Proof mining in L_1-approximation

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Abstract

In this paper we present another case study in the general project of proof mining which means the logical analysis of prima facie non-effective proofs with the aim of extracting new computationally relevant data. We use techniques based on monotone functional interpretation (developed in [17]) to analyze Cheney's simplification [6] of Jackson's original proof [10] from 1921 of the uniqueness of the best L_1-approximation of continuous functions $f \in C[0,1]$ by polynomials $p \in P_n$ of degree $\leq n$. Cheney's proof is non-effective in the sense that it is based on classical logic and on the non-computational principle **WKL** (binary König's lemma). The result of our analysis provides the first effective (in all parameters) uniform modulus of uniqueness (a concept which generalizes 'strong uniqueness' studied extensively in approximation theory). Moreover, the extracted modulus has the optimal ε-dependency as follows from Kroó [22]. The paper also describes how the uniform modulus of uniqueness can be used to compute the best L_1-approximations of a fixed $f \in C[0,1]$ with arbitrary precision. The second author uses this result to give a complexity upper bound on the computation of the best L_1-approximation in [25].

Key words: L_1-approximation, strong unicity, proof theory, proof mining, computable analysis, constructive mathematics

1991 MSC: 03F10, 03F60, 41A10, 41A52, 41A50, 26E40

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[1] **BRICS** – Basic Research in Computer Science, funded by the Danish National Research Foundation.

1 Introduction

This paper is another case study in the general project of proof mining which means the logical analysis of prima facie non-effective proofs with the aim of extracting new computationally relevant data [2]. At the same time we obtain new results in approximation theory. More specifically, we analyze a non-effective proof of the uniqueness of best approximations of continuous functions $f \in C[0,1]$ by polynomials $p \in P_n$ of degree $\leq n$ with respect to the L_1-norm [3]

$$\|f\|_1 :\equiv \int_0^1 |f(x)|dx.$$

In [15], the first author showed how a quite general class of (non-effective) proofs of uniqueness theorems in analysis can be analyzed such that an effective so-called modulus of uniqueness can be extracted which generalises the concept of *strong unicity* [4]. In [15] and [16] this technique has been applied to the case of best Chebycheff approximation yielding new uniform bounds on constants of strong unicity and a new quantitative version of the alternation theorem. In this paper we apply this logical approach to investigate the quantitative rate of strong unicity for the quite different case of best L_1-approximation. Like Chebycheff approximation, L_1-approximation, also called 'approximation in the mean', is a classical topic in numerical mathematics and was considered already by Chebycheff in 1859 and has been investigated ever since (see [26] for a comprehensive survey). The uniqueness of the best L_1-approximation of $f \in C[0,1]$ by polynomials of degree $\leq n$ was first proved in [10]. This proof uses measure theoretic arguments. A new uniqueness proof which avoids this and only uses the Riemann integral instead was given in 1965 by Cheney (see [6],[7]). Because of this feature, Cheney called his proof 'elementary'. From a logical point of view, however, it is highly non-constructive relying both on classical logic and non-computational analytical principles which correspond – in logical terminology – to the so-called binary ('weak') König's lemma, a principle which has received considerable attention in various parts of logic in recent years (see [27]). In this paper we carry out a complete logical analysis of Cheney's proof and show how the explicit modulus mentioned above can be extracted from this (seemingly) hopelessly non-

[2] See [15], [16], [19], [20] and [21] for other case studies as well as more information on proof mining in general.
[3] For $f \in L_1$ uniqueness in general fails.
[4] The term *strong unicity* was introduced by Newman and Shapiro [24] in 1963 and has been studied extensively in approximation theory. See e.g. the introduction in [2] and the references given there for a discussion of the crucial importance of estimates of strong unicity for the convergence analysis of iterative algorithms and for stability analysis.

constructive proof. Consequently, our result, like Cheney's proof, does not require any measure theory.

The main result of the present paper is the following effective strong uniqueness theorem:

Main result (Theorem 4.1) *Let* $\Phi(\omega, n, \varepsilon) :\equiv \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \frac{c_n\varepsilon}{2}\omega_{f,n}(\frac{c_n\varepsilon}{2})\}$, where

$$c_n :\equiv \frac{\lfloor n/2 \rfloor ! \lceil n/2 \rceil !}{2^{4n+3}(n+1)^{3n+1}} \text{ and } \omega_n(\varepsilon) :\equiv \min\{\omega(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4 \lceil \frac{1}{\omega(1)} \rceil}\}.$$

The functional Φ *is a uniform modulus of uniqueness for the best* L_1*-approximation of any function* f *in* $C[0,1]$ *having modulus of uniform continuity* ω *from* P_n*, i.e.*

$$\begin{cases} \forall n \in \mathbb{N}; p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}^*_+ \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) < \Phi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_1 \le \varepsilon \Big), \end{cases}$$

where $dist_1(f, P_n) :\equiv \inf_{p \in P_n} \|f - p\|_1$ *and* $\omega : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ *is a modulus of uniform continuity for* $f \in C[0,1]$ *if* [5]

$$\forall x, y \in [0,1]; \varepsilon \in \mathbb{Q}^*_+ (|x - y| < \omega(\varepsilon) \to |f(x) - f(y)| < \varepsilon).$$

Moreover, this theorem can be proved in Heyting Arithmetic $\mathbf{HA}^\omega$ in all finite types, and consequently holds in constructive mathematics in the sense of Bishop. Such a "constructivization", however, is not necessary for the extraction of Φ which is done from the ineffective proof. In fact, our verification of Φ is also done in $\mathbf{E\text{-}PA}^\omega + \mathbf{WKL}$. The fact that Φ can be verified in $\mathbf{HA}^\omega$ then follows from a conservation result due to the first author.

The technical details of this analysis are mainly due to the second author who is using the results in a subsequent paper [25] to determine a complexity upper bound for the sequence $(p_{b,n})_{n \in \mathbb{N}}$ of best approximating polynomials for poly-time computable functions $f \in C[0,1]$ (in the sense of [11], [12]).

[5] Note that this notion – used also in constructive mathematics and computable and feasible analysis – differs from the concept of modulus of continuity used in numerical analysis which we will discuss further below.

Before going into the details of the analysis we need to recall some general logical background from [15] [6]. First we introduce a little amount of logical terminology:

Let $\mathcal{A}^\omega$ be a (sub-)system of classical arithmetic in all finite types (like $\mathbf{E\text{-}PA}^\omega$ from [28] or Feferman's fragment $\mathbf{E\text{-}PRA}^\omega$ with quantifier-free induction and primitive recursion on the type 0 only [8]). Let $\mathcal{A}^\omega_*$ denote the extension of $\mathcal{A}^\omega$ by the schema

$$\mathbf{QF\text{-}AC} \quad : \quad \forall f^1 \exists x^0 A_{qf}(f,x) \rightarrow \exists F^2 \forall f^1 A_{qf}(f, F(f))$$

of quantifier-free choice from functions to numbers (where A_{qf} is quantifier-free) plus certain analytical principles Γ which – described in analytical terms – correspond to applications of Heine-Borel compactness of e.g. $[0,1]^d$. In logical terms, these principles correspond to the so-called binary ('weak') König's lemma **WKL** which suffices to derive a substantial amount of mathematics relative to weak fragments of arithmetic (see [27]) [7]. In this paper the only genuine analytical tool Γ (which goes beyond $\mathbf{E\text{-}PA}^\omega + \mathbf{QF\text{-}AC}$) is the attainment of the infimum of continuous functions on compact intervals

$$\forall f \in C[0,1] \exists x \in [0,1] \Big(f(x) = \inf_{y \in [0,1]} f(y) \Big). \tag{1}$$

(1) is known to fail in computable analysis and even for poly-time computable f there will be in general no computable $x \in [0,1]$ satisfying (1) (see [12]) [8].

Now, let X be a Polish space, K a compact Polish space and $F : X \times K \rightarrow \mathbb{R}$ a continuous function (moreover all these objects have to be explicitly representable in $\mathcal{A}^\omega$) and assume that we can prove in $\mathcal{A}^\omega_*$ that for every $f \in X$, $F(f, \cdot)$ has at most one root in K, i.e. [9]

$$\forall f \in X \forall x_1, x_2 \in K \Big(\bigwedge_{i=1}^{2} F(f, x_i) = 0 \rightarrow x_1 = x_2 \Big).$$

[6] Readers only interested in the numerical results but not in the general process of proof mining might skip this passage.

[7] $\mathbf{E\text{-}PRA}^\omega + \mathbf{QF\text{-}AC} + \mathbf{WKL}$ is a finite type extension of the system $\mathbf{WKL}_0$ used in reverse mathematics and is (like the latter) Π^0_2-conservative over primitive recursive arithmetic **PRA** (see [1], [14]).

[8] The principle (1) is known to be equivalent to **WKL** over systems like $\mathbf{E\text{-}PRA}^\omega +$ **QF-AC** even when f is given together with a modulus of uniform continuity, see [27].

[9] We may even have functions $F : X \times Y \rightarrow \mathbb{R}$, where X, Y are general Polish spaces and can allow constructively definable families $(K_f)_{f \in X}$ of compact subspaces of Y which are parametrised by $f \in X$ instead of a fixed K. See [15] for details.

Then by a general logical meta-theorem proved in [15] (Theorem 4.3) one can extract from such a proof an explicit bound $\Phi(f,k)$ (given by a closed term of the underlying arithmetical system $\mathcal{A}^\omega$) such that

$$\begin{cases} \forall f \in X \forall k \in \mathbb{N} \forall x_1, x_2 \in K \\ \quad \Big(\bigwedge\limits_{i=1}^{2} (|F(f,x_i)| < 2^{-\Phi(f,k)}) \to d_K(x_1,x_2) < 2^{-k} \Big), \end{cases} \tag{2}$$

where d_K denotes the metric on K. Moreover, (2) can be proved without using **WKL** and even in the intuitionistic variant $\mathcal{A}_i^\omega$ of $\mathcal{A}^\omega$ (and hence in constructive analysis in the sense of Bishop).

The proof of this meta-theorem provides an algorithm for actually extracting Φ. This algorithm is based on the proof-theoretic technique of monotone functional interpretation [17]. It is important to note that $\Phi(f,k)$ does **not** depend on $x_1, x_2 \in K$. Because of this fact, $\Phi(f,k)$ – which we call a **modulus of uniqueness** – can be used to compute the unique root (if existent) from any algorithm $\Psi(f,k)$ computing approximate so-called $\varepsilon(=2^{-k})$-roots of $F(f,\cdot)$:

$$\forall f \in X \forall k \in \mathbb{N} \Big(\Psi(f,k) \in K \wedge |F(f,\Psi(f,k))| < 2^{-k} \Big). \tag{3}$$

One easily verifies that (2) and (3) imply that $\Psi(f,\Phi(f,k))$ is a Cauchy sequence in K which converges with rate of convergence 2^{-k} to the unique root $x \in K$ of $F(f,\cdot)$. So $x = \lim_{k\to\infty} \Psi(f,\Phi(f,k))$ can be computed with arbitrarily prescribed precision (which can also be proved in $\mathcal{A}_i^\omega$, see [15], Theorem 4.4) and the computational complexity of x can be estimated in terms of the complexities of Φ and Ψ (cf. [25]).

Remark 1.1 (Important!) *As usual in computable analysis (see [29]), the functionals $\Phi(f,k)$ and $\Psi(f,k)$ will depend not only on $f \in X$ in the set theoretic sense but on a (computationally meaningful) representation of f. In the case of $f \in C[0,1]$, the representation of $C[0,1]$ as a Polish space $(C[0,1], \|\cdot\|_\infty)$ in $\mathcal{A}^\omega$ requires that f is endowed with a modulus of uniform continuity ω_f. So when we write $\Phi(f,k)$ we tacitly understand that f is given as a pair (f,ω_f). Actually, it now suffices to use the restriction f_r of f to the rational numbers in $[0,1]$ (which can be enumerated so that f_r can be represented as a number theoretic function), since f can be reconstructed from f_r with the help of ω_f. In this way, the representation (f_r,ω_f) of f can be viewed as an object of type 1 so that computability on f reduces to the well-known type-2 notion of computability (see again [29] for more information on this).*

Let us now move to the case of best L_1-approximation treated in the present paper. The uniqueness of the best approximation can be written as follows

$$\begin{cases} \forall n \in \mathbb{N} \forall f \in C[0,1] \forall p_1, p_2 \in P_n \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 = dist_1(f, P_n)) \to p_1 = p_2 \Big). \end{cases} \tag{4}$$

Note that in (4) we can without loss of generality replace the non-compact subspace P_n of $C[0,1]$ with the compact one $\tilde{K}_{f,n} :\equiv \{p \in P_n : \|p\|_1 \leq 2\|f\|_1\}$ since any best approximation p_b has to satisfy $\|f - p_b\|_1 \leq \|f\|_1$ because otherwise the zero polynomial would be a better approximation. As a consequence of this, $dist_1(f, P_n) = dist_1(f, \tilde{K}_{f,n})$ can easily be seen to be computable (uniformly in f as represented above and n). We use the slightly larger space $K_{f,n} :\equiv \left\{p \in P_n : \|p\|_1 \leq \frac{5}{2}\|f\|_1\right\}$ in (4) since a modulus of uniqueness for $K_{f,n}$ can be extended to P_n in a particular convenient way.

In this paper we analyze the above mentioned proof of Cheney for (4) as given in [6],[7] [10] which uses the non-computational principle (1) (together with classical logic) but which can be formalized in $\mathcal{A}^\omega_*$ (as was shown in [13]). So the above mentioned result on the extractability of a modulus of uniqueness is applicable, i.e. the extractability of a (primitive recursive in the sense of Gödel's T) functional Φ satisfying

$$\begin{cases} \forall n, k \in \mathbb{N} \forall f \in C[0,1] \forall p_1, p_2 \in K_{f,n} \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) < 2^{-\Phi(f,n,k)}) \to \|p_1 - p_2\|_1 < 2^{-k} \Big) \end{cases} \tag{5}$$

is guaranteed. Moreover, a simple trick (used also in [15] in the Chebycheff case) allows to replace $K_{f,n}$ with P_n so that

$$\begin{cases} \forall n, k \in \mathbb{N} \forall f \in C[0,1] \forall p_1, p_2 \in P_n \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) < 2^{-\Phi(f,n,k)}) \to \|p_1 - p_2\|_1 < 2^{-k} \Big) \end{cases}$$

Remark 1.2 *Markov inequality states that for any polynomial p of degree $\leq n$, $\|p'\|_\infty \leq 2n^2\|p\|_\infty$, where p' denotes the first derivative of p. Using this inequality one can show that for any polynomial $p \in P_n$, $\|p\|_\infty \leq 2(n+1)^2\|p\|_1$. Hence, any upper bound on $\|p_1 - p_2\|_1$ gives also an upper bound on $\|p_1 - p_2\|_\infty$ and we can use this to get a bound on the coefficients of $p_1 - p_2$. Namely, if*

[10] This result was first proved in [10] and is also called Jackson's Theorem. Cheney's proof (which applies to arbitrary Chebycheff systems) is a simplification of Jackson's proof.

