On uniform weak König's lemma

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Dedicated to Anne S. Troelstra for his 60th Birthday

Abstract

The so-called weak König's lemma WKL asserts the existence of an infinite path b in any infinite binary tree (given by a representing function f). Based on this principle one can formulate subsystems of higher-order arithmetic which allow to carry out very substantial parts of classical mathematics but are Π^0_2-conservative over primitive recursive arithmetic PRA (and even weaker fragments of arithmetic). In [10] we established such conservation results relative to finite type extensions PRA^ω of PRA (together with a quantifier-free axiom of choice schema which – relative to PRA^ω – implies the schema of Σ^0_1-induction). In this setting one can consider also a uniform version UWKL of WKL which asserts the existence of a functional Φ which selects uniformly in a given infinite binary tree f an infinite path Φf of that tree. This uniform version of WKL is of interest in the context of explicit mathematics as developed by S. Feferman. The elimination process in [10] actually can be used to eliminate even this uniform weak König's lemma provided that PRA^ω only has a quantifier-free rule of extensionality QF-ER instead of the full axioms (E) of extensionality for all finite types. In this paper we show that in the

*Basic Research in Computer Science, Centre of the Danish National Research Foundation.

presence of (E), UWKL is much stronger than WKL: whereas WKL remains to be Π^0_2-conservative over PRA, $\mathrm{PRA}^\omega + (E)$+UWKL contains (and is conservative over) full Peano arithmetic PA.
We also investigate the proof-theoretic as well as the computational strength of UWKL relative to the intuitionistic variant of PRA^ω both with and without the Markov principle.

1 Introduction

The binary (so-called 'weak') König's lemma WKL plays an important role in the formulation of mathematically strong but proof-theoretically weak subsystems of analysis. In particular the fragment (WKL_0) of second-order arithmetic which is based on recursive comprehension (with set parameters), Σ^0_1-induction (with set parameters) and WKL occurs prominently in the context of reverse mathematics (see [18]). Although (WKL_0) allows to carry out a great deal of classical mathematics, it is Π^0_2-conservative over primitive recursive arithmetic PRA, as was shown first by H. Friedman using a model-theoretic argument. In [17] a proof-theoretic argument is given for a variant of (WKL_0) which uses function variables instead of set variables. In [10] we established various conservation results for WKL relative to subsystems of arithmetic in all finite types. As a special case these results yield that

(1) $\mathrm{E\text{-}PRA}^\omega$+$\mathrm{QF\text{-}AC}^{1,0}$+$\mathrm{QF\text{-}AC}^{0,1}$+WKL is Π^0_2-conservative over PRA,

where $\mathrm{E\text{-}PRA}^\omega$+$\mathrm{QF\text{-}AC}^{1,0}$+$\mathrm{QF\text{-}AC}^{0,1}$+WKL is a finite type extension of (WKL_0) (see below for a precise definition). The proof of this fact relies on a combination of Gödel's functional interpretation with elimination of extensionality (see [14]), negative translation and Howard's [8] majorization technique. The first step of the proof reduces the case with the full axiom of extensionality to a subsystem $\mathrm{WE\text{-}PRA}^\omega$+$\mathrm{QF\text{-}AC}^{1,0}$+$\mathrm{QF\text{-}AC}^{0,1}$+WKL which is based on a weaker quantifier-free rule of extensionality only (see below) which was introduced in Spector [19]. From this system, WKL is then eliminated. This elimination actually eliminates WKL via a strong uniform version of WKL, called UWKL below, which states the existence of a functional which selects uniformly in a given infinite binary tree an infinite path from that tree. This yields the following conservation result (which isn't stated explicitly in [10] but which can be obtained from the proofs in section 4 of that paper, see below):

(2) $\mathrm{WE\text{-}PRA}^\omega$+QF-AC+UWKL is Π^0_2-conservative over PRA.

In this weakly extensional context based on a quantifier-free rule of extensionality '+' must be understood in the sense that the axioms QF-AC and WKL must not be used in the proof of a premise of an application of the extensionality rule.[1] For WE-PA$^\omega$ we get the following result (with the same convention on + as above)

(3) WE-PA$^\omega$+QF-AC+UWKL is conservative over PA,

where PA denotes full first-order Peano arithmetic.

(2) is of interest in the context of so-called explicit mathematics as developed by S. Feferman (starting with [3]) and further investigated also by A. Cantini, G. Jäger and T. Strahm among others, since the uniform weak König's lemma UWKL seems to be a very natural 'explicit' formulation of WKL. We have been asked about the status of UWKL in the presence of full extensionality. In this note we give a surprisingly simple answer to this question showing, in particular, that

(4)E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+UWKL contains (and is conservative over) PA

and

(5)E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+UWKL has the same strength as $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$.

