Real Growth in Standard Parts of Analysis*

Ulrich Kohlenbach

Fachbereich Mathematik

J.W. Goethe Universität

Robert-Mayer-Str. 6-10

60054 Frankfurt am Main, Germany

e-mail: kohlenb@math.uni-frankfurt.de

May 1995

*This paper is a slightly revised version of my Habilitationsschrift which was presented to J.W. Goethe-Universität Frankfurt in January 1995.

To Gabriele

Contents

Introduction*

It is known since the twenties of this century (mainly due to the work of D. Hilbert which is reported in [25]) that the part of mathematics which usually is called classical analysis can be developed to a great extent in formal systems $\mathfrak{A}$ of the following type:
Let PA^2 denote the extension of the usual first–order Peano arithmetic PA by variables $X, Y, Z, \ldots$ for sets of natural numbers together with quantifiers over these variables and their usual logical axioms and rules. In this language one can formulate the axiom schema of comprehension over numbers:

$$\mathrm{CA}^{set} \ : \ \bigvee X \bigwedge x \big(x \in X \leftrightarrow A(x)\big),$$

where x is a number variable and A is an arbitrary formula (not containing X free) of PA^2. In particular A may contain set quantifiers.
Now $\mathfrak{A}$ is defined as PA^2+CA^{set}.
For the formalization of notions and proofs in analysis it is more convenient to have (besides variables $x^0, y^0, z^0, \ldots$ over numbers) also variables $x^1, y^1, f^1, g^1, \ldots$ over functions $\mathbb{N} \to \mathbb{N}$ and variables $x^2, y^2, z^2, \Phi^2, \Psi^2, \ldots$ over function(al)s : $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ of such functions and so on. More generally $x^{\rho(\tau)}$ is a function which maps objects of type τ into objects of type ρ.[1] Let us denote the corresponding functional version of PA^2 by PA^ω. In PA^ω sets are given by their characteristic function. In the language of functionals of finite type the schema of comprehension corresponding to CA^{set} now reads as follows:

$$\mathrm{CA}^{func} \ : \ \bigwedge x^0 \bigvee! y^0 A(x, y) \to \bigvee f^1 \bigwedge x^0 A(x, fx),$$

where A is an arbitrary formula of PA^ω.
For some theorems in analysis, e.g. the equivalence between ε–δ– and sequential continuity of $f : \mathbb{R} \to \mathbb{R}$ in $x \in \mathbb{R}$ one needs a weak form of the axiom of choice

$$\mathrm{AC}^{0,1} \ : \ \bigwedge x^0 \bigvee f^1 A(x, f) \to \bigvee g^{1(0)} \bigwedge x^0 A(x, gx).$$

Let $\mathfrak{A}^\omega$ denote the theory PA^ω+CA^{func}+$\mathrm{AC}^{0,1}$.

Now let us consider the following situation:
Let $A_0(\underline{x}^0, y^0)$ be a quantifier–free and therefore decidable[2] formula of $\mathfrak{A}^\omega$, where $\underline{x} = x_1^0, \ldots, x_k^0$ and y^0 are all free variables of A_0 and suppose that

$$(1)\ \mathfrak{A}^\omega \vdash \bigwedge \underline{x}^0 \bigvee y^0 A_0(\underline{x}, y).$$

A_0 defines a partial recursive function in $\underline{x}$:

$$f\underline{x} := \begin{cases} \min y[A_0(\underline{x}, y)], \text{ if } \bigvee y\, A_0(\underline{x}, y) \\ \text{undefined, otherwise.} \end{cases}$$

By (1), $\mathfrak{A}^\omega$ proves that f is in fact a **total** recursive function. This is the reason why f is called **provably recursive** (or provably total) in $\mathfrak{A}^\omega$.

*I am grateful to Prof. H. Luckhardt for stimulating discussions on the subject of this paper and for helpful suggestions for the presentation of the results.

[1] 1 (resp. 2) abbreviates the type 0(0) (resp. 0(0(0))).

[2] We only have equality $=_0$ between numbers as a primitive notion. Higher type equality is defined extensionally. Throughout this paper $A_0, B_0, C_0, \ldots$ denote quantifier–free formulas.

What do we know about the rate of growth of this function if we know that (1) holds?

It is well–known that for systems like $\mathfrak{A}^\omega$ the rate of growth may be really huge and goes far beyond the rate of growth occuring in usual mathematics. In particular it may grow much faster than e.g. the Ackermann function and even faster than every ε_0–recursive function. A description of the provably recursive functions of $\mathfrak{A}^\omega$ in terms of recursion schemas was given by C. Spector in [64] by means of so–called bar recursion.

Although beginning in 1977 a few examples of simple $\bigwedge x^0 \bigvee y^0 A_0(x,y)$–sentences of concrete combinatorial or number theoretic nature were found such that $fx := \min y\, A_0(x,y)$ is of enormous rate of growth (see [50],[28],[19], [62]) this phenomenon seems to be extremely rare in concrete mathematics (especially in analysis). In fact the growth of f in these examples is due to the fact that A_0 indirectly expresses certain properties of ordinals.
This observation indicates that $\mathfrak{A}^\omega$ is much to strong to capture faithfully the reasoning used in actual proofs in analysis. Most parts of analysis in fact can be developed in small fragments of $\mathfrak{A}^\omega$. This was noticed already by mathematicians like Poincare, Borel and above all H. Weyl in his influential monograph 'Das Kontinuum'([71]) where he developes analysis on the basis of so–called predicative comprehension (due to B. Russell) which imposes a restriction on the schema of comprehension:

$$\mathrm{CA}_{ar}^{func}\ :\ \bigwedge x^0 \bigvee ! y^0 A(x,y) \to \bigvee f^1 \bigwedge x^0 A(x,fx),$$

where A contains only **quantifiers over type–0–objects, i.e. over numbers**. We call such a formula A **arithmetical**.
Although the concept of predicativity was formulated because of foundational questions concerning the consistency of unrestricted comprehension[3] it also has an impact on our question:

Let $\mathfrak{A}^\omega_{ar}$:=PA$^\omega$+CA$_{ar}^{func}$+AC^{0,1}–qf, where AC^{0,1}–qf is the restriction of AC^{0,1} to quantifier–free formulas[4].
The rate of growth of provably recursive functions of $\mathfrak{A}^\omega_{ar}$ is much lower compared to $\mathfrak{A}^\omega$ (put in technically terms the provably recursive functions of $\mathfrak{A}^\omega_{ar}$ are just the $\alpha(<\varepsilon_{\varepsilon_0})$–recursive functions, see [11]) but still is tremendous.

In the late 70ies G. Takeuti (see [65]) noticed that almost the same portion of analysis can be carried out in a more restricted system, where the full schema of induction

$$\mathbf{IA}\ :\ A(0) \wedge \bigwedge x^0 \big(A(x) \to A(x+1)\big) \to \bigwedge x^0 A(x)$$

is available only for arithmetical formulas A.[5] Let us denote the corresponding restriction of $\mathfrak{A}^\omega_{ar}$ (resp. PA$^\omega$) by $\widehat{\mathfrak{A}}^\omega_{ar}\!\restriction$ (resp. $\widehat{\mathrm{PA}}^\omega\!\restriction$).[6] In the presence of CA$_{ar}^{func}$ this restricted schema of induction follows from the **axiom of quantifier–free induction**

$$\mathbf{QF\text{–}IA}\ :\ \bigwedge f^1 \big(f0=0 \wedge \bigwedge x\big(fx=0 \to f(x+1)=0\big) \to \bigwedge x(fx=0)\big).$$

[3]For detailed information on predicativity and proof–theoretical investigations of formal systems for predicative mathematics see e.g. [10].

[4]Using CA$_{ar}^{func}$ and AC^{0,1}–qf one easily can derive AC^{0,1} for arbitrary arithmetical formulas.

[5]See also [17],[18] and [11] for results in this direction.

[6]Also the Gödel recursor constants R_ρ are replaced by the predicative Kleene recursors $\widehat{R}_\rho$. The system used by Takeuti differs in various respects from $\widehat{\mathfrak{A}}^\omega_{ar}\!\restriction$ but this is not important for our discussion. Takeuti also discusses a second system with a variant of the first one.

The most interesting fact about $\widehat{\mathfrak{A}}^{\omega}_{ar}\!\upharpoonright$ is that it is conservative over first–order Peano arithmetic (see e.g. [11]). In particular this implies that the provably recursive functions of $\widehat{\mathfrak{A}}^{\omega}_{ar}\!\upharpoonright$ are $\alpha(<\varepsilon_0)$–recursive.

In [17],[60], [8] and [56] it is shown that various important theorems of analysis are already provable in a second–order fragment (WKL_0) of $\widehat{\mathrm{PA}}^{\omega}\!\upharpoonright$+$\mathrm{AC}^{0,0}$–qf +WKL, where WKL is the binary ('weak') König's lemma.
Friedman showed (in an unpublished manuscript) model–theoretically that (WKL_0) is Π^0_2–conservative over the primitive recursive arithmetic PRA.[7]
In [32],[33] we developed a proof–theoretical method which **extracts** primitive recursive bounds from proofs of $\bigwedge\bigvee$–sentences in the extension $\widehat{\mathrm{PA}}^{\omega}\!\upharpoonright$+AC–qf+WKL of ($\mathrm{WKL}_0$) to all finite types.
In fact we showed much more:

$$(*)\quad\begin{cases}\text{From a proof } \widehat{\mathrm{PA}}^{\omega}\!\upharpoonright\!+\mathrm{AC}\text{–qf }+\mathrm{WKL}\vdash \bigwedge u^1\bigwedge v\leq_\tau tu\bigvee w^\gamma A_0(u,v,w)\\ \text{one can extract a primitive recursive (in the sense of [30]) bound } \Phi \text{ such that}\\ \widehat{\mathrm{PA}}^{\omega}_i\!\upharpoonright\,\vdash \bigwedge u^1\bigwedge v\leq_\tau tu\bigvee w\leq_\gamma \Phi u\, A_0(u,v,w),\end{cases}$$

where A_0 is a quantifier–free formula containing only u,v,w free, τ is arbitrary, $\gamma\leq 2$ and t is a closed term of $\widehat{\mathrm{PA}}^{\omega}\!\upharpoonright$[8]. Note that the bound Φu does not depend on v.

The mathematical significance of this result in particular rests on the fact that in applications in analysis one quite often is interested in **uniform** bounds Φ which do not depend on input data $x\in K$ where K is a compact metric space. Since compact metric spaces have standard representations by sets of functions having the form $\{f^1 : f\leq_1 t\}$, $(*)$ provides such uniform bounds. The fact that Φu is only a **bound** on $\bigvee w$ is no essential weakening since $\bigwedge u^1\bigwedge v\leq_\tau tu\bigvee w^0$–sentences in analysis usually are **monotone** w.r.t. w^0 and thus every bound on $\bigvee w^0$ in fact provides a realization of $\bigvee w^0$, i.e. $\bigwedge u^1\bigwedge v\leq_\tau tu\, A_0(u,v,\Phi u)$ (see [39] for a discussion of this phenomenon).

In [32],[37],[38] this method is applied to concrete (ineffective) proofs in approximation theory yielding new a–priori estimates for numerically relevant data as constants of strong unicity and others which improve known estimates significantly.

In analyzing these applications we developed in [39] a new monotone functional interpretation which has important advantages over the method from [33] and provides a particular perspicuous procedure of analyzing ineffective proofs in analysis.

The starting point for the investigation carried out in the present paper are the following problems:

(I) Whereas the general meta–theorem $(*)$ only guarantees the existence of a primitive recursive bound Φ, the bounds which are actually obtained in our applications to approximation theory have a very low rate of growth which is polynomial (of degree ≤ 2) relatively to the growth

[7]For a proof–theoretic treatment of this result using cut–elimination see [57]. In [58] and [59] also Π^0_2–conservativity of WKL over elementary recursive arithmetic is shown. But note that the proof for Π^1_1–conservation given in [58],[59] is incorrect (see [35] for a discussion of this point).

[8]$\widehat{\mathrm{PA}}^{\omega}_i\!\upharpoonright$ denotes the intuitionistic variant of $\widehat{\mathrm{PA}}^{\omega}\!\upharpoonright$. Instead of u^1,w^γ one may also have tuples of variables of type ≤ 1 resp. ≤ 2. In particular, instead of the quantifier–free A_0 one may have $A_1\in\Sigma^0_1$.

of the data of the problem. Thus the problem arises to close the still large gap between polynomial and primitive recursive growth.

(II) Although in a theory like $\widehat{\mathrm{PA}}^{\omega}\!\upharpoonright$+AC–qf+WKL one can carry out a substantial portion of analysis there are important analytical principles, e.g. the Bolzano–Weierstraß principle for bounded sequences in $\mathbb{R}$, the Arzelà–Ascoli lemma for bounded sequences of equicontinuous functions $f \in C[0,1]$ or the existence of $\limsup, \liminf$ for bounded sequences in $\mathbb{R}$, which are not provable by this means. In fact these principles are known to be equivalent to CA_{ar}^{func} (relatively to $\widehat{\mathrm{PA}}^{\omega}\!\upharpoonright$+AC–qf). Thus the problem arises to impose mathematical natural restrictions on the use of these principles and to prove that under these restrictions one can extract bounds of more reasonable growth.

(III) So far we have considered the question of extracting bounds

$$\bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w \leq_\gamma \Phi u\, A_0(u,v,w)$$

for sentences

$$\bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w^\gamma A_0(u,v,w)$$

with **quantifier–free** A_0.
It is natural to ask for bounds for more general and even arbitrary formulas A instead of A_0. The problem is that in the presence of full classical logic there are simple logically valid sentences $\bigwedge x^0 \bigvee y^0 \bigwedge z^0 A_0(x,y,z)$ such that there is no computable bound on $\bigvee y$ at all.
If however analytical principles (even non–constructive ones) are used only relatively to **intuitionistic** arithmetical reasoning, then it might be possible (and in fact is possible for many non–constructive analytical theorems as we will show in chapter 8) to extract bounds for very general formulas A.

In order to address the problems formulated in (I)–(III) we first introduce a hierarchy $\left(\mathrm{G}_n\mathrm{A}^\omega\right)_{n\in\mathbb{N}}$ of subsystems of $\widehat{\mathrm{PA}}^{\omega}\!\upharpoonright$ and investigate the rate of growth caused by various analytical principles relatively to $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf. The definable functionals $t^{1(1)}$ in $\mathrm{G}_n\mathrm{A}^\omega$ are of increasing order of growth:

If $n=1$, then tf^1x^0 is bounded by a linear function in f^M, x,

if $n=2$, then tf^1x^0 is bounded by a polynomial in f^M, x;

if $n=3$, then tf^1x^0 is bounded by an elementary recursive (i.e. a (fixed) finitely iterated exponential) function in f^M, x,

where $f^M := \lambda x^0.\max(f0,\ldots,fx)$ and Φfx is called linear (polynomial, elementary recursive) in f,x if $\bigwedge f^1, x^0(\Phi fx =_0 \widehat{\Phi}[f,x])$ for a term $\widehat{\Phi}[f,x]$ which is built up from $0^0, x^0, f^1, S^1, +$ (respectively $0^0, x^0, f^1, S^1, +, \cdot$ and $0^0, x^0, f^1, S^1, +, \cdot, \lambda x^0, y^0.x^y$) only.

Let us motivate this notion for the polynomial case:
If Φfx is a polynomial in f^1, x^0, then in particular for every polynomial $p \in \mathbb{N}[x]$ the function $\lambda x^0.\Phi px$ can be written as a polynomial in $\mathbb{N}[x]$. Moreover there exists a polynomial $q \in \mathbb{N}[x]$

(depending only on the term structure of Φ) such that

$$\begin{cases} \text{For every polynomial } p \in \mathbb{N}[x] \\ \text{one can construct a polynomial } r \in \mathbb{N}[x] \text{ such that} \\ \bigwedge f^1 \big(f \leq_1 p \to \bigwedge x^0 (\Phi f x \leq_0 r(x))\big) \text{ and } deg(r) \leq q(deg(p)). \end{cases}$$

Since every closed term $t^{1(1)}$ in $\mathrm{G}_2\mathrm{A}^\omega$ is bounded by a polynomial $\Phi f^M x$ in f^M, x and $f \leq_1 p \to f^M \leq_1 p$ (since p is monotone) this holds also for tfx instead of Φfx.

In particular every closed term t^1 ($t^0 \overbrace{(0)\cdots(0)}^{k}$) of $\mathrm{G}_2\mathrm{A}^\omega$ is bounded by a polynomial $p_t \in \mathbb{N}[x]$ (resp. a polynomial $p_t \in \mathbb{N}[x_1, \ldots, x_k]$).
For general $n \in \mathbb{N}$, every closed term t^1 of $\mathrm{G}_n\mathrm{A}^\omega$ is bounded by some function $f_t \in \mathcal{E}^n$ where $\mathcal{E}^n$ denotes the n–th level of the Grzegorczyk hierarchy.