$p_1(x) - p_2(x) = a_n x^n + \ldots + a_1 x + a_0$ and $\|p_1 - p_2\|_1 < M$ then $|a_i| \leq \frac{(2(n+1)^2)^{i+1}}{i!} M$. We present the complete proof in Section 3.5.

The importance of the modulus of uniqueness $\Phi(f, k)$ can also be illustrated by the fact that $\Phi + 1$ is automatically a modulus of pointwise continuity for the operator which maps $f \in X$ to its unique best approximation $f_b \in E \subset X$ (see [15]). For the special cases of Chebycheff resp. L_1-approximation this was shown first in [7] resp. [3]. Therefore,

$$\begin{cases} \forall n, k \in \mathbb{N} \forall f, \tilde{f} \in C[0,1] \\ \quad \left(\|f - \tilde{f}\|_1 < 2^{-\Phi(f,n,k)-1} \to \|\mathcal{P}(f,n) - \mathcal{P}(\tilde{f},n)\|_1 < 2^{-k} \right), \end{cases}$$

where $\mathcal{P}(f, n)$ is the unique best L_1-approximation of $f \in C[0,1]$ from P_n.

Since $(C[0,1], \|\cdot\|_1)$ is not a Polish space we have to represent $C[0,1]$ as the space $(C[0,1], \|\cdot\|_\infty)$ to apply the logical meta-theorem mentioned above. As we discussed already, this amounts to enriching the input f by a modulus of uniform continuity ω_f so that Φ will also depend on ω_f.

Note that if $C[0,1]$ is replaced by the (pre-)compact (w.r.t. $\|\cdot\|_\infty$) set $\mathcal{K}_{\omega,M}$ of all functions $f \in C[0,1]$ which have the common modulus of uniform continuity ω and the common bound $\|f\|_\infty \leq M$, then the same logical meta-theorem guarantees the extractability of a modulus of uniqueness Φ which only depends on $\mathcal{K}_{\omega,M}$ i.e. on ω, M (in addition to n, k). Moreover, even the M-dependency can be eliminated as the approximation problem for f can be reduced to that for $\tilde{f}(x) :\equiv f(x) - f(0)$ so that only a bound $N \geq \sup_{x \in [0,1]} |f(x) - f(0)|$ is required, which can easily be computed from ω (e.g take $N :\equiv \lceil \frac{1}{\omega(1)} \rceil$). Therefore, from the logical meta-theorem and the fact that Cheney's proof can be formalized in $\textbf{E-PA}^\omega + \textbf{QF-AC} + \textbf{WKL}$ we obtain already the extractability of a primitive recursive (in the sense of Gödel's T) modulus of uniqueness Φ which only depends on ω_f, n and k: **a-priori information.** Of course, only the actual extraction of Φ by applying the algorithm provided by the logical meta-theorem gives the detailed mathematical form of Φ as presented above: **a-posteriori information.**

2 Analysing proofs in analysis

The algorithm to be used for proof mining applied in cases like Cheney's proof of Jackson's Theorem (as treated in this paper) is based on the proof theoretic technique of monotone functional interpretation combined with negative translation as developed in [17]. Whereas the meta-mathematical details of this process are of importance to establish general meta-theorems on proof mining, this is not necessary for applications to specific proofs since here all numerical data will explicitly be exhibited and verified. This is because monotone functional interpretation explicitly transforms a given proof into another numerically enriched proof (in the normal mathematical sense). It is the strategy to find that proof (and to guarantee its existence) which is provided by the logical technique.

To approach the problem of proof mining applied to a logically involved proof as Cheney's, one starts off by splitting the proof into small pieces which are analyzed separately. As a consequence of the modularity of monotone functional interpretation one can easily combine the results obtained from the analysis of the pieces into a global result (this only requires functional application and λ-abstraction). Applications of monotone functional interpretation to the lemmas in the given proof at hand consist mostly of two steps,

1) transforming a given lemma L into a variant L^* which has the form

$$\forall n \in \mathbb{N} \forall x \in X \forall y \in K \exists k A_1(n, x, y, k), \tag{6}$$

where X is a Polish space, K a compact Polish space and $A_1 \in \Sigma_1^0$, and

2) extracting a bound $\Phi(n, x)$ for k which is independent of y.

It turns out that all the main lemmas to be analyzed have the form of (6). Because of this it is worthwhile to formulate the application of monotone functional interpretation to lemmas of this form as a special meta-theorem (2.1 below) which allows us to avoid having to go into the details of the underlying mechanism of functional interpretation each time. Although in the following we perform the transformation $L \mapsto L^*$ "by hand" one should note that this transformation is also usually automatically provided by functional interpretation.

Theorem 2.1 ([15], Theorem 4.1) *Let X, K be $\mathcal{A}^\omega$-definable Polish spaces, K compact and consider a sentence which can be written (when formalized in the language of $\mathcal{A}^\omega$) in the form*

$$A :\equiv \forall n \in \mathbb{N} \forall x \in X \forall y \in K \exists k \in \mathbb{N}\, A_1(n, x, y, k),$$

where A_1 is a purely existential. Then the following rule holds:[11]

$$\begin{cases} \mathcal{A}^\omega_* \vdash \forall n \in \mathbb{N} \forall x \in X \forall y \in K \exists k \in \mathbb{N}\, A_1(n, x, y, k) \\ \textit{then one can extract an } \mathcal{A}^\omega\textit{-definable functional } \Phi \textit{ s.t.} \\ \mathcal{A}^\omega_i \vdash \forall n \in \mathbb{N} \forall x \in X \forall y \in K \exists k \leq \Phi(n, x)\, A_1(n, x, y, k). \end{cases}$$

[11] As the theorem shows the conclusion can be proved already in $\mathcal{A}^\omega_i$ instead of $\mathcal{A}^\omega_*$. This, however, is not important for the applied aspect of the present paper where only the construction of Φ matters.

In particular, if

$$\mathcal{A}_i^\omega \vdash (k \leq \tilde{k} \wedge A_1(n,x,y,k)) \to A_1(n,x,y,\tilde{k})$$

then

$$\mathcal{A}_i^\omega \vdash \forall n \in \mathbb{N} \forall x \in X \forall y \in K\, A_1(n,x,y,\Phi(n,x)).$$

Again it is important to note that Φ does not depend on $y \in K$ [12] *.*

It is important to observe that real numbers are represented as Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ of rational number with fixed rate of convergence (say 2^{-n}) i.e. $\forall k, \tilde{k} \geq n(|a_k - a_{\tilde{k}}| \leq 2^{-n})$. In this way, equality $=_\mathbb{R}$ (similarly $\leq_\mathbb{R}$ and $\geq_\mathbb{R}$) between real numbers is a $\forall$-statement (for any point $k+1$ in the Cauchy sequence the approximants are close by 2^{-k}) and strict inequality $<_\mathbb{R}$ is a $\exists$-statement (there exists a point $k+1$ in the sequence such that the approximants are distant by 2^{-k}). We call those: 'hidden quantifiers'. For example, let $a, b \in \mathbb{R}$, then $a <_\mathbb{R} b$ is an abbreviation for $\exists k \in \mathbb{N}(a_{k+1} + 2^{-k} <_\mathbb{Q} b_{k+1})$. When observing whether a lemma has the logical form of A above also the hidden quantifiers have to be taken into consideration. We can, however, avoid going into the representation of the real numbers by observing that $a <_\mathbb{R} b$ can be written either as $\exists r \in \mathbb{Q}_+^*(a <_\mathbb{R} b + r)$ or $\exists r \in \mathbb{Q}_+^*(a \leq_\mathbb{R} b + r)$. The idea is that, if $a <_\mathbb{R} b$ occurs positively we write it as $\exists r \in \mathbb{Q}_+^*(a <_\mathbb{R} b + r)$ and if it occurs negatively we write it as $\exists r \in \mathbb{Q}_+^*(a \leq_\mathbb{R} b + r)$, in this way after prenexing these quantifiers the matrix is purely existential and (given that the prenexed quantifiers have a $\forall\exists$ form as described in Theorem 2.1) we can apply our meta-theorem 2.1.

Moreover, the extractability of a Φ such that (5) holds can be also justifying by an application of the meta-theorem above. We just have to write (4) (after presenting the hidden quantifiers) as,

$$\begin{cases} \forall n \in \mathbb{N}; f \in C[0,1]; p_1, p_2 \in K_{f,n}; k \in \mathbb{N} \exists l \in \mathbb{N} \\ \quad \Big(\bigwedge_{i=1}^2 \|f - p_i\|_1 \leq dist_1(f, P_n) + 2^{-l} \to \|p_1 - p_2\|_1 < 2^{-k} \Big), \end{cases}$$

which has the form A above. In [13] it is shown that Cheney's proof can be formalized in the system **E-PA**$^\omega$+**QF-AC**+**WKL**, and since (as we will show) $K_{f,n}$ can be replaced by P_n the functional Φ realizing $\exists l$ in the formula above is in fact a uniform modulus of uniqueness for L_1-approximation of functions in $C[0,1]$ by polynomials in P_n. Therefore, from the meta-theorem 2.1 and previous discussions we obtain the following corollary (see [15], Theorems 4.1 and 5.1).

[12] Recall that $\Phi(n,x)$ will depend on the representation of $x \in X$.

Corollary 2.1 *A functional $\Phi(f,n,k)$ given by a closed term of* **E-PA**$^\omega$ *(i.e. a primitive recursive functional Φ in the sense of Gödel [9]) can be extracted from Cheney's proof of Jackson's Theorem so that,*

$$\begin{cases} \textbf{(E-)HA}^\omega \vdash \forall n \in \mathbb{N}; f \in C[0,1]; p_1, p_2 \in P_n; k \in \mathbb{N} \\ \quad \Big(\bigwedge\limits_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n)) \leq 2^{-\Phi(f,n,k)} \to \|p_1 - p_2\|_1 < 2^{-k} \Big). \end{cases}$$

Moreover, using the Φ above, a primitive recursive functional Ψ can be constructed such that,

$$\begin{cases} \textbf{(E-)HA}^\omega \vdash \forall n \in \mathbb{N}; f \in C[0,1] \\ \quad \Big(\Psi(f,n) \in P_n \wedge \|f - \Psi(f,n)\|_1 = dist_1(f, P_n) \Big). \end{cases}$$

In this paper we carry out the extraction of a modulus of uniqueness Φ from Cheney's proof of Jackson's theorem. We shall try to keep as separate as possible the mathematical and the logical parts of the analysis. Readers interested in the mathematical results can focus upon the claims together with their proofs. Meanwhile, for readers interested in the process of proof mining we try to explain how the various steps in our concrete 'mining' correspond to steps in the monotone functional interpretation (as used in the general meta-theorems). Those explanations usually precede the treatment of each lemma. This is important to serve the twofold goal of this paper, namely not only to prove new quantitative results in L_1-approximation theory but also to get further insights into the process of proof mining in general.

3 Analysis of Cheney's proof of Jackson's theorem

3.1 Logical preliminaries on Cheney's proof

In this section we sketch how a slight modification of Cheney's proof can be seen to be formalizable in basic arithmetic like $\mathcal{A}^\omega :\equiv$ **E-PA**$^\omega$ plus the already mentioned analytical principle (1), i.e. **WKL**. The only part of the proof which cannot be directly formalized in $\mathcal{A}^\omega$ is the so-called 'Lemma 1' (see [7], p. 219) which reads as follows

Lemma 3.1 ([7], Lemma 1) *Let $f, h \in C[0,1]$. If f has at most finitely many roots and if $\int_0^1 h\, sgn(f) \neq 0$, then for some $\lambda \in \mathbb{R}$, $\int_0^1 |f - \lambda h| < \int_0^1 |f|$,* *where*

$$sgn(f)(x) \stackrel{\mathbb{N}}{=} \begin{cases} 1, \ \textit{if} \ f(x) >_{\mathbb{R}} 0 \\ 0, \ \textit{if} \ f(x) =_{\mathbb{R}} 0 \\ -1, \ \textit{if} \ f(x) <_{\mathbb{R}} 0. \end{cases}$$

In the context of the Cheney's proof of Jackson's theorem, h will be a polynomial in P_n. Moreover, it will be shown that if f (for the particular f at hand) has only less than $n+1$ roots one can construct an h such that $\int_0^1 h \, sgn(f) \neq 0$. So we only need the lemma with the stronger assumption that f has fewer than $n+1$ roots. The existence of $sgn(f)$ relies on the existence of the characteristic function $\chi_{=_{\mathbb{R}}}$ for equality between reals which in turn is equivalent to the existence of Feferman's ([8]) non-constructive μ-operator (see [18]) and hence to a strong form of arithmetical comprehension which is not available in $\mathcal{A}^\omega_* :\equiv \mathcal{A}^\omega + \mathbf{WKL}$. However, the use of sgn can be eliminated as follows: if f has less than $n+1$ roots then there exist points $x_0 < \ldots < x_{n+1}$ in $[0,1]$ (where $x_0 = 0$ and $x_{n+1} = 1$) which contain all the roots of f. By classical logic and induction one shows in $\mathcal{A}^\omega$ the existence of a vector $(\sigma_1, \ldots, \sigma_{n+1}) \in \{-1, 1\}^{n+1}$ such that

$$\sigma_i =_0 \begin{cases} 1, \ \text{if } f \text{ is positive on } (x_{i-1}, x_i), \\ -1, \ \text{if } f \text{ is negative on } (x_{i-1}, x_i) \end{cases}$$

for $i = 1, \ldots, n+1$. Therefore, $\int_0^1 h \, sgn(f)$ can be written as $\sum_{i=1}^{n+1} \sigma_i \int_{x_{i-1}}^{x_i} h$. In Section 3.10 we shall see that this reformulation of Lemma 1 plays a crucial role in the analysis of Cheney's proof. Monotone functional interpretation of (the negative translation of) our version of Lemma 1 will automatically introduce the main notion needed for the quantitative analysis of the proof, namely the concept of so-called 'r-clusters of δ-roots'. This concept, furthermore, is the key for the elimination of the use of (1) (i.e. **WKL**) on which Cheney's proof of Lemma 1 relies [13].