In the final section we investigate the status of UWKL in the context of the intuitionistic variant E-P(R)A$^\omega_i$ of E-P(R)A$^\omega$. In [12] we have shown that many non-constructive function(al) existence principles can be added to systems like E-PRA$^\omega_i$ without changing the growth rates of the provable (not only the provably recursive) functions of the system. This is true although the proof-theoretic strength of the resulting 'hybrid' systems is as strong as that of their counterpart with full classical logic. We apply this to UWKL and show that if a sentence $\forall x^0 \exists y^0 A(x,y)$ is provable in E-PRA$^\omega_i$+AC+UWKL, then one can construct a primitive recursive bounding function $\forall x \exists y \le p(x) A(x,y)$ (here A is of arbitrary logical complexity). Moreover, this system is closed under the so-called fan rule. This even holds in the presence of a strong independence-of-premise principle IP$_\neg$ for negated formulas but fails if the Markov principle M for numbers is added:

[1]See [10] (where we use a special symbol '$\oplus$' to emphasize this point) for details on this, and [13], where we show that without this restriction weakly extensional systems would violate the deduction theorem already for closed Π^0_1-axioms. Actually it is sufficient to impose this restriction on the use of the additional axioms for UWKL only.
The conservation results in [10] are much more general than the one we mentioned. This makes the proofs more involved than is needed for the special (Π^0_2-)case relevant here. A corresponding simplification of our argument has been worked out in [1].

Every $\alpha(<\varepsilon)$-recursive function is provably recursive in E-PRA$^\omega_i$+UWKL+M.

Acknowledgement: This paper was prompted by discussions the author has had with Gerhard Jäger and Thomas Strahm who asked him about the status of the uniform weak König's lemma in a fully extensional (classical) context.

2 Preliminaries

The set $\mathbf{T}$ of all finite types is defined inductively by

$$(i)\ 0 \in \mathbf{T} \text{ and } (ii)\ \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type ρ to objects of type τ.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

$$(i)\ 0 \in \mathbf{P} \text{ and } (ii)\ \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}.$$

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write $0(00)$ instead of $0(0(0))$. Furthermore we write for short $\tau\rho_k\ldots\rho_1$ instead of $\tau(\rho_k)\ldots(\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n+1$. The types $0, 00, 0(00), 0(0(00))\ldots$ are so represented by $0,1,2,3\ldots$ For arbitrary types $\rho \in \mathbf{T}$ the degree of ρ (for short deg(ρ)) is defined by deg(0) $:= 0$ and deg($\tau(\rho)$) $:=$ max(deg(τ),deg(ρ) + 1). For pure types the degree is just the number which represents this type.

The system E-PRA$^\omega$ is formulated in the language of functionals of all finite types and contains $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$-combinators for all types (which allows one to define λ-abstraction) and all primitive recursive functionals in the sense of Kleene (i.e. primitive recursion is available only on the type 0). Furthermore, E-PRA$^\omega$ contains the schema of quantifier-free induction

$$\text{QF-IA: } A_0(0) \wedge \forall x(A_0(x) \to A_0(x')) \to \forall x A_0(x),$$

where A_0 is quantifier-free, as well as the axioms of extensionality

$$(E):\ \forall x^\rho, y^\rho, z^{\tau\rho}(x =_\rho y \to zx =_\tau zy)$$

for all finite types (where for $\rho = 0\rho_k\ldots\rho_1$, $x =_\rho y$ is defined as $\forall z_1^{\rho_1}, \ldots, z_k^{\rho_k}(xz_1\ldots z_k =_0 yz_1\ldots z_k)$).[2] We only include equality $=_0$ between num-

[2] We deviate slightly from our notation in [11]. The system denoted by E-PRA$^\omega$ in the present paper results from the corresponding system in [11] if we replace the universal axioms 9) in the definition of the latter by the schema of quantifier-free induction.

bers as a primitive predicate.
So E-PRA$^\omega$ essentially is $\widehat{\mathrm{PA}}^{\omega}\restriction + (E)$, where $\widehat{\mathrm{PA}}^{\omega}\restriction$ is Feferman's system from [4].

E-PA$^\omega$ is the extension of E-PRA$^\omega$ obtained by the addition of the schema of full induction and all (impredicative) primitive recursive functionals in the sense of Gödel [6] and coincides with Troelstra's [20] system (E-HA$^\omega$)^c.

