On the asymptotic behavior of odd operators

Ulrich Kohlenbach*

Department of Mathematics
Technische Universität Darmstadt
Schlossgartenstraße 7, 64289 Darmstadt, Germany

3. May 2011, some typos corrected 10. Dec. 2013

Abstract

We give quantitative versions of strong convergence results due to Baillon, Bruck and Reich for iterations of nonexpansive odd (and more general) operators in uniformly convex Banach spaces.

Keywords: Nonexpansive mappings, odd operators, fixed points, metastability, functional interpretation, proof mining.
Mathematics Subject Classification (2010): 03F10; 47H09; 47H10

1 Introduction

Let X be a uniformly convex Banach space and $C \subseteq X$ a closed convex subset satisfying $C = -C$. In [2], Baillon, Bruck and Reich showed (among many other things) that the iteration $(T^n x)$ of an odd nonexpansive mapping $T : C \to C$ that is asymptotically regular at $x \in C$ strongly converges to a fixed point of T. By a famous result due to Ishikawa [4] the averaged mapping $T_\lambda x := (1-\lambda)x + \lambda T(x)$ with $\lambda \in (0,1)$ of a nonexpansive mapping $T : C \to C$ always is asympotically regular provided that $(T^n_\lambda x)$ is bounded (in fact – by another result from [2] – it suffices that $(\|T^n_\lambda x\|/n)$ converges to 0).
With T also T_λ is nonexpansive and odd and so the sequence (x_n) defined by $x_n := T^n_\lambda x$ (which trivially is bounded) converges strongly towards a fixed point $p \in C$ of T.

We first observe that the condition of T being nonexpansive and odd can be weakened to the condition

$$(W): \ \forall x, y \in C \ \big(\|Tx + Ty\| \leq \|x + y\|\big)$$

studied in [14] which also makes the assumption $C = -C$ superfluous.

It is easy to show that there is no computable (in the data at hand) rate of convergence even for $X := \mathbb{R}, C := [0,1], \lambda := 1/2, x := 1$ in the sense that there is a computable sequence

*The author has been supported by the German Science Foundation (DFG Project KO 1737/5-1).

(f_l) of odd nonexpansive functions $f_l : [-1,1] \to [-1,1]$ such that there is no computable function $\delta : \mathbb{N} \to \mathbb{N}$ such that for $x^l_n := (f_l)^n_{\frac{1}{2}}(1)$

$$(+)\ \forall m \geq \delta(l)\, (|x^l_m - x^l_{\delta(l)}| \leq \frac{1}{2}).$$

Define $f_l(x) := a_l \cdot x$, where $a_l := \sum\limits_{i=0}^{\infty} g(l,i) \cdot 2^{-i-1} \in [0,1]$ with

$$g(l,n) := \begin{cases} 1, & \text{if } \neg T(l,l,n) \\ 0, & \text{otherwise,} \end{cases}$$

Here T denotes the Kleene T-predicate.

Now observe that

$$(++)\ a_l = 1 \to x^l_{\delta(l)} = 1 \text{ and } a_l < 1 \to x^l_{\delta(l)} \in [0, 1/2].$$

While the first implication is immediate from the definition of x^l_n, the second follows using $(+)$ and the fact that (by – an essentially trivial use of – Ishikawa's theorem [4]) (x^l_n) converges towards the unique fixed point 0 of f_l.

By $(++)$ the computability of δ would allow us to decide whether $a_l = 1$ or $a_l < 1$ and so whether or not $\exists n \in \mathbb{N}\, T(l,l,n)$ contradicting the undecidability of the (special) Halting problem.

While we do not know whether for single computable operators $T : C \to C$ in effective uniformly convex spaces, the iteration $x_n := T^n_\lambda x$ (for computable $x \in C, \lambda \in (0,1)$) might have no computable rate of convergence, we show that the rate is computable iff the norm $\|p\|$ of the strong limit p of (x_n) is computable.