3.2 Analysing the structure of the proof

The main goal of the paper is to extract from Cheney's proof [7] of Jackson's theorem [10] an effective modulus of uniqueness which can be used, as it will be shown in Section 5, to compute the best L_1-approximation, p_b, from P_n of a given function $f \in C[0,1]$ with arbitrary precision [14]. In order to carry

[13] It is the argument that 'δ', in the middle of page 219 in [7], is strictly positive which uses (1). See Section 3.10 and Remark 3.10.3 for more information.
[14] P_n is a Haar subspace of $C[0,1]$ of dimension $n+1$.

out the analysis we need to formalize Cheney's proof. The first step we take in this direction is to list the main formulas used in the proof and to show how they are combined into lemmas. As mentioned before, each lemma will be analyzed separately. The functional interpretation of the lemma shows which functionals *can be* extracted from the proof of the lemma. But not all the functionals need to be presented, since some of them will disappear in the analysis of the proof (see the treatment of modus pones in the soundness of functional interpretation, e.g. in [17]). By analyzing the structure of the whole proof we can see which functionals are relevant and *need to be* extracted in order to obtain the final result. Then we construct such functionals and prove that they realize the lemma. In Section 4 we show how the final modulus Φ is obtained by combining these functionals.

In the propositions $A - K$ below we omitted the parameters f, n, p_1 and p_2, therefore, instead of A one should read $A(f, n, p_1, p_2)$, where n ranges over $\mathbb{N}$, $f \in C[0,1]$ and $p_1, p_2 \in P_n$, and the same holds for all the others propositions. We also use here and for the rest of this paper the defined functions $p(x) :\equiv \frac{p_1(x)+p_2(x)}{2}$ and $f_0(x) :\equiv f(x) - p(x)$ as shorthand notation. In the formulas and in the sketch of the proof presented below we use $\overline{x} :\equiv x_1, \ldots, x_n$ and $\overline{\sigma} :\equiv \sigma_1, \ldots, \sigma_{n+1}$. The following formulas are used in Cheney's proof:

$A :\equiv \bigwedge_{i=1}^{2}(\|f - p_i\|_1 - dist_1(f, P_n) = 0)$, i.e.
p_1 and p_2 are best L_1-*approximations* of f from P_n.
$B :\equiv \|f - p\|_1 - dist_1(f, P_n) = 0$, i.e. p is a best L_1-*approximation* of f.
$C :\equiv \|f_0\|_1 = \frac{1}{2}\|f - p_1\|_1 + \frac{1}{2}\|f - p_2\|_1$.
$C_1 :\equiv \forall \varepsilon \in \mathbb{Q}_+^* \exists \delta \in \mathbb{Q}_+^* \forall x, y \in [0,1](|x - y| < \delta \to |g(x) - g(y)| < \varepsilon)$,
where $g(x) :\equiv |f_0(x)| - \frac{1}{2}|f(x) - p_1(x)| - \frac{1}{2}|f(x) - p_2(x)|$.
The formula C_1 states that g is uniformly continuous.
$D :\equiv \forall x \in [0,1](|f_0(x)| = \frac{1}{2}(|f(x) - p_1(x)| + |f(x) - p_2(x)|))$.
$E :\equiv \exists x_0, \ldots, x_n \in [0,1]\Big(\bigwedge_{i=0}^{n} f_0(x_i) = 0 \wedge \bigwedge_{i=1}^{n} x_{i-1} < x_i\Big)$, i.e.
f_0 has at least $n+1$ distinct roots.
$F :\equiv \exists x_0, \ldots, x_n \in [0,1]\Big(\bigwedge_{i=0}^{n} p_1(x_i) = p_2(x_i) \wedge \bigwedge_{i=1}^{n} x_{i-1} < x_i\Big)$, i.e.
$p_1 - p_2$ has at least $n+1$ distinct roots.
$G :\equiv \forall x \in [0,1](p_1(x) = p_2(x))$, alternatively, $\|p_1 - p_2\|_1 = 0$ or $p_1 = p_2$.
$H(h) :\equiv \|f_0 - h\|_1 \geq \|f_0\|_1$.
$I(\overline{x}, \overline{\sigma}, h) :\equiv \sum_{i=1}^{n+1} \sigma_i \int_{x_{i-1}}^{x_i} h(x)dx > 0$, where $x_0 :\equiv 0$ and $x_{n+1} :\equiv 1$.
$J(\overline{x}) :\equiv \exists y \in [0,1](f_0(y) = 0 \wedge \bigwedge_{i=0}^{n+1} x_i \neq y)$, where $x_0 :\equiv 0$ and $x_{n+1} :\equiv 1$.
$K :\equiv \forall x \in [0,1](f_0(x) = 0 \to p_1(x) = p_2(x))$.

The first part of the proof (which we call derivation $\mathcal{D}_1$) is very simple and derives K from the assumption A,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A] \quad \dfrac{[A] \quad A \to B}{B}}{A \wedge B} \quad A\wedge B \to C}{C} \quad C_1}{C \wedge C_1} \quad C\wedge C_1 \to D}{D} \quad D \to K}{K}$$

The most involved part of the proof (which includes the application of Lemma 1) is when we want to prove that f_0 has $n+1$ distinct roots. In the derivations below we use $\overline{\sigma}' :\equiv \sigma'_1, \ldots, \sigma'_{n+1}$, where $\sigma'_i :\equiv sgn\,(f_0)(\frac{x_{i-1}+x_i}{2})$. Moreover, $\forall\overline{x} :\equiv \forall x_1 \leq \ldots \leq x_n$, where $\forall x_1 \leq \ldots \leq x_n\; Q(\overline{x})$ is an abreviation for $\forall x_1, \ldots, x_n\; (x_1 \leq \ldots \leq x_n \to Q(\overline{x}))$. Let the following derivation

$$\dfrac{\forall\overline{x},\overline{\sigma}\exists\tilde{h}_{\overline{x},\overline{\sigma}} I(\overline{x},\overline{\sigma},\tilde{h}_{\overline{x},\overline{\sigma}}) \quad \dfrac{\forall\overline{x},h\,(\forall\lambda H(\lambda h)\wedge I(\overline{x},\overline{\sigma}',h)\to J(\overline{x}))}{\forall\overline{x}\,(\forall\lambda H(\lambda\tilde{h}_{\overline{x},\overline{\sigma}'})\wedge I(\overline{x},\overline{\sigma}',\tilde{h}_{\overline{x},\overline{\sigma}'})\to J(\overline{x}))}}{\forall\lambda H(\lambda\tilde{h}_{\overline{x},\overline{\sigma}'})\to\forall\overline{x}J(\overline{x})}$$

be named $\mathcal{D}_2$. Using $\mathcal{D}_2$ from the assumption A we can derive that f_0 has $n+1$ distinct roots.

$$\dfrac{\dfrac{\dfrac{[A]\quad A\to B}{B}\quad B\to\forall h\,H(h)}{\forall h\,H(h)} \quad \dfrac{\mathcal{D}_2}{\forall\lambda H(\lambda\tilde{h}_{\overline{x},\overline{\sigma}'})\to\forall\overline{x}J(\overline{x})}}{\forall\overline{x}\,J(\overline{x})}$$

We call this derivation $\mathcal{D}_3$. An outline of the whole proof in the form of an informal natural deduction derivation is presented below,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{K}\quad\dfrac{\dfrac{\mathcal{D}_3}{\forall\overline{x}\,J(\overline{x})}\quad\forall\overline{x}\,J(\overline{x})\to E}{E}}{K\wedge E}\quad K\wedge E\to F}{F}\quad F\to G}{G}}{A\to G}\,[A]$$

Remark 3.1 *In general, we can only apply our meta-theorem 2.1 if P_n is replaced by $K_{f,n}$. As it happened, only in Section 3.5 this limitation really matters. Nonetheless, as we discussed already, at the end of the article we show that the final result actually holds for P_n.*

3.3 Lemma $A \to B$ [Triangle inequality]

The first lemma states,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n} \\ \quad \big(\bigwedge_{i=1}^{2} \|f - p_i\|_1 = dist_1(f, P_n) \to \|f - p\|_1 = dist_1(f, P_n)\big). \end{cases}$$

As described in the previous section, the first step is to present the hidden quantifiers,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n} \\ \quad \big(\forall \delta \in \mathbb{Q}_+^*(\bigwedge_{i=1}^{2} \|f - p_i\|_1 - dist_1(f, P_n) \leq \delta) \to \\ \qquad\qquad \forall \varepsilon \in \mathbb{Q}_+^*(\|f - p\|_1 - dist_1(f, P_n) < \varepsilon)\big). \end{cases}$$

Then we look at the functional interpretation of the lemma,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \varepsilon \in \mathbb{Q}_+^* \exists \delta \in \mathbb{Q}_+^* \\ \quad \big(\bigwedge_{i=1}^{2} \|f - p_i\|_1 - dist_1(f, P_n) \leq \delta \to \\ \qquad\qquad \|f - p\|_1 - dist_1(f, P_n) < \varepsilon\big). \end{cases} \tag{7}$$

We see now that (7) has the same structure as the formula A in Theorem 2.1. Therefore, we are sure to find a functional Φ_1, depending at most on n, f and ε, such that,[15]

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \varepsilon \in \mathbb{Q}_+^* \exists \delta \geq \Phi_1(f, n, \varepsilon) \\ \quad \big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) < \delta) \to \\ \qquad\qquad \|f - p\|_1 - dist_1(f, P_n) < \varepsilon\big). \end{cases} \tag{8}$$

Since we have monotonicity in δ the functional Φ_1 actually realizes δ. The same phenomenon will happen in all the following lemmas, i.e. the lower bounds will always be realizing functionals for the variables they bound. Here, it is obvious how to construct Φ_1,

[15] Since in Theorem 2.1 we used 2^{-k} (with $k \in \mathbb{N}$) instead of $\delta \in \mathbb{Q}_+^*$, the upper bound on k guaranteed by the meta-theorem gives a lower bound on δ.

Claim 3.1 *The functional* $\boxed{\Phi_1(f,n,\varepsilon) :\equiv \Phi_1(\varepsilon) :\equiv \varepsilon}$ *does the job*[16] .

Proof. Suppose (i) $\|f-p_1\|_1 - dist_1(f,P_n) < \varepsilon$ and (ii) $\|f-p_2\|_1 - dist_1(f,P_n) < \varepsilon$. Multiplying (i) and (ii) by $1/2$ and adding them together we get $1/2(\|f-p_1\|_1 + \|f-p_2\|_1) - dist_1(f,P_n) < \varepsilon$. By the triangle inequality for the L_1-*norm*, $1/2(\|2f-p_1-p_2\|_1) - dist_1(f,P_n) < \varepsilon$, i.e. $\|f-p\|_1 - dist_1(f,P_n) < \varepsilon$. □

Remark 3.2 *The reader may have noticed that from* (7) *to* (8) *we changed from* $\leq$ *to* $<$ *in the premise of the implication. The reason we wrote* $\leq$ *first was just to show that the lemma could be written in the form of A (from Theorem 2.1) and that a functional realizing* δ *was guaranteed by our meta-theorem. Since* $a \leq b/2$ *implies* $a < b$ *(and the reverse implication holds without the factor* $1/2$*) we normally write the relation that yields the optimal bound. When analysing the following lemmas we often claim that some sentence is an instance of our meta-theorem 2.1 without bothering to write it explicitly in the form of A. We hope the reader can see that through the implications mentioned above these lemmas could in fact be written in the form of A.*

3.4 Lemma $A \wedge B \to C$ [Basic norm property]

The lemma states,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n} \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f-p_i\|_1 = dist_1(f,P_n)) \to \\ \qquad\qquad \|f-p\|_1 - 1/2\|f-p_1\|_1 - 1/2\|f-p_2\|_1 = 0 \Big). \end{cases}$$

After presenting the hidden quantifiers and performing the functional interpretation we come again to the same logical structure of the formula in Theorem 2.1, and again we know that there must exist a functional Φ_2 depending at most on n, f and ε such that,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \varepsilon \in \mathbb{Q}_+^* \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f-p_i\|_1 - dist_1(f,P_n) < \Phi_2(f,n,\varepsilon)) \to \\ \qquad\qquad |\, \|f-p\|_1 - 1/2\|f-p_1\|_1 - 1/2\|f-p_2\|_1 \,| < \varepsilon \Big). \end{cases}$$

Again, the choice of Φ_2 is simple,

Claim 3.2 *The functional* $\boxed{\Phi_2(f,n,\varepsilon) :\equiv \Phi_2(\varepsilon) :\equiv \varepsilon}$ *does the job.*

[16] Note that in fact Φ_1 is independent of n and f. We adopt the convention that parameters not used in the definition of the functionals will be dropped.

Proof. Suppose (i) $\|f-p_1\|_1 - dist_1(f,P_n) < \varepsilon$ and (ii) $\|f-p_2\|_1 - dist_1(f,P_n) < \varepsilon$. By previous lemma we have (iii) $\|f-p\|_1 - dist_1(f,P_n) < \varepsilon$. And $\frac{(i)+(ii)}{2}$ gives (iv) $1/2(\|f-p_1\|_1 + \|f-p_2\|_1) - dist_1(f,P_n) < \varepsilon$. From (iii) and (iv), we have, $|\ \|f-p\|_1 - 1/2\|f-p_1\|_1 - 1/2\|f-p_2\|_1\ | < \varepsilon$, since if $a \in [0,m)$ and $b \in [0,m)$ then $|a-b| \in [0,m)$. $\square$

3.5 Lemma C_1 [Continuity of $g(x)$]

Let $g(x) :\equiv |f_0(x)| - \frac{1}{2}|f(x)-p_1(x)| - \frac{1}{2}|f(x)-p_2(x)|$. Based on the continuity of f, p_1 and p_2 we derive that g is continuous,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \varepsilon \in \mathbb{Q}^*_+; x, y \in [0,1] \exists \delta \in \mathbb{Q}^*_+ \\ \quad \Big(|x-y| \le \delta \to |g(x)-g(y)| < \varepsilon\Big). \end{cases}$$

Note that here we can again apply the meta-theorem 2.1 and we are sure to find a function Δ depending only f, n and ε such that,[17]

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \varepsilon \in \mathbb{Q}^*_+; x, y \in [0,1] \\ \quad \Big(|x-y| < \Delta(f,n,\varepsilon) \to |g(x)-g(y)| < \varepsilon\Big). \end{cases}$$

We write $\Delta(f,n,\varepsilon)$ as $\omega_{f,n}(\varepsilon)$. In this section we show how the modulus of continuity $\omega_{f,n}(\varepsilon)$ can be computed using only n, the modulus of continuity of f, ω_f, and an upper bound $M_f \ge \|f\|_\infty$ (in Section 4 we show that we just need a bound M_f on $\sup_{x\in[0,1]}|f(x)-f(0)|$, for instance $\lceil \frac{1}{\omega_f(1)} \rceil$, so that the final result only depends on ω_f and n). It should be noted that the modulus of continuity of a function is not unique, therefore when in the following we write $\omega_f(\varepsilon) :\equiv \ldots$ we mean that $\ldots$ can be taken as the modulus of continuity of the function f.