It turns out that many basic concepts of real analysis can be defined already in $\mathrm{G}_2\mathrm{A}^\omega$: e.g. rational numbers, real numbers (with their usual arithmetical operations and inequality relations), d–tuples of real numbers, sequences and series of reals, continuous functions $F : \mathbb{R}^d \to \mathbb{R}$ and uniformly continuous functions $F : [a,b]^d \to \mathbb{R}$, the supremum of $F \in C([a,b]^d, \mathbb{R})$ on $[a,b]^d$, the Riemann integral of $F \in C[a,b]$. Furthermore the trigonometric functions $\sin, \cos, \tan, \arcsin, \arccos, \arctan$ and π as well as the restriction $\exp_k$ ($\ln_k$) of the exponential function (logarithm) to $[-k,k]$ for every **fixed** number k can be introduced in $\mathrm{G}_2\mathrm{A}^\omega$ (The unrestricted functions exp and ln can be defined in $\mathrm{G}_3\mathrm{A}^\omega$).
$\mathrm{G}_2\mathrm{A}^\omega$+AC–qf (and even its intuitionistic version $\mathrm{G}_2\mathrm{A}^\omega_i$+AC–qf) proves many of the basic properties of these objects.

Thus it is reasonable to consider proofs of sentences

$$(+)\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^0 A_0, \text{ where } \underline{u} = u_1^1, \ldots, u_l^1,\ \underline{k} = k_1^0, \ldots, k_i^0$$

which use relatively to $\mathrm{G}_2\mathrm{A}^\omega$+AC–qf various higher analytical theorems Γ (which usually will not be provable in $\mathrm{G}_2\mathrm{A}^\omega$).

In view of the problem (I) formulated above we now ask:

What do we know about the rate of growth of bounds Φ

$$(++)\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_0 \Phi \underline{u}\,\underline{k}\, A_0$$

which can be extracted from a given proof[9]

$$\mathrm{G}_2\mathrm{A}^\omega\text{+AC–qf} + \Gamma \vdash\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^0 A_0\ ?$$

Let Γ consist of theorems choosen from the following list

[9]Such bounds can be extracted also for tuples $\underline{w}$ of variables $w_i^{\gamma_i}$ with $\gamma_i \leq 2$. For simplicity we discuss here only the (most important) case type$(w) = 0$.

- The fundamental theorem of calculus
- Fejér's theorem on the uniform approximation of 2π–periodic continuous functions by trigonometric polynomials
- The equivalence (local and global) of ε–δ –continuity and sequential continuity of $F : \mathbb{R} \to \mathbb{R}$ in $x \in \mathbb{R}$.
- Attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$[10] on $[0,1]^d$
- Mean value theorem for integrals
- Mean value theorem of differentiation
- Cauchy–Peano existence theorem for ordinary differential equations
- Brouwer's fixed point theorem for continuous functions $f : [0,1]^d \to [0,1]^d$
- Every pointwise continuous function $G : [0,1]^d \to \mathbb{R}$ is uniformly continuous on $[0,1]^d$ and possesses a modulus of uniform continuity
- $[0,1]^d \subset \mathbb{R}^d$ has the (sequential form of the) Heine–Borel covering property
- Dini's theorem: Every sequence G_n of pointwise continuous functions $:[0,1]^d \to \mathbb{R}$ which increases pointwise to a pointwise continuous function $G : [0,1]^d \to \mathbb{R}$ converges uniformly on $[0,1]^d$ to G and there exists a modulus of uniform convergence
- Every strictly increasing pointwise continuous function $G : [0,1] \to \mathbb{R}$ possesses a uniformly continuous strictly increasing inverse function $G^{-1} : [G0, G1] \to [0,1]$ together with a modulus of uniform continuity

then one can extract a bound Φ which is (bounded by) a polynomial in $\underline{u}^M, \underline{k}^0$ in the sense above.

¿From a proof of (+) in $\mathrm{G_3A^\omega}$+AC–qf+Γ one can extract a bound Φ which is (bounded by) an elementary recursive (i.e. finitely iterated exponential) function in $\underline{u}^M, \underline{k}^0$.
Let us consider the important case where the proof uses besides tools which are available already in $\mathrm{G_2A^\omega}$+AC-qf+Γ only certain fixed functions f^1 of simple exponential growth as e.g. $fx := 2^x$ or $fx := x!$. Since the proof may use a (fixed) finite number of iterations of f (either explicitly by forming terms like $f(fx)$ or implicitly by a logical circumscription of such a substitution, e.g. in its most simple form $\bigwedge x \bigvee y, z(y = fx \wedge z = fy)$), in general only an elementary recursive bound is guaranteed. If however the proof does not iterate f (not even implicitly)[11] as is often the case in practice, our method will yield a bound which is built up from $\underline{u}^M, \underline{k}^0, 0^0, S^0, +, \cdot, f$ with f–depth 1 and thus (for polynomially bounded $\underline{u}$) is essentially simple exponential in $\underline{k}$ (more precisely bounded by $2^{p(\underline{k})}$ where $p \in \mathbb{N}[\underline{k}]$). So our result that analytical theorems Γ from the list above do not cause any non–polynomial growth is of relevance also in the presence of certain functions having exponential growth.

[10] Here and below we may have also $[a_1, b_1] \times \ldots \times [a_d, b_d]$ for variable real numbers $a_i < b_i$ $(i = 1, \ldots, d)$ instead of $[0,1]^d$.

[11] Of course whether a proof implicitly makes use of an iteration of f or not is not always possible to recognize in advance but may become transparent only by the process of the extraction of Φ itself.

¿From these results it is clear that for the part of analysis outlined so far it is the **arithmetical** reasoning used in a given proof which is decisive for the growth of bounds. We now discuss an arithmetical principle used in analysis which may contribute significantly to the growth of extractable bounds:

$$PCM1 \ : \begin{cases} \text{Every decreasing sequence } (x_n)_{n\in\mathbb{N}} \text{ of positive real numbers } x_n \text{ is a} \\ \text{Cauchy sequence} \end{cases}$$

(The restriction to the special lower bound 0 is convenient for our discussion but of course not essential).
This principle (which is not provable in $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf for any n) may contribute to the growth of bounds which can be extracted from a proof in $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+$\Gamma + PCM1$ by a functional Ψ such that

$$(+++)\ \bigwedge k^0, g^1 \bigvee n \leq_0 \Psi((x_n),k,g)\big(gn >_0 n \to x_n - x_{gn} \leq \frac{1}{k+1}\big),$$

(+++) is satisfied by

$$\Psi((x_n),k,g) := \max_{i<C(x_0)(k+1)} \big(g^i(0)\big),$$

Where $\mathbb{N} \ni C(x_0) \geq x_0$ and $g^i(0)$ is the i–th iteration $g(\ldots(g(0))\ldots)$ of g (starting with 0).
Since Ψ essentially is the iteration functional $\Phi gx := g^x(0)$ and since Φ can be used (relatively to $\mathrm{G}_2\mathrm{A}^\omega$) to define every primitive recursive function, the use of (+++) in a proof has the consequence that (in general) only the existence of a primitive recursive bound is guaranteed. This is unavoidable since we can show that $\mathrm{G}_2\mathrm{A}^\omega + PCM1$ proves the schema of Σ^0_1–induction

$$\Sigma^0_1\text{–IA} \ : \ A(0) \wedge \bigwedge x\big(A(x) \to A(x+1)\big) \to \bigwedge x\, A(x),$$

where $A \in \Sigma^0_1$, and the provably recursive functions of $\mathrm{G}_2\mathrm{A}^\omega+\Sigma^0_1$–IA are just the primitive recursive ones.
However in the important special case where $(+++)$ is applied only to $g := S$ we still have polynomial growth: $\Psi((x_n),k,S) \leq C(x_0)(k+1)$. Furthermore for special sequences (x_n) there may be much simpler bounds (+++) than Ψ.

We now come to our results concerning problem (II). Let us illustrate the general type of these results for the most simple example namely for the **analytical** strengthening $PCM2$ of $PCM1$ which asserts the existence of a Cauchy modulus **function** for every decreasing sequence of positive real numbers, i.e.

$$PCM2 \ : \begin{cases} \text{For every decreasing sequence } (x_n)_{n\in\mathbb{N}} \text{ of positive real numbers } x_n \text{ there exists} \\ \text{a function } h^1 \text{ such that } \bigwedge k^0, m^0\big(m \geq_0 hk \to x_{hk} - x_m \leq \frac{1}{k+1}\big). \end{cases}$$

In particular $PCM2$ easily implies the existence of a limit of $(x_n)_{n\in\mathbb{N}}$ (together with a modulus of convergence). The existence of a limit does not follow from $PCM1$ (relatively to $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf) since within $\mathrm{G}_n\mathrm{A}^\omega$ (as in algorithmic numerical analysis and complexity theory for real analysis) real numbers are always given by Cauchy sequences of rational numbers with **fixed Cauchy rate** (See chapter 3 for an extensive discussion on enrichment of data).
The proof

$$\mathrm{G}_2\mathrm{A}^\omega \vdash PCM1 \to \Sigma^0_1\text{–IA}$$

mentioned above yields

$$\mathrm{G}_2\mathrm{A}^\omega + \mathrm{AC{-}qf} \;\vdash PCM2 \to \; \mathrm{CA}_{ar}^{func}.$$

Hence every $\alpha(<\varepsilon_0)$–recursive function is provably recursive in $\mathrm{G}_2\mathrm{A}^\omega$+AC–qf+$PCM2$.
In contrast to this general result, we show that if $PCM2$ is applied only to **single instances** or more general **single sequences of instances** of $PCM2$ in a proof of a sentence (+) (where these instances may depend on the parameters $\underline{u}, \underline{k}, v$ of (+)) then the contribution of PCM2 to the bound Φ is just Ψ above applied to (majorants of) these instances. In particular the facts on $PCM1$ mentioned above apply and the existence of a primitive recursive bound is guaranteed. Again if Ψ is applied only to $g := S$, then one has a polynomial bound.

In a similar way single (sequences of) instances of the following principles

- The existence of a greatest lower bound for every sequence of real numbers which is bounded from below
- The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$
- The Arzelà–Ascoli lemma

can be reduced to single (sequences of) instances of $PCM1$ in a given proof of (+) relatively to $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+$\Gamma$ for $n \geq 2$ (resp. in the case of the Arzelà–Ascoli lemma for $n \geq 3$), where Γ is the set of analytical theorems from above.

Hence these principles contribute to the growth of bounds in the same way as $PCM2$.

Finally we investigate

- the existence $\exists \limsup(x_n)$ of the $\limsup$ for bounded sequences (x_n) in $\mathbb{R}$

w.r.t. its impact on the growth of bounds (likewise for $\liminf$):
Single instances of $\exists \limsup(x_n)$ in proofs of sentences (+) (relatively to $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+$\Gamma$ for $n \geq 2$) can be reduced to a certain **arithmetical** sentence $L(x_n) \in \Pi^0_5$. $L(x_n)$ can be proved in $\mathrm{G}_3\mathrm{A}^\omega+\Sigma^0_2$–IA (but seems to be unprovable in $\mathrm{G}_3\mathrm{A}^\omega + \Sigma^0_1$–IA). In contrast to $\mathrm{G}_n\mathrm{A}^\omega + \Sigma^0_1$–IA, the theory $\mathrm{G}_3\mathrm{A}^\omega + \Sigma^0_2$–IA suffices to prove the totality of the Ackermann function.
Thus $\exists \limsup(x_n)$ is the strongest (w.r.t. its impact on growth) principle used in the standard parts of classical analysis.
Note however that sometimes $\limsup x_n$ is used only to abbreviate a certain proposition which can be expressed also without assuming the existence of $\limsup x_n$, e.g. '$\limsup x_n \leq c$' can be paraphrased simply as '$\bigwedge k^0 \bigvee n^0 \bigwedge m > n(x_m \leq c + \frac{1}{k+1})$'. There are also important applications of the Bolzano–Weierstraß principle and the Arzelà–Ascoli lemma which do not contribute to the growth of bounds since they are used just to derive theorems which e.g. have a simple monotone functional interpretation (e.g. the theorem on the attainment of the maximum of $f \in C[0,1]$ and the Cauchy–Peano existence theorem discussed in (I) are usually proved using these principles respectively).

In this paper we are interested in the determination of the rate of growth of bounds which can be extracted from proofs in various parts of analysis and in most perspicuous methods for carrying out such extractions but not in the proof–theoretic strength of the tools needed to verify these bounds. We are satisfied with their classical truth, i.e. the truth in the full set–theoretic type structure $\mathcal{S}^\omega$ (where set–theoretic is meant in the sense of e.g. ZFC).[12]

Concerning problem (III) we show in particular the following results:
Let $A(\underline{u}^1, \underline{k}^0, v^\tau, w^0)$ be an **arbitrary** formula (containing only $\underline{u}, \underline{k}, v, w$ as free variables) and let us consider the **intuitionistic** version $\mathrm{G}_n\mathrm{A}^\omega_i$ of $\mathrm{G}_n\mathrm{A}^\omega$.
If the sentence

$$(1)\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\ \underline{k} \bigvee w^0 A(\underline{u}, \underline{k}, v, w)$$

is proved in

$$\mathrm{G}_n\mathrm{A}^\omega_i + \mathrm{AC} + \tilde{\Gamma},$$

where AC is the axiom schema of full choice and $\tilde{\Gamma}$ is a set of analytical principles taken from the following list[13]

- The fundamental theorem of calculus
- Fejér's theorem on the uniform approximation of 2π–periodic continuous functions by trigonometric polynomials
- Attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$ on $[0,1]^d$
- Mean value theorem for integrals
- Cauchy–Peano existence theorem for ordinary differential equations
- Brouwer's fixed point theorem for continuous functions $f : [0,1]^d \to [0,1]^d$
- The axiom schema of comprehension for negated formulas

$$\mathrm{CA}^\rho_\neg \ :\ \bigvee \Phi \leq_{0\rho} \lambda x^\rho.1^0 \bigwedge y^\rho \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big)$$

then one can extract from the proof a bound Φ such that

$$(2)\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\ \underline{k} \bigvee w \leq_0 \Phi\underline{u}\ \underline{k}\ A(\underline{u}, \underline{k}, v, w)$$

is true in the full type structure $\mathcal{S}^\omega$ and

(i) Φ is a polynomial in $\underline{u}^M, \underline{k}$ (in the sense above), if $n = 2$,

(ii) Φ is elementary recursive in $\underline{u}^M, \underline{k}$, if $n = 3$.

[12] It should be noted however that for the results discussed so far a verification in e.g. PA^ω is possible in principle: Our methods developed in this paper yield verification proofs in $\mathrm{PA}^\omega_i + \Delta$+b–AC, where Δ is a set of sentences having the form $(*)\ \bigwedge x^1 \bigvee y \leq_1 sx \bigwedge z^0 A_0$ and b–AC the schema of bounded choice from [34] (see also chapter 2 below). Using results from [33] one can reduce $(*)$ to an ε–weakening, which is provable in PA^ω for our examples, and thereby eliminate b–AC from the verification proof. Since also all universal axioms we use are provable in PA^ω we obtain a verification in PA^ω. We will not go into details of this in the present paper.

[13] Here continuous functions $f : [0,1]^d \to \mathbb{R}$ are always understood to be endowed with a modulus of uniform continuity.

Thus even in the presence of the highly non–constructive and impredicative comprehension schema $\mathrm{CA}_\neg$ (note that $\mathrm{G}_2\mathrm{A}_i^\omega$+$\mathrm{CA}_\neg$ has the proof–theoretic strength of full classical simple type theory as can be seen via negative translation) one obtains reasonable bounds as long as the underlying arithmetical theory only uses intutionistic logic[14] (This is in contrast to the corresponding classical theory which has the same provably recursive functions as simple type theory).

If A is restricted to a certain set Γ which in particular includes all sentences in prenex normal form where the universal quantifiers have types ≤ 1 and the existence quantifiers have types ≤ 2 and if $\tau \leq 1$ and $\mathrm{CA}_\neg$ is replaced by

$$\mathrm{CA}^\rho_{\vee f} \;:\; \bigvee \Phi \leq_{0\rho} \lambda x^\rho.1^0 \bigwedge y^\rho \big(\Phi y =_0 0 \leftrightarrow B(y)\big), \text{ where } B \text{ is } \bigvee\text{–free},$$

then the result above also holds if the following principles are added to $\tilde{\Gamma}$

- Every pointwise continuous function $G : [0,1]^d \to \mathbb{R}$ is uniformly continuous on $[0,1]^d$ and possesses a modulus of uniform continuity
- $[0,1]^d \subset \mathbb{R}^d$ has the (sequential form of the) Heine–Borel covering property
- Dini's theorem: Every sequence G_n of pointwise continuous functions $:[0,1]^d \to \mathbb{R}$ which increases pointwise to a pointwise continuous function $G : [0,1]^d \to \mathbb{R}$ converges uniformly on $[0,1]^d$ to G and there exists a modulus of uniform convergence

The last two principles may even be strengthened by allowing arbitrary (not necessarily open balls) in the Heine–Borel property and omitting the monotonicity assumption in Dini's theorem. These strengthened versions which can easily be refuted **classically** do not have **constructive** counterexamples.
These results cannot be extended to intuitionistic proofs relative to $PCM1$ (or even $PCM2$ and the other principles discussed under (II)) since $PCM1$ itself is a $\bigwedge\bigvee$–sentence $\in \Gamma_1$ but (for general (x_n)) there is no computable bound on $\bigvee$ in $PCM1$.

We now indicate very briefly the **proof–theoretic methods** used in the proofs of the results sketched so far.