Things are much better for a reformulation of the convergence property known in logic as the no-counterexample interpretation of the former ([7, 8], see also [5]) which recently has been popularized under the name of 'metastability' by T. Tao (see [11, 12]). Here one considers the statement

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists k \in \mathbb{N}\, \forall i,j \in [k; k+g(k)]\, \big(\|T^i x - T^j x\| < \varepsilon\big)$$

which, ineffectively, is equivalent to the strong convergence of $(T^n x)$. Here $[k; k+m] := \{k, k+1, k+2, \ldots, k+m\}$.

We then give an explicit effective (in fact even primitive recursive) and highly uniform rate $\Phi(b, \alpha, \varepsilon, g)$ of metastability of $(T^n x)$

$$\begin{aligned} \forall \varepsilon \in (0,2]\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall b \in \mathbb{N}^*\, \forall x \in C_b\, \exists n \leq \Phi(b, \alpha, \varepsilon, g) \\ \forall i,j \in [n; n+g(n)]\, \big(\|T^i x - T^j x\| < \varepsilon\big) \end{aligned}$$

that (in addition to ε and g) only depends on a norm upper bound $b \geq \|x\|$ of x and a uniform rate α of asymptotic regularity of T on $C_b := \{x \in C : \|x\| \leq b\}$, i.e.

$$\forall \varepsilon > 0\, \forall b \in \mathbb{N}^*\, \forall x \in C_b\, \forall n \geq \alpha(b, \varepsilon)\, \big(\|T^{n+1} x - T^n x\| < \varepsilon\big).$$

In fact, instead of α a (uniform) rate on the metastable version of asymptotic regularity, i.e. a φ such that

$$\forall \varepsilon > 0 \, \forall f : \mathbb{N} \to \mathbb{N} \, \forall b \in \mathbb{N}^* \, \forall x \in C_b \, \exists k \leq \varphi(b, f, \varepsilon) \, \forall i \in [k; k + f(k)] \, \big(\|T^{i+1}x - T^i x\| < \varepsilon\big),$$

is sufficient.
The bound Φ is independent of X (and C) except for a modulus of uniform convexity η of X (and an upper bound b on $\|x\|$). The extraction of this bound is an instance of a general logical metatheorem which not only guarantees the extractability of such bounds for large classes of proofs but also provides an algorithm for the actual construction of the bound from a given proof. This then results again in an ordinary proof that no longer relies on any facts from logic (see [5], in particular Chapters 17 and 18, for all this).

Using the optimal rate of asymptotic regularity α for T_λ from [1] this gives an effective (and even primitive recursive) rate of metastability for the strong convergence of (x_n) (as defined above) that only depends on ε, g and b.

A primitive recursive rate on the metastability of the **Cesàro means** (i.e. ergodic averages) of operators in Hilbert space satisfying Wittmann's condition was recently extracted from Wittmann's [14] proof of strong convergence of these means by Safarik [10]. For another quantitative strong nonlinear ergodic theorem see [6]. Again, these results have been obtained using the aforementioned proof-theoretic approach.

2 Results

In the following, let X be a uniformly convex Banach space with a modulus of convexity $\eta : (0, 2] \to (0, 1]$, i.e.

$$\forall x, y \in B_1(0) \, \forall \varepsilon \in (0, 2] \, \left(\left\| \frac{x + y}{2} \right\| > 1 - \eta(\varepsilon) \to \|x - y\| < \varepsilon \right),$$

where $B_d(0)$ denotes the closed ball with center 0 and radius d in X.

