and lemma 1.2. $\Box$

Remark 3.6 *Note that the proof of the corollary only uses the elementary lemma 1.2 but not the deep Browder-Göhde-Kirk fixed point theorem which implies the existence of a fixed point of f in C under the assumptions of the corollary (if, moreover, X is complete and C is closed).*

Examples: It is well-known that the Banach spaces L_p with $1 < p < \infty$ are uniformly convex ([4], see also [14]). For $p \geq 2$, $\frac{\varepsilon^p}{p2^p}$ is a modulus of convexity ([11], see also [17]). Since

$$
\frac{\varepsilon^p}{p2^p} = \varepsilon \cdot \tilde{\eta}_p(\varepsilon)
$$

we get

$$
\tilde{\eta}_p(\varepsilon) = \frac{\varepsilon^{p-1}}{p2^p}
$$

satisfying (21) in the theorem above. So – disregarding constants depending on p, d only – we get $\gamma(\varepsilon^p)$ as rate of asymptotic regularity for L_p.

For the case $X := \mathbb{R}$ with the Euclidean norm we can choose $\tilde{\eta}(\varepsilon) := \frac{1}{2}$ (since $\varepsilon/2$ is a modulus of convexity) which gives the rate $\gamma(\varepsilon)$. For L_2 and $\mathbb{R}$ these rates are known to be optimal even in the case of constant $\lambda_k := \frac{1}{2}$, where they were first obtained in [13].

Remark 3.7 In [17] we already treated the case $\lambda_k = \frac{1}{2}$ for uniformly convex spaces by a logical analysis of the usual asymptotic regularity proof which goes back to [20] for the case of compact C and [3] for the case of bounded and closed C (and general $\lambda \in (0,1)$). For Hilbert spaces this was improved in [22] where weak convergence is shown. The analysis of that proof yielded basically the same bound as was obtained in [13] (for the case of general uniformly convex spaces) but with a completely elementary proof (since only approximate fixed points are used) whereas the proof in [13] is based on the Browder-Göhde-Kirk fixed point theorem (to get an actual fixed point). We also showed in [17] that a logically motivated modification of that proof allows to take into account the property (21) from the theorem above which is shared by many moduli of convexity. This yielded in the special cases of $X = L_p$ and $X = \mathbb{R}$ the improved bounds mentioned above. We subsequently learned that the similarly modified proof was used in [10] to prove asymptotic regularity for general sequences (λ_k) satisfying condition (4) which suggested the possibility to extend our quantitative analysis from [17] to this case. Our proof above shows that this indeed can be carried out. Again we don't need the existence of an actual fixed point (but only approximate fixed points) which allows us to state the result in greater generality than Groetsch's theorem.

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