3.5.1 Modulus of the sum

Given the moduli of continuity ω_f and ω_g for the functions f and g respectively, we find the modulus of continuity for $f+g$, ω_{f+g}, in the following way. We have,

$$|x-y| < \omega_f(\varepsilon/2) \to |f(x)-f(y)| < \varepsilon/2.$$
$$|x-y| < \omega_g(\varepsilon/2) \to |g(x)-g(y)| < \varepsilon/2.$$

[17] Here it is fundamental that p_1 and p_2 live in the compact space $K_{f,n}$ otherwise the modulus of continuity for g would depend also on these elements and we would be unable to get a uniform modulus of uniqueness at the end.

Therefore,

$$|x-y| < \min\{\omega_f(\varepsilon/2), \omega_g(\varepsilon/2)\} \to \\ (|f(x)-f(y)| < \varepsilon/2 \wedge |g(x)-g(y)| < \varepsilon/2).$$
$$|x-y| < \min\{\omega_f(\varepsilon/2), \omega_g(\varepsilon/2)\} \to |f(x)+g(x)-f(y)-g(y)| < \varepsilon.$$

Hence, $\omega_{f+g}(\varepsilon) :\equiv \min\{\omega_f(\varepsilon/2), \omega_g(\varepsilon/2)\}$.

3.5.2 Modulus of a constant times a function

We show that $\omega_{af}(\varepsilon) :\equiv \omega_f(\frac{\varepsilon}{a})$. For all $a \in \mathbb{Q}_+^*$, if $|x-y| < \omega_f(\frac{\varepsilon}{a})$ then $|f(x)-f(y)| < \frac{\varepsilon}{a}$, and therefore, $|af(x)-af(y)| < \varepsilon$.

3.5.3 Modulus of p_1 and p_2

Let $p_i \in K_{f,n}$. Then $\|p_i\|_1 \leq \frac{5}{2}\|f\|_1 \leq \frac{5}{2}\|f\|_\infty$. If $p_i(x) = a_n x^n + \ldots + a_1 x + a_0$ and $p_i^*(x) = \frac{a_n x^{n+1}}{n+1} + \ldots + \frac{a_1 x^2}{2} + a_0 x$ then for all $x \in [0,1]$ we have,

$$|p_i^*(x)| = |\int_0^x p_i(x)dx| \leq \int_0^x |p_i(x)|dx \leq \|p_i\|_1 \leq \frac{5}{2}\|f\|_\infty,$$

i.e. $\|p_i^*\|_\infty \leq \|p_i\|_1 \leq \frac{5}{2}\|f\|_\infty$. By Markov inequality (see e.g. [7]),

$$\|p_i\|_\infty = \|(p_i^*)'\|_\infty \leq 2(n+1)^2\|p_i^*\|_\infty \leq 2(n+1)^2(\frac{5}{2}\|f\|_\infty) = 5(n+1)^2\|f\|_\infty.$$

If we apply Markov inequality once more we get,

$$\|p_i'\|_\infty \leq 2n^2 5(n+1)^2\|f\|_\infty < 10(n+1)^4\|f\|_\infty.$$

By the mean value theorem this implies that p_i has Lipschitz constant $10(n+1)^4\|f\|_\infty$ on $[0,1]$, i.e. $\frac{\varepsilon}{10(n+1)^4\|f\|_\infty}$ is a modulus of uniform continuity for p_i on $[0,1]$. Given an upper bound M_f on $\|f\|_\infty$ we have,[18]

$$\omega_{p_i}(\varepsilon) :\equiv \frac{\varepsilon}{10(n+1)^4 M_f}.$$

Remark 3.3 *Here we present how one gets a bound on the coefficients of p given $\|p\|_1$ (or some bound on $\|p\|_1$). Let p^i denote the i-th derivative of p. Above we have shown that $\|p\|_\infty \leq 2(n+1)^2\|p\|_1$ which by Markov inequality yields (+) $\|p^i\|_\infty \leq (2(n+1)^2)^{i+1}\|p\|_1$. Since $p^i(x) = \frac{n!}{(n-i)!}a_n x^{n-i} +$*

[18] It should be clear that given f together with its modulus of continuity, ω_f, there is a simple algorithm to compute M_f, just take for instance $M_f :\equiv \max\{|f(i.\omega_f(1))| : 0 \leq i \leq \lfloor\frac{1}{\omega_f(1)}\rfloor\} + 1$.

$\ldots + i!a_i$, *from* $(+)$ *we get* $|i!\ a_i| \leq (2(n+1)^2)^{i+1}\|p\|_1$ *which implies* $|a_i| \leq \frac{(2(n+1)^2)^{i+1}}{i!}\|p\|_1$.

3.5.4 The modulus of continuity $\omega_{f,n}$

Now we can present $\omega_{f,n}$ as a function of ω_f and n (note that we can take $\omega_{|f|} :\equiv \omega_f$),

$$\begin{aligned}\omega_{f,n}(\varepsilon) &= \min\{\omega_{|f-p|}(\varepsilon/2), \omega_{1/2|f-p_1|}(\varepsilon/4), \omega_{1/2|f-p_2|}(\varepsilon/4)\}\\ &= \min\{\omega_{f-p}(\varepsilon/2), \omega_{f-p_1}(\varepsilon/2), \omega_{f-p_2}(\varepsilon/2)\}\\ &= \min\{\omega_f(\varepsilon/4), \omega_{p_1}(\varepsilon/4), \omega_{p_2}(\varepsilon/4)\}\\ &= \min\{\omega_f(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4 M_f}\}.\end{aligned}$$

3.6 Lemma $C \wedge C_1 \to D$ *[Integrand is* ≤ 0 *and continuous]*

Let $g(x) :\equiv |f(x) - p(x)| - 1/2|f(x) - p_1(x)| - 1/2|f(x) - p_2(x)|$. The lemma says,

$$\forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}\Big(\int_0^1 g(x)dx = 0 \to \forall x \in [0,1](g(x) = 0)\Big).$$

After presenting the hidden quantifiers and applying functional interpretation we observe that again we can apply Theorem 2.1, and we are guaranteed to find a functional $\Phi_3(f, n, \varepsilon)$ such that,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \varepsilon \in \mathbb{Q}^*_+ \\ \quad \Big(|\int_0^1 g(x)dx| \leq \Phi_3(f,n,\varepsilon) \to \|g\|_\infty \leq \varepsilon\Big). \end{cases}$$

Let $\omega_{f,n} : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ denote the modulus of uniform continuity of the function $g \in C[0,1]$, proved to exist in the analysis of lemma C_1 (Section 3.5).

Claim 3.3 *The functional* $\boxed{\Phi_3(f,n,\varepsilon) :\equiv \Phi_3(\omega_{f,n},\varepsilon) :\equiv \frac{\varepsilon}{2}\cdot\min\{\frac{1}{2},\ \omega_{f,n}(\frac{\varepsilon}{2})\}}$ *does the job.*

Proof. Assume $\|g\|_\infty > \varepsilon$, since $\forall x \in [0,1](g(x) \leq 0)$ we conclude $\exists x_0 \in [0,1](g(x_0) \leq -\varepsilon)$. By the continuity of g we also have,

$$\forall x \in [0,1]\Big(|x - x_0| < \omega_{f,n}(\varepsilon/2) \to g(x) < -\varepsilon/2\Big).$$

If $x_0 < 1/2$ then,

$$|\int_0^1 g(x)dx| > |\int_{x_0}^{\min\{1,x_0+\omega_{f,n}(\varepsilon/2)\}} -\varepsilon/2\ dx| \geq \frac{\varepsilon}{2}\min\{\frac{1}{2},\omega_{f,n}(\frac{\varepsilon}{2})\},$$

otherwise ($x_0 \geq 1/2$),

$$|\int_0^1 g(x)dx| > |\int_{\max\{0,x_0-\omega_{f,n}(\varepsilon/2)\}}^{x_0} -\varepsilon/2\ dx| \geq \frac{\varepsilon}{2}\min\{\frac{1}{2},\omega_{f,n}(\frac{\varepsilon}{2})\}.$$

From this we conclude,

$$|\int_0^1 g(x)dx| > \frac{\varepsilon}{2}\min\{\frac{1}{2},\omega_{f,n}(\frac{\varepsilon}{2})\}. \qquad \square$$

3.7 Lemma $D \to K$ [If $f_0(x) = 0$ then $p_1(x) = p_2(x)$]

Let $f_1(x) :\equiv 1/2(|f(x) - p_1(x)| + |f(x) - p_2(x)|)$, the lemma says,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; x \in [0,1] \\ \quad \big(\|\,|f_0| - f_1\|_\infty = 0 \to (|f_0(x)| = 0 \to p_1(x) = p_2(x))\big). \end{cases}$$

Again we are sure to find functionals $\Phi_4(f,n,\varepsilon)$ and $\Phi_5(f,n,\varepsilon)$ such that,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; x \in [0,1]; \varepsilon \in \mathbb{Q}_+^* \\ \quad \big(\|\,|f_0| - f_1\|_\infty \leq \Phi_4(f,n,\varepsilon) \to \\ \qquad\qquad (|f_0(x)| \leq \Phi_5(f,n,\varepsilon) \to |p_1(x) - p_2(x)| \leq \varepsilon)\big). \end{cases}$$

Claim 3.4 *The functionals* $\boxed{\Phi_4(f,n,\varepsilon) :\equiv \Phi_4(\varepsilon) :\equiv \varepsilon/8}$ *and* $\boxed{\Phi_5(f,n,\varepsilon) :\equiv \Phi_5(\varepsilon) :\equiv \varepsilon/8}$ *do the job.*

Proof. Trivial. $\square$

3.8 Lemma $F \to G$ [If p has $n+1$ roots then $p = 0$]

The lemma states that if the polynomial $p_1(x) - p_2(x)$ has $n+1$ distinct roots in the interval $[0,1]$ then $p_1(x)$ and $p_2(x)$ are actually identical,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n} \forall x_0, \ldots, x_n \in [0,1] \\ \quad \big(\bigwedge_{i=1}^n (x_i < x_{i+1}) \wedge \bigwedge_{i=0}^n (p_1(x_i) = p_2(x_i)) \to \|p_1 - p_2\|_\infty = 0\big). \end{cases}$$

Then we present the hidden quantifiers and apply functional interpretation,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; r, \varepsilon \in \mathbb{Q}_+^*; x_0, \dots, x_n \in [0,1] \exists \delta \in \mathbb{Q}_+^* \\ \quad \Big(\bigwedge_{i=1}^n (x_{i-1} + r \le x_i) \wedge \bigwedge_{i=0}^n (|p_1(x_i) - p_2(x_i)| \le \delta) \to \|p_1 - p_2\|_\infty \le \varepsilon \Big). \end{cases}$$

By Theorem 2.1 we are sure to find a functional Φ_6 realizing δ.

Claim 3.5 *The functional* $\boxed{\Phi_6(f, n, r, \varepsilon) :\equiv \Phi_6(n, r, \varepsilon) :\equiv \frac{\lfloor n/2 \rfloor ! \lceil n/2 \rceil ! r^n}{(n+1)} \varepsilon}$ *does the job.*

Proof. See [15], pages 82–83. □

Remark 3.4 *In fact, the functional Φ_6 does the job for $p_1, p_2 \in P_n$ (not only for $p_1, p_2 \in K_{f,n}$).*

3.9 Lemma $B \to \forall h\ H(h)$ [Definition of best L_1-approximation]

This lemma is a trivial consequence of the definition of $dist_1$,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n} \\ \quad \big(\|f_0\|_1 = dist_1(f, P_n) \to \forall h \in P_n (\|f_0 - h\|_1 \ge \|f_0\|_1) \big). \end{cases}$$

We can easily find a functional $\Phi_7(f, n, \varepsilon)$ s.t.,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \varepsilon \in \mathbb{Q}_+^* \\ \quad \big(\|f_0\|_1 - dist_1(f, P_n) \le \Phi_7(f, n, \varepsilon) \to \\ \qquad\qquad \forall h \in P_n (\|f_0 - h\|_1 + \varepsilon \ge \|f_0\|_1) \big). \end{cases}$$

Claim 3.6 *The functional* $\boxed{\Phi_7(f, n, \varepsilon) :\equiv \Phi_7(\varepsilon) :\equiv \varepsilon}$ *does the job.*

Proof. Assume (i) $\|f_0\|_1 - dist_1(f, P_n) \le \varepsilon$. By the definition of $dist_1$ we have for any $h \in P_n$ (ii) $\|f_0 - h\|_1 = \|f - (p + h)\|_1 \ge dist_1(f, P_n)$. From (i) and (ii) we have $\|f_0 - h\|_1 + \varepsilon \ge \|f_0\|_1$. □

3.10 Lemma $\forall \overline{x}, h\ (\forall \lambda H(\lambda h) \wedge I(\overline{x}, \overline{\sigma}', h) \to J(\overline{x}))$ [Lemma 1]

This is the most intricate lemma used in the proof, hence we analyze it in greater detail. We first rewrite the lemma as it is stated in [7]. The contraposition of Lemma 1 is used in the proof.