Lemma 2.1. *Let $x, y \in B_d(0) \subset X$ with $0 < d \leq b \in \mathbb{N}$. Then*

$$\forall \varepsilon \in (0, 2] \, \left(\left\| \frac{x + y}{2} \right\| > d(1 - \eta(\varepsilon/b)) \to \|x - y\| < \varepsilon \right).$$

Proof: Define $\tilde{x} := x/d, \tilde{y} := y/d$ so that $\tilde{x}, \tilde{y} \in B_1(0)$.
Assume that $\left\|\frac{x+y}{2}\right\| > d(1 - \eta(\varepsilon/b))$. Then

$$\left\| \frac{\tilde{x} + \tilde{y}}{2} \right\| = \frac{1}{d} \left\| \frac{x + y}{2} \right\| > 1 - \eta(\varepsilon/b)$$

and so $\frac{1}{d}\|x - y\| = \|\tilde{x} - \tilde{y}\| < \frac{\varepsilon}{b}$. Hence $\|x - y\| < \frac{d \cdot \varepsilon}{b} \leq \varepsilon$. $\square$

Notation: For $b \in \mathbb{N}^*$ define $C_b := \{x \in C : \|x\| \leq b\}$.
For $n, m \in \mathbb{N}$ we define $n \dot{-} m := n - m$ if $n \geq m$ and $:= 0$, otherwise.

Theorem 2.2. *Let $C \subseteq X$ be any nonempty subset of X and $T : C \to C$ a selfmapping of C that satisfies Wittmann's [14] condition*

$$(W): \ \forall x, y \in C \left(\|Tx + Ty\| \le \|x + y\| \right).$$

Moreover, assume that for each $0 < b \in \mathbb{N}$ the mapping T is (uniformly on C_b) asymptotically regular with a rate $\alpha : \mathbb{N} \times \mathbb{R}_+^* \to \mathbb{N}$, i.e.

$$\forall \varepsilon > 0 \, \forall b \in \mathbb{N}^* \, \forall x \in C_b \, \forall n \ge \alpha(b, \varepsilon) \, \left(\|T^{n+1}x - T^n x\| < \varepsilon \right).$$

Then $(T^n x)_{n\in\mathbb{N}}$ converges strongly with the following rate of metastability

$$\begin{aligned} &\forall \varepsilon \in (0, 2] \, \forall g : \mathbb{N} \to \mathbb{N} \, \forall b \in \mathbb{N}^* \, \forall x \in C_b \, \exists n \le \Phi(b, \alpha, \varepsilon, g) \\ &\qquad\qquad \forall i, j \in [n; n + g(n)] \, \left(\|T^i x - T^j x\| < \varepsilon \right), \end{aligned}$$

where

$$\begin{aligned} &\Phi(b, \alpha, \varepsilon, g) := \Psi(b, h_{b,\alpha,\varepsilon,g}, \tfrac{\delta_b(\varepsilon)}{2}) \text{ with} \\ &h_{b,\alpha,\varepsilon,g}(n) := h(n) := \max\left\{ \alpha\left(b, \tfrac{\delta_b(\varepsilon)}{\max\{g(n),1\}}\right) \dot{-} n, g(n) \right\} \text{ and} \\ &\Psi(b, f, \delta) := \tilde{f}^{(\lceil b/\delta \rceil)}(0) \text{ with } \tilde{f}(n) := n + f(n) \text{ for } f : \mathbb{N} \to \mathbb{N}, \\ &\delta_b(\varepsilon) := \tfrac{\varepsilon}{2} \cdot \eta(\varepsilon/b). \end{aligned}$$

If T is continuous and C closed, then the strong limit of $(T^n x)_{n\in\mathbb{N}}$ is a fixed point of T. For the metastability statement the completeness of X is not needed.