The main proof–theoretic tool used for the results on (I) is a monotone version of Gödel's functional interpretation which is based on a suitable notion of majorizability. This method was introduced in [39] for PA^ω and is now applied to our theories $\mathrm{G}_n\mathrm{A}^\omega$. In addition to the features of this method developed in [39] we make essential use of the fact that this interpretation allows to extract bounds Φ which have a very simple term structure. This fact (which is also of central importance for (II)) enables us to measure the growth of these bounds in usual mathematical terms using only **logical**

[14]In addition to pure intuitionistic logic one may use the so–called independence–of–premise schema for negated formluas

$$IP_\neg \;:\; (\neg A \to \bigvee y^\rho B) \to \bigvee y^\rho(\neg A \to B) \text{ } (y \text{ not free in } A),$$

which does not hold intuitionistically. Also note that $\mathrm{CA}_\neg$ implies the tertium–non–datur schema for negated formulas.
A different kind of a theory which adds a non–constructive principle (more precisely the so–called limited principle of omniscience) to an intuitionistic theory is presented in [18]. Friedman shows that his system ALPO is conservative over Peano arithmetic PA.

normalization (i.e. λ–reductions).
Within G_2A^ω we develope a special representation of real numbers and continuous functions which has the property that many basic facts for these notions can be expressed as purely universal sentences $\bigwedge \underline{u}^1, \underline{k}^0 A_0(\underline{u}, \underline{k})$ (which sometimes requires strengthened quantitative versions of these facts together which an enrichment of data). Since universal sentences have a very simple monotone functional interpretation they can be treated simply as axioms. In particular such sentences contribute to the growth of bounds at most via majorants for the terms used in their formalization but not by their proofs.
At some occasions we introduce new constants c of type 1 or 1(0) to G_2A^ω (e.g. for $\sin, \cos$) together with universal axioms. Since these constants have majorants c^* by closed terms in G_2A^ω and 'c^* majorizes c' is a purely universal sentence (for the types $1, 1(0)$), the addition of such constants contributes to the growth of bounds only via c^*.
One of the most important properties of the monotone functional interpretation is that sentences having the form

$$(*)\ \bigwedge x^\delta \bigvee y \leq_\rho sx \bigwedge z^\tau A_0$$

also have a very simple direct (i.e. even without negative translation) monotone functional interpretation (whereas they usually do not have a direct Gödel functional interpretation by any computable functionals and even the Gödel functional interpretation of their negative translation[15] cannot be satisfied by primitive recursive functionals in the extended sense of Gödel's calculus T). The relevance of this is due to the fact that some central theorems of analysis, e.g. the attainment of the maximum of $F \in C([0,1]^d, \mathbb{R})$ on $[0,1]^d$, are not purely universal but can be expressed in the logical form $(*)$.

Nevertheless there still are important analytical theorems, e.g. Dini's theorem, which do not have the form $(*)$. In order to treat such theorems in the context of G_2A^ω we introduce a new axiom F^- which has the form $(*)$ and implies combined with $AC^{1,0}$–qf the following **principle of Σ^0_1–boundedness**

$$\Sigma^0_1\text{–UB}^- : \begin{cases} \bigwedge y^{1(0)} \big(\bigwedge k^0 \bigwedge x \leq_1 yk \bigvee z^0\ A(x,y,k,z) \to \bigvee \chi^1 \bigwedge k^0, x^1, n^0 \\ \qquad\qquad \big(\bigwedge\limits_{i<_0 n} (xi \leq_0 yki) \to \bigvee z \leq_0 \chi k\ A((\overline{x,n}), y, k, z)\big)\big), \end{cases}$$

where $A \in \Sigma^0_1$ and

$$(\overline{x,n})(k) := \begin{cases} xk, & \text{if } k < n \\ 0^0, & \text{otherwise.} \end{cases}$$

Using Σ^0_1–UB$^-$ one can give very short proofs (even more simple than the usual ones) of Dini's theorem (together with a modulus of convergence), the uniform continuity of every pointwise continuous function $F : [0,1]^d \to \mathbb{R}$ (together with a modulus of uniform continuity), the (sequential) Heine–Borel property of $[0,1]^d$ and the existence of a continuous strictly increasing inverse function for every strictly increasing continuous function $F : [0,1] \to \mathbb{R}$.
F^- is not true in the full type structure $\mathcal{S}^\omega$ of all set–theoretic functionals but only in the type structure $\mathcal{M}^\omega$ of all so-called strongly majorizable functionals which was introduced in [4]. However

[15] Note that the negative translation of $(*)$ is a weakening of $(*)$ (intuitionistically).

F^- can be eliminated proof–theoretically from the verification of a bound Φ extracted from a proof which uses F^-. For $\tau \leq 2$

$$\mathcal{M}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_0 \Phi \underline{u}\,\underline{k}\, A(\underline{u}, \underline{k}, v, w)$$

implies

$$\mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_0 \Phi \underline{u}\,\underline{k}\, A(\underline{u}, \underline{k}, v, w)$$

and thus for a classical verification of Φ no F^-–elimination is needed.
This also holds for a strengthened version F of F^- which proves the uniform continuity of **every** function $F : [0,1]^d \to \mathbb{R}$ which is **given by a functional** $\Phi^{1(1)}$. This does not contradict the existence of discontinuous functions since the existence of a **functional** $\Phi^{1(1)}$ which represents a discontinuous function requires comprehension over functions which is not available in our systems (Of course within $\mathrm{G}_n\mathrm{A}^\omega$ one can express discontinuous functional dependencies
$\bigwedge x \in \mathbb{R} \bigvee ! y \in \mathbb{R}\, A(x,y)$ which describe uniquely determined discontinuous functions).

The proofs of our results on (II), i.e. on $PCM2$, the Bolzano–Weierstraß principle and so on, form the proof–theoretically most complicated part of this paper. Let us motivate what proof–theoretic tools are needed for these results for the most simple example $PCM2$:

The reduction of an instance of $PCM2$ to the corresponding instance of $PCM1$ in a proof of a $\bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w^0 A_0$–sentence requires the transformation of a given proof of

$$(1)\ \bigwedge u^1 \bigwedge v \leq_\tau tu \big(\bigvee h^1 \bigwedge k^0 \bigwedge m, \tilde{m} > hk (|(\xi uv)_m - (\xi uv)_{\tilde{m}}| \leq \frac{1}{k+1}) \to \bigvee w^0 A_0(u,v,w)\big)$$

into a proof (within a theory which is not stronger w.r.t. the growth of extractable bounds) of

$$(2)\ \bigwedge u^1 \bigwedge v \leq_\tau tu \big(\bigwedge k^0 \bigvee n^0 \bigwedge m, \tilde{m} > n (|(\xi uv)_m - (\xi uv)_{\tilde{m}}| \leq \frac{1}{k+1}) \to \bigvee w^0 A_0(u,v,w)\big),$$

where $\big((\xi uv)_n\big)_{n\in\mathbb{N}}$ is a (bounded monotone) sequence in $\mathbb{R}$.[16]
More general we are looking for a proof–theoretic procedure which produces a proof for

$$(3)\ A \equiv \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee y_1^0 \bigwedge x_1^0 \ldots \bigvee y_k^0 \bigwedge x_k^0 \bigvee w^\gamma A_0(u,v,y_1,x_1,\ldots,y_k,x_k,w),$$

from a given proof of the Herbrand normal form A^H of A, where

$$(4)\ A^H :\equiv \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge h_1,\ldots,h_k \bigvee y_1^0,\ldots,y_k^0, w^\gamma \underbrace{A_0(u,v,y_1,h_1y_1,\ldots,y_k,h_ky_1\ldots y_k,w)}_{A_0^H :\equiv}$$

(A_0 is quantifier–free and contains only $u, v, \underline{y}, \underline{x}, w$ free, t is a closed term of $\mathrm{G}_n\mathrm{A}^\omega$ and τ, γ are arbitrary finite types).
For

$$(5)\ B :\equiv \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee k \bigwedge n \bigvee m, \tilde{m}, w \big((m,\tilde{m} > n \to |(\xi uv)_m - (\xi uv)_{\tilde{m}}| \leq \frac{1}{k+1}) \to B_0(u,v,w)\big)$$

[16]This transformation is possible for an arbitrary sequence ξuv of real numbers. The assumption that this sequence is bounded and monotone is used only to ensure the constructability of a functional which satisfies the (negative translation of the) monotone functional interpretation of the implicative premise in (2).

(which is just a prenex normal form of (2)) this would yield the passage from (1) to (2).
However such a proof–theoretic procedure does not exist. In fact for every fixed number k one can construct an arithmetical sentence A such that

$$\mathrm{G}_2\mathrm{A}^\omega \vdash A^H \text{ but } \mathrm{G}_k\mathrm{A}^\omega + \Gamma + \text{AC–qf} \nvdash A, \text{ where } \Gamma \text{ is as above.}$$

This phenomenon (a special case of which was noticed firstly in [35]) will be studied in detail in chapter 10 below.
On the other hand, if A satisfies a **monotonicity condition**

$$Mon(A) :\equiv \begin{cases} \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge x_1, \tilde{x}_1, \ldots, x_k, \tilde{x_k}, y_1, \tilde{y}_1, \ldots y_k, \tilde{y}_k \\ \qquad \Big(\bigwedge\limits_{i=1}^{k} (\tilde{x}_i \leq_0 x_i \wedge \tilde{y}_i \geq_0 y_i) \wedge \bigvee w^\gamma A_0(u, v, y_1, x_1, \ldots, y_k, x_k, w) \\ \qquad\qquad \to \bigvee w^\gamma A_0(u, v, \tilde{y}_1, \tilde{x}_1, \ldots, \tilde{y}_k, \tilde{x}_k, w) \Big), \end{cases}$$

then such a transformation is possible. In fact in chapter 10 we will show

$$(6)\ \mathrm{G}_n\mathrm{A}^\omega + Mon(A) \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge h_1, \ldots, h_k \bigvee y_1 \leq_0 \Psi_1 u\underline{h} \ldots \bigvee y_k \leq_0 \Psi_k u\underline{h} \bigvee w^\gamma A_0^H \to A,$$

where $\Psi_1, \ldots, \Psi_k$ are arbitrary closed terms (of suitable types) of $\mathrm{G}_n\mathrm{A}^\omega$.

Thus if A^H is proved within a theory $\mathcal{T}^\omega$ for which the extractability of such bounds $\Psi_1, \ldots, \Psi_k$ on $\bigvee y_1, \ldots, y_k$ is guaranteed, e.g. for $\mathcal{T}^\omega := \mathrm{G}_n\mathrm{A}^\omega + \Gamma + \text{AC–qf}$, then one can construct a proof of A (in a theory which is closely related to $\mathcal{T}^\omega$).
The relevance of this result follows from the fact that

$$\mathrm{G}_2\mathrm{A}^\omega \vdash Mon(B) \text{ for } B \text{ from (5) above}^{17},$$

and thus a proof of (2) can be transformed into a proof of (1) thereby replacing the **analytical** implicative premise

$$\bigvee h^1 \bigwedge k^0 \bigwedge m, \tilde{m} > hk \big(|(\xi uv)_m - (\xi uv)_{\tilde{m}}| \leq \frac{1}{k+1} \big)$$

by the **arithmetical** premise

$$\bigwedge k^0 \bigvee n^0 \bigwedge m, \tilde{m} > n \big(|(\xi uv)_m - (\xi uv)_{\tilde{m}}| \leq \frac{1}{k+1} \big).$$

It is not always as obvious as in the case of $PCM2$ to what arithmetical principle a certain analytical premise may be reducible. E.g. for $\exists \limsup(x_n)$ the construction of the monotone arithmetical principle $L(x_n) \in \Pi^0_5$ is quite complicated. Nevertheless the reduction of $\exists \limsup(x_n)$ to $L(x_n)$ is faithfull since $\mathrm{G}_2\mathrm{A}^\omega \vdash \exists \limsup(x_n) \to L(x_n)$.

For the Bolzano–Weierstraß principle BW things are even more complicated since we are not able to construct a **monotone** arithmetical sentence whose Skolem normal form implies BW and which is implied by BW. In order to capture BW we first investigate the axiom Π^0_1–CA of Π^0_1–comprehension. We show that every single instance Π^0_1–CA(g) of this axiom follows (in $\mathrm{G}_2\mathrm{A}^\omega$) from a suitable instance $PCM2(t(g))$ of $PCM2$ and therefore can be reduced to $PCM1(t(g))$. Using a suitable sequence of instances of Π^0_1–CA combined with the axiom F^- discussed above we are able

[17] $m, \tilde{m}, w$ can be coded into a single variable w' of type γ.

to prove every single sequence of instances of BW.
The treatment of the Arzelà–Ascoli lemma is similar although technically more involved. The determination of the growth caused (potentionally) by the use of instances of the last two principles thus uses results from almost all parts of this paper.

Our results on (III), i.e. on analytical proofs relatively to the intuitionistic theories $\mathrm{G}_n\mathrm{A}^\omega_i$ are based on new **monotone** versions of the well–known 'modified realizability' and 'modified realizability with truth' interpretations.

In **chapter 1** the theories $\mathrm{G}_n\mathrm{A}^\omega$ and several variants and extensions are introduced. Furthermore the growth of the definable functionals of these theories is measured.
Chapter 2 developes the method of monotone functional interpretation for $\mathrm{G}_n\mathrm{A}^\omega$ and applies it for the extraction of uniform bounds from proofs in (analytical extensions of) $\mathrm{G}_n\mathrm{A}^\omega$.
Chapter 3 deals with the representation in $\mathrm{G}_2\mathrm{A}^\omega$ of the basic objects and concepts of analysis as e.g. real numbers, continuous functions $f : \mathbb{R}^d \to \mathbb{R}$, uniformly continuous functions $f : [0,1]^d \to \mathbb{R}$, maximum and sum of variable length for sequences of real numbers, $\sup\limits_{x\in[0,1]} fx$ and $\int\limits_0^x f(x)dx$ $(x \in [0,1])$ for $f \in C[0,1]$. We discuss the impact of enrichments of data on the logical form of the basic properties of these objects and quantification over them (A summary of these results can be found at the end of chapter 3).
In **chapter 4** we show that various criteria for convergence of series can be proved in $\mathrm{G}_2\mathrm{A}^\omega$ even in quantitative versions. **Chapter 5** treats (in the context of $\mathrm{G}_2\mathrm{A}^\omega$) the trigonometric functions $\sin, \cos, \tan, \arcsin, \arccos, \arctan$ as well as the restrictions $\exp_k$ and $\ln_k$ of exp and ln to the interval $[-k,k]$ for every fixed number k (The unristricted versions of these functions can be introduced in $\mathrm{G}_3\mathrm{A}^\omega$).
In **chapter 6** we investigate in the context of $\mathrm{G}_2\mathrm{A}^\omega + \mathrm{AC}^{1,0}$–qf the fundamental theorem of calculus, Fejér's theorem, and the (local and global) equivalence between sequential and ε–δ continuity of real functions.
Chapter 7 shows that various important non–constructive theorems of analysis as e.g. the attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$, Brouwer's fixed point theorem, Cauchy–Peano's existence theorem and mean value theorems have monotone functional interpretations which can be fulfilled by terms of $\mathrm{G}_2\mathrm{A}^\omega$. Furthermore the axioms F and F^- are introduced. These axioms combined with $\mathrm{AC}^{1,0}$–qf yield principles of uniform–Σ^0_1–boundedness which are used to derive e.g. Dini's theorem, the (sequential) Heine–Borel property for $[0,1]^d$, the existence of an inverse function for every strictly increasing function $f \in C[0,1]$ and so on. Also we introduce a generalization WKL^2_{seq} of the binary König's lemma WKL to sequences of trees in a higher type formulation which can be used in $\mathrm{G}_2\mathrm{A}^\omega$ (The usual formulation of WKL in the literature uses already for its formulation a coding functional which is available only in $\mathrm{G}_3\mathrm{A}^\omega$). WKL^2_{seq} can be derived in $\mathrm{G}_2\mathrm{A}^\omega + F^- + \mathrm{AC}^{1,0}$–qf. Finally we show how to eliminate F^- from proofs of $\bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w^\gamma A_0$–sentences which yields conservation results for WKL^2_{seq}.
Chapter 8 applies monotone versions of modified realizability interpretations for the proof of the results on (III) discussed above.
In **chapter 9** we study versions of induction which go beyond quantifier–free induction. E.g.

we discuss the rule of Σ^0_1–induction. Furthermore we show the equivalence of the axiom of Σ^0_1–induction Σ^0_1–IA and $PCM1$. In particular we construct a functional χ in $\mathrm{G}_2\mathrm{A}^\omega$ such that $\mathrm{G}_2\mathrm{A}^\omega_i \vdash PCM1(\chi(g)) \to \Sigma^0_1\text{–IA}(g)$ and determine the rate of growth caused (potentionally) by the use of $PCM1$.
Chapter 10 is devoted to the elimination of Skolem functions from monotone premises in given proofs thereby replacing analytical premises by suitable arithmetical ones. In particular we prove (6) above.
In **chapter 11** results from the chapters 1,2,3,4,7,9 are combined with the method developed in chapter 10 to determine the contribution to the growth of bounds by single (sequences of) instances of $PCM2$, the existence of a greatest lower bound for every sequence of reals which is bounded from below, Π^0_1–CA and Π^0_1–AC (and their arithmetical consequences Δ^0_2–IA and Σ^0_2–collection), the Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$, the Arzelà–Ascoli lemma and the existence of lim sup for bounded sequences in $\mathbb{R}$.
In **chapter 12** we first notice that our results on Π^0_1–CA imply as a corollary the fact that the restriction Π^0_1–CA$^-$ of Π^0_1–CA without function parameters produces only primitive recursive growth (relative to $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf). We show that various theorems on Π^0_1–CA$^-$ stated by Mints and Sieg in [46], [57] are incorrect. A discussion of the errors in their proofs exhibits that our result on Π^0_1–CA$^-$ cannot be obtained (at least not straightforeward) from their proofs.