Lemma 3.2 (Lemma 1) *Let $f \in C[0,1]$, $n \in \mathbb{N}$ and $h, p_1, p_2 \in P_n$. If f_0 has at most n roots then either $\int_0^1 (h(x)\, sgn(f_0)(x))dx = 0$ or there exists a $\lambda \in \mathbb{R}$ such that $\int_0^1 |f_0(x) - \lambda h(x)|dx < \int_0^1 |f_0(x)|dx$.*

Proof. Assume that all the roots of f_0 are among $0 = x_0 \le x_1 \le \ldots \le x_{n+1} = 1$ and w.l.g. assume that $\int_0^1 (h(x)\, sgn(f_0)(x))dx > 0$. Let $B' :\equiv \bigcup_{i=0}^{n+1}(x_i - r, x_i + r)$ and $B :\equiv B' \bigcap [0,1]$. Let $A :\equiv [0,1] \backslash B$. Make r small enough so that $\int_A (h(x)\, sgn(f_0)(x))dx > \int_B |h(x)|dx$. Note that A is a finite union of closed intervals which contain no roots of f_0, therefore $\delta :\equiv \min\{|f_0(x)| : x \in A\}$ is positive. Hence we can find a λ such that $0 < \lambda \|h\|_\infty < \delta$, and for points $x \in A$, $sgn(f_0 - \lambda h)(x) = sgn(f_0)(x)$, which implies (see [7] or the proof of Claim 3.7 for details) that $\int_0^1 |f_0(x) - \lambda h(x)|dx < \int_0^1 |f_0(x)|dx$. $\Box$

3.10.1 Logical analysis of Lemma 1

The Lemma 1 as it is presented above does not have the logical form to which we can apply the meta-theorem 2.1. We can, however, show that a variation of the Lemma 1, which can be used in Cheney's proof does have that logical form. Let $B' :\equiv \bigcup_{i=0}^{n+1}(x_i - r, x_i + r)$, $B :\equiv B' \bigcap [0,1]$ and $A :\equiv [0,1] \backslash B$, where $x_0 :\equiv 0$ and $x_{n+1} :\equiv 1$. Note that A can be written as the union of smaller intervals [19] $A_i :\equiv [x_{i-1} + \min\{r, \frac{x_i - x_{i-1}}{2}\}, x_i - \min\{r, \frac{x_i - x_{i-1}}{2}\}]$, for $1 \le i \le n+1$. For the rest of Section 3 we use x_0, x_{n+1}, A, B and A_i as defined above and we mention explicitly which r we are using when this is not clear from the context. The version of Lemma 1 we consider is: For all $f \in C[0,1]$ and $n \in \mathbb{N}$

$$\begin{cases} \forall p_1, p_2 \in P_n; x_1 \le \ldots \le x_n \in [0,1]; h \in P_n; r \in \mathbb{Q}_+^* \\ \quad \big(\forall y \in A(fy \neq 0) \wedge \int_A h\, sgn(f) > \int_B |h| \rightarrow \\ \qquad\qquad \exists \lambda \in \mathbb{R}(\|f - \lambda h\|_1 < \|f\|_1)\big) \end{cases} \tag{9}$$

where A, B depend on $x_1 \le \ldots \le x_n$ and r.

First we show how (9) can be used in Cheney's proof. Since f will be taken to be f_0 we can prove $\forall \lambda \in \mathbb{R}; h \in C[0,1](\|f_0 - \lambda h\|_1 \ge \|f_0\|_1)$ which leaves, for all $f \in C[0,1]$ and $n \in \mathbb{N}$

$$\begin{cases} \forall p_1, p_2 \in P_n; x_1 \le \ldots \le x_n \in [0,1]; h \in P_n; r \in \mathbb{Q}_+^* \\ \quad \big(\exists y \in A(f_0(y) = 0) \vee \int_A h\, sgn(f_0) \le \int_B |h|\big) \end{cases}$$

[19] Note that the intervals $\bigcup A_i$ and A only differ on at most a finite number of points. Clearly, however, the integrations $\sum \int_{A_i}$ and $\int_A$ coincide.

but we can easily prove

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in P_n; x_1 \leq \ldots \leq x_n \in [0,1] \\ \quad \exists h \in C[0,1]; r \in \mathbb{Q}_+^* \big(\forall y \in A(f_0(y) \neq 0) \to \int_A h\, sgn(f_0) > \int_B |h| \big) \end{cases}$$

from which we can obtain the existence of $n+1$ roots by induction.

Now we can replace P_n with $K_{f,n}$ in (9) and rewrite the integral of $h\, sgn(f_0)$ over the intervals A as a sum of integrals over smaller intervals A_i (which are guaranteed by the premise to contain no root of f_0) as described in Section 3.1. Hence Lemma 1 can be formally written as, for all $f \in C[0,1]$ and $n \in \mathbb{N}$

$$\begin{cases} \forall p_1, p_2 \in K_{f,n}; x_1 \leq \ldots \leq x_n \in [0,1]; h \in P_n; r \in \mathbb{Q}_+^* \\ \quad \big(\forall y \in A(f_0(y) \neq 0) \wedge \sum_{i=1}^{n+1} \sigma_i \int_{A_i} h > \int_B |h| \to \\ \qquad \qquad \exists \lambda \in \mathbb{R}(\|f_0 - \lambda h\|_1 < \|f_0\|_1) \big) \end{cases}$$

where $\sigma_i :\equiv sgn(f_0)(\frac{x_{i-1}+x_i}{2})$, $x_0 :\equiv 0$ and $x_{n+1} :\equiv 1$. Presenting the hidden quantifiers we obtain [20], for all $f \in C[0,1]$ and $n \in \mathbb{N}$

$$\begin{cases} \forall p_1, p_2 \in K_{f,n}; x_1 \leq \ldots \leq x_n \in [0,1]; h \in P_n; \delta, r, \eta \in \mathbb{Q}_+^* \; \exists l \in \mathbb{Q}_+^* \\ \quad \big(\forall y \in A(|f_0(y)| \geq \delta) \wedge \sum_{i=1}^{n+1} \sigma_i \int_{A_i} h \geq \int_B |h| + \eta \to \\ \qquad \qquad \exists \lambda \in \mathbb{R}(\|f_0 - \lambda h\|_1 + l < \|f_0\|_1) \big). \end{cases}$$

This last step can be viewed as a weakening of the Lemma 1 since we replace $\forall y \in A(f_0(y) \neq 0)$ by the stronger statement $\exists \delta \in \mathbb{Q}_+^* \forall y \in A(|f_0(y)| \geq \delta)$ in the premise. In view of **WKL**, however, we have that the above formula actually implies the original Lemma 1. Note that we can take $\eta = 1$ w.l.g. since $h/\eta \in P_n$. Hence, get for all $f \in C[0,1]$ and $n \in \mathbb{N}$

$$\begin{cases} \forall p_1, p_2 \in K_{f,n}; x_1 \leq \ldots \leq x_n \in [0,1]; h \in P_n; \delta, r \in \mathbb{Q}_+^* \; \exists l \in \mathbb{Q}_+^* \\ \quad \big(\forall y \in A(|f_0(y)| \geq \delta) \wedge \sum_{i=1}^{n+1} \sigma_i \int_{A_i} h \geq \int_B |h| + 1 \to \\ \qquad \qquad \exists \lambda \in \mathbb{R}(\|f_0 - \lambda h\|_1 + l < \|f_0\|_1) \big) \end{cases} \tag{10}$$

3.10.2 Functional realizing Lemma 1

By observing that (10) has (relative to $\mathbf{E\text{-}PA}^\omega$) the same logical form as the formula A in the meta-theorem 2.1 [21] we are sure to find a functional

[20] Using that by **WKL**, $\forall y \in A(f_0(y) \neq 0) \leftrightarrow \exists \delta \in \mathbb{Q}_+^* \forall y \in A(|f_0(y)| \geq \delta)$.

[21] Note that we can treat σ_i as $\forall \sigma_1, \ldots, \sigma_{n+1} \in \{-1, 1\}$ with the purely universal assumption

$\Phi_8(f, n, \delta, r, h)$ such that, for all $f \in C[0,1]$ and $n \in \mathbb{N}$

$$\begin{cases} \forall p_1, p_2 \in K_{f,n}; x_1 \leq \ldots \leq x_n \in [0,1]; h \in P_n; \delta, r \in \mathbb{Q}^*_+ \\ \quad \Big(\forall y \in A(|f_0(y)| > \delta) \wedge \sum_{i=1}^{n+1} \sigma_i \int_{A_i} h > \int_B |h| + 1 \rightarrow \\ \qquad\qquad \exists \lambda \in \mathbb{R}(\|f_0 - \lambda h\|_1 + \Phi_8(f, n, \delta, r, h) < \|f_0\|_1)\Big). \end{cases}$$

Claim 3.7 *The functional* $\boxed{\Phi_8(f, n, \delta, r, h) :\equiv \Phi_8(n, \delta, h) :\equiv \frac{\delta}{\|h\|_\infty}}$ *does the job.*

Proof. We have to prove that, for all $f \in C[0,1]$ and $n \in \mathbb{N}$

$$\begin{cases} \forall p_1, p_2 \in K_{f,n}; x_1 \leq \ldots \leq x_n \in [0,1]; h \in P_n; \delta, r \in \mathbb{Q}^*_+ \\ \quad \Big(\forall y \in A(|f_0(y)| > \delta) \wedge \sum_{i=1}^{n+1} \sigma_i \int_{A_i} h > \int_B |h| + 1 \rightarrow \\ \qquad\qquad \exists \lambda \in \mathbb{R}(\|f_0 - \lambda h\|_1 + \frac{\delta}{\|h\|_\infty} < \|f_0\|_1)\Big). \end{cases}$$

Let $f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; h \in P_n; \delta, r \in \mathbb{Q}^*_+$ be fixed. Note that now we not only require f_0 not to have roots in A but not even δ-roots (i.e. $|f_0(y)| > \delta$). As a consequence y has to be 'r-apart' from all x_i. We say that y does not belong to the (x_i, r)-clusters[22]. Now we follow the original proof. Take n points, $x_1, \ldots, x_n$, such that (i) $0 = x_0 \leq x_1 \leq \ldots \leq x_{n+1} = 1$ and suppose that (ii) all δ-roots of f_0 belong to at least one of the (x_i, r)-clusters. Moreover, suppose that (iii) $\sum_{i=1}^{n+1} \sigma_i \int_{A_i} h > \int_B |h| + 1$, where $\sigma_i = sgn(f_0)(\frac{x_{i-1}+x_i}{2})$. By assumption (ii) we have $\sigma_i = sgn(f_0)(x)$, for $x \in A_i$ and then $\sum_{i=1}^{n+1} \sigma_i \int_{A_i} h(x)\, dx = \int_A (h(x)\, sgn(f_0)(x))\, dx$. By (ii) we have $f_0(x) > \delta$ for all $x \in A$. Therefore, taking $\lambda :\equiv \frac{\delta}{\|h\|_\infty}$ we have (iv) $sgn(f_0 - \lambda h)(x) = sgn(f_0)(x)$, for $x \in A$. Hence,

---

$\bigwedge_{i=1}^{n+1} (\sigma_i = 1 \rightarrow sgn(f_0)(\frac{x_i - x_{i-1}}{2}) \geq 0 \wedge \sigma_i = -1 \rightarrow sgn(f_0)(\frac{x_i - x_{i-1}}{2}) \leq 0)$,
since the case where $sgn(f_0)(\frac{x_i - x_{i-1}}{2}) = 0$ does not matter.

[22] This is fundamental to the elimination of the **WKL**, as mentioned in Section 3.1. We discuss this point in more details in Section 3.10.3.

$$
\begin{aligned}
\|f_0 - \lambda h\|_1 &= \int_A |f_0 - \lambda h| + \int_B |f_0 - \lambda h| \\
&\overset{(iv)}{=} \int_A (f_0 - \lambda h)\, sgn(f_0) + \int_B |f_0 - \lambda h| \\
&= \int_A f_0\, sgn(f_0) - \lambda \int_A h\, sgn(f_0) + \int_B |f_0 - \lambda h| \\
&\leq \int_A f_0\, sgn(f_0) - \lambda \int_A h\, sgn(f_0) + \int_B |f_0| + \lambda \int_B |h| \\
&= \int_A |f_0| + \int_B |f_0| + \lambda \int_B |h| - \lambda \int_A h\, sgn(f_0) \\
&= \int_0^1 |f_0| + \lambda \int_B |h| - \lambda \int_A h\, sgn(f_0).
\end{aligned}
$$

Now we can add $\frac{\delta}{\|h\|_\infty}$ on both sides of the inequality and put $\lambda = \frac{\delta}{\|h\|_\infty}$ in evidence to get,

$$
\begin{aligned}
\|f_0 - \lambda h\|_1 + \frac{\delta}{\|h\|_\infty} &\leq \|f_0\|_1 + \frac{\delta}{\|h\|_\infty}(1 + \int_B |h| - \int_A h\, sgn(f_0)) \\
&\overset{(iii)}{<} \|f_0\|_1. \qquad \Box
\end{aligned}
$$

Remark 3.5 *In order to be precise we should have written* $\max\{1, \|h\|_\infty\}$ *instead of* $\|h\|_\infty$ *in the definition of* Φ_8*, so that it is always defined. This can be seen to be not necessary because we only apply these functionals to an* h *with uniform norm different from zero (see Section 3.12). Moreover, the functional* Φ_8 *should range over* $\mathbb{Q}_+^*$*, but* $\|h\|_\infty \in \mathbb{R}_+$*. Therefore, we should have also written* $\|h\|_{\infty,\mathbb{Q}}$ *instead of* $\|h\|_\infty$ *in the definition of* Φ_8*, where* $\|h\|_{\infty,\mathbb{Q}}$ *is a rational upper bound on* $\|h\|_\infty$*.*

Remark 3.6 *As it turned out the functional* Φ_8 *can be given independently of* r*. This independency can be explained by fact that (as we will see in Section 3.11)* r *is taken to be a function of* $\|h\|_\infty$*, and such dependency already appears in* Φ_8*.*

3.10.3 Elimination of **WKL**

As we discussed already in the introduction, the logical method of monotone functional interpretation upon which the proof of the general logical meta-theorem is based not only provides an algorithm for the extraction of the modulus of uniqueness Φ but also a constructive verification of Φ which can be formalized in intuitionistic arithmetic in all finite types $\mathbf{HA}^\omega$. In particular, we get from this that Jackson's theorem is provable in $\mathbf{HA}^\omega$ despite the fact that Cheney's proof heavily relies on classical logic and the non-computational binary König's lemma **WKL**. We will not carry out the details of this intuitionistic **verification** since we focus in this paper on the **applied** aspect of

constructing Φ, which is, as a special feature of monotone functional interpretation, largely independent from the "constructivization" part. However, in 3.10.2 above we can see already how the constructivisation of Cheney's proof comes out of our analysis: as said before, **WKL** is used in the equivalent (see [27]) [23] form of

$$\forall f \in C[0,1] \forall a, b \in [0,1](a < b \to \exists x_0 \in [a,b](f(x_0) = \inf_{x_0 \in [a,b]} f(x))) \quad (11)$$

to conclude

$$\forall x \in [x_{i-1}+r, x_i - r](f(x) > 0) \to \inf_{x \in [x_{i-1}+r, x_i - r]} f(x) > 0.$$

After our replacement of 'roots x_i' by 'r-clusters of δ-roots' this transforms into

$$\forall x \in [x_{i-1}+r, x_i - r](f(x) > \delta) \to \inf_{x \in [x_{i-1}+r, x_i - r]} f(x) \geq \delta$$

which follows from the constructively valid 'ε-weakening'

$$\begin{cases} \forall f \in C[0,1] \forall a, b \in [0,1] \\ \quad (a < b \to \forall \varepsilon > 0 \exists x_0 \in [a,b](f(x_0) - \inf_{x_0 \in [a,b]} f(x) < \varepsilon)) \end{cases}$$

version of (11) which eliminates the use of **WKL**. Also the use of classical logic to find σ_i such that

$$\sigma_i =_0 0 \leftrightarrow f(\frac{x_{i-1}+x_i}{2}) \geq_{\mathbb{R}} 0$$

is no longer necessary since we now have that

$$f(\frac{x_{i-1}+x_i}{2}) \geq_{\mathbb{R}} \delta \vee f(\frac{x_{i-1}+x_i}{2}) \leq_{\mathbb{R}} -\delta$$

which can easily be decided since $\delta \in \mathbb{Q}^*_+$.