The 'weakly extensional'[3] versions WE-PRA$^\omega$ and WE-PA$^\omega$ of these systems result if we replace the extensionality axioms (E) by a quantifier-free rule of extensionality (due to Spector [19])

$$\text{QF-ER: } \frac{A_0 \to s =_\rho t}{A_0 \to r[s] =_\tau r[t]},$$

where A_0 is quantifier-free, $s^\rho, t^\rho, r[x^\rho]^\tau$ are arbitrary terms of the system and $\rho, \tau \in$ are arbitrary types.
Note that QF-ER allows one to derive the extensionality axiom for type 0, but already the extensionality axiom for type-1-arguments, i.e.

$$\forall z^2 \forall x^1, y^1 (x =_1 y \to zx =_0 zy)$$

is underivable in WE-PA$^\omega$ (see [8]).

In the last section of this paper we will also need the intuitionistic versions WE-PRA$_i^\omega$ and E-PRA$_i^\omega$ of WE-PRA$^\omega$ and E-PRA$^\omega$.

The schema of choice is given by

$$\mathrm{AC}^{\rho,\tau} \ : \ \forall x^\rho \exists y^\tau A(x,y) \to \exists Y^{\tau(\rho)} \forall x^\rho A(x, Yx), \quad \mathrm{AC} \ := \bigcup_{\rho,\tau \in \mathbf{T}} \{\, \mathrm{AC}^{\rho,\tau}\},$$

where A is an arbitrary formula.

The restriction of AC to quantifier-free formulas A_0 is denoted by QF-AC.

Remark 2.1

WE-PRA$^\omega$+QF-AC$^{0,0} \vdash \Sigma^0_1$-IA, Δ^0_1-CA, where

$$\Sigma^0_1\text{-IA: } \exists y^0 A_0(0,y) \wedge \forall x^0 (\exists y^0 A_0(x,y) \to \exists y^0 A_0(x',y)) \to \forall x \exists y A_0(x,y),$$

[3] This terminology is due to [20].

and

$$\Delta^0_1\text{-CA: } \forall x^0(\exists y^0 A_0(x,y) \leftrightarrow \forall y^0 B_0(x,y)) \to \exists f^1 \forall x(fx = 0 \leftrightarrow \exists y A_0(x,y)),$$

with A_0, B_0 quantifier-free (parameters of arbitrary types allowed).
So the system RCA_0 from reverse mathematics (see [18]) can be viewed as a subsystem of WE-PRA$^\omega$+QF-AC^{0,0} by indentifying sets $X \subseteq \mathbb{N}$ with their characteristic function.

In the following we use the formal definition of the binary ('weak') König's lemma as given in [21] (see also [22]; here $*, \overline{b}x, lth(n)$ refer to the primitive recursive coding of finite sequences from [20]):

Definition 2.2 (Troelstra(74))
$Tf :\equiv \forall n^0, m^0(f(n*m) =_0 0 \to fn =_0 0) \wedge \forall n^0, x^0(f(n*\langle x\rangle) =_0 0 \to x \leq_0 1)$
(i.e. $T(f)$ asserts that f represents a binary tree),
$T^\infty(f) :\equiv T(f) \wedge \forall x^0 \exists n^0(lth(n) = x \wedge fn = 0)$,
(i.e. $T^\infty(f)$ expresses that f represents an infinite binary tree),
WKL $:\equiv \forall f^1(T^\infty(f) \to \exists b^1 \forall x^0(f(\overline{b}x) = 0))$.

Definition 2.3 *The uniform weak König's lemma* UWKL *is defined as*

$$\mathrm{UWKL} :\equiv \exists \Phi^{1(1)} \forall f^1(T^\infty(f) \to \forall x^0(f((\overline{\Phi f})x) = 0)).$$

Instead of the full uniform version UWKL of WKL one can also consider a sequentially uniform version WKL_{seq} which asserts the existence of a sequence of infinite paths b_i in f_i for a sequence of infinite binary trees $(f_i)_{i\in\mathbb{N}}$:

Definition 2.4 $\mathrm{WKL}_{seq} :\equiv \forall f^{1(0)}_{(\cdot)}(\forall i^0 T^\infty(f_i) \to \exists b^{1(0)}_{(\cdot)} \forall i, x(f_i(\overline{b_i}x) = 0))$.

However, WKL_{seq} is (in contrast to UWKL) derivable in E-PRA$^\omega$+WKL (see proposition 3.1 below).

3 Results in the classical case

We first show that WKL_{seq} is not stronger than WKL relative to WE-PRA$^\omega$:

Proposition 3.1 WE-PRA$^\omega \vdash$ WKL$\to$ WKL_{seq}.