1 Subsystems of primitive recursive arithmetic in all finite types

1.1 Classical and intuitionistic predicate logic PL^ω and HL^ω in the language of all finite types

The set $\mathbf{T}$ of all finite types is defined inductively by

(i) $0 \in \mathbf{T}$ and (ii) $\rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}$.

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type ρ to objects of type τ.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

(i) $0 \in \mathbf{P}$ and (ii) $\rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}$.

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write $0(00)$ instead of $0(0(0))$. Furthermore we write for short $\tau\rho_k \ldots \rho_1$ instead of $\tau(\rho_k) \ldots (\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n+1$. The types $0, 00, 0(00), 0(0(00)) \ldots$ are so represented by $0, 1, 2, 3 \ldots$. For arbitrary types $\rho \in \mathbf{T}$ the degree of ρ (for short $\deg(\rho)$) is defined by $\deg(0) := 0$ and $\deg(\tau(\rho)) := \max(\deg(\tau),\deg(\rho) + 1)$. For pure types the degree is just the number which represents this type. Functions having a type whose degree is > 1 are usually called functionals.
The language $\mathcal{L}(\mathrm{HL}^\omega)$ of HL^ω contains variables $x^\rho, y^\rho, z^\rho, \ldots$ for each type $\rho \in \mathbf{T}$ together with corresponding quantifiers $\bigwedge x^\rho, \bigvee y^\rho$ as well as the logical constants $\wedge, \vee, \rightarrow$ and an equality relation $=_0$ between objects of type 0. Furthermore we have a propositional constant $\perp$ (”falsum”). Negation as a defined notion: $\neg A :\equiv A \rightarrow \perp$. Finally $\mathcal{L}(\mathrm{HL}^\omega)$ contains 'logical' combinators $\Pi_{\rho,\tau}$ and $\Sigma_{\delta,\rho,\tau}$ of type $\rho\tau\rho$ and $\tau\delta(\rho\delta)(\tau\rho\delta)$ for all $\rho, \tau, \delta \in \mathbf{T}$.
HL^ω has the usual axioms and rules of intuitionistic predicate logic (for all sorts of variables) plus the equality axioms for $=_0$ (e.g. see [67]). Equations $s =_\rho t$ between terms of higher type $\rho = 0\rho_k \ldots \rho_1$ are abbreviations for the formula $\bigwedge x_1^{\rho_1}, \ldots, x_k^{\rho_k}(sx_1 \ldots x_k =_0 tx_1 \ldots x_k)$.
$\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$ are characterized by the corresponding axioms of typed combinatory logic:

$\Pi_{\rho,\tau} x^\rho y^\tau =_\rho x$ and $\Sigma_{\delta,\rho,\tau} xyz =_\tau xz(yz)$ where $x \in \tau\rho\delta, y \in \rho\delta, z \in \delta$.

Furthermore we have the following quantifier–free rule of extensionality

$$\text{QF–ER} : \ \frac{A_0 \rightarrow s =_\rho t}{A_0 \rightarrow r[s] =_\tau r[t]}, \text{ where } A_0 \text{ is quantifier–free.}$$

Classical predicate logic in all finite types PL^ω results if the tertium–non–datur schema $A \vee \neg A$ is added to HL^ω. The enrichment of HL^ω (resp. PL^ω) obtained by adding the extensionality **axiom**

$(E_\rho) : \bigwedge x^\rho, y^\rho, z^{\tau\rho}(x =_\rho y \rightarrow zx =_\tau zy)$

for every type ρ is denoted by E–HL^ω (resp. E–PL^ω).

Remark 1.1.1 *Using $\Pi_{\rho,\tau}$ and $\Sigma_{\delta,\rho,\tau}$ one defines (e.g. as in [67]) λ–terms $\lambda x^\rho.t^\tau[x]$ for each term $t^\tau[x^\rho]$ such that*
$HL^\omega \vdash \big(\lambda x^\rho.t^\tau[x]\big)s^\rho =_\tau t[s]$. In particular one can define a combinator $\Pi'_{\rho,\tau} = \lambda x^\rho, y^\tau.y$ such that $\Pi'_{\rho,\tau} x^\rho y^\tau =_\tau y$ (E.g. take $\Pi' := \Pi(\Sigma\Pi\Pi)$ for Σ, Π of suitable types).

Notational convention: Throughout this paper $A_0, B_0, C_0, \ldots$ always denote quantifier–free formulas.

1.2 Subsystems of arithmetic in all finite types corresponding to the Grzegorczyk hierarchy

In the following we extend PL^ω and HL^ω by adding certain computable functionals and the schema of quantifier–free induction. The following definition from [54] is a variant of a definition due to [1] and can be used for a perspicuous definition of the well–known Grzegorczyk hierarchy from [22] (see def.1.2.26).

Definition 1.2.1 *For each $n \in \mathbb{N}$ we define (by recursion on n from the outside) the n-th branch of the Ackermann function $A_n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by*

$$A_0(x,y) := y' \quad \textit{(Here and in the following } x' \textit{ stands for the successor } Sx \textit{ of } x\textit{)},$$

$$A_{n+1}(x,0) := \begin{cases} x, & \text{if } n=0 \\ 0, & \text{if } n=1 \\ 1, & \text{if } n \geq 2, \end{cases}$$

$$A_{n+1}(x,y') := A_n(x, A_{n+1}(x,y)).$$

Remark 1.2.2 *1) $A_1(x,y) = x+y,\ A_2(x,y) = x \cdot y,\ A_3(x,y) = x^y,\ A_4(x,y) = x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}$ (y times).*

2) For each fixed $n \in \mathbb{N}$ the function A_n is primitive recursive. But: $A(x) := A_x(x,x)$ is not primitive recursive.

We now define the **Grzegorczyk arithmetic $\mathrm{G}_n\mathrm{A}^\omega$ of level n in all finite types** and their intuitionistic variant $\mathbf{G}_n\mathbf{A}^\omega_i$:

$\mathcal{L}(\mathrm{G}_n\mathrm{A}^\omega)$ is defined as the extension of $\mathcal{L}(\mathrm{PL})^\omega)$ by the addition of function constants S^{00} (successor), $\max_0^{000}, \min_0^{000}, A_0^{000}, \ldots, A_n^{000}$ and functional constants $\Phi_1^{001}, \ldots, \Phi_n^{001}, \mu_b^{001}$ (bounded μ–operator), $\tilde{R}_\rho \in \rho(\rho 0)(\rho 00)\rho 0$ (for each $\rho \in \mathbf{T}$). Furthermore we have a predicate symbol $\leq_0$.
In addition to the axioms and rules of PL^ω the theory $\mathrm{G}_n\mathrm{A}^\omega$ contains the following:

1) $\leq_0$–axioms: $x \leq_0 x,\ x \leq_0 y \vee y \leq_0 x,\ x \leq_0 y \wedge y \leq_0 z \to x \leq_0 z$.

2) S–axioms: $Sx =_0 Sy \to x =_0 y,\ \neg 0 =_0 Sx,\ x \leq_0 Sx$.

3) (max) : $\max_0(x,y) \geq_0 x,\ \max_0(x,y) \geq_0 y,\ \max_0(x,y) =_0 x \vee \max_0(x,y) =_0 y$.

4) (min) : $\min_0(x,y) \leq_0 x,\ \min_0(x,y) \leq_0 y,\ \min_0(x,y) =_0 x \vee \min_0(x,y) =_0 y$.

5) The defining recursion equations for $A_0, \ldots, A_n$ from the definition 1.2.1 above.

6) Defining recursion equations for $\Phi_1, \ldots, \Phi_n$:

$$\begin{cases} \Phi_i f 0 =_0 f0 \\ \Phi_i f x' =_0 A_{i-1}(fx', \Phi_i f x) \quad \text{for } i \geq 2 \end{cases}$$

and

$$\begin{cases} \Phi_1 f 0 =_0 f0 \\ \Phi_1 f x' =_0 \max_0(fx', \Phi_1 f x). \end{cases}$$

(For $i \geq 1$, Φ_i is the iteration of the $i-1$-th branch A_{i-1} of the Ackermann function on the f–values $f0, \ldots, fx$ for variable x).

7) $$(\mu_b) : \begin{cases} y \leq_0 x \wedge f^{000}xy =_0 0 \rightarrow fx(\mu_b fx) =_0 0, \\ y <_0 \mu_b fx \rightarrow fxy \neq 0, \\ \mu_b fx =_0 0 \vee (fx(\mu_b fx) =_0 0 \wedge \mu_b fx \leq_0 x) \end{cases}$$

(These axioms express that $\mu_b fx = \min y \leq_0 x(fxy =_0 0)$ if such an $y \leq x$ exists and $= 0$ otherwise).

8) Defining recursion equations for $\tilde{R}_\rho$ (bounded and predicative recursion, since only type–0–values are used in the recursion):

$$\begin{cases} \tilde{R}_\rho 0yzv\underline{w} =_0 y\underline{w} \\ \tilde{R}_\rho x'yzv\underline{w} =_0 \min_0(z(\tilde{R}_\rho xyzv\underline{w})x\underline{w}, vx\underline{w}), \end{cases}$$

where $y \in \rho = 0\rho_k \ldots \rho_1,\ \underline{w} = w_1^{\rho_1} \ldots w_k^{\rho_k},\ z \in \rho 00,\ v \in \rho 0.$

9) All $\mathbb{N}, \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$–true purely universal sentences $\bigwedge \underline{x} A_0(\underline{x})$, where $\underline{x}$ is a tuple of variables whose types have a degree ≤ 2 (Here B^A denotes the set of all set–theoretic functions : $A \rightarrow B$).

$\mathrm{G}_n\mathrm{A}_i^\omega$ is the variant of $\mathrm{G}_n\mathrm{A}^\omega$ with intuitionistic logic only.
If we add $(E) = \bigcup_\rho \{(E_\rho)\}$ to $\mathrm{G}_n\mathrm{A}^\omega$,$\mathrm{G}_n\mathrm{A}_i^\omega$ we obtain theories which are denoted by E–$\mathrm{G}_n\mathrm{A}^\omega$, E–$\mathrm{G}_n\mathrm{A}_i^\omega$.
$\mathrm{G}_n\mathrm{R}^\omega$ denotes the set of all closed terms on $\mathrm{G}_n\mathrm{A}^\omega$.

Remark 1.2.3 *1) The functionals Φ_1, Φ_2 and Φ_3 have the following meaning:*
$\Phi_1 fx = \max(f0, f1, \ldots, fx),\ \Phi_2 fx = \sum_{y=0}^x fy,\ \Phi_3 fx = \prod_{y=0}^x fy.$

2) Our definition of G_nA^ω contains some redundances (which however we want to remain for greater flexibility of our language): E.g. Φ_i $(i > 1)$ can be defined from $A_i, \tilde{R}, \min_0$ and Φ_1: With $f^M := \lambda x.\Phi_1 fx$ prop.1.2.16 and 1.2.18 below imply $\Phi_i fx \leq \Phi_i f^M x \leq A_i(f^M+1, x+1)$. Hence Φ_i can be defined by $\tilde{R}$ using $A_i(f^M+1, x+1)$ as boundary function v.

3) The axiom of quantifier–free induction

$$(1)\ \bigwedge f^1, x^0\big(f0 =_0 0 \wedge \bigwedge y < x(fy =_0 0 \rightarrow fy' =_0 0) \rightarrow fx =_0 0\big)$$

can be expressed as an universal sentence $\bigwedge f^1, x^0 A_0$ by prop.1.2.6 below and thus is an axiom of $G_nA_i^\omega$. (1) implies every instance (with parameters of arbitrary type) of the schema of quantifier–free induction

$$QF\text{–}IA\ :\ \bigwedge x^0\big(A_0(0) \wedge \bigwedge y < x(A_0(y) \rightarrow A_0(y')) \rightarrow A_0(x)\big)$$

since again by prop.1.2.6 there exists a term t such that $tx =_0 0 \leftrightarrow A_0(x)$: QF–IA now follows from (1) applied to $f := t$.

4) *Because of the axioms in 9), our theories are not recursively enumerable. The motivation for the addition of these sentences as axioms is two–fold:*
(i) As G. Kreisel has pointed out in various papers, proofs of $\mathbb{N}$*–true universal lemmas have no impact on bounds extracted from proofs using such lemmas. For the methods we use for the extraction of bounds (e.g. our monotone functional interpretation) this applies even for arbitrary universal sentences* $\bigwedge x^\rho A_0$ *where* ρ *may be an arbitrary type. Taking such sentences as axioms usually simplifies the process of the extraction of bounds enormously. The reason for our restriction to those sentences for which* $\rho \leq 2$ *is that on some places in this paper we deal with principles which are valid only in the type structure* $\mathcal{M}^\omega$ *of the so–called majorizable functionals (see chapter 7 below) but not in the full type structure* $\mathcal{S}^\omega$ *of all set–theoretic functionals. Since both type structures coincide up to type 1 and for the type 2 the inclusion* $\mathcal{M}_2^\omega \subset \mathcal{S}_2^\omega$ *holds, the implication* $\mathcal{S}^\omega \models \bigwedge x^\rho A_0 \Rightarrow \mathcal{M}^\omega \models \bigwedge x^\rho A_0$ *is obvious* **if** $\rho \leq 2$*. The same holds if we replace* $\mathcal{M}^\omega$ *by the type structure ECF of all extensional continuous functionals over* $\mathbb{N}^\mathbb{N}$ *(see [67] for details on ECF).*
(ii) Many important primitive recursive functions as $sg, \overline{sg}, |x-y|$ *and so on are already definable in* G_1A^ω*. However the usual proofs for their characteristic properties (which can be expressed as universal sentences) often make use of functions which are not definable in* G_1A^ω *(as e.g.* $x \cdot y$*). Thus we would have to carry out the boring details of a proof for these properties in* G_1A^ω*.*

Using $\tilde{R}_0$ the following primitive recursive functions can be defined easily in $\mathrm{G_1A^\omega}$:

Definition 1.2.4

1) $$\begin{cases} prd(0) =_0 0 \\ prd(x') =_0 x \quad \textit{(predecessor)}, \end{cases}$$

2) $$\begin{cases} sg(0) =_0 0 & \overline{sg}(0) =_0 1 \quad (1 := S0) \\ sg(x') =_0 1, & \overline{sg}(x') =_0 0, \end{cases}$$

3) $$\begin{cases} x \dot{-} 0 =_0 x \\ x \dot{-} y' =_0 prd(x \dot{-} y), \end{cases}$$

4) $|x-y| =_0 \max(x \dot{-} y, y \dot{-} x)$ *(symmetrical difference),*

5) $\varepsilon(x,y) =_0 sg(|x-y|)$ *(characteristic function for* $=_0$*),*

6) $\delta(x,y) =_0 \overline{sg}(|x-y|)$ *(characteristic function for* $\neq$*).*

Remark 1.2.5 *Because of the universal axioms in 9), the theory* $G_1A_i^\omega$ *proves the usual properties of the functions* $\max, \min, prd, sg, \overline{sg}, \dot{-}, |x-y|, \varepsilon$ *and* δ*, e.g.*
$sg(x) = 0 \leftrightarrow x = 0,\ \overline{sg}(x) = 0 \leftrightarrow x \neq 0,\ sg(x) \leq 1,\ \overline{sg}(x) \leq 1,\ prd(x) \leq x \dot{-} 1,$
$|x-y| = 0 \leftrightarrow x = y,\ x = 0 \vee x = S(prd(x)),\ \max(x,y) = 0 \leftrightarrow x = 0 \wedge y = 0,$
$\min(x,y) = 0 \leftrightarrow x = 0 \vee y = 0.$

Proposition: 1.2.6 *Let n be ≥ 1. For each formula $A \in \mathcal{L}(G_nA^\omega)$ which contains no quantifiers except for bounded quantifiers of type 0 one can construct a closed term t_A in G_nA^ω such that*

$$G_nA_i^\omega \vdash \bigwedge x_1^{\rho_1}, \ldots, x_k^{\rho_k}\big(t_A x_1 \ldots x_k =_0 0 \leftrightarrow A(x_1, \ldots, x_k)\big),$$

where $x_1, \ldots, x_k$ are all free variables of A.