3.11 Lemma $\forall \overline{x}, \overline{\sigma} \exists h \; I(\overline{x}, \overline{\sigma}, h)$

In the second part of Cheney's proof he considers the case where f_0 has less than $n+1$ roots, from this assumption he arrives at a contradiction (using Lemma 1) when assuming that for any $h \in P_n$, $\int h \; sgn(f_0) = 0$. We have indicated above that a contradiction is also obtained by assuming $\exists r \in \mathbb{Q}^*_+(\int_A h \; sgn(f) > \int_B |h|)$. Here we show that for any given n points

[23] Note that $f \in C[0,1]$ is given together with a modulus of uniform continuity ω_f.

$x_1 \leq \ldots \leq x_n$ in the interval $[0,1]$ and for any $\sigma_1, \ldots, \sigma_{n+1} \in \{-1,1\}$ (where σ_i will denote the sign of the function f_0 in the interval A_i) it is possible to find a function $h \in P_n$ and $r \in \mathbb{Q}_+^*$ such that $\sum_{i=1}^{n+1} \sigma_i \int_{A_i} h > \int_B |h|$, where $x_0 = 0$ and $x_{n+1} = 1$. Formally,

$$\begin{cases} \forall n \in \mathbb{N}; x_1 \leq \ldots \leq x_n \in [0,1]; \sigma_1, \ldots, \sigma_{n+1} \in \{-1,1\} \exists h \in P_n; r \in \mathbb{Q}_+^* \\ \quad \Big(\sum_{i=1}^{n+1} \sigma_i \int_{A_i} h > \int_B |h| \Big). \end{cases}$$

In the same way as we did in Section 3.10.1 we present the hidden quantifier η in the inequality and since $h/\eta \in P_n$ we have,

$$\begin{cases} \forall n \in \mathbb{N}; x_1 \leq \ldots \leq x_n \in [0,1]; \sigma_1, \ldots, \sigma_{n+1} \in \{-1,1\} \exists h \in P_n; r \in \mathbb{Q}_+^* \\ \quad \Big(\sum_{i=1}^{n+1} \sigma_i \int_{A_i} h > \int_B |h| + 1 \Big). \end{cases}$$

The sentence above states the existence of an $r \in \mathbb{Q}_+^*$ and a function $h \in P_n$. Therefore, there exists also a $k \in \mathbb{Q}_+^*$ such that $k \geq \|h\|_\infty$. Here we can again apply our meta-theorem 2.1 and we are sure to find functions Φ_9 and Φ_{10} depending only on n such that,[24]

$$\begin{cases} \forall n \in \mathbb{N}; x_1 \leq \ldots \leq x_n \in [0,1]; \sigma_1, \ldots, \sigma_{n+1} \in \{-1,1\} \exists h \in P_n; r \geq \Phi_9(n) \\ \quad \Big(\sum_{i=1}^{n+1} \sigma_i \int_{A_i} h > \int_B |h| + 1 \wedge \Phi_{10}(n) \geq \|h\|_\infty \Big), \end{cases}$$

where A and B are defined as before.

Claim 3.8 *The functions* $\boxed{\Phi_9(n) :\equiv \frac{1}{16(n+1)^3}}$ *and* $\boxed{\Phi_{10}(n) :\equiv 8(n+1)^2}$ *do the job.*

Proof. Let $0 = x_0 \leq x_1 \leq \ldots \leq x_{n+1} = 1$ and $\sigma_1, \ldots, \sigma_{n+1} \in \{-1,1\}$ be given. Ignore all the points x_j such that $x_i = x_j$ and $i < j$. We are left with $\tilde{n}+2$ points $0 = x_{a_0} < x_{a_1} < \ldots < x_{a_{\tilde{n}+1}} = 1$ where $a_{i-1} < a_i$, $a_i \in \{0, \ldots, n+1\}$ and $\tilde{n} \leq n$. Let $\tilde{x}_i :\equiv x_{a_i}$ and $\tilde{\sigma}_i :\equiv \sigma_{a_i}$. Since we have eliminated just empty intervals we have for any function $h \in P_n$, $\sum_{i=1}^{n+1} \sigma_i \int_{x_{i-1}}^{x_i} h(x)\,dx = \sum_{i=1}^{\tilde{n}+1} \tilde{\sigma}_i \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} h(x)\,dx$. Among the points $\tilde{x}_1, \ldots, \tilde{x}_{\tilde{n}}$ pick only the points $\tilde{x}_i$ for which $\tilde{\sigma}_i \neq \tilde{\sigma}_{i+1}$. Finally, we are left with $m+2$ points $0 = \tilde{x}_{b_0} < \tilde{x}_{b_1} < \ldots < \tilde{x}_{b_{m+1}} = 1$ where $b_{i-1} < b_i$, $b_i \in \{0, \ldots, \tilde{n}+1\}$ and $m \leq \tilde{n}$. Let $y_i :\equiv \tilde{x}_{b_i}$ and $\sigma_i^* :\equiv \tilde{\sigma}_{b_i}$. Again we have $\sum_{i=1}^{\tilde{n}+1} \tilde{\sigma}_i \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} h(x)dx = \sum_{i=1}^{m+1} \sigma_i^* \int_{y_{i-1}}^{y_i} h(x)dx$, for any $h \in P_n$. Then we define $\tilde{h}(x) :\equiv (x - y_1) \ldots (x - y_m)$ and

$$h(x) :\equiv \frac{+/-8(n+1)^2}{\|\tilde{h}\|_\infty} \tilde{h}(x).$$

[24] Note that Φ_9 and Φ_{10} do not depend on the points $x_1, \ldots, x_n$ nor on $\sigma_1, \ldots, \sigma_{n+1}$ since they are elements from the compact spaces $[0,1]$ and $\{-1,1\}$, respectively, and $\bigwedge_{i=1}^{n-1} x_i \leq x_{i+1}$ is purely universal.

Choose $+/-$ so that $\sum_{i=1}^{m+1}\sigma_i^*\int_{y_{i-1}}^{y_i}h(x)dx=\sum_{i=1}^{m+1}\int_{y_{i-1}}^{y_i}|h(x)|dx$. Hence,

$$\sum_{i=1}^{n+1}\sigma_i\int_{x_{i-1}}^{x_i}h(x)\,dx=\int_0^1|h(x)|\,dx.$$

Moreover, it is clear from the definition of h that $\|h\|_\infty=8(n+1)^2$. Therefore, from Remark 1.2 (cf. also Section 3.5.3) we get

$$\int_0^1|h(x)|\,dx=\|h\|_1\geq\frac{\|h\|_\infty}{2(n+1)^2}=4.$$

Let $r:\equiv\Phi_9(n)$. It is clear that the intervals B as a whole (as defined above) have length at most $\frac{1}{8(n+1)^2}$. Therefore, $\int_B|h(x)|\,dx\leq 1$. Hence,

$$\sum_{i=1}^{n+1}\sigma_i\int_{A_i}h(x)\,dx=\int_A|h(x)|\,dx>\int_B|h(x)|\,dx+1.$$

$\square$

Remark 3.7 *Note that (as follows from the result above) we can even allow σ_i to range over $\{-1,0,1\}$ as long as $\sigma_i=0$ only when $x_i-x_{i-1}\leq 2\Phi_9(n)$. In such cases the value of σ_i has no influence on the result.*

3.12 Eliminating the polynomial h in Lemma 1

We have just shown that,

$$\begin{cases}\forall x_1\leq\ldots\leq x_n\in[0,1];\sigma_1,\ldots,\sigma_{n+1}\in\{-1,1\}\exists h\in P_n\\ \quad\Big(\sum_{i=1}^{n+1}\sigma_i\int_{A_i}h>\int_B|h|+1\wedge\Phi_{10}(n)\geq\|h\|_\infty\Big),\end{cases}\tag{12}$$

where A_i and B are defined with r replaced by $\Phi_9(n)$. We can take $r=\Phi_9(n)$ because h is taken (cf. proof of Claim 3.8) in such way that $\sum_i\sigma_i\int_{A_i}h=\int_A|h|$ which makes the matrix of the lemma monotone on $\exists r$.

Let $f\in C[0,1], n\in\mathbb{N}, p_1,p_2\in K_{f,n}$ and $x_1\leq\ldots\leq x_n\in[0,1]$ be fixed, and let $\tilde h$ be the function from (12) when $\sigma_i:\equiv f_0(\frac{x_{i-1}+x_i}{2})$, where $x_0:\equiv 0$ and $x_{n+1}:\equiv 1$. Note that here σ_i can be 0 (cf. Remark 3.7). Applying Lemma 1 to $\tilde h$ and $\Phi_9(n)$ (i.e. taking $h=\tilde h$ and $r=\Phi_9(n)$) we get,

$$\begin{cases}\forall f\in C[0,1];n\in\mathbb{N};p_1,p_2\in K_{f,n};x_1\leq\ldots\leq x_n\in[0,1];\delta\in\mathbb{Q}_+^*\\ \quad\Big(\forall\lambda\in\mathbb{R}(\|f_0-\lambda\tilde h\|_1+\Phi_8(n,\delta,\tilde h)\geq\|f_0\|_1)\to\exists y\in A(|f_0(y)|\leq\delta)\Big).\end{cases}$$

Having in mind that we have $\|\tilde{h}\|_\infty \le 8(n+1)^2$ we take $\boxed{\tilde{\Phi_8}(n,\delta) :\equiv \frac{\delta}{8(n+1)^2}}$.

By the monotonicity of the functional Φ_8 in $\|h\|_\infty$ we have $\tilde{\Phi_8}(n,\delta) \le \Phi_8(n,\delta,\tilde{h})$. Then,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; x_1 \le \ldots \le x_n \in [0,1]; \delta \in \mathbb{Q}^*_+ \\ \quad \Big(\forall \lambda \in \mathbb{R}(\|f_0 - \lambda\tilde{h}\|_1 + \tilde{\Phi_8}(n,\delta) \ge \|f_0\|_1) \to \exists y \in A(|f_0(y)| \le \delta)\Big). \end{cases}$$

We can then conclude,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \delta \in \mathbb{Q}^*_+ \\ \quad \Big(\forall h \in P_n(\|f_0 - h\|_1 + \tilde{\Phi_8}(n,\delta) \ge \|f_0\|_1) \to \\ \qquad\qquad \forall x_1 \le \ldots \le x_n \in [0,1] \exists y \in A(|f_0(y)| < \delta)\Big). \end{cases}$$

We can actually replace the conclusion of the implication above with the actual existence of $n+1$ roots in the following way (lemma $\forall \overline{x} J(\overline{x}) \to E$). Assume

$$\forall x_1 \le \ldots \le x_n \in [0,1] \exists y \in [0,1]\Big(|f_0(y)| < \delta \wedge \bigwedge_{i=0}^{n+1} |x_i - y| \ge \Phi_9(n))\Big). \tag{13}$$

If $m < n+1$ is the biggest number of δ-roots of f_0 which are pairwise apart from each other by at least $\Phi_9(n)$ then by (13) we have a contradiction. Hence,

$$\exists x_0, \ldots, x_n \in [0,1](\bigwedge_{i=0}^{n} |f_0(x_i)| < \delta \wedge \bigwedge_{i=1}^{n} (x_{i-1} + \Phi_9(n) \le x_i)).$$

Therefore, we have,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \delta \in \mathbb{Q}^*_+ \\ \quad \Big(\forall h \in P_n(\|f_0 - h\|_1 + \tilde{\Phi_8}(n,\delta) \ge \|f_0\|_1) \to \\ \qquad\qquad \exists x_0, \ldots, x_n \in [0,1](\bigwedge_{i=0}^{n} |f_0(x_i)| < \delta \wedge \bigwedge_{i=1}^{n} x_{i-1} + \Phi_9(n) \le x_i)\Big). \end{cases}$$

4 The uniform modulus of uniqueness for L_1-approximation

In this section we show how the computed functionals are combined in order to obtain the uniform modulus of uniqueness. Let $f \in C[0,1]$, $n \in \mathbb{N}$, $p_1, p_2 \in K_{f,n}$

and $\varepsilon \in \mathbb{Q}^*_+$ be fixed. Assume (for $i \in \{1,2\}$),

$$\begin{cases} \|f - p_i\|_1 - dist_1(f, P_n) < \\ \quad \min\{\Phi_1(\Phi_7(\tilde{\Phi_8}(n, \Phi_5(\Phi_6(n, \Phi_9(n), \varepsilon))))), \\ \qquad\qquad \Phi_1(\Phi_2(\Phi_3(\omega_{f,n}, \Phi_4(\Phi_6(n, \Phi_9(n), \varepsilon)))))\}. \end{cases} \tag{14}$$

By Section 3.3 we have, (where $f_0(x) = f(x) - \frac{p_1(x)+p_2(x)}{2}$)

$$\|f_0\|_1 - dist_1(f, P_n) < \Phi_2(\Phi_3(\omega_{f,n}, \Phi_4(\Phi_6(n, \Phi_9(n), \varepsilon)))).$$

By Section 3.4 (and since Φ_1 is the identity),

$$|\ \|f_0\|_1 - 1/2\|f - p_1\|_1 - 1/2\|f - p_2\|_1\ | < \Phi_3(\omega_{f,n}, \Phi_4(\Phi_6(n, \Phi_9(n), \varepsilon))).$$