Proof: It suffices to prove the metastability statement which (ineffectively) implies the strong Cauchy property of the sequence (and so using the completeness of X its convergence). That for continuous T (and closed C) the limit is a fixed point of T then trivially follows from the asymptotic regularity of T.
Let $\varepsilon \in (0, 2], b \in \mathbb{N}^*, g : \mathbb{N} \to \mathbb{N}$ and C, T, x be as in the theorem. By the condition (W) the sequence $(\|T^n x\|)_{n\in\mathbb{N}}$ is nonincreasing and hence convergent. By [5] (Proposition 2.27, Remark 2.29) it follows that Ψ is a rate of metastability for this sequence, i.e.

$$\forall \delta > 0 \, \forall f : \mathbb{N} \to \mathbb{N} \, \exists n \le \Psi(b, f, \delta) \, \forall i, j \in [n; n + f(n)] \left(\left| \|T^i x\| - \|T^j x\| \right| < \delta \right).$$

For $\delta := \frac{\delta_b(\varepsilon)}{2}$ and $f := h := h_{b,\alpha,\varepsilon,g}$ let $n \in \mathbb{N}$ be such a number.
Define $d := \|T^n x\| = \max\{\|T^k x\| : k \in [n; n + h(n)]\} \le b$. Then

$$(1) \ \forall k \in [n; n + h(n)] \left(d - \frac{\delta_b(\varepsilon)}{2} < \|T^k x\| \le d \right).$$

From the assumption on α we get

$$(2) \ \forall i \in \mathbb{N}^* \, \forall \varepsilon > 0 \, \forall k \ge \alpha(b, \varepsilon/i) \, \forall j \le i \, \left(\|T^k x - T^{k+j} x\| < \varepsilon \right),$$

since

$$\|T^k x - T^{k+j} x\| \le \sum_{l=0}^{j-1} \|T^{k+l} x - T^{k+l+1} x\| < \sum_{l=0}^{j-1} \frac{\varepsilon}{i} \le \varepsilon$$

for all $0 < j \le i$ and $k \ge \alpha(b, \varepsilon/i)$.

For $k := n + h(n) \ge \alpha\left(b, \frac{\delta_b(\varepsilon)}{\max\{g(n),1\}}\right)$ we get from (1) and (2) that for all $i \in [n; n+g(n)] \subseteq [n; k]$:

$$\begin{aligned} \forall j \le g(n)\ \big(2(d - \tfrac{\delta_b(\varepsilon)}{2}) \le 2\|T^k x\| &\le \|T^{k+j}x + T^k x\| + \|T^k x - T^{k+j}x\| \\ &< \|T^{k+j}x + T^k x\| + \delta_b(\varepsilon) \\ &\overset{(W)}{\le} \|T^{i+j}x + T^i x\| + \delta_b(\varepsilon)\big). \end{aligned}$$

Hence

$$(3)\ \forall i, j \in [n; n+g(n)]\ \left(d - \delta_b(\varepsilon) < \left\|\frac{T^i x + T^j x}{2}\right\|\right).$$

Case 1: $d := \|T^n x\| < \frac{\varepsilon}{2}$. Then

$$\forall i, j \in [n; n+g(n)]\ \big(\|T^i x - T^j x\| \le \|T^i x\| + \|T^j x\| \le 2\|T^n x\| < \varepsilon\big)$$

and so we are done.

Case 2: $d \ge \frac{\varepsilon}{2}$. Then by the definition of $\delta_b(\varepsilon)$ and (3) we have

$$(4)\ \forall i, j \in [n; n+g(n)]\ \left(d(1 - \eta(\varepsilon/b)) < \left\|\frac{T^i x + T^j x}{2}\right\|\right).$$

Using (1), (4) and lemma 2.1 yields that

$$\forall i, j \in [n; n+g(n)]\ \big(\|T^i x - T^j x\| < \varepsilon\big).$$

$\square$

Remark 2.3. *If $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $0 < \varepsilon_1 \le \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \le \tilde{\eta}(\varepsilon_2)$, then we can replace $\delta_b(\varepsilon)$ in the bound in theorem 2.2 by $\delta_b(\varepsilon) := \varepsilon \cdot \tilde{\eta}(\varepsilon/b)$. In particular, in the case of a Hilbert space X (where one can take $\eta(\varepsilon) := \varepsilon^2/8$, see e.g. [6]), this yields $\delta_b(\varepsilon) := \frac{\varepsilon^2}{8b}$.*