Proof:[4] Let $f^{1(0)}_{(\cdot)}$ be such that $\forall i\, T^\infty(f_i)$. Using the Cantor pairing function j we define

$$\tilde{f}(n) = \begin{cases} 0^0, \text{ if } \forall i((n)_i \leq 1) \wedge \forall i,k(j(i,k) \leq lth(n) \rightarrow f_i(\overline{\lambda l.(n)_{j(i,l)}}(k)) = 0), \\ 1^0, \text{ otherwise.} \end{cases}$$

Since '$\forall i$' and '$\forall i, k$' can be bounded primitive recursively in n, $\tilde{f}$ can be uniformly defined in f by a closed term of WE-PRA$^\omega$. It is easy to show (using basic properties of j) that $T^\infty(\tilde{f})$. Hence WKL yields a function $b \leq \lambda x.1$ such that $\forall x(\tilde{f}(\overline{b}x) = 0)$. This implies (using again basic properties of j) that

$$\forall i, x(f_i(\overline{\lambda l.b(j(i,l))})(x)) = 0)$$

and so $\lambda i, l.b(j(i,l))$ satisfies WKL$_{seq}$. □

We now switch to the uniform weak König's lemma, UWKL, which is not derivable in E-PA$^\omega$+QF-AC+WKL already for continuity reasons: the type structure ECF of all extensional continuous functionals as defined in [20] forms a model of E-PA$^\omega$+QF-AC+WKL (see [20][2.6.5,2.6.20]), whereas UWKL implies the existence of a non-continuous functional (see the proof of proposition 3.4 below). Nevertheless, for the **weakly** extensional systems WE-PRA$^\omega$ and WE-PA$^\omega$ we have the following conservation results for UWKL:

Theorem 3.2 *1)* WE-PRA$^\omega$+QF-AC+UWKL *is* Π^0_2*-conservative over* PRA.

2) WE-PA$^\omega$+QF-AC+UWKL *is conservative over* PA.
(Here, again, + must be understood in the sense of (2) *in section 1).*

Proof: 1) In [10] (4.2-4.7), we constructed a primitive recursive functional $f^1, g^1 \mapsto \zeta fg := (\widehat{\widehat{f_g}})$ such that

$$(1)\ \text{WE-PRA}^\omega \vdash \forall f, g\, T^\infty(\zeta fg)$$

and

$$(2)\ \text{WE-PRA}^\omega \vdash \forall f(T^\infty(f) \rightarrow \exists g(f =_1 \zeta fg)).$$

[4]See also the proof of theorem IV.1.8 in [18] where a similar argument is used.

By the proof of theorem 4.8 in [10] (and the fact that WE-PRA$^\omega$ is Π^0_2-conservative over PRA), it follows that

$$\text{WE-PRA}^\omega\text{+QF-AC+UWKL}^* \text{ is } \Pi^0_2\text{-conservative over PRA},$$

where

$$\text{UWKL}^* :\equiv \exists B \forall f, g, x((\zeta f g)((\overline{Bfg})x) =_0 0).$$

It remains to show that

$$\text{WE-PRA}^\omega \vdash \text{UWKL}^* \to \text{UWKL}.$$

The proof of (2) in [10](4.7) shows that g can be primitive recursively defined in f as

$$\tilde{f}(x) := \begin{cases} \min n \leq \overline{1^1}x[lth(n) = x \wedge f(n) = 0], & \text{if such an } n \text{ exists} \\ 0^0, & \text{otherwise.} \end{cases}$$

Thus for $\xi f := \zeta(f, \tilde{f})$

$$(2)' \text{ WE-PRA}^\omega \vdash \forall f(T^\infty(f) \to f =_1 \xi f).$$

Define $\Phi f := B(f, \tilde{f})$ for B satisfying UWKL*. Then

$$\forall x((\xi f)((\overline{\Phi f})x) =_0 0)$$

and so for f such that $T^\infty(f)$ (which implies $f =_1 \xi f$)

$$\forall x(f((\overline{\Phi f})x) =_0 0),$$

i.e. Φ satisfies UWKL.
2) As in 1) we obtain from the proof of 4.8 in [10] that

$$\text{WE-PA}^\omega\text{+QF-AC+UWKL is } \forall \underline{x}^{\underline{\rho}} \exists \underline{y}^0 A_0(\underline{x}, \underline{y})\text{-conservative over WE-PA}^\omega,$$

where $\underline{x}^{\underline{\rho}}$ is a tuple of variables of type levels ≤ 1, A_0 is quantifier-free and contains only $\underline{x}, \underline{y}$ as free variables. Now let A be a sentence in the language of PA which can be assumed to be in prenex normal form and assume that