By Section 3.6 [25],

$$\|\ |f_0| - 1/2|f - p_1| - 1/2|f - p_2|\ \|_\infty \le \Phi_4(\Phi_6(n, \Phi_9(n), \varepsilon)).$$

Hence, by Section 3.7,

$$\begin{cases} \forall x \in [0,1] \\ \quad (|f_0(x)| \le \Phi_5(\Phi_6(n, \Phi_9(n), \varepsilon)) \to \\ \qquad\qquad |p_1(x) - p_2(x)| \le \Phi_6(n, \Phi_9(n), \varepsilon)). \end{cases} \tag{15}$$

By the same assumption (14) and Section 3.3 we also have,

$$\|f_0\|_1 - dist_1(f, P_n) < \Phi_7(\tilde{\Phi_8}(n, \Phi_5(\Phi_6(n, \Phi_9(n), \varepsilon)))).$$

And by Section 3.9,

$$\forall h \in P_n \Big(\|f_0 - h\|_1 + \tilde{\Phi_8}(n, \Phi_5(\Phi_6(n, \Phi_9(n), \varepsilon))) \ge \|f_0\|_1\Big).$$

Hence, by Section 3.12 (taking $\delta = \Phi_5(\Phi_6(n, \Phi_9(n), \varepsilon))$),

$$\begin{cases} \exists x_0, \ldots, x_n \in [0,1] \\ \quad (\bigwedge_{i=0}^n |f_0(x_i)| < \Phi_5(\Phi_6(n, \Phi_9(n), \varepsilon)) \wedge \bigwedge_{i=1}^n x_{i-1} + \Phi_9(n) \le x_i). \end{cases}$$

And by (15),

$$\begin{cases} \exists x_0, \ldots, x_n \in [0,1] \\ \quad (\bigwedge_{i=0}^n |p_1(x_i) - p_2(x_i)| \le \Phi_6(n, \Phi_9(n), \varepsilon) \wedge \bigwedge_{i=1}^n x_{i-1} + \Phi_9(n) \le x_i). \end{cases}$$

[25] Since $\int |f_0| - \frac{1}{2}|f - p_1| - \frac{1}{2}|f - p_2| = \|f_0\|_1 - \frac{1}{2}\|f - p_1\|_1 - \frac{1}{2}\|f - p_2\|_1$.

Therefore, by Section 3.8 (taking $r = \Phi_9(n)$) we conclude,

$$\|p_1 - p_2\|_\infty \le \varepsilon. \tag{16}$$

If we substitute the linear functionals, $\Phi_1, \Phi_2, \Phi_4, \Phi_5$ and Φ_7, to make the conclusion more legible, we have (14) $\to$ (16),

$$\begin{cases} \|f - p_i\|_1 - dist_1(f, P_n) < \\ \quad \min\{\tilde{\Phi_8}(n, \frac{\Phi_6(n,\Phi_9(n),\varepsilon)}{8}), \Phi_3(\omega_{f,n}, \frac{\Phi_6(n,\Phi_9(n),\varepsilon)}{8})\} \to \|p_1 - p_2\|_\infty \le \varepsilon. \end{cases}$$

After applying $\tilde{\Phi_8}$ and Φ_9 we get,

$$\begin{cases} \|f - p_i\|_1 - dist_1(f, P_n) < \\ \quad \min\{\frac{\Phi_6(n,\frac{1}{16(n+1)^3},\varepsilon)}{64(n+1)^2}, \Phi_3(\omega_{f,n}, \frac{\Phi_6(n,\frac{1}{16(n+1)^3},\varepsilon)}{8})\} \to \|p_1 - p_2\|_\infty \le \varepsilon. \end{cases}$$

Then we apply Φ_6,

$$\begin{cases} \|f - p_i\|_1 - dist_1(f, P_n) < \\ \quad \min\{\frac{\frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2^{4n+3}(n+1)^{3n+1}}\varepsilon}{8(n+1)^2}, \Phi_3(\omega_{f,n}, \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2^{4n+3}(n+1)^{3n+1}}\varepsilon)\} \to \|p_1 - p_2\|_\infty \le \varepsilon. \end{cases}$$

Let $c_n :\equiv \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2^{4n+3}(n+1)^{3n+1}}$ then we can rewrite the above formula as,

$$\begin{cases} \|f - p_i\|_1 - dist_1(f, P_n) < \\ \quad \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \Phi_3(\omega_{f,n}, c_n\varepsilon)\} \to \|p_1 - p_2\|_\infty \le \varepsilon. \end{cases}$$

And finally we apply the definition of Φ_3,

$$\begin{cases} \|f - p_i\|_1 - dist_1(f, P_n) < \\ \quad \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \frac{c_n\varepsilon}{2}\omega_{f,n}(\frac{c_n\varepsilon}{2})\} \to \|p_1 - p_2\|_\infty \le \varepsilon. \end{cases}$$

Let $\tilde{\Phi}(f, n, \varepsilon) :\equiv \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \frac{c_n\varepsilon}{2}\omega_{f,n}(\frac{c_n\varepsilon}{2})\}$, where

$$\omega_{f,n} :\equiv \min\{\omega_f(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4 M_f}\}$$

and M_f is a bound on $\|f\|_\infty$. We have shown that,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n} \\ \quad \big(\bigwedge_{i=1}^{2}(\|f - p_i\|_1 - dist_1(f, P_n) < \tilde{\Phi}(f, n, \varepsilon)) \to \|p_1 - p_2\|_\infty \le \varepsilon\big). \end{cases}$$

Proposition 1 *The functional $\tilde{\Phi}(f,n,\varepsilon)$ is a uniform modulus of uniqueness for the best L_1-approximation of $C[0,1]$ from $K_{f,n}$, i.e.*

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1, p_2 \in K_{f,n}; \varepsilon \in \mathbb{Q}^*_+ \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) < \tilde{\Phi}(f,n,\varepsilon)) \to \|p_1 - p_2\|_1 \leq \varepsilon \Big). \end{cases}$$

Proof. Above, plus the fact that $\|p_1 - p_2\|_1 \leq \|p_1 - p_2\|_\infty$. $\Box$

Claim 4.1 $\tilde{\Phi}(f,n,\varepsilon) \leq \frac{\varepsilon}{8}$

Proof. Trivial. $\Box$

Now we show that Proposition 1 can be generalised to the whole space P_n (i.e. we can replace $K_{f,n}$ with P_n). Moreover, we notice that the dependency on particular values of the function f can be eliminated so that the modulus of uniqueness depends on f only through its modulus of continuity.

Theorem 4.1 *Let $\Phi(\omega,n,\varepsilon) :\equiv \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \frac{c_n\varepsilon}{2}\omega_{f,n}(\frac{c_n\varepsilon}{2})\}$, where the constant $c_n :\equiv \frac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2^{4n+3}(n+1)^{3n+1}}$ and $\omega_n(\varepsilon) :\equiv \min\{\omega(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4\lceil\frac{1}{\omega(1)}\rceil}\}$. For all $f \in C[0,1]$ with modulus of continuity ω*

$$\begin{cases} \forall n \in \mathbb{N}; p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}^*_+ \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) < \Phi(\omega,n,\varepsilon)) \to \|p_1 - p_2\|_1 \leq \varepsilon \Big). \end{cases}$$

Proof. Actually, we prove the stronger version of the theorem where instead of $\lceil\frac{1}{\omega(1)}\rceil$ in the definition of ω_n we have any upper bound on $\sup_{x\in[0,1]}|f(x) - f(0)|$. First we show that in Proposition 1 we can replace $K_{f,n}$ with P_n. Suppose without loss of generality that $p_1 \in P_n \backslash K_{f,n}$. Then $\|p_1\|_1 > \frac{5}{2}\|f\|_1$ and hence $\|f - p_1\|_1 > \frac{3}{2}\|f\|_1 \geq \frac{3}{2}dist_1(f,P_n)$. Assume that $\|f - p_i\|_1 < dist_1(f,P_n) + \tilde{\Phi}(f,n,\varepsilon)$. By Claim 4.1, $\|f - p_i\|_1 < dist_1(f,P_n) + \frac{\varepsilon}{8}$. Then, $\frac{\varepsilon}{8} > \frac{1}{2}dist_1(f,P_n)$, i.e. $dist_1(f,P_n) < \frac{\varepsilon}{4}$. Therefore $\|f - p_i\|_1 < dist_1(f,P_n) + \frac{\varepsilon}{8} < \frac{\varepsilon}{2}$ and we have $\|p_1 - p_2\|_1 \leq \varepsilon$. The second point is that some upper bound $M_f \geq \|f\|_\infty$ is used to define $\omega_{f,n}$ in Proposition 1. We claim that an upper bound $N_f \geq \sup_{x\in[0,1]}|f(x) - f(0)|$ is sufficient. For any function $f \in C[0,1]$ and polynomials $p_1, p_2 \in P_n$ let $\tilde{f}$, $\tilde{p_1}$ and $\tilde{p_2}$ be the functions obtained by the transposition of f, p_1 and p_2 respectively by $f(0)$ (i.e. $\tilde{f}(x) :\equiv f(x) - f(0)$ and $\tilde{p_i}(x) :\equiv p_i(x) - f(0)$). It is clear that

(*i*) $\|f - p_i\|_1 = \|\tilde{f} - \tilde{p_i}\|_1$,
(*ii*) $dist_1(f,P_n) = dist_1(\tilde{f},P_n)$ and
(*iii*) $\|p_1 - p_2\|_1 = \|\tilde{p_1} - \tilde{p_2}\|_1$.

Let ω be the modulus of continuity for f and assume $\|f - p_i\|_1 < dist(f, P_n) + \Phi(\omega, n, \varepsilon)$. By (i) and (ii) we have, $\|\tilde{f} - \tilde{p_i}\|_1 < dist(\tilde{f}, P_n) + \Phi(\omega, n, \varepsilon)$. Since ω is also a modulus of continuity for $\tilde{f}$ and $\|\tilde{f}\|_\infty = \sup_{x\in[0,1]} |f(x) - f(0)| \le N_f$ we have $\tilde{\Phi}(\tilde{f}, n, \varepsilon) = \Phi(\omega, n, \varepsilon)$, therefore,

$$\|\tilde{f} - \tilde{p_i}\|_1 < dist(\tilde{f}, P_n) + \tilde{\Phi}(\tilde{f}, n, \varepsilon),$$

which implies, by Proposition 1, the first part of this proof and (iii), $\|p_1 - p_2\|_1 \le \varepsilon$. Since $\lceil \frac{1}{\omega(1)} \rceil \ge \sup_{x\in[0,1]} |f(x) - f(0)|$ if ω is a modulus of uniform continuity for f the theorem follows. $\Box$

As mentioned in Remark 3.3, the function $\Psi(n) :\equiv \frac{n!}{2^{n+1}(n+1)^{2n+2}}$ relates the L_1-norm of a polynomial $p \in P_n$ to its actual coefficients, i.e.

$$\forall n \in \mathbb{N} \forall p \in P_n \Big(\|p\|_1 \le \Psi(n) \cdot \varepsilon \to \|p\|_{\max} \le \varepsilon \Big),$$

where $\|p\|_{\max}$ denotes the maximum absolute value of the coefficients of p. Therefore, we obtain the following corollary.

Corollary 4.1 *Let $\Phi(\omega, n, \varepsilon)$ be as defined above. For all $f \in C[0,1]$ with modulus of continuity ω*

$$\begin{cases} \forall n \in \mathbb{N}; p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}^*_+ \\ \quad \Big(\bigwedge_{i=1}^2 (\|f - p_i\|_1 - dist_1(f, P_n) < \Phi(\omega, n, \Psi(n) \cdot \varepsilon)) \to \|p_1 - p_2\|_{\max} \le \varepsilon \Big). \end{cases}$$

A function $f \in C[0,1]$ is said to *be Lipschitz continuous with Lipschitz constant* $\lambda \in \mathbb{R}^*_+$ if $|f(x) - f(y)| \le \lambda |x - y|$ (i.e. $\frac{\varepsilon}{\lambda}$ is a modulus of continuity for f) and is *Lipschitz-α continuous with constant* λ, $0 < \alpha \le 1$, if $|f(x) - f(y)| \le \lambda |x - y|^\alpha$ (equivalently, $(\frac{\varepsilon}{\lambda})^{1/\alpha}$ is a modulus of continuity in our sense for f)[26]. In this way, if a function f is Lipschitz continuous (or Lipschitz-α continuous) with constant λ then $\sup_{x\in[0,1]} |f(x) - f(0)| \le \lambda$ (and we can take λ instead of $\lceil \frac{1}{\omega(1)} \rceil$ in Theorem 4.1).

Corollary 4.2 *For any $f \in C[0,1]$,*

i) let $\Phi_L(\lambda, n, \varepsilon) :\equiv \min\{\frac{c_n \varepsilon}{8(n+1)^2}, \frac{c_n^2 \varepsilon^2}{160(n+1)^4 \lambda}\}$. If f is Lipschitz continuous with constant λ then the functional Φ_L is a modulus of uniqueness for f.

ii) let $\Phi_{L_\alpha}(\lambda, \alpha, n, \varepsilon) :\equiv \min\{\frac{c_n \varepsilon}{8(n+1)^2}, \frac{c_n \varepsilon}{2} (\frac{c_n \varepsilon}{8\lambda})^{1/\alpha}, \frac{c_n^2 \varepsilon^2}{160(n+1)^4 \lambda}\}$. If f is Lipschitz-α continuous with constant λ then the functional Φ_{L_α} is a modulus of uniqueness for f.