Proof: Induction on the logical structure of A_0 using the remark above. Bounded quantifiers are captured by μ_b:

$$\mathrm{G}_n\mathrm{A}_i^\omega \vdash \bigvee y \leq_0 x A_0(x, y, \underline{a}) \overset{(\mu_b)}{\leftrightarrow} A_0(x, \mu_b(\lambda y.t_{A_0} xy\underline{a}, x), \underline{a}).$$

Proposition: 1.2.7 *Let $n \geq 1$, $A_0(\underline{x}) \in \mathcal{L}(G_nA^\omega)$, where $\underline{x} = x_1^{\rho_1} \ldots x_k^{\rho_k}$ are all free variables of A_0, and $t_1^{0\rho_k \ldots \rho_1}, t_2^{0\rho_k \ldots \rho_1}$ are closed terms of G_nA^ω. Then there exists a closed term $\Phi^{0\rho_k \ldots \rho_1}$ in G_nA^ω such that*

$$G_nA_i^\omega \vdash \bigwedge \underline{x} \left(\Phi\underline{x} =_0 \left\{ \begin{array}{l} t_1\underline{x}, \text{ if } A_0(\underline{x}) \\ t_2\underline{x}, \text{ if } \neg A_0(\underline{x}). \end{array} \right\} \right)$$

Proof: Define $t_2' := \lambda y^0, u^0.t_2\underline{x}$, $t_2'' := \lambda u^0.t_2\underline{x}$. One easily verifies that $\Phi := \lambda\underline{x}.\tilde{R}_\rho(t_{A_0}\underline{x})t_1 t_2' t_2''\underline{x}$ with t_{A_0} as in the previous proposition and $\rho = 0\rho_k \ldots \rho_1$ fulfils our claim.

Definition 1.2.8 (and lemma) *For* $\mathbf{n \geq 2}$ *we can define the surjective Cantor pairing function j ('diagonal counting from below') with its projections[18] in G_nR^ω:*

$$j(x^0, y^0) := \begin{cases} \min u \leq_0 (x+y)^2 + 3x + y[2u =_0 (x+y)^2 + 3x + y] \text{ if existent} \\ 0^0, \text{ otherwise,}^{19} \end{cases}$$
$$j_1 z := \min x \leq_0 z[\bigvee y \leq z(j(x, y) = z)],$$
$$j_2 z := \min y \leq_0 z[\bigvee x \leq z(j(x, y) = z)].$$

Using j, j_1, j_2 we can define a coding of k–tuples for every **fixed** *number k by*

$$\nu^1(x_0) := x_0,\ \nu^2(x_0, x_1) := j(x_0, x_1),\ \nu^{k+1}(x_0, \ldots, x_k) := j(x_0, \nu^k(x_1, \ldots, x_k)),$$
$$\nu_i^k(x_1, \ldots, x_k) := \begin{cases} j_1 \circ (j_2)^{i-1}(x), \text{ if } 1 \leq i < k \\ (j_2)^{k-1}(x), \text{ if } 1 < i = k \end{cases} \quad (\text{if } k > 1)$$

One easily verifies that $\nu_i^k(\nu^k(x_1, \ldots, x_k)) = x_i$ for $1 \leq i \leq k$ and $\nu^k(\nu_1^k(x), \ldots, \nu_k^k(x)) = x$. Finite sequences are coded (following [67]) by

$$\langle\rangle := 0,\ \langle x_0, \ldots, x_k\rangle := S(\nu^2(k, \nu^{k+1}(x_0, \ldots, x_k))).$$

Using $\tilde{R}$ one can define functions $lth, \Pi(k, y) \in G_nR^\omega$ such that for every fixed k, n

$$lth(\langle\rangle) = 0,\ lth(\langle x_0, \ldots, x_k\rangle) = k+1,\ \Pi(x, y) = \begin{cases} x_y, \text{ if } y \leq m \\ 0^0, \text{ otherwise} \end{cases} \text{ if } x = \langle x_0, \ldots, x_m\rangle.$$

[18]For detailed information on this as well as various other codings see [63] and also [13] (where j is called 'Cauchy's pairing function').

[19]One easily shows that $(x+y)^2 + 3x + y$ is always even (This can be expressed as a purely universal sentence, i.e. as an axiom in $\mathrm{G}_n\mathrm{A}^\omega$). Hence the case 'otherwise' never occurs and therefore $2j(x, y) = (x+y)^2 + 3x + y$ for all x, y.

Define

$$lth(x) := \begin{cases} 0^0, & \textit{if } x =_0 0 \\ j_1(x \dot{-} 1) + 1, & \textit{otherwise}, \end{cases}$$

$$\Pi(x,y) =_0 \begin{cases} 0^0, \textit{if } lthx = 0 \\ j_1 \circ (j_2)^y (x \dot{-} 1), & \textit{if } 1 \le y < lthx \\ (j_2)^{lthx}(y), & \textit{if } 1 < y = lthx \end{cases}$$

We usually write $(x)_y$ *instead of* $\Pi(x,y)$.
In order to verify that $\Pi(x,y)$ *is definable in* G_2R^ω *it suffices to show that the variable iteration* $\varphi xy = (j_2)^y(x)$ *of* j_2 *is definable in* G_2R^ω. *This however follows from the fact that* $\varphi xy \le x$ *for all* x, y. *Thus we can define* φxy *by* $\tilde{R}$ *using* $\lambda y.x$ *as bounding function.*
For $\mathbf{n \ge 3}$ *we can code initial segments of* **variable** *length of a function* f *in* G_nA^ω, *i.e. there is a functional* $\Phi_{\langle\rangle} \in G_3R^\omega$ *such that* $\Phi_{\langle\rangle} fx = \langle f0, \ldots, f(x \dot{-} 1)\rangle$.[20]
As an intermediate step we first show the definability of

$$\begin{cases} \tilde{f}0 = f0 \\ \tilde{f}x' = \tilde{j}(\tilde{f}x, fx'), & \textit{where } \tilde{j}(x,y) := j(y,x) \end{cases}$$

in G_3R^ω: *One easily verifies (using* $j(x,x) \le 4x^2$*) that* $\tilde{f}x \le 4^{3^x}\left(f^M x\right)^{2^x}$ *for all* x. *Hence the definition of* $\tilde{f}$ *can be carried out by* $\tilde{R}$ *using* $\lambda x.4^{3^{x'}}\left(f^M x'\right)^{2^{x'}} \in G_3R^\omega$ *as bounding function.* $\tilde{f}x$ *means* $\tilde{j}(\ldots\tilde{j}(\tilde{j}(f0,f1),f2)\ldots fx)$. *Hence* $\widehat{f}x := (\lambda y.\widetilde{f(x \dot{-} y)})x$ *has the meaning* $j(f0, j(f1, f2)), \ldots, fx)) \ldots)$. *We are now able to define* $\Phi_{\langle\rangle} \in G_3R^\omega$:

$$\Phi_{\langle\rangle} fx := \begin{cases} 0^0, & \textit{if } x = 0 \\ \widehat{(f_x)}x + 1, & \textit{otherwise}, \end{cases}$$

where

$$f_x y := \begin{cases} x, & \textit{if } y = 0 \\ f(y \dot{-} 1), & \textit{otherwise}. \end{cases}$$

We usually write $\overline{f}x$ *for* $\Phi_{\langle\rangle} fx$. *Furthermore one can define a function* $*$ *in* G_3R^ω *such that*

$$\langle x_0, \ldots, x_k\rangle * \langle y_0, \ldots, y_m\rangle = \langle x_0, \ldots, x_k, y_0, \ldots, y_m\rangle.$$

Define

$$n * m := \Phi_{\langle\rangle}(fnm)(lth(n) + lth(m)), \ \textit{where}$$

$$(fnm)(k) := \begin{cases} (n)_k, & \textit{if } k < lth(n) \\ (m)_{k \dot{-} lthn}, \textit{otherwise}. \end{cases}$$

[20] Of course we cannot write $\langle f0, \ldots, f(x \dot{-} 1)\rangle$ for variable x. However the meaning of $\Phi_{\langle\rangle} fx$ can be expressed via $(\Phi_{\langle\rangle} fx)_y = fy$ for all $y < x$ (and $= 0$ for $y \ge x$).

Note that $\Phi_{\langle\rangle}$ and $*$ are not definable in G_2R^ω since their definitions involve an iteration of the polynomial j.

Definition 1.2.9 *Between functionals of type ρ we define relations $\leq_\rho$ ('less or equal') and s–maj$_\rho$ ('strongly majorizes') by induction on the type:*

$$\begin{cases} x_1 \leq_0 x_2 :\equiv (x_1 \leq_0 x_2), \\ x_1 \leq_{\tau\rho} x_2 :\equiv \bigwedge y^\rho (x_1 y \leq_\tau x_2 y); \end{cases}$$

$$\begin{cases} x^* \ s\text{–}maj_0\ x :\equiv x^* \geq_0 x, \\ x^* \ s\text{–}maj_{\tau\rho}\ x :\equiv \bigwedge y^{*\rho}, y^\rho (y^* \ s\text{–}maj_\rho\ y \to x^* y^* \ s\text{–}maj_\tau\ x^* y,\ xy). \end{cases}$$

Remark 1.2.10 *'s–maj' is a variant of W.A. Howard's relation 'maj' from [26] which is due to [4]. For more details see [34].*

Lemma: 1.2.11 *$G_1 A_i^\omega$ proves the following facts:*

1) $\tilde{x}^* =_\rho x^* \wedge \tilde{x} =_\rho x \wedge x^* \ s\text{–}maj_\rho\ x \to \tilde{x}^* \ s\text{–}maj_\rho\ \tilde{x}.$

2) $x^* \ s\text{–}maj_\rho\ x \to x^* \ s\text{–}maj_\rho\ x^*$ ([4]).

3) $x_1 \ s\text{–}maj_\rho\ x_2 \wedge x_2 \ s\text{–}maj_\rho\ x_3 \to x_1 \ s\text{–}maj_\rho\ x_3$ ([4]).

4) $x^* \ s\text{–}maj_\rho\ x \wedge x \geq_\rho y \to x^* \ s\text{–}maj_\rho\ y.$

5) *For $\rho = \tau\rho_k \ldots \rho_1$ we have*

$$x^* \ s\text{–}maj_\rho\ x \leftrightarrow \bigwedge y_1^*, y_1, \ldots, y_k^*, y_k \Big(\bigwedge_{i=1}^{k} (y_i^* \ s\text{–}maj_{\rho_i}\ y_i) \to x^* y_1^* \ldots y_k^* \ s\text{–}maj_\tau\ x^* y_1 \ldots y_k, x y_1 \ldots y_k \Big).$$

6) $x^* \ s\text{–}maj_1\ x \leftrightarrow x^* \text{ monotone } \wedge x^* \geq_1 x,$

where x^* is monotone iff $\bigwedge u, v (u \leq_0 v \to x^* u \leq_0 x^* v)$.

7) $x^* \ s\text{–}maj_2\ x \to \lambda y^1 . x^*(\Phi_1 y) \geq_2 x.$

Proof: 1)–4) follow easily by induction on the type (in the proof of 3) one has to use 2)). 5) follows by induction on k using 2) (for details see [34]). 6) is trivial. 7) follows from $\bigwedge y^1 (\Phi_1 y$ s–maj$_1 y)$.

Remark 1.2.12 *In contrast to $\geq_\rho$ the relation s–maj$_\rho$ has a nice behaviour w.r.t. substitution (see 5) of the lemma above). This makes it possible to prove results on majorization of complex terms simply by induction on the term structure. For types ≤ 2 (which are used in our applications to analysis) we can infer from a majorant to a 'real' $\geq$–bound by 6) and 7) of lemma 1.2.11.*

Next we need some basic properties of A_j which are formulated in the following lemmas (since these properties are purely universal we only have to verify their truth in order to ensure their provability in $\mathrm{G}_n\mathrm{A}_i^\omega$ for $j \leq n$):

Lemma: 1.2.13 *Assume $j \geq 1$. Then $\bigwedge x \bigwedge y \geq 1 \big(A_j(x,y) \geq x\big)$.*

Proof: j–Induction: $j = 1:\ A_1(x,y) = x + y \geq x$.
$j \mapsto j+1:$ y–induction: $A_{j+1}(x,1) = A_j(x, A_{j+1}(x,0)) =$

$$= \begin{cases} A_j(x,0) = x + 0 \geq x, \text{ if } j = 1 \\ A_j(x,1) \overset{j-I.H.}{\geq} x, \text{ if } j \geq 2. \end{cases}$$

$$y \mapsto y+1: \quad A_{j+1}(x,y+1) = A_j(x, \underbrace{A_{j+1}(x,y)}_{\geq x\ (y-I.H.)}) \overset{j-I.H.}{\geq} x.$$

Lemma: 1.2.14 *For all $j \in \mathbb{N}$ the following holds:*

$$\bigwedge x, \tilde{x}, y, \tilde{y} \big(\tilde{x} \geq x \geq 1 \wedge \tilde{y} \geq y \to A_j(\tilde{x}, \tilde{y}) \geq A_j(x,y)\big).$$

Proof: j–Induction. For $j = 0, 1, 2$ the lemma is trivial. $j \mapsto j+1$: To begin with we verify (for $x \geq 1$) by y–induction

$$(*)\ \bigwedge y \big(A_{j+1}(x, y+1) \geq A_{j+1}(x,y)\big):$$

I. $A_{j+1}(x,1) \overset{1.2.13}{\geq} x \geq 1 \overset{j \geq 2}{=} A_{j+1}(x,0).)$
II. $y \mapsto y+1: \quad A_{j+1}(x,y+2) = A_j(x, \underbrace{A_{j+1}(x,y+1)}_{\overset{y-I.H.}{\geq A_{j+1}(x,y)}}) \overset{j-I.H.}{\geq} A_j(x, A_{j+1}(x,y)) = A_{j+1}(x,y+1).$

$(*)$ implies

$$(**)\ \bigwedge y \bigwedge \tilde{y} \geq y (A_{j+1}(x, \tilde{y}) \geq A_{j+1}(x,y)).$$

Again by y–induction we show (for $\tilde{x} \geq x \geq 1$):

$$(***)\ \bigwedge y (A_{j+1}(\tilde{x}, y) \geq A_{j+1}(x,y)):$$

$y = 0: \quad A_{j+1}$–definition! $y \mapsto y+1:$
$A_{j+1}(\tilde{x}, y+1) = A_j(\tilde{x}, \underbrace{A_{j+1}(\tilde{x},y)}_{\geq A_{j+1}(x,y)\ (y-I.H.)}) \overset{j-I.H.}{\geq} A_j(x, A_{j+1}(x,y)) = A_{j+1}(x,y+1).$
$(**)$ and $(***)$ yield the claim for $j+1$.

Lemma: 1.2.15 *If $j \geq 2$, then $\bigwedge y (A_j(0,y) \leq 1)$.*

Proof: j–Induction: The case $j = 2$ is clear.
$A_{j+1}(0,0) = 1,\ A_{j+1}(0,y+1) = A_j(0, A_{j+1}(0,y)) \overset{j-I.H.}{\leq} 1.$

Proposition: 1.2.16 *Φ_j s–maj Φ_j for all $j \geq 1$.*

Proof: Assume f^* s–maj$_1 f \wedge x^* \geq_0 x$. $j=1$:
$\Phi_1 f^* x^* = \max\limits_{y \leq x^*} f^* y \geq \max\limits_{y \leq x} fy = \Phi_1 fx$.
$j \geq 2$: By induction on x^* we show

$$\bigwedge x^* \bigwedge x \leq x^* (\Phi_j f^* x^* \geq_0 \Phi_j fx):$$

$x^* = 0$: $\Phi_j f^* 0 = f^* 0 \geq f0 = \Phi_j f0$.

$$\Phi_j f^*(x^*+1) = A_{j-1}(f^*(x^*+1), \Phi_j f^* x^*) \begin{cases} \overset{1.2.13, I.H.}{\geq} \Phi_j f0 \\ \overset{!}{\geq} A_{j-1}(f(x+1), \Phi_j fx) = \Phi_j f(x+1) \end{cases}.$$

Ad!: x^*–I.H. yields $\Phi_j f^* x^* \geq \Phi_j fx$. Because of f^* s–maj f it follows that $f^*(x^*+1) \geq f(x+1)$.
Case 1: $f(x+1) \geq 1$. Then '!' follows from 1.2.14 .
Case 2: $f(x+1) = 0$: 2.1 $f^*(x^*+1) = 0$. Then $\bigwedge y \leq x^*+1 (f^* y = fy = 0)$ and therefore $\Phi_j f^*(x^*+1) = \Phi_j f(x+1)$.
2.2 $f^*(x^*+1) \geq 1$: For $j \geq 3$ lemma 1.2.15 yields $A_{j-1}(f(x+1), \Phi_j fx) \leq 1$.
By lemma 1.2.13 we have $A_{j-1}(f^*(x^*+1), \Phi_j f^* x^*) \geq 1$, if $\Phi_j f^* x^* \geq 1$ (If $0 = \Phi_j f^* x^* \geq \Phi_j fx$, then $A_{j-1}(f(x+1), \Phi_j fx) \leq A_{j-1}(f^*(x^*+1), \Phi_j f^* x^*)$ follows immediately from the definition of A_{j-1}).
The case $j = 2$ is trivial.

Lemma: 1.2.17 *For every $j \geq 1$ the following holds:*

$$\bigwedge f \big(f \text{ monotone } \wedge f \geq 1 \to \bigwedge x (A_j(fx, x+1) \geq_0 \Phi_j fx)\big).$$

Proof: The case $j = 1$ is trivial. Assume $j \geq 2$. We proceed by induction on x:

$$A_j(f0, 1) = A_{j-1}(f0, A_j(f0, 0)) = \begin{cases} f0 = \Phi_j f0 \text{ for } j = 2 \\ A_{j-1}(f0, 1) \overset{1.2.13}{\geq} f0 = \Phi_j f0 \text{ for } j > 2. \end{cases}$$

$$A_j(f(x+1), x+2) = A_{j-1}(f(x+1), A_j(f(x+1), x+1)) \overset{fx' \geq fx \geq 1}{\geq} A_{j-1}(f(x+1), A_j(fx, x+1))(1.2.14)$$
$$\overset{I.H., 1.2.14}{\geq} A_{j-1}(f(x+1), \Phi_j fx) = \Phi_j f(x+1).$$

Proposition: 1.2.18 *For all $j \geq 1$: $\lambda f, x.A_j(fx+1, x+1)$ s–maj Φ_j*[21] .