Proof: With $\delta_b(\varepsilon) := \varepsilon \cdot \tilde{\eta}(\varepsilon/b)$ one gets instead of (4) in the proof of theorem 2.2

$$(4)' \left\{ \begin{array}{l} \forall i, j \in [n; n+g(n)] \\ \left(d(1 - \eta(\varepsilon/d)) = d(1 - \frac{\varepsilon}{d} \cdot \tilde{\eta}(\varepsilon/d)) \le d(1 - \frac{\varepsilon}{d} \cdot \tilde{\eta}(\varepsilon/b)) < \left\|\frac{T^i x + T^j x}{2}\right\|\right). \end{array}\right.$$

The claim now follows using lemma 2.1 since $T^i x, T^j x \in B_d(0)$ for $i, j \in [n; n+g(n)]$. $\square$

The above extraction of the rate of metastability Φ from the proof given in [2] (and also the fact that Φ only depends on the arguments $b, \alpha, \varepsilon, g$) is an instance of a general logical metatheorem (see [3] Theorem 6.3.2 or [5] Theorem 17.69.2 and note that the condition (W) is purely universal and implies that T is majorized by the identity function). In fact, that metatheorem even guarantees such a bound when the rate of asymptotic regularity α is replaced by a weaker rate of metastability φ instead, i.e.

$(*)\ \forall \varepsilon > 0\, \forall f : \mathbb{N} \to \mathbb{N}\, \forall b \in \mathbb{N}^*\, \forall x \in C_b\, \exists k \leq \varphi(b, f, \varepsilon)\, \forall i \in [k; k+f(k)]\ \big(\|T^{i+1}x - T^i x\| < \varepsilon\big).$

We will briefly demonstrate this now. In fact, one only needs φ for constant-c functions ($c \in \mathbb{N}$) that we also denote by c. Modifying φ to $\varphi'(b, c, l, \varepsilon) := \varphi(b, c+l, \varepsilon) + l$ one gets for each $l \in \mathbb{N}$

$$(**)\ \exists k \leq \varphi'(b, c, l, \varepsilon)\, \forall i \in [k; k+c]\ \big(k \geq l \wedge \|T^{i+1}x - T^i x\| < \varepsilon\big).$$

Now define (using $(**)$) $\alpha_{n,g}(b, \varepsilon)$ as the least $k \leq \varphi'(b, g(n), n + g(n), \varepsilon)$ such that

$$\forall i \in [k; k + g(n)]\ \big(k \geq n + g(n) \wedge \|T^{i+1}x - T^i x\| < \varepsilon\big)\,.$$

Then theorem 2.2 holds with α and $h_{b,\alpha,\varepsilon,g}$ being replaced by $\alpha_{n,g}$ and $h_{b,\varphi,\varepsilon,g}(n) := \alpha_{n,g}\left(b, \frac{\delta_b(\varepsilon)}{\max\{g(n),1\}}\right) - n$ respectively. Replacing $h_{b,\varphi,\varepsilon,g}$ by the monotone upper bound

$$h^*_{b,\varphi,\varepsilon,g}(n) := \max\{\varphi'(b, g(m), m + g(m), \delta_b(\varepsilon)/\max\{g(m), 1\}) - m \,:\, m \leq n\}$$

yields an upper bound

$$\Phi(b, \varphi, \varepsilon, g) := \Psi(b, h^*_{b,\varphi,\varepsilon,g}, \frac{\delta_b(\varepsilon)}{2}) \geq \Psi(b, h_{b,\varphi,\varepsilon,g}, \frac{\delta_b(\varepsilon)}{2})$$

satisfying theorem 2.2. This yields the following qualitative improvement of theorem 2.2

Corollary 2.4. *For the strong convergence of $(T^n x)$ in theorem 2.2 one can weaken the asymptotic regularity assumption to*

$$\forall \varepsilon > 0\, \forall c \in \mathbb{N}\, \forall x \in C\, \exists k \in \mathbb{N}\, \forall i \in [k; k+c]\ \big(\|T^{i+1}x - T^i x\| < \varepsilon\big).$$

If T is continuous and C is closed, then the limit of $(T^n x)$ is a fixed point of T.