$$\text{WE-PA}^\omega\text{+QF-AC+UWKL} \vdash A.$$

Then a fortiori

$$\text{WE-PA}^\omega\text{+QF-AC+UWKL} \vdash A^H,$$

where A^H is the Herbrand normal form of A. By the conservation result just mentioned we get

$$\text{WE-PA}^\omega \vdash A^H$$

and therefore by [9](theorem 4.1)

$$\text{PA} \vdash A.$$

□

Remark 3.3 *The passage from the provability of A^H to that of A used in the proof of 2) above does not apply to* WE-PRA$^\omega$ *and* PRA *(see* [9] *for a counterexample). Indeed, already* WE-PRA$^\omega$+QF-AC^{0,0} *is not* Π^0_3*-conservative over* PRA*: the former theory proves the schema of* Σ^0_1*-collection* Σ^0_1-CP*, but it is known that there are instances of* Σ^0_1-CP *(which always can be prenexed as* Π^0_3*-sentences)*[5] *which are unprovable in* PRA *(see* [16]*).*

We now show that the picture changes completely if we consider the systems E-PRA$^\omega$ and E-PA$^\omega$ with full extensionality instead of WE-PRA$^\omega$, WE-PA$^\omega$. This phenomenon is due to the following

Proposition 3.4
E-PRA$^\omega$ $\vdash$ UWKL $\leftrightarrow \exists\varphi^2\forall f^1(\varphi f =_0 0 \leftrightarrow \exists x^0(fx =_0 0))$.

Proof: 1) '$\rightarrow$': We first show that any Φ satisfying UWKL is – provably in E-PRA$^\omega$ – (effectively) discontinuous[6], i.e.

$$\text{E-PRA}^\omega \vdash \begin{cases} \forall\Phi^{1(1)}(\forall f^1(T^\infty(f) \rightarrow \forall x^0(f((\overline{\Phi f})x) =_0 0)) \rightarrow \\ \qquad \exists g^{1(0)}_{(\cdot)}, g^1(T^\infty(g) \wedge \forall i T^\infty(g_i) \wedge \forall i \forall j \geq i(g_j(i) =_0 g(i)) \\ \qquad\qquad \wedge \forall i, j(\Phi(g_i, 0) = \Phi(g_j, 0) \neq \Phi(g, 0)))) \end{cases}$$

[5]Here PRA is understood not as a quantifier-free theory but with full first-order predicate logic.

[6]The term 'effectively discontinuous' is due to [7] on which we rely in the second part of our proof.

and, moreover, $g_{(\cdot)}, g$ can be computed uniformly in Φ by closed terms of E-PRA$^\omega$. Define g primitive recursively such that

$$g(k) = \begin{cases} 0, \text{ if } \forall m < lth(k)((k)_m = 0) \vee \forall m < lth(k)((k)_m = 1) \\ 1, \text{otherwise.} \end{cases}$$

g represents a tree with two infinite paths, corresponding to an infinite sequence of 0's and an infinite sequence of 1's. So it is clear that (provably in E-PRA$^\omega$) $T^\infty(g)$. Now let $\Phi^{1(1)}$ be such that

$$\forall f^1(T^\infty(f) \to \forall x(f((\overline{\Phi f})x) =_0 0)).$$

<u>Case 1</u>: $\Phi(g,0) = 0$. Define a primitive recursive function $\lambda i, k.g_i(k)$ such that

$$g_i(k) = \begin{cases} 0, \text{ if } [lth(k) \leq i \wedge \forall m < lth(k)((k)_m = 0)] \vee [\forall m < lth(k)((k)_m = 1)] \\ 1, \text{otherwise.} \end{cases}$$

g_i represents the same tree as g except that the left branch has been truncated at level i. So again we easily verify within E-PRA$^\omega$ that $\forall i T^\infty(g_i)$. From the construction of g_i and g it is clear that

$$\forall k \forall l \geq lth(k)(g_l(k) = g(k)).$$

Since our coding has the property that $lth(k) \leq k$, we get

$$\forall k \forall l \geq k(g_l(k) = g(k)).$$

Since $\lambda x.1$ is the only infinite path of the binary tree represented by g_i, it follows that

$$\forall i(\Phi(g_i, 0) = 1).$$

<u>Case 2</u>: $\Phi(g,0) = 1$. The proof is analogous to case 1 with

$$g_i(k) := \begin{cases} 0, \text{ if } [lth(k) \leq i \wedge \forall m < lth(k)((k)_m = 1)] \vee [\forall m < lth(k)((k)_m = 0)] \\ 1, \text{otherwise.} \end{cases}$$

This finishes the proof of the discontinuity of Φ. We now show – using an argument from [7] known as 'Grilliot's trick' [7] – that the functional φ^2 defined by

[7] This argument plays an important role in the context of the Kleene/Kreisel countable functionals. See [15], whose formulation of it we adopt here.