Proof: Assume f^* s–maj f and $x^* \geq_0 x$. By prop.1.2.16 we know $\Phi_j(f^*+1)x^* \geq_0 \Phi_j fx$.
L.1.2.11 6) yields that f^*+1 is monotone. Hence – by l.1.2.17 ,1.2.14 – $A_j(f^*(x^*)+1, x^*+1) \geq A_j(fx+1, x+1), \Phi_j(f^*+1)x^*$.

Lemma: 1.2.19 *If $A_j^*(x,y) := \max(A_j(x,y), 1)$. Then A_j^* s–maj A_j.*

Proof: For $j \leq 2$ the lemma is trivial. Assume $j \geq 3$: We have to show

$$\bigwedge x, \tilde{x}, y, \tilde{y} \big(\tilde{x} \geq x \wedge \tilde{y} \geq y \to A_j^*(\tilde{x}, \tilde{y}) \geq A_j^*(x,y), A_j(x,y)\big):$$

If $x \geq 1$ this follows from l.1.2.14.
Assume $x = 0$. By l.1.2.15 $\bigwedge y (A_j^*(0,y), A_j(0,y) \leq 1)$ and therefore
$\bigwedge \tilde{x}, \tilde{y}, y \big(A_j^*(\tilde{x}, \tilde{y}) \geq A_j^*(0,y), A_j(0,y)\big)$ (since $A_j^*(\tilde{x}, \tilde{y}) \geq 1$).

[21] For $j = 1$ the more simple functional $\lambda f, x.fx$ already majorizes Φ_1.

Definition 1.2.20 *1) The subset $G_nR^\omega_- \subset G_nR^\omega$ denotes the set of all terms which are built up from $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}, A_0, \ldots, A_n, 0^0, S, prd, \min_0$ and $\max_0$ only (i.e. without $\Phi_1, \ldots, \Phi_n, \tilde{R}_\rho$ or μ_b).*

2) $G_nR^\omega_-[\Phi_1]$ is the set of all term built up from $G_nR^\omega_-$ plus Φ_1.

Proposition: 1.2.21 *For all $n \geq 1$ the following holds: To each term $t^\rho \in G_nR^\omega$ one can construct by induction on the structure of t (without normalization) a term $t^{*\rho} \in G_nR^\omega_-$ such that*

$$G_nA^\omega_i \vdash t^* \text{ s–}maj_\rho\ t.$$

Proof: 1. Replace every occurrence of $\tilde{R}_\rho$ in t by G_ρ, where

$$G_\rho := \lambda x, y, z, v, \underline{w}. \max_0(y\underline{w}, v(prd(x), \underline{w})).$$

G_ρ is built up from Π, Σ (which are used for defining the λ–operator) and the monotone functions $\max_0$ and prd. One easily verifies that

$$(i)\ G_\rho \geq \tilde{R}_\rho \text{ and } (ii)\ G_\rho \text{ s–maj } G_\rho.$$

Together with l.1.2.11 (i) and (ii) imply G_ρ s–maj $\tilde{R}_\rho$.
2. Replace all occurrences of $\Phi_1, \ldots, \Phi_n, \mu_b$ in t by

$$\Phi^*_1 := \lambda f, x.fx,\ \Phi^*_j := \lambda f, x.A_j(fx+1, x+1) \text{ for } i \geq 2,\ \mu^*_b := \lambda f, x.x.$$

By prop. 1.2.18 we conclude

$$\mathrm{G}_n\mathrm{A}^\omega_i \vdash \Phi^*_j \text{ s–maj } \Phi_j \wedge \mu^*_b \text{ s–maj } \mu_b.$$

3. Replace all occurrences of $A_0, \ldots, A_n$ in t by $A^*_0, \ldots, A^*_n$.
The term t^* which results after having carried out 1.–3. is $\in \mathrm{G}_n\mathrm{R}^\omega_-$. t^* is constructed by replacing every constant c in t by a closed term s^*_c such that s^*_c s–maj c. Since t is built up from constants only this implies (using lemma 1.2.11.1),5) t^* s–maj t.

Corollary to the proof:
Since $\lambda x^0.x^0$ s–maj$_1$ prd and A_1 s–maj $\max_0, \min_0$, the term t^* can be constructed even without $prd, \max_0$ and $\min_0$. However estimating $\max_0$ by A_1 may give away interesting numerical information. For the extraction of bounds from actually given proofs we may use not only max or min but any further functions which are convenient for the construction of a majorant which is numerically as sharp as possible.

The majorizing term t^* constructed in prop.1.2.21 will have (in general) a much simpler form than t since t^* does not contain any higher mathematical functional but only the 'logical' functionals Π and Σ. In the following we show that if t^* has a type ρ with $\deg(\rho) \leq 2$, than it can be simplified further by eliminating even these logical functionals. This will allow the exact calibration of the rate of growth of the definable functions of $\mathrm{G}_n\mathrm{A}^\omega$ and will be crucial also for our elimination of monotone Skolem functions in chapters 10 and 11 below.

Proposition: 1.2.22 *Assume $deg(\rho) \leq 2$ (i.e. $\rho = 0\rho_k \ldots \rho_1$ where $deg(\rho_i) \leq 1$ for $i = 1, \ldots, k$) and $t^\rho \in G_nR^\omega_-$. Then one can construct (by 'logical' normalization, i.e. by carrying out all possible Π, Σ–reductions) a term $\widehat{t}[x_1^{\rho_1}, \ldots, x_k^{\rho_k}]$ such that*

1) $\widehat{t}[x_1,\ldots,x_k]$ *contains at most* $x_1\ldots,x_k$ *as free variables,*

2) $\widehat{t}[x_1,\ldots,x_k]$ *is built up only from* $x_1,\ldots,x_k,A_0,\ldots,A_n,S^1,0^0,prd,\min_0,\max_0,$

3) $G_nA_i^\omega\vdash\bigwedge x_1^{\rho_1},\ldots,x_k^{\rho_k}(\widehat{t}[x_1,\ldots,x_k]=_0 tx_1\ldots x_k).$

Proof: We carry out reductions $\Pi st\rightsquigarrow s$ and $\Sigma str\rightsquigarrow sr(tr)$ in $tx_1\ldots x_k$ as long as no further such reduction is possible and denote the resulting term by $\widehat{t}[x_1,\ldots,x_k]$. The well–known strong normalization theorem for typed combinatory logic ensures that this situation will always occur after a finite number of reduction steps. Since $\Pi xy=x$ and $\Sigma xyz=xz(yz)$ are axioms of $\mathrm{G}_n\mathrm{A}_i^\omega$ the quantifier–free rule of extensionality yields

$$\mathrm{G}_n\mathrm{A}_i^\omega\vdash\bigwedge x_1^{\rho_1},\ldots,x_k^{\rho_k}(\widehat{t}[x_1,\ldots,x_k]=_0 tx_1\ldots x_k).$$

It remains to show that $\widehat{t}[x_1,\ldots,x_k]$ does not contain the combinators Π, Σ anymore:
Assume that $\widehat{t}[x_1,\ldots,x_k]$ contains an occurrence of Σ (resp. Π). Then Σ (Π) must occur in the form $\Sigma,\Sigma t_1$ or Σt_1t_2 $(\Pi,\Pi t_1)$ but not in the form $\Sigma t_1t_2t_3$ (resp. Πt_1t_2) since in the later case we could have carried out the reduction $\Sigma t_1t_2t_3\rightsquigarrow t_1t_3(t_2t_3)$ (resp. $\Pi t_1t_2\rightsquigarrow t_1$) contradicting the construction of $\widehat{t}$. All the terms $s=\Sigma,\Sigma t_1,\Sigma t_1t_2,\Pi,\Pi t_1$ have a type whose degree is ≥ 1. Hence s can occur in $\widehat{t}$ only in the form $r(s)$, where $r=\Sigma,\Sigma t_4,\Sigma t_4t_5,\Pi$ or Πt_4 since these terms are the only reduced ones requiring an argument of type ≥ 1, which can be built up from $x_1^{\rho_1},\ldots,x_k^{\rho_k},\Sigma,\Pi,A_i,S^1,0^0$ and $\max_0$ (because of $\deg(\rho_i)\leq 1$). Now the cases $r=\Sigma t_4t_5$ and $r=\Pi t_4$ can not occur since otherwise r(s) would allow a reduction of Σ resp. Π. Hence $r(s)$ is again a Π,Σ–term having a type of degree ≥ 1 and therefore has to occur within a term r' for which the same reasoning as for r applies etc. .... Thus we obtain a contradiction to the finite structure of $\widehat{t}$.

Remark 1.2.23 *Proposition1.2.22 becomes false if deg(ρ) = 3: Define* $\rho:=0(0(000))$ *and* $t^\rho:=\lambda x^{0(000)}.x(\Pi_{0,0})$. *Then* $tx=_0 x(\Pi_{0,0})$ *contains* Π *but no* Π*–reduction applies.*

Corollary 1.2.24 *Assume* $deg(\rho)\leq 2$ *(i.e.* $\rho=0\rho_k\ldots\rho_1$ *where* $deg(\rho_i)\leq 1$ *for* $i=1,\ldots,k$*) and* $t^\rho\in G_nR^\omega$. *Then one can construct (by majorization and subsequent 'logical' normalization) a term* $t^*[x_1^{\rho_1},\ldots,x_k^{\rho_k}]$ *such that*

1) $t^*[x_1,\ldots,x_k]$ *contains at most* $x_1\ldots,x_k$ *as free variables,*

2) $t^*[x_1,\ldots,x_k]$ *is built up only from* $x_1,\ldots,x_k,A_0,\ldots,A_n,S^1,0^0,prd,\min_0,\max_0,$

3) $G_nA_i^\omega\vdash\lambda x_1,\ldots,x_k.t^*[x_1,\ldots,x_k]$ *s–maj* t.

Proof: The corollary follows immediately from prop.1.2.21 and prop.1.2.22 (using lemma 1.2.11 (1)).

The use of the concept of majorization combined with logical normalization has enabled us to majorize a term t of type ≤ 2 by a term t^* which does not contain any functionals of type >1. This allows the calibration of the rate of growth of the functions given by $t^1\in\mathrm{G}_n\mathrm{R}^\omega$ in usual mathematical terms **without any computation of recursor terms** (which would require the reduction of closed number terms to numerals):

Definition 1.2.25 ([22] ,[54]) *The function $f(\underline{x}, y)$ is defined from $g(\underline{x}), h(\underline{x}, y, z)$ and $j(\underline{x}, y)$ by* **limited recursion** *if*

$$\begin{cases} f(\underline{x}, 0) =_0 g(\underline{x}) \\ f(\underline{x}, y+1) =_0 h(\underline{x}, y, f(\underline{x}, y)) \\ f(\underline{x}, y) \leq_0 j(\underline{x}, y). \end{cases}$$

Definition 1.2.26 (n-th level of the Grzegorczyk hierarchie) *For each $n \geq 0$, $\mathcal{E}^n$ is defined to be the smallest class of functions containing the successor function S, the constant–zero function, the projections $U_i^n(x_1, \ldots, x_n) = x_i$, and $A_n(x, y)$ which is closed under substitutions and limited recursion.*

Remark 1.2.27 *Grzegorczyk's original definition of $\mathcal{E}^n$ uses somewhat different functions $g_n(x, y)$ instead of $A_n(x, y)$. Ritchie ([54]) showed that the same class of $\mathcal{E}^n$ of functions results if the g_n are replaced by the (more natural) A_n (which are denoted by f_n in [54]). See also [13] for a proof of this result.*

Proposition: 1.2.28 *Assume $t^1 \in G_nR^\omega$. Then one can construct a function $f_t \in \mathcal{E}^n$ such that $\bigwedge x^0(tx \leq_0 f_t x)$ and every function $f \in \mathcal{E}^n$ can be defined in G_nR^ω, i.e. there is a term $t^1_f \in G_nR^\omega$ such that $\bigwedge x^0(fx = tx)$.*
In particular for $n = 1, 2, 3$ the following holds:

$$\begin{cases} t^1 \in G_1R^\omega \Rightarrow \exists c_1, c_2 \in \mathbb{N}: \ G_1A_i^\omega \vdash \bigwedge x^0(tx \leq_0 c_1x + c_2) \ \textit{(linear growth)}, \\ t^1 \in G_2R^\omega \Rightarrow \exists k, c_1, c_2 \in \mathbb{N}: \ G_2A_i^\omega \vdash \bigwedge x^0(tx \leq_0 c_1x^k + c_2) \ \textit{(polynomial groth)}, \\ \ t^1 \in G_3R^\omega \Rightarrow \exists k, c \in \mathbb{N}: \ G_3A_i^\omega \vdash \bigwedge x^0(tx \leq_0 2_k^{cx}), \ \textit{where } 2_0^a = a, 2_{k'}^a = 2^{2_k^a} \\ \qquad\qquad\qquad\qquad \textit{(finitely iterated exponential growth).} \end{cases}$$

More generally, if t^ρ (where $\rho = 0\underbrace{(0)\ldots(0)}_{m-times}$), defines an m–ary function:

$$\begin{cases} t^\rho \in G_1R^\omega \Rightarrow \exists c_1, \ldots, c_{m+1} \in \mathbb{N}: \ G_1A_i^\omega \vdash \bigwedge x_1^0, \ldots, x_m^0(t\underline{x} \leq_0 c_1x_1 + \ldots + c_mx_m + c_{m+1}), \\ t^\rho \in G_2R^\omega \Rightarrow \exists p \in \mathbb{N}[x_1, \ldots, x_m]: \ G_2A_i^\omega \vdash \bigwedge \underline{x}(t\underline{x} \leq_0 p\underline{x}), \\ t^\rho \in G_3R^\omega \Rightarrow \exists k, c_1, \ldots, x_m \in \mathbb{N}: \ G_3A_i^\omega \vdash \bigwedge \underline{x}(t\underline{x} \leq_0 2_k^{c_1x_1+\ldots+c_mx_m}). \end{cases}$$

The constants $c_i, k \in \mathbb{N}$ in can be effectively written down for each given term t.

Proof: To t^1 we construct $\widehat{t}[x]$ (according to cor.1.2.24 and the corollary to the proof of 1.2.21) such that $\widehat{t}[x]$ is built up from $x^0, 0^0$ and $A_0, \ldots, A_n$, and $\lambda x.\widehat{t}[x]$ s–maj$_1$ tx. The later property implies $\bigwedge x^0(\widehat{t}[x] \geq_0 tx)$. By [54] (p. 1037) we know that $A_0, \ldots, A_n \in \mathcal{E}^n$. Since $\mathcal{E}^n$ is closed under substitution it follows that $f_t := \lambda x.\widehat{t}[x] \in \mathcal{E}^n$.
For the other direction assume $f \in \mathcal{E}^n$. Since $\mathrm{G}_n\mathrm{R}^\omega$ contains $S, \lambda x.0^0$, the projections U_i^k and A_n, and it is closed under substitution (because λ–abstraction is available) and limited recursion (because of $\tilde{R}$) it follows that f is definable in $\mathrm{G}_n\mathrm{R}^\omega$.
We now consider the special cases $n = 1, 2, 3$:

$n=1$: Assume $t^\rho \in \mathrm{G}_1\mathrm{R}^\omega$ where $\rho = 0\underbrace{(0)\ldots(0)}_{m}$. $\widehat{t}[x_1^0,\ldots,x_m^0]$ is built up from $0^0, A_0$ and A_1 only.

Both $A_0(x_1,x_2) = 0\cdot x_1 + 1\cdot x_2 + 1$ and $A_1(x_1,x_2) = 1\cdot x_1 + 1\cdot x_2 + 0$ are functions having the form $c_1x_1 + c_2x_2 + c_3$ or – more generally – $c_1x_1 + \ldots + c_kx_k + c_{k+1}$. Since substitution of such functions again yields a function which can be written in this form it follows that $\widehat{t}[x_1,\ldots,x_m] = c_1x_1 + \ldots + c_mx_m + c_{m+1}$ for suitable constants $c_1,\ldots,c_{m+1}$.
$n=2$: Assume $t^\rho \in \mathrm{G}_2\mathrm{R}^\omega$. $\widehat{t}[x_1,\ldots,x_m]$ is built up from $0^0, A_0, A_1, A_2$. Since A_0, A_1 and A_2 are polynomials (in two variables) and substitution of polynomials in several variables yields a function which can be written again as a polynomial, it is clear that $\widehat{t}[x_1,\ldots,x_m] = p(x_1,\ldots,x_m)$ for a suitable polynomial in $\mathbb{N}[x_1,\ldots,x_m]$. In the case $m=1$, $p(x)$ can be bounded by $c_1x^k + c_2$ for suitable numbers c_1, c_2.
$n=3$: Assume $t^\rho \in \mathrm{G}_3\mathrm{R}^\omega$. For all x,y the following inequalities hold:
$(*)$ $A_3(x+2,y+2) \geq A_2(x,y), A_1(x,y), A_0(x,y), 2$. Replace in $\widehat{t}[x_1,\ldots,x_m]$ all occurrences of 0 by 2 and all occurrences of $A_i(x,y)$ with $i \leq 2$ by $A_3(x+2,y+2)$ and denote the resulting term by $\tilde{t}[x_1,\ldots,x_m]$.
$(*)$ together with the monotonicity of $A_3(x,y)$ in x,y for $x,y \geq 2$ yields

$$\bigwedge x_1,\ldots,x_m\big(\tilde{t}[x_1,\ldots,x_m] \geq \widehat{t}[x_1,\ldots,x_m] \geq tx_1\ldots x_m\big).$$

$\tilde{t}[x_1,\ldots,x_m]$ is built up from $x_1,\ldots,x_m$,$+2$ and A_3 only. Let k be the number of A_3–occurrences in $\tilde{t}[x_1,\ldots,x_m]$. Then $\tilde{t}[x_1,\ldots,x_m]$ can be bounded by y_k, where $y_0 := 0,\ y_{k'} := y^{y_k}$ and $y := \max(x_1,\ldots,x_m)+2$. By [44] we have $y_k \leq 2^{\underline{x}}_k$, where $2^{\underline{x}}_1 := x_1 + \ldots + x_m$ and $2^{\underline{x}}_{k'} = 2^{2^{\underline{x}}_k}$. Hence $\bigwedge \underline{x}(2^{\underline{x}}_k \geq t\underline{x})$.