Proof: By the reasoning above, the sequence $(T^n x)$ is metastable (note that for metastability in the point x we also only need the above weak form of asymptotic regularity in x) and hence is strongly Cauchy. For closed C the limit is in C and – for continuous T – a fixed point of T as the condition in the corollary implies that

$$\forall \varepsilon > 0\, \forall n \in \mathbb{N}\, \exists k \geq n\ (\|T^{k+1}x - T^k x\| < \varepsilon).$$

$\square$

In the following, we apply theorem 2.2 to averages mappings for which effective (full) rates of asymptotical regularity are known (here 'π' denotes the constant π):

Theorem 2.5. *Let X be a uniformly convex Banach space and $C \subseteq X$ a closed and convex subset. Assume that $T : C \to C$ satisfies (W) and is nonexpansive. Let $\lambda \in (0, 1)$ and define $T_\lambda x := (1 - \lambda)x + \lambda T x$, $x_n := T_\lambda^n x$ for $x \in C$. Then $(x_n)_{n\in\mathbb{N}}$ strongly converges to a fixed point $p \in C$ of T and the following rate of metastability holds:*

$$\forall \varepsilon \in (0, 2]\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall b \in \mathbb{N}^*\, \forall x \in C_b\, \exists n \leq \Phi(b, \alpha, \varepsilon, g)$$
$$\forall i, j \in [n; n + g(n)]\ \big(\|x_i - x_j\| < \varepsilon\big),$$

where Φ is as in theorem 2.2 and $\alpha(b,\varepsilon) := \left\lceil \frac{b^2\cdot\lambda}{\pi(1-\lambda)\varepsilon^2} \right\rceil$.
For the last statement no completeness of X or closedness of C is needed.

Proof: For $x \in C_b$ it follows from a deep result due to Baillon and Bruck [1] (and using that $\lambda\|x_n - T(x_n)\| = \|T_\lambda^{n+1}x - T_\lambda^n x\|$) that α is a rate of asymptotic regularity for T_λ (this result even holds in arbitrary normed spaces).[1] With T also T_λ satisfies (W) since

$$\begin{aligned}\|T_\lambda x + T_\lambda y\| &= \|(1-\lambda)x + \lambda Tx + (1-\lambda)y + \lambda Ty\| \\ &\le (1-\lambda)\|x+y\| + \lambda\|Tx+Ty\| \\ &\le (1-\lambda)\|x+y\| + \lambda\|x+y\| = \|x+y\|.\end{aligned}$$

Hence the corollary follows from theorem 2.2 applied to T_λ (note that the proof for the metastability statement did not use the completeness of X nor the closedness of C). $\square$

Remark 2.6. *For nonexpansive T the condition (W), in particular, holds when $C = -C$ and T is odd, i.e. $T(-x) = -T(x)$.*

The proof of theorem 2.2 (and theorem 2.5) immediately yields an effective rate of convergence of $(T^n x)_{n\in\mathbb{N}}$ (instead of a rate of metastability only) provided one has a rate $\Psi_{x,T}$ of convergence for $(\|T^n x\|)_{n\in\mathbb{N}}$ given, i.e. for $d := \lim\limits_{n\to\infty} \|T^n x\|$

$$\forall \varepsilon > 0\, \forall n \ge \Psi_{x,T}(\varepsilon)\ \left(\|T^n x\| - d < \varepsilon\right).$$

Then $\Psi_{x,T}\left(\frac{\delta_b(\varepsilon)}{2}\right)$ is a rate of convergence of $(T^n x)_{n\in\mathbb{N}}$. This leads to the following (using the notion of computability for Banach spaces and mappings between Banach spaces from [9] and [13]).