$(+)\forall f^1(\varphi f =_0 0 \leftrightarrow \exists x(fx =_0 0))$ can be defined primitive recursively in Φ in such a way that $(+)$ holds provably in E-PRA$^\omega$:
We can construct a closed term $t^{1(1)}$ of E-PRA$^\omega$ such that (provably in E-PRA$^\omega$) we have

$$thi = \begin{cases} g_j(i), & \text{for the least } j < i \text{ such that } h(j) > 0, \text{ if such a } j \text{ exists} \\ g_i(i), & \text{otherwise.} \end{cases}$$

Together with $\forall i \forall j \geq i(g_j(i) = g_i(i))$ this yields

$$\exists j(h(j) > 0) \rightarrow th =_1 g_j \text{ for the least such } j$$

and together with $\forall i(g_i(i) = g(i))$

$$\forall j(h(j) = 0) \rightarrow th =_1 g.$$

Hence using the extensionality axiom for type-2-functionals we get[8]

$$\forall j(h(j) = 0) \leftrightarrow \Phi(th, 0) =_0 \Phi(g, 0).$$

So $\varphi := \lambda h^1.\overline{sg} \circ |\Phi(t(\overline{sg} \circ h), 0) - \Phi(g, 0)|$ where $\overline{sg}(x) := 0$ for $x \neq 0$ and $\overline{sg}(x) := 1$ otherwise, does the job.
We now combine the two constructions of φ corresponding to the two cases above into a single functional which defines φ primitive recursively in Φ: Let χ be a closed term such that

$$\text{E-PRA}^\omega \vdash \forall x^0((x =_0 0 \rightarrow \chi x =_{1(1)} t) \wedge (x \neq 0 \rightarrow \chi x =_{1(1)} \tilde{t})),$$

where t is defined as above with g_i from case 1 whereas $\tilde{t}$ is defined analogously but with g_i as in case 2. Then define $\varphi := \lambda h^1.\overline{sg} \circ |\Phi((\chi(\Phi(g, 0))(\overline{sg} \circ h), 0) - \Phi(g, 0)|$.
2) '$\leftarrow$': Primitive recursively in φ one can easily compute a functional Φ which selects an infinite branch of an infinite binary tree (for example, the leftmost infinite branch, in particular). $\square$

Corollary to the proof of proposition 3.4: One can construct closed terms t_1, t_2 of E-PRA$^\omega$ such that

$$\text{E-PRA}^\omega \vdash \begin{cases} \forall \Phi^{1(1)}(\forall f^1(T^\infty(f) \rightarrow \forall x^0(f((\overline{\Phi f})x) = 0)) \rightarrow \\ \qquad \forall f^1((t_1\Phi)f =_0 0 \leftrightarrow \exists x(fx = 0))) \end{cases}$$

[8]It is the direction '$\rightarrow$' which needs (E). The direction '$\leftarrow$' can be shown using only QF-ER, since free variables are allowed to occur in premises A_0 of QF-ER.

and

$$\text{WE-PRA}^\omega \vdash \begin{cases} \forall \varphi^2 (\forall f^1 (\varphi f = 0 \leftrightarrow \exists x (fx = 0)) \to \\ \qquad \forall f^1 (T^\infty (f) \to \forall x^0 (f((\overline{t_2 \varphi f})x) = 0))). \end{cases}$$

Corollary 3.5
E-PRA$^\omega$ + QF-AC^{1,0} $\vdash$ UWKL $\leftrightarrow \exists \mu^2 \forall f^1 (\exists x^0 (fx = 0) \to f(\mu f) = 0)$.

Proof: The existence of μ obviously implies the existence of φ in proposition 3.4 and hence of Φ. For the other direction we only have to observe that the existence of φ implies the existence of μ be applying QF-AC^{1,0} to

$$\forall f \exists x (\varphi(f) = 0 \to fx = 0).$$

□

Remark 3.6 In contrast to the corollary to the proof of proposition 3.4 above there exists no closed term t in E-PRA$^\omega$ which computes μ in Φ, i.e.

$$\mathcal{S}^\omega \not\models \forall \Phi^{1(1)} (\forall f^1 (T^\infty (f) \to \forall n^0 (f(\overline{\Phi f})n = 0)) \to \forall f^1 (\exists x (fx = 0) \to f(t \Phi f) = 0)$$

for every closed term t (of appropriate type) of E-PRA$^\omega$, since – by [8] – every such term has a majorant t^*, Φ is majorized by $\lambda f^1, x^0.1$ and so μ would have a majorant $\lambda f^M.t^*(1^{1(1)}, f^M)$ (where $f^M(x) := \max(f0, \ldots, fx)$), which contradicts the easy observation that μ has not even a majorant in $\mathcal{S}^\omega$ (here $\mathcal{S}^\omega$ denotes the full set-theoretic type structure).