Remark 1.2.29 *This proposition provides a quite perspicuous characterization of the rate of growth of the functions which are definable in G_nA^ω. Of course for concrete terms t the bounds given for $n=1,2,3$ may be to rough. To obtain better estimates one will use combinations of any convenient functions like e.g.* $\max, \min$ *(instead of replacing them by $x+y$) and (for $n=3$) the growth of t will be expressed using* $\max, \min, A_0, A_1, A_2$ *and* A_3 *and not* A_3 *allone. Thus one can treat also all intermediate levels between e.g. polynomial and iterated exponential growth.*

The estimates for $n=1,2,3$ generalize to function parameters as follows: Let $t^{1(1)} \in \mathrm{G}_n\mathrm{R}^\omega$, then tf^1 can be bounded by a linear (polynomial resp. elementary recursive) function **in** f^* where
f^* s–maj f. By **'$\mathbf{tf^1x^0}$ is linear (polynomial, elementary recursive) in $\mathbf{f,x}$'** we mean that $tfx =_0 \tilde{t}[x,f]$ for all x,f, where $\tilde{t}[x,f]$ is a term which is built up only from $x, f, 0^0, S^1, +$ ($x, f, 0^0, S^1, +, \cdot$ resp. $x, f, 0^0, S^1, +, \cdot, (\cdot)^{(\cdot)}$). In particular this implies that if f^* is a linear (polynomial, elementary recursive) function then tf^* can be written again as a linear (polynomial, elementary recursive) function. This holds even uniformly in the following sense (which we formulate here explicitly only for the most interesting polynomial case):

Proposition: 1.2.30 *Let $t^{1(1)} \in G_2R^\omega$. Then one can construct a polynomial $q \in \mathbb{N}[x]$ such that*

$$\begin{cases} \textit{For every polynomial } p \in \mathbb{N}[x] \\ \textit{one can construct a polynomial } r \in \mathbb{N}[x] \textit{ such that} \\ \bigwedge f^1\big(f \leq_1 p \to \bigwedge x^0(tfx \leq_0 r(x))\big) \textit{ and } deg(r) \leq q(deg(p)) \end{cases}$$

This extends to the case where t has tuples $f_1^1, \ldots, f_k^1, x_1^0, \ldots, x_l^0$ of arguments with $f_1, \ldots, f_k \leq_1 p$ and $r \in \mathbb{N}[x_1, \ldots, x_l]$.

Proof: Let $p \in \mathbb{N}[x]$. Since p is monotone, $f \leq p$ implies p s–maj f. Let $\widehat{t}[f, x]$ be constructed to tf according to prop.1.2.22 and the corollary to the proof of prop.1.2.21. Then $\widehat{t}[p, x] \geq_0 tfx$ for all $f \leq_1 p$ and $\widehat{t}[p, x]$ is built up from $x, 0^0, A_0, A_1$ and p only. As in the proof of prop.1.2.28 one concludes that $\widehat{t}[p, x]$ can be written as a polynomial r in x. The existence of the polynomial q bounding the degree of r in the degree of p follows from the fact that the degree of a polynomial $p_1 \in \mathbb{N}[x_1, \ldots, x_m]$ obtained by substitution of a polynomial p_2 for one variable in a polynomial p_3 is $\leq \deg(p_2) \cdot \deg(p_3)$.

1.3 Extensions of $\mathbf{G}_n\mathbf{A}^\omega$

Definition 1.3.1 *1) Let $G_\infty A^\omega$ denote the union of the theories $G_n A^\omega$ for all $n \geq 1$ and $G_\infty A_i^\omega$ its intuitionistic variant.*
E–$G_\infty A^\omega$ and E–$G_\infty A_i^\omega$ are the corresponding theories with full extensionality.
$G_\infty R^\omega$ is the set of all closed terms of these theories, i.e. $G_\infty R^\omega := \bigcup_{n \in \mathbb{N}} G_n R^\omega$.

2) PRA^ω is the theory obtained from $G_\infty A^\omega$ by adding the Kleene–recursor operators $\widehat{R}_\rho$ (on which S. Feferman's theory $\widehat{PA}^\omega\!\upharpoonright$ is based on; see [11]):

$$\begin{cases} \widehat{R}_\rho 0yz\underline{v} =_0 y\underline{v} \\ \widehat{R}_\rho (Sx)yz\underline{v} =_0 z(\widehat{R}_\rho xyz\underline{v})x\underline{v}, \end{cases}$$

where $y \in \rho, z \in \rho 00$ and $\underline{v} = v_1^{\rho_1} \ldots v_k^{\rho_k}$ are such that $y\underline{v}$ is of type 0.
Correspondingly we have the theories PRA_i^ω, E–PRA^ω and E–PRA_i^ω.
The set of all closed terms of PRA^ω is denoted by $\widehat{PR}^\omega$.

Thus PRA^ω is equivalent to $\widehat{PA}^\omega\!\upharpoonright$+all true $\bigwedge x\, A_0$–sentences for $\rho \leq 2$. We now show that the same theory results if we only add the (unrestricted) iteration functional Φ_{it} together with the axioms

$$\begin{cases} \Phi_{it} 0yf =_0 y \\ \Phi_{it} x'yf =_0 f(\Phi_{it} xyf) \ \ i.e. \Phi_{it} xyf = f^x y \end{cases}$$

instead of $\widehat{R}$:
We define $\widehat{R}_\rho$ through one intermediate step:
Firstly we show that $\widehat{R}_\rho$ can be defined from $\tilde{\Phi}$, where

$$\begin{cases} \tilde{\Phi} 0yf =_0 y \\ \tilde{\Phi} x'yf =_0 f(\tilde{\Phi} xyf)x \quad (f \in 0(0)(0)). \end{cases}$$

One easily verifies that $\widehat{R}_\rho$ can be defined as

$$\widehat{R}_\rho := \lambda x, y, z, \underline{v}.\tilde{\Phi} x(y\underline{v})(\lambda x_1^0, x_2^0. zx_1x_2\underline{v}).$$

$\tilde{\Phi}$ in turn is definable using Φ_{it}: This follows from the fact that for $\tilde{f}x := \max(\Phi_1(\lambda y_1.\Phi_1(\lambda y_2.fy_1y_2)x)x, x')(= \max\limits_{y_1,y_2\leq_0 x}(fy_1y_2, x'))$ one has $\Phi_{it}xy\tilde{f} \geq_0 \tilde{\Phi}xyf$ for all x, y, f. Thus using Φ_{it} as a bound in the recursion one can define $\tilde{\Phi}$ by the bounded recursor operator $\tilde{R}$. Put together we have shown that $\widehat{R}_\rho$ is definable in PRA$^\omega$. Since on the other hand Φ_{it} is trivially definable using $\widehat{R}$ our claim follows.

On the level of type 1 the theories PRA$^\omega$ and G$_\infty$A$^\omega$ coincide: The functions given by the closed terms of type level 1 of both theories are just the primitive recursive ones: For PRA$^\omega$ this follows from [11]. Since G$_\infty$A$^\omega$ is a subtheory of PRA$^\omega$ it suffices to verify that all primitive recursive functions are definable in it. This however follows immediately from prop.1.2.28 and the well–know fact (due to Grzegorczyk) that the class of all primitive recursive functions is just the union of all $\mathcal{E}^n$.
In contrast to this, both theories differ already on the type–2–level:

Proposition: 1.3.2 *The functional Φ_{it} is not definable in $G_\infty A^\omega$, i.e. there is no term $t \in G_\infty R^\omega$ such that t satifies (provable in $G_\infty A^\omega$) the defining equations of Φ_{it}.*

Proof: Assume that Φ_{it} is definable in G$_\infty$A$^\omega$. Then there exists an n such that Φ_{it} is already definable in G$_n$A$^\omega$. On the hand from the proof above we know that within G$_n$A$^\omega$ + Φ_{it} the unbounded recursors $\widehat{R}_\rho$ and therefore all primitive recursive functions (in particular A_{n+1}) are definable. Hence A_{n+1} could be defined in G$_n$A$^\omega$ contradicting prop.1.2.28, since A_{n+1} cannot be bounded by a function from $\mathcal{E}^n$ (see [54]).

Finally we introduce the theory PA$^\omega$ which results from PRA$^\omega$ if

1) $\widehat{R}_\rho$ is replaced by the Gödel–recursor operators R_ρ characterized by

$$\begin{cases} R_\rho 0yz =_\rho y \\ R_\rho x'yz =_\rho z(R_\rho xyz)x, \quad \text{where } y \in \rho, z \in \rho 0 \rho, \end{cases}$$

2) the schema of full induction

$$(\mathrm{IA}) \ : \ A(0) \wedge \bigwedge x(A(x) \to A(x')) \to \bigwedge x A(x)$$

is added.
The set of all closed terms of PA$^\omega$ is denoted by T (following Gödel).
PA$^\omega_i$ is the intuitionistic variant of PA$^\omega$. E–PA$^\omega$, E–PA$^\omega_i$ are the corresponding theories with full extensionality (E).

In this chapter we have introduced a hierarchy G$_1$A$^\omega$, G$_2$A$^\omega$, ..., PRA$^\omega$ of subsystems of arithmetic in all finite types PA$^\omega$. Furthermore we have determined the growth of the functionals $t^{1(1)}$ which are definable in these theories. In particular for $n \leq 3$ it turned out that t can be majorized by a term t^* of type 1(1) such that

$t^*f^1x^0$ is a linear function in f, x, if $n = 1$,

$t^*f^1x^0$ is a polynomial function in f, x, if $n = 2$,

$t^*f^1x^0$ is an elementary recursive function in f, x, if $n = 3$, and in the case $n = 2$, for every polynomial p^1 there is a polynomial r^1 such that $t^* f x \leq_0 r x$ for all $f \leq_1 p$.

In the following chapters these theories (in particular $\mathrm{G}_2\mathrm{A}^\omega$) will be used as base theories to measure the impact on the growth of provably recursive functionals of many analytical principles.

2 Monotone functional interpretation of $\mathbf{G}_n\mathbf{A}^\omega$, $\mathbf{PRA}^\omega$,$\mathbf{PA}^\omega$ and their extensions by analytical axioms: the rate of growth of provable function(al)s

2.1 Gödel functional interpretation

Definition 2.1.1 *The schema of the quantifier–free axiom of choice is given by*

$$AC^{\rho,\tau}\text{–}qf\ :\ \bigwedge x^\rho \bigvee y^\tau A_0(x,y) \to \bigvee Y^{\tau\rho} \bigwedge x^\rho A_0(x,Yx),$$

where A_0 is a quantifier–free formula of the respective theory.

$$AC\text{–}qf := \bigcup_{\rho,\tau\in\mathbf{T}} \{AC^{\rho,\tau}\text{–}qf\}.$$

If

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \bigwedge x^\rho \bigvee y^\tau A_0(x,y),$$

then

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{\rho,\tau}\text{–qf}\ \vdash \bigvee Y^{\tau\rho} \bigwedge x^\rho A_0(x,Yx).$$

In order to determine the growth which is implicit in the functional dependency '$\bigwedge x^\rho \bigvee y^\tau$' we have to determine the rate of growth of a functional term which realizes (or bounds) '$\bigvee Y^{\tau\rho}$'. Let A' denote one of the well–known negative translations of A (see [43] for a systematical treatment) and A^D be the Gödel functional interpretation of A (as defined in [43] or [67]).
A^D has the logical form

$$\bigvee \underline{x} \bigwedge \underline{y} A_D(\underline{x},\underline{y},\underline{a}),$$

where A_D is quantifier–free, $\underline{x},\underline{y}$ are tuples of variables of finite type and $\underline{a}$ is the tuple of all free variables of A. For our theories this functional interpretation holds:

Theorem 2.1.2 *Let Γ be a set of purely universal sentences $F \equiv \bigwedge u^\gamma F_0(u) \in \mathcal{L}(G_nA^\omega)$ and $n \in \mathbb{N}\cup\{\infty\}$ $(n \geq 1)$. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Gamma + AC\text{–}qf\ \vdash A \Rightarrow \exists \underline{t} \in G_nR^\omega \text{ such that} \\ G_nA_i^\omega + \Gamma \vdash \bigwedge \underline{y}\left((A')_D(\underline{t}\underline{a},\underline{y},\underline{a})\right). \end{cases}$$

$\underline{t}$ can be extracted from a given proof
(An analogous result holds if G_nA^ω, G_nR^ω, $G_nA_i^\omega$ are replaced by PRA^ω, $\widehat{PR}^\omega$, PRA_i^ω or PA^ω, T, PA_i^ω).

Proof: For PA^ω the proof is given e.g. in [67]. The interpretation of the logical axioms and rules only requires the closure under λ–abstraction, definition by cases and the existence of characteristic functionals for the prime formulas. All this holds in $\mathrm{G}_n\mathrm{R}^\omega$ and $\widehat{PR}^\omega$. The interpretation of the universal axioms is trivial.

Corollary 2.1.3 *Let Γ be as above and $A_0(\underline{x}, \underline{y})$ is a quantifier–free formula which has only $\underline{x}, \underline{y}$ as free variables. Then*

$$\begin{cases} G_nA^\omega + \Gamma + AC\text{–}qf \vdash \bigwedge \underline{x} \bigvee \underline{y} A_0(\underline{x}, \underline{y}) \Rightarrow \exists \underline{t} \in \ G_nR^\omega \ \textit{such that}: \\ G_nA_i^\omega + \Gamma \vdash \bigwedge \underline{x} A_0(\underline{x}, \underline{tx}) \end{cases}$$

(Analogously for PRA^ω and PA^ω).

By the well–known elimination procedure for the extensionality axiom (E) one may replace $\mathrm{G}_n\mathrm{A}^\omega$ by E–$\mathrm{G}_n\mathrm{A}^\omega$ if the types of $\underline{x}$ are ≤ 1 and the types in AC–qf are somewhat restricted:

Corollary 2.1.4 *Assume that $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$, and $\underline{x} = x_1^{\rho_1}, \ldots, x_k^{\rho_k}$ where $\rho_i \leq 1$ for $i = 1, \ldots, k$. Then*

$$\begin{cases} E\text{–}G_nA^\omega + \Gamma + AC^{\alpha,\beta}\text{–}qf \ \vdash \bigwedge \underline{x} \bigvee \underline{y} A_0(\underline{x}, \underline{y}) \Rightarrow \exists \underline{t} \in G_nR^\omega \ \textit{such that}: \\ G_nA_i^\omega + \Gamma \vdash \bigwedge \underline{x} A_0(\underline{x}, \underline{tx}) \end{cases}$$

(Analogously for E–PRA^ω and E–PA^ω).

Proof: The corollary follows from the previous corollary using the elimination of extensionality procedure as carried out in [43] and observing the following facts:

1) The hereditary extensionality of $\tilde{R}_\rho$ (i.e. $\mathrm{Ex}(\tilde{R})$ in the notation of [43]) can be proved by (QF–IA). Similarly for Φ_1. The heriditary extensionality of μ_b follows easily from the axioms μ_b.

2) $(\mathrm{AC}^{1,0}\text{–qf})_e$ is provable by bounded search using μ_b and prop. 1.2.6 .

3) For $F \in \Gamma$ the implication $F \rightarrow F_e$ holds logically.