And as a corollary of Proposition 5.4 from [15] and Theorem 4.1 above we get,

[26] In analysis the condition '$|f(x) - f(y)| \le \lambda |x - y|^\alpha$ for some λ' is called *Hölder condition with exponent α.*

Theorem 4.2 *Let $\mathcal{P}(f,n)$ denote the operator which assigns to any given function $f \in C[0,1]$ and any $n \in \mathbb{N}$ the best L_1-approximation of $f \in C[0,1]$ from P_n. Then $\Phi_P(\omega_f,n,\varepsilon) :\equiv \frac{\Phi(\omega_f,n,\varepsilon)}{2}$, Φ as defined in Theorem 4.1, is a modulus of pointwise continuity for the operator $\mathcal{P}(f,n)$, i.e.,*

$$\begin{cases} \forall f,\tilde{f} \in C[0,1]; n \in \mathbb{N}; \varepsilon \in \mathbb{Q}_+^* \\ \quad (\|f-\tilde{f}\|_1 < \Phi_P(\omega_f,n,\varepsilon) \to \|\mathcal{P}(f,n)-\mathcal{P}(\tilde{f},n)\|_1 \leq \varepsilon). \end{cases}$$

Proof. For completeness we reproduce here the proof as given in [15]. One easily verifies that $dist_1(f,P_n)$ is Lipschitz continuous in f (with respect to the L_1-norm) with $\lambda = 1$, i.e.

$$\|f-\tilde{f}\|_1 < \varepsilon \to |dist_1(f,P_n) - dist_1(\tilde{f},P_n)| < \varepsilon. \tag{17}$$

Assume now that $\|f-\tilde{f}\|_1 < \Phi_P(\omega_f,n,\varepsilon) = \frac{1}{2}\Phi(\omega_f,n,\varepsilon)$. Then,

$$\begin{aligned} \|f-\mathcal{P}(\tilde{f},n)\|_1 &\leq \|\tilde{f}-\mathcal{P}(\tilde{f},n)\|_1 + \|f-\tilde{f}\|_1 = dist_1(\tilde{f},P_n) + \|f-\tilde{f}\|_1 \\ &\overset{(17)}{<} dist_1(f,P_n) + \frac{1}{2}\Phi(\omega_f,n,\varepsilon) + \|f-\tilde{f}\|_1 \\ &< dist_1(f,P_n) + \Phi(\omega_f,n,\varepsilon). \end{aligned}$$

Since, furthermore, $\|f-\mathcal{P}(f,n)\|_1 = dist_1(f,P_n)$, we obtain from Theorem 4.1 that $\|\mathcal{P}(f,n)-\mathcal{P}(\tilde{f},n)\|_1 \leq \varepsilon$. $\square$

5 Computing the sequence $(p_{b,n})_{n\in\mathbb{N}}$

An operator $B_{f,n} : \mathbb{Q}_+^* \to P_n$ *computes the unique best L_1-approximation, $p_b \in P_n$, of a function $f \in C[0,1]$ (given with a modulus of uniform continuity ω_f) from P_n* if for any given $\varepsilon \in \mathbb{Q}_+^*$ it generates a polynomial of degree $\leq n$ with rational coefficients (i.e. a $n+1$-vector of rational coefficients) $B_{f,n}(\varepsilon)$ such that, $\|B_{f,n}(\varepsilon) - p_b\|_1 \leq \varepsilon$. We indicate how this can be achieved using the uniform modulus of uniqueness, $\Phi(\omega_f,n,\varepsilon)$,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p_1,p_2 \in P_n; \varepsilon \in \mathbb{Q}_+^* \\ \quad \Big(\bigwedge_{i=1}^2(\|f-p_i\|_1 - dist_1(f,P_n) < \Phi(\omega_f,n,\varepsilon)) \to \|p_1-p_2\|_1 \leq \varepsilon\Big). \end{cases}$$

First we substitute p for p_1 and p_b for p_2,

$$\begin{cases} \forall f \in C[0,1]; n \in \mathbb{N}; p \in P_n; \varepsilon \in \mathbb{Q}_+^* \\ \quad \Big(\|f-p\|_1 - dist_1(f,P_n) < \Phi(\omega_f,n,\varepsilon) \to \|p-p_b\|_1 \leq \varepsilon\Big). \end{cases}$$

Now we just need to find a $B_{f,n}(\varepsilon)$ such that, $\|f - B_{f,n}(\varepsilon)\|_1 - dist_1(f, P_n) < \Phi(\omega_f, n, \varepsilon)$. Note that now there is no explicit reference to p_b, only implicit in $dist_1(f, P_n)$.

A set $N_\varepsilon :\equiv \{p_1, p_2, \ldots, p_{n_\varepsilon}\} \subset P_n$ is said to be an *ε-net of $K_{f,n}$* if $\forall p \in K_{f,n} \exists p_i \in N_\varepsilon(\|p - p_i\|_1 \leq \varepsilon)$. The algorithm for computing p_b consists in evaluating $\|f - p_i\|_1$ for each p_i in some $\Phi(\omega_f, n, \varepsilon)$-net of $K_{f,n}$ and taking the p_i which gives the minimum value. Although the elements of the net N_ε are taken to be polynomials with rational coefficients, the value of $\|f - p_i\|_1$ will in general be a real number. Therefore, we only compute $\|f - p_i\|_1$ up to some precision. By an appropriate choice of the precision the minimum value returned by the search will in fact be close the the actual minimum.

The complexity analysis of the whole algorithm has been carried out in [25] and the following result was obtained.

Theorem 5.1 ([25]) *For polynomial time computable $f \in C[0,1]$ the sequence $(p_{b,n})_{n\in\mathbb{N}}$ is strongly* **NP** *computable in* $\mathbf{NP}[B_f]$*, where B_f is an oracle deciding left cuts for integration.*

6 Related results

The first proof of the uniqueness of the best L_1-approximation of $f \in C[0,1]$ by polynomials in P_n was given in 1921 by Jackson [10]. The proof we analysed was published by Cheney [6] in 1965 and reprinted in his book [7] from 1966. Only in 1975 Björnestål [3], by analyzing the qualitative (relative to the dependencies) aspect of the continuity of the projection operator for arbitrary normed linear spaces X into a closed linear subspace of X, obtained the following result.

Theorem 6.1 (Björnestål, 75) *Let $f \in C[0,1]$ and the modulus Ω_f be defined as*

$$\Omega_f(\varepsilon) :\equiv \sup_{|x-y|<\varepsilon} |f(x) - p_b(x) - f(y) + p_b(y)|,$$

where p_b is the best L_1-approximation of f from P_n. Then, for $p \in P_n$, ε sufficiently small and for some constant c depending on f and n,

$$\|p - p_b\|_1 \geq \varepsilon \to \|f - p\|_1 - \|f - p_b\|_1 \geq 2 \int_0^{\Omega_f^{-1}(c\,\varepsilon)} c\,\varepsilon - \Omega_f(x)\, dx,$$

where $\Omega_f^{-1}(\varepsilon)$ *is defined as*[27]

$$\Omega_f^{-1}(\varepsilon) :\equiv \inf\{\delta : \Omega_f(\delta) = \varepsilon\}.$$

We show that our Theorem 4.1 implies an effective version of Björnestål's theorem. First we can rewrite his theorem in the form we have been working with,

$$\begin{cases} \|f - p\|_1 < dist_1(f, P_n) + 2\int_0^{\Omega_f^{-1}(c\,\varepsilon)} c\,\varepsilon - \Omega_f(x)\, dx \to \\ \quad \|p - p_b\|_1 < \varepsilon. \end{cases} \tag{18}$$

First we show that $\int_0^{\Omega_f^{-1}(c\,\varepsilon)} c\,\varepsilon - \Omega_f(x)\, dx$ can be written as $c'\,\varepsilon\Omega_f^{-1}(c'\,\varepsilon)$, for some constant $\frac{c}{2} \le c' \le c$. For that purpose note that,

$$\int_0^{\Omega_f^{-1}(c\,\varepsilon)} c\,\varepsilon - \Omega_f(x)\, dx \le \int_0^{\Omega_f^{-1}(c\,\varepsilon)} c\,\varepsilon\, dx = c\,\varepsilon\Omega_f^{-1}(c\,\varepsilon).$$

On the other hand we have,

$$\begin{aligned} \int_0^{\Omega_f^{-1}(c\,\varepsilon)} c\,\varepsilon - \Omega_f(x)\, dx &\ge \int_0^{\Omega_f^{-1}(\frac{c}{2}\,\varepsilon)} c\,\varepsilon - \Omega_f(x)\, dx \\ &\ge \int_0^{\Omega_f^{-1}(\frac{c}{2}\,\varepsilon)} \frac{c}{2}\,\varepsilon\, dx = \frac{c}{2}\,\varepsilon\Omega_f^{-1}(\frac{c}{2}\,\varepsilon). \end{aligned}$$

Therefore, for some $\frac{c}{2} \le c' \le c$, (18) is equivalent to,

$$\|f - p\|_1 < dist_1(f, P_n) + 2c'\,\varepsilon\Omega_f^{-1}(c'\,\varepsilon) \to \|p - p_b\|_1 < \varepsilon.$$

The constant c, however, is not presented by Björnestål and moreover the function Ω_f^{-1} is usually non-computable. We can give an effective modulus of continuity for $f - p_b$ following Section 3.5 (and taking $M_f = \lceil \frac{1}{\omega_f(1)} \rceil$ as suggested in the proof of Theorem 4.1),

$$\begin{aligned} \omega_{f-p_b}(\varepsilon) &\ge \min\{\omega_f(\frac{\varepsilon}{2}), \omega_{p_b}(\frac{\varepsilon}{2})\} \\ &\ge \min\{\omega_f(\frac{\varepsilon}{2}), \frac{\varepsilon}{20(n+1)^4\lceil\frac{1}{\omega_f(1)}\rceil}\} \end{aligned}$$

Therefore, let $\omega^*_{f-p_b}(\varepsilon) :\equiv \min\{\omega_f(\frac{\varepsilon}{2}), \frac{\varepsilon}{20(n+1)^4\lceil\frac{1}{\omega_f(1)}\rceil}\}$, we can restate our Theorem 4.1 and see how it relates to Björnestål's result:

[27] Note that $\Omega_f^{-1}(\varepsilon)$ (for ε small enough so that $\Omega_f^{-1}(\varepsilon)$ is defined) is a special modulus of continuity for $f - p_b$ in our sense.

Corollary 6.1 *Let $f \in C[0,1]$, ω_f be some modulus of uniform continuity of f, and $p \in P_n$. Then for $\varepsilon \leq 1$,*

$$\|f-p\|_1 < dist_1(f,P_n) + \tilde{c}_n\,\varepsilon\,\omega^*_{f-p_b}(\tilde{c}_n\,\varepsilon) \to \|p-p_b\|_1 \leq \varepsilon,$$

where $\tilde{c}_n :\equiv \frac{c_n}{8(n+1)^2}$ and $c_n :\equiv \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2^{4n+3}(n+1)^{3n+1}}$.

Proof. From Theorem 4.1 we have,

$$\begin{cases} \|f-p\|_1 < dist_1(f,P_n) + \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \frac{c_n\varepsilon}{2}\omega^*_{f-p_b}(\frac{c_n\varepsilon}{4})\} \to \\ \quad \|p-p_b\|_1 \leq \varepsilon, \end{cases}$$

which implies,

$$\begin{cases} \|f-p\|_1 < dist_1(f,P_n) + \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \frac{c_n\varepsilon}{8(n+1)^2}\omega^*_{f-p_b}(\frac{c_n\varepsilon}{4})\} \to \\ \quad \|p-p_b\|_1 \leq \varepsilon. \end{cases}$$

For $\varepsilon \leq 1$ we have $\omega^*_{f-p_b}(\frac{c_n\varepsilon}{4}) \leq 1$. Hence,

$$\|f-p\|_1 < dist_1(f,P_n) + \frac{c_n\varepsilon}{8(n+1)^2}\omega^*_{f-p_b}(\frac{c_n\varepsilon}{4}) \to \|p-p_b\|_1 \leq \varepsilon.$$

Since $8(n+1)^2 > 4$ we get our result. $\square$

Some years later, in 1978, Kroó [22] showed that the constant c in Björnestål's result needed not to depend on any particular point of the function f but only on its modulus of continuity [28]. We got an effective version of Björnestål's result where our constant c is completely independent of the function f and only depends on the dimension of the space P_n.

Remark 6.1 *In Kroó [22] the problem of L_1-approximation of continuous functions is considered for arbitrary Haar subspaces of $C[0,1]$ containing the constant functions. Kroó [23] treats uniqueness subspaces of $C[0,1]$ but in that case the constant c also depends on values of the function f and not only on its modulus of continuity. Since Cheney's proof which we analyzed works for arbitrary Haar subspaces we are also guaranteed to extract uniform moduli of uniqueness in the general setting. As done by Jackson [10] in his original proof, in the present work we focused on the specific Haar subspace P_n in order to get fully explicit results. One can observe that only Section 3.8 (Lagrange interpolation formula used to show that P_n is a Haar space), Section 3.5 (Markov inequality used to show that $K_{f,n}$ is compact by constructing a common modulus of uniform continuity) and Section 3.11 (Markov inequality plus the construction of a polynomial which changes sign in each x_i) made*

[28] As in Björnestål [3], Kroó does not present the actual constant.

reference to the particular Haar space P_n. From results in [4](Lemma 4.3), [5](lemma) and [13](after Lemma 9.32) it follows that there exist effective and quantitative substitutes for each of these constructions for arbitrary (effectively given) Haar spaces. So it is clear that the analysis carried out in this paper can be extended to general Haar spaces H containing the constant functions[29] .

7 Concluding remarks on the extraction of Φ

We emphasize again the two important roles played by logic in the extraction of the modulus of uniqueness for best L_1-approximation presented here. First, by showing that Cheney's proof could be formalized in the system $\mathcal{A}_*^\omega$ (and by the logical meta-theorem 2.1) we were guaranteed that such a modulus Φ would exist and that it could be extracted from the mentioned proof. Moreover, the fact that Φ depends only on ω_f, n and ε (which was proved by Kroó years after Cheney's proof) is obtained immediately from the meta-theorem 2.1. The second important role is that logic not only guaranteed the existence of the modulus but it went even further and supplied a procedure (monotone functional interpretation) to extract the modulus, which enabled us to provide for the first time an explicit dependency on n and ω_f. And, as it happened, the extracted modulus of uniqueness has the optimal ε-dependency established by Kroó.

We hope it is transparent that all the mathematical tools used in our analysis were already present in Cheney's proof,[30] which can be noticed for instance in the analysis of Lemma 1 (Section 3.10) where in order to prove that the functionals presented realized the lemma (see Claim 3.7) we followed line by line the original proof from [7], the only difference being that we considered the ε-version of the propositions. This visibly shows that the uniform modulus of uniqueness here extracted was really implicitly present in Cheney's proof but could only be made explicit with the help of logic. The difficulty to extract ad hoc such information can be understood because Cheney's proof (although very simple from the mathematical point of view and even called 'elementary' by the author) is logically very intricate due to the use of proof by contradiction and principles that fail in computable analysis.

[29] We only need the constant functions to belong to H if we want to get rid of the f dependency in c, i.e. obtain a constant c in the uniform modulus of uniqueness depending only on n and ω_f.

[30] Except Markov inequality which was used to show that the set $K_{f,n}$ is compact (and also in Section 3.11) and Lagrange interpolation formula used to prove that P_n is a Haar space. These tools, however, are standard in approximation theory.

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