Theorem 3.7 *1)* E-PRA$^\omega$+UWKL *contains Peano arithmetic* PA*.*

2) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+UWKL *is conservative over* PA*.*

3) E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+UWKL *proves the consistency of* PA *and has the same proof-theoretic strength as (and is* Π^1_2*-conservative over) the second order system* $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$.

Proof: 1) Using φ from proposition 3.4 one easily gets characteristic functions for all arithmetical formulas $A(\underline{x})$. By applying the quantifier-free induction axiom of E-PRA$^\omega$ to them, one obtains every arithmetical instance of induction.
2) This follows from corollary 3.5 and the conservation of E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+μ over PA, which is due to [4] (note that the usual elimination of extensionality

procedure – which applies to the existence of μ but not to UWKL – yields a reduction of E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} $+\,\mu$ to its variant where the extensionality axioms for types > 0 are dropped, see [14] for details on this).
3) follows from [4],[5] using again corollary 3.5 above and elimination of extensionality. □

Remark 3.8 *1) The functionals φ and μ from proposition 3.4 and corollary 3.5 provide uniform versions (in the same sense in which* UWKL *is a uniform version of* WKL*) of*

$$(1)\ \Pi^0_1\text{-CA}:\ \forall f \exists g \forall x^0 (g(x) =_0 0 \leftrightarrow \exists y^0 (f(x,y) =_0 0))$$

respectively of

$$(2)\ \Pi^0_1\text{-}\widehat{\text{CA}}:\ \forall f \exists g \forall x^0, z^0 (f(x,gx) =_0 0 \vee f(x,z) \neq 0),$$

but yet φ, μ are not stronger than $(1), (2)$ *relative to* E-PRA$^\omega$ *(but only relative to* E-PA$^\omega$*) as Feferman's results cited in the proof above show. The reason for this is that* E-PRA$^\omega$ *is too weak to iterate φ or μ uniformly since this would require a primitive recursion of type level 1. In contrast to this fact,* UWKL *is stronger than* WKL *already relative to* E-PRA$^\omega$*.*

2) One might ask whether UWKL *becomes weaker if we allow $\Phi^{1(1)}$ to be a partial functional which is required to be defined only on those functions f which represent an infinite binary tree. However the construction ξ (used in the proof of theorem 3.2) such that*

$$(1)\ \forall f^1 T^\infty(\xi f)$$

and

$$(2)\ \forall f^1 (T^\infty(f) \rightarrow \xi f =_1 f)$$

shows that any such partial Φ could be easily extended to a total one.

4 Results in the intuitionistic case

We first show that in the intuitionistic context of E-PRA$^\omega_i$, UWKL is proof-theoretically as strong as in the classical context but does not contribute to the growth of provable function(al)s. In the previous section we have seen that in sharp contrast to this, UWKL does contribute to the growth of provably recursive functions when classical logic is allowed. In proposition 4.8 below we observe that this already happens in the presence of the Markov principle.

Definition 4.1 *1) Independence-of-premise principle for negated formulas:*

$$\mathrm{IP}_\neg:\ (\neg A \to \exists x^\rho B) \to \exists x^\rho(\neg A \to B),$$

where A, B are arbitrary formulas, x not free in A and ρ arbitrary.

2) Markov principle for numbers :

$$\mathrm{M}:\ \neg\neg\exists x^0 A_0 \to \exists x^0 A_0,$$

where A_0 is quantifier-free.

Proposition 4.2
$\text{E-PRA}^\omega_i \vdash \text{UWKL} \leftrightarrow \exists\tilde{\varphi}^2\forall f^1(\tilde{\varphi}f =_0 0 \leftrightarrow \forall x^0(fx =_0 0))$.

Proof: The proposition follows from the corollary to the proof of proposition 3.4 by negative translation and some easy intuitionistic reasoning (using the decidability of $=_0$). □

As a corollary we get the following strengthened version of 3.4:

Corollary 4.3 $\text{E-PRA}^\omega_i+\text{M} \vdash \text{UWKL} \leftrightarrow \exists\varphi^2\forall f^1(\varphi f =_0 0 \leftrightarrow \exists x^0(fx =_0 0))$.