2.2 Monotone functional interpretation

In [39] we introduced a new **monotone functional interpretation** which extracts instead of a realizing term t for $\bigvee y$ in cor.2.1.3 a 'bound' t^* for t (in the sense of s–maj, which for types ≤ 2 provides a $\geq$–bound by lemma 1.2.11.7). This is sufficient in order to estimate the rate of growth of t. The construction of t^* does not cause any rate of growth in addition to that actually involved in a given proof since besides the terms from the proof only the functionals $\max_\rho$[22] and Φ_1 are used (For the theories $\mathrm{G}_n\mathrm{A}^\omega$ even Φ_1 is not necessary for the construction of t^* but only for the very simple transformation of t^* into a $\geq$–bound for type ≤ 2 by lemma 1.2.11). This has been confirmed in applications to concrete proofs in approximation theory where t^* could be used to improve known estimates significantly (see [37] ,[38] ,[39]). In most applications in analysis the formula $\bigwedge x \bigvee y A(x, y)$ $(A \in \Sigma_1^0)$ will be monotone w.r.t. y, i.e.

$$\bigwedge x, y_1, y_2 \big(y_2 \geq y_1 \wedge A(x, y_1) \rightarrow A(x, y_2)\big),$$

and thus the bound t^* in fact also realizes '$\bigvee y$' (This phenomenon is discussed in [39]).
The monotone functional interpretation has various properties which are important for the following but do not hold for the usual functional interpretation:

[22] $\max_{\tau\rho}(x_1^{\tau\rho}, x_2^{\tau\rho}) := \lambda y^\rho . \max_\tau(x_1 y, x_2 y)$.

1) The extraction of t^* by monotone functional interpretation from a given proof is much easier than the extraction of t provided by the usual functional interpretation: E.g. no decision of prime formulas and no functionals defined by cases are needed for the construction of t^* (but only for its verification) since the logical axioms $A \to A \wedge A$ and $A \vee A \to A$ have a simple **monotone** functional interpretation (whereas these axioms are the difficult ones for the usual functional interpretation). Because of this also the structure of the term t^* is more simple than that of t, in particular $t^* \in \mathrm{G}_n\mathrm{R}^\omega_-$ whereas $t \in \mathrm{G}_n\mathrm{R}^\omega$.

2) The bound t^* obtained by monotone functional interpretation for $\bigvee z^\tau$ in sentences $\bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee z^\tau A_0(x,y,z)$ **does not depend on** y, i.e. $\bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee z \leq_\tau t^*x\ A_0(x,y,z)$ (Here $\tau \leq 2$ and s is a closed term).

   The **most important property of our monotone functional interpretation** however is the following

3) Sentences of the form

   $$(*)\ \bigwedge x^\gamma \bigvee y \leq_\delta sx \bigwedge z^\eta A_0(x,y,z)$$

   have a simple monotone functional interpretation which is fulfilled by any term s^* such that s^* s–maj s (see [39]). This means that sentences $(*)$ although covering many strong non–constructive analytical theorems which usually do not have a functional interpretation in the usual sense (not even in T) (as we will see in chapter 7 below) do not contribute to the growth of the bound t^* by their proofs but only by the term s and therefore can be treated simply as axioms.

Definition 2.2.1 (bounded choice) *The schema of 'bounded' choice is defined as*

$$(b\text{–}AC^{\delta,\rho})\ :\ \bigwedge Z^{\rho\delta}\big(\bigwedge x^\delta \bigvee y \leq_\rho Zx\ A(x,y,Z) \to \bigvee Y \leq_{\rho\delta} Z \bigwedge x A(x,Yx,Z)\big),$$

$$b\text{–}AC\ :=\ \bigcup_{\delta,\rho\in\mathbf{T}} \Big\{(b\text{–}AC^{\delta,\rho})\Big\}.$$

(a discussion of this principle can be found in [34]).

Theorem 2.2.2 *Let Δ be a set of sentences having the form $\bigwedge u^\gamma \bigvee v \leq_\delta tu \bigwedge w^\eta F_0(u,v,w)$, where $t \in G_nR^\omega$. Then the following rule holds*

$$\begin{cases} \text{From a proof } G_nA^\omega + \Delta + AC\text{–}qf \vdash (A)' \\ \text{one can extract by monotone functional interpretation a tuple } \underline{\Psi} \in G_nR^\omega_- \text{ such that} \\ G_nA^\omega_i + \Delta + b\text{-}AC \vdash (\underline{\Psi} \text{ satisfies the monotone functional interpretation of } (A)'), \end{cases}$$

where $(A)'$ denotes the negative translation of A.
In particular for $A_0(x,y,z)$ containing only x,y,z free and $s \in G_nR^\omega$ the following rule holds for $\tau \leq 2$:

$$\begin{cases} \text{From a proof } G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee z^\tau A_0(x,y,z) \\ \text{by monotone functional interpretation one can extract a } \Psi \in G_nR^\omega_-[\Phi_1] \text{ such that} \\ G_nA^\omega_i + \Delta + b\text{–}AC \vdash \bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee z \leq_\tau \Psi x\ A_0(x,y,z). \end{cases}$$

Ψ is built up from $0^0, 1^0, \max_\rho, \Phi_1$ and majorizing terms[23] *(for terms t occurring in those quantifier axioms $\bigwedge xGx \to Gt$ and $Gt \to \bigvee xGx$ which are used in the given proof) by use of λ–abstraction and substitution.*
If $\tau \le 1$ (resp. $\tau = 2$) then Ψ has the form $\Psi \equiv \lambda x^1.\Psi_0 x^M$ (resp. $\Psi \equiv \lambda x^1, y^1.\Psi_0 x^M y^M$), where $x^M := \Phi_1 x$ and Ψ_0 does not contain Φ_1
(An analogous result holds for PRA^ω, PA^ω).

Corollary 2.2.3 *For $1 \le n \le 3$ the following holds (for $A_0(x^0, y^\rho, z^0)$ containing only x, y, z free)*

$$G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge x^0 \bigwedge y \le_\rho sx \bigvee z^0 A_0(x,y,z) \Rightarrow$$

$$\begin{cases} \exists c_1, c_2 \in \mathbb{N} : G_1A_i^\omega + \Delta + b\text{–}AC \vdash \bigwedge x^0 \bigwedge y \le_\rho sx \bigvee z \le_0 c_1x + c_2 \; A_0(x,y,z), & \text{if } n = 1 \\ \exists k, c_1, c_2 \in \mathbb{N} : G_2A_i^\omega + \Delta + b\text{–}AC \vdash \bigwedge x^0 \bigwedge y \le_\rho sx \bigvee z \le_0 c_1x^k + c_2 \; A_0(x,y,z), & \text{if } n = 2 \\ \exists k, c \in \mathbb{N} : G_3A_i^\omega + \Delta + b\text{–}AC \vdash \bigwedge x^0 \bigwedge y \le_\rho sx \bigvee z \le_0 2_k^{cx} A_0(x,y,z), & \text{if } n = 3. \end{cases}$$

This generalizes to the case $\bigwedge x^0, \tilde{x}^1 \bigwedge y \le_\rho sx\tilde{x} \bigvee z^0 A_0$: One obtains a bound which linear (polynomially, elementary recursive) in $x^0, \tilde{x}^M$ in the sense of chapter 1 for $n = 1$ ($n = 2$,$n = 3$) and for $n = 2$ prop.1.2.30 applies.

Remark 2.2.4 *1) For $\delta, \rho \le 1$ the theory G_nA^ω may be strengthened to $E\text{–}G_nA^\omega$ in thm.2.2.2 and cor.2.2.3 if AC–qf is restricted as in 2.1.4 .*

2) Theorem 2.2.2 and cor.2.2.3 generalize immediately to tuples $\underline{x}, \underline{y}, \underline{z}$ of variables instead of x, y, z, if b–AC is formulated for tuples. Furthermore instead of $\bigvee w^\tau A_0$ we may also have $\bigvee z^\tau \bigvee z' A_0$ where z' is of arbitrary type: It still is possible to bound $\bigvee z^\tau$.

Remark 2.2.5 *Cor.2.2.3 is a considerable generalization of a theorem due to Parikh ([49]): Parikh shows for a subsystem (called PB) of the first order fragment of G_2A^ω: If PB$\vdash \bigwedge x \bigvee A(x,y)$ (where A contains only bounded quantifiers and only x, y as free variables) then there is a polynomial p such that PB$\vdash \bigwedge x \bigvee y \le p(x) \; A(x,y)$.*

Proof of thm.2.2.2 : For PA$^\omega$ the theorem is proved in [39] . We only recall the treatment of Δ: The negative translation $\neg\neg\bigwedge u^\gamma \neg\neg\bigvee v \le_\delta tu \bigwedge w^\eta \neg\neg F_0$ of $D :\equiv \bigwedge u \bigvee v \le tu \bigwedge w \, F_0$ is intuitionistically implied by D. The functional interpretation transforms D into
$D^D :\equiv \bigvee V \le t \bigwedge u, w \; F_0(u, Vu, w)$. Let t^* be such that t^* s–maj t. Then (by lemma1.2.11.4) $V \le t \to t^*$ s–maj V. Hence t^* satisfies the monotone functional interpretation of D (provable by D^D and thus in the presence of b-AC by D). The same proof applies to PRA$^\omega$. For G$_n$A$^\omega$ one has to use prop.1.2.21 to show that the majorizing terms for the terms occuring in the quantifier axioms can be choosen in G$_n$R$^\omega_-$ (and not only in G$_n$R$^\omega$).

Proof of cor.2.2.3 : The corollary follows immediately from thm.2.2.2 and prop.1.2.28 using the embedding $x^0 \mapsto \lambda y^0.x^0$ of type 0 into type 1. The assertion for the case $\bigwedge x^0, \tilde{x}^1 \bigwedge y \le_\rho sx\tilde{x} \bigvee z^0 A_0$ follows using prop.1.2.21, the corollary to its proof and prop.1.2.22.

[23]Here $t^*[\,\underline{a}\,]$ is called a majorizing term if $\lambda\underline{a}.t^*$ s–maj $\lambda\underline{a}.t$, where $\underline{a}$ are all free variables of t.

Remark 2.2.6 *The size of the numbers k, c_1, c_2, c in the cor.2.2.3 above depends on the depth of nestings of the functions $+, \cdot$ resp. x^y occuring in the given proof. Such nestings may occur explicitly by the formation of terms like $(x \cdot (x \cdot (\ldots)))$ by substitution or are logically circumscribed. In the later case they are made explicit by the (logical) normalization of the bound extracted by monotone functional interpretation. The process of normalization may increase the term depth enormously (In fact by an example due to [55] even non–elementary recursively in the type degree of the term). This corresponds to the fact that there are proofs of $\bigvee x^0 A_0(x)$–sentences such that the term complexity of a realizing term for $\bigvee x^0$ is not elementary recursive in the size of the proof (see [72]). However such a tremendous term complexity is very unlikely to occur in concrete proofs from mathematical practice: Firstly the parameter which is crucial for this complexity (the quantifier–complexity resp. the type degree of the modus ponens formulas) is very small in practice, lets say ≤ 3. Secondly even complex modus ponens formulas are able to cause an explosion of the term complexity only under very special circumstances which describe logically the iteration of a substitution process as in the example from [72] (we intend to discuss this matter in detail in another paper). Hence if a given proof does not involve such an iterated substitution process the degree of the polynomial bound in cor.2.2.3 will essentially be of the order of the degrees of the polynomials occuring in the proof and if the proof uses the exponential function 2^x (without applying it to itself) it will be a polynomial in 2^x. Hence the results of this paper which establish that main parts of analysis can be developed in a system whose provable growth is polynomial bounded also apply in a relativised form to proofs using e.g. the exponential function.*

¿From the proof of thm.2.2.2 it follows that b–AC is needed only to derive $\tilde{F} :\equiv \bigvee V \leq_{\delta\gamma} t \bigwedge u^\gamma, w^\eta F_0(u, Vu, w)$ from $F :\equiv \bigwedge u^\gamma \bigvee v \leq_\delta tu \bigwedge w^\eta F_0(u, v, w)$.[24] Hence if in the conclusion Δ is replaced by $\tilde{\Delta} := \left\{\tilde{F} : F \in \Delta\right\}$ then b–AC can be omitted. In particular this is the case if each $F \in \Delta$ has the form $\bigvee v \leq t \bigwedge w\ F_0(v, w)$ since $\tilde{F} \equiv F$ for such sentences.

Combining the proof of thm.2.2.2 with the proof of thm.2.9 from [33] one can strenghten the theorem by weakening b–AC($-\bigwedge$) to b–AC–qf, i.e. b–AC restricted to quantifier–free formulas: As in the proof of thm.2.9 in [33] one shows that

$$\mathrm{G}_n\mathrm{A}^\omega + \text{AC–qf} \vdash \bigwedge u^\gamma, W^{\eta\delta} \bigvee v \leq_\delta tu\ F_0(u, v, Wv) \to \bigwedge u^\gamma \bigvee v \leq_\delta tu \bigwedge w^\eta F_0.$$

Thus Δ can be replaced by $\widehat{\Delta} := \left\{\bigwedge u, W \bigvee v \leq tu\ F_0 : F \in \Delta\right\}$ without weakening of the theory. Since the implication

$$\bigwedge u, W \bigvee v \leq tuF_0(u, v, Wv) \to \bigvee V \leq \lambda u, W.tu \bigwedge u, WF_0(u, VuW, W(VuW))$$

can be proved by b–AC–qf (u, W can be coded into a single variable in $\mathrm{G}_n\mathrm{A}^\omega$ for $n \geq 2$)[25] the proof of the conclusion of thm.2.2.2 can be carried out in

$$\mathrm{G}_n\mathrm{A}_i^\omega + \widehat{\Delta} + \text{b–AC–qf}$$

and thus a fortiori in

$$\mathrm{G}_n\mathrm{A}_i^\omega + \Delta + \text{b–AC–qf}.$$

[24]Thus in particular only b–AC restricted to universal formulas (b–AC–$\bigwedge$) is used.
[25]For $n = 1$ one has to formulate b–AC–qf for tuples of variables.

However replacing Δ by $\widehat{\Delta}$ may make the extraction of a bound more complicated since it causes a raising of the types involved. Since we are interested in an extraction method which is as practical as possible and yields bounds which are numerically as good as possible but not (primarily) in the proof–theoretic strength of the theory used to verify these bounds we prefer the more simple extraction from thm.2.2.2 .
Similarly to thm. 2.12 in [33] we have the following generalization of thm.2.2.2 to a larger class of formulas:

Theorem 2.2.7 *Let Δ be as in thm.2.2.2 , $\rho_1, \rho_2 \in \mathbf{T}$ arbitrary types, $\tau_1, \tau_2 \leq 2$, $A_0(x,y,z,a,b)$ a quantifier–free formula containing at most x,y,z,a,b free and $s,r \in G_nR^\omega$. Then the following rule holds:*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge x^1 \bigwedge y \leq_{\rho_1} sx \bigvee z^{\tau_1} \bigwedge a \leq_{\rho_2} rxz \bigvee b^{\tau_2} A_0(x,y,z,a,b) \\ \Rightarrow \text{ by monotone functional interpretation } \exists \Psi_1, \Psi_2 \in G_nR^\omega_{-}[\Phi_1]: \\ E\text{–}G_nA^\omega + \Delta + b\text{–}AC \vdash \bigwedge x^1 \bigwedge y \leq_{\rho_1} sx \bigvee z \leq_{\tau_1} \Psi_1 x \bigwedge a \leq_{\rho_2} rxz \bigvee b \leq_{\tau_2} \Psi_2 x \; A_0(x,y,z,a,b). \end{cases}$$

Ψ_1, Ψ_2 are built up as Ψ in thm.2.2.2 . (An analogous result holds for PRA^ω and PA^ω).

Proof: Since the implication

$$\bigwedge x^1 \bigwedge y \leq_{\rho_1} sx \bigvee z^{\tau_1} \bigwedge a \leq_{\rho_2} rxz \bigvee b^{\tau_2} A_0(x,y,z,a,b) \rightarrow$$
$$\bigwedge x^1 \bigwedge y \leq_{\rho_1} sx \bigwedge A \leq_{\rho_2\tau_1} rx \bigvee z^{\tau_1}, b^{\tau_2} A_0(x,y,z,Az,b)$$

holds logically the assumption of the theorem implies

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC\text{–}qf} \vdash \bigwedge x^1 \bigwedge y \leq_{\rho_1} sx \bigwedge A \leq_{\rho_2\tau_1} rx \bigvee z^{\tau_1}, b^{\tau_2} A_0(x,y,z,Az,b).$$

By thm.2.2.2 and remark 2.2.4 2) one can extract (by monotone functional interpretation) terms $\Psi_1, \Psi_2 \in \mathrm{G}_n\mathrm{R}^\omega_{-}[\Phi_1]$ such that

$$\bigwedge x^1 \bigwedge y \leq_{\rho_1} sx \bigwedge A \leq_{\rho_2\tau_1} rx \bigvee z \leq_{\tau_1} \Psi_1 x \bigvee b \leq_{\tau_2} \Psi_2 x \; A_0(x,y,z,Az,b).$$

As in the proof of 2.12 in [33] (using the fact that lemma 2.11 from [33] also holds for E–$\mathrm{G}_n\mathrm{A}^\omega_i$+ b–AC) one concludes the assertion of the theorem.

Theorem 2.2.8 *All of our results on G_nA^ω ($G_nA^\omega_i$, E–G_nA^ω, E–$G_nA^\omega_i$) and G_nR^ω remain valid if these theories are replaced by $G_nA^\omega[\underline{\chi}]$ ($G_nA^\omega_i[\underline{\chi}]$, E–$G_nA^\omega[\underline{\chi}]$, E–$G_nA^\omega_i[\underline{\chi}]$) and $G_nR^\omega[\underline{\chi}]$, where for a theory $\mathcal{T}$, $\mathcal{T}[\underline{\chi}]$ is defined as the extension obtained by adding a tuple $\underline{\chi}$ of function symbols $\chi_i^{\rho_i}$ with deg(ρ_i) ≤ 1 together with*

(1) *arbitrary purely