Corollary 2.7. *Let X be a computable uniformly convex Banach space with a computable modulus of uniform convexity η and C be a closed and convex subset. Let $T : C \to C$ be a computable nonexpansive mapping satisfying condition (W) and $x \in C$ be a computable point. Finally, let $\lambda \in (0,1)$ be computable. Then $(T_\lambda^n x)_{n\in\mathbb{N}}$ converges effectively (i.e. with a computable rate of convergence) to its limit $p := \lim\limits_{n\to\infty} T_\lambda^n x$ if and only if $\|p\|$ is computable.*

Proof: The assumptions yields that (x_n) with $x_n := T_\lambda^n x$ is a computable sequence in X. If (x_n) converges effectively, then also p and hence $\|p\|$ is computable. Conversely, suppose that $\|p\|$ is computable. Then there is a computable function $\rho : \mathbb{Q}_+^* \to \mathbb{N}$ such that

$$\forall q \in \mathbb{Q}_+^*\ \left(\|T_\lambda^{\rho(q)} x\| - \|p\| < q\right)$$

since '$\|T_\lambda^n x\| - \|p\| < q$' is computably enumerable in n, q. Since $(\|T^n x\|)_{n\in\mathbb{N}}$ is nonincreasing, ρ in fact is a rate of convergence. The comments preceding this corollary now yield a computable rate of convergence for $(T_\lambda^n x)_{n\in\mathbb{N}}$. $\square$

[1]The bound in [1] is stated for sequences in $B_1(0)$ but can easily be adapted to $B_b(0)$ by switching to the norm $\|x\|_b := \frac{1}{b}\cdot\|x\|$. The α in our theorem results from this adaptation.

References

[1] Baillon, J., Bruck, R.E., The rate of asymptotic regularity is $0(\frac{1}{\sqrt{n}})$. Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math. 178, pp. 51-81, Dekker, New York, 1996.

[2] Baillon, J.B., Bruck, R.E., Reich, S., On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces. Houston J. Math. **4**, pp. 1-9 (1978).

[3] Gerhardy, P., Kohlenbach, U., General logical metatheorems for functional analysis. Trans. Amer. Math. Soc. **360**, pp. 2615-2660 (2008).

[4] Ishikawa, S., Fixed points and iterations of a nonexpansive mapping in a Banach space. Proc. Amer. Math. Soc. **59**, pp. 65-71 (1976).

[5] Kohlenbach, U., Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. xx+536pp., Springer Heidelberg-Berlin, 2008.

[6] Kohlenbach, U.,On quantitative versions of theorems due to F.E. Browder and R. Wittmann. Advances in Mathematics **226**, pp. 2764-2795 (2011).

[7] Kreisel, G., On the interpretation of non-finitist proofs, part I. J. Symbolic Logic **16**, pp.241-267 (1951).

[8] Kreisel, G., On the interpretation of non-finitist proofs, part II: Interpretation of number theory, applications. J. Symbolic Logic **17**, pp. 43-58 (1952).

[9] Pour-El, M.B., Richards, J.I., Computability in Analysis and Physics. Springer-Verlag, Berlin Heidelberg, 1989.

[10] Safarik, P., A quantitative nonlinear strong ergodic theorem for Hilbert spaces. Preprint 2011.

[11] Tao, T., Soft analysis, hard analysis, and the finite convergence principle. Essay posted May 23, 2007. Appeared in: 'T. Tao, Structure and Randomness: Pages from Year One of a Mathematical Blog. AMS, 298pp., 2008'.

[12] Tao, T., Norm convergence of multiple ergodic averages for commuting transformations. Ergodic Theory and Dynamical Systems **28**, pp. 657-688 (2008).

[13] Weihrauch, K., Computable Analysis. Springer, Berlin 2000.

[14] Wittmann, R., Mean ergodic theorems for nonlinear operators. Proc. Amer. Math. Soc. **108**, pp. 781-788 (1990).