Theorem 4.4
E-PRA^ω_i+UWKL *has the same proof-theoretic strength as its classical version* E-PRA^ω+UWKL *(which by theorem 3.7 is that of* PA*).*

Proof: The theorem follows by negative translation using easy intuitionistic reasoning and the decidability of $=_0$. □

In contrast to this we have the following result that UWKL does not contribute to the growth of provable function(al)s relative to E-PRA^ω_i.

Theorem 4.5 *Let $A(u^\delta, v^\rho, w^\tau)$ be an arbitrary formula containing only u, v, w as free variables, $\delta \leq 1, \tau \leq 2$ and t a closed term (of suitable type). Then the following rule holds for $\mathcal{T}_i$:=*$\text{E-PRA}^\omega_i+\text{IP}_\neg+\text{AC}+\text{UWKL}$:

$$\begin{cases} \mathcal{T}_i \vdash \forall u^\delta \forall v \leq_\rho tu \exists w^\tau A(u,v,w) \\ \text{then one can extract a closed term } \Phi \text{ of E-PRA}^\omega_i \text{ s.t.} \\ \mathcal{T}_i \vdash \forall u^\delta \forall v \leq_\rho tu \exists w \leq_\tau \Phi(u)\, A(u,v,w), \end{cases}$$

where $x_1 \leq_{0(\rho_k)\ldots(\rho_1)} x_2 :\equiv \forall y_1^{\rho_1},\ldots,y_k^{\rho_k}(x_1\underline{y} \leq_0 x_2\underline{y})$.
Similarly for E-PRA$_i^\omega$ *replaced by* E-PA$_i^\omega$ *in the definition of* $\mathcal{T}_i$. *Then* Φ *is a closed term of* E-PA$_i^\omega$, *i.e. a primitive recursive functional in the sense of Gödel's* T.

Proof: This result follows from theorem 3.3 of [12] and the fact that UWKL is equivalent (relative to E-PRA$_i^\omega$) to an axiom having the form $\forall x^\gamma(C \to \exists y \leq_\eta sx\neg B)$ (as required in that theorem): take '$\forall x$' as dummy quantifier, $s := \lambda x.1$, $C :\equiv (0 = 0)$ and notice that the formula $\forall f^1(T^\infty(f) \to \forall x^0(f((\overline{\Phi f})x) =_0 0))$ is intuitionistically equivalent to its double negated version since $=_0$ is decidable. □

Corollary 4.6 *Let* $\mathcal{T}_i$ *be as in theorem 4.5. Then* $\mathcal{T}_i$ *is closed under the fan rule, i.e.*

$$\mathcal{T}_i \vdash \forall f \leq_1 g \exists n^0 A(f,n) \quad \Rightarrow \mathcal{T}_i \vdash \exists m^0 \forall f \leq_1 g \exists n \leq m A(f,n),$$

where A *contains only free variables of type* ≤ 1.
This result also holds for E-PA$_i^\omega$ *instead of* E-PRA$_i^\omega$ *in the definition of* $\mathcal{T}_i$ *and for the systems where one or both of the principles* AC *and* IP$_\neg$ *are omitted from* $\mathcal{T}_i$.

Corollary 4.7 *Let* $\mathcal{T}_i$ *be as in theorem 4.5 and* $A(f^1,n^0)$ *be an arbitrary formula of* E-PRA$_i^\omega$ *containing only* f^1, n^0 *as free variables. Then the following rule holds:*

$$\begin{cases} \mathcal{T}_i \vdash \forall f \exists n A(f,n) \\ \Rightarrow \exists \text{ a primitive recursive (in the sense of Kleene) functional s.t.} \\ \mathcal{T}_i \vdash \forall f \exists n \leq \Phi(f)\, A(f,n). \end{cases}$$

In particular: For $A(m^0,n^0)$ *instead of* $A(f,n)$, Φm *is a primitive recursive function in* m.

Proof: This follows as a special case from theorem 4.5 using the well-known fact that the functions definable by closed terms of E-PRA$_i^\omega$ are just the ordinary primitive recursive functions. □

In contrast to this, the addition of the Markov principle M yields the same provable recursive functions than in the classical case:

Proposition 4.8
Every $\alpha(<\varepsilon_0)$*-recursive function is provably recursive in* E-PRA$_i^\omega$+M+UWKL.

Proof: The proposition follows from theorem 3.7, the well-known fact that every $\alpha(<\varepsilon_0)$-recursive function is provably recursive in PA and an application of negative translation which yields that E-PRA$^\omega$+UWKL is Π_2^0-conservative over E-PRA$_i^\omega$+M+UWKL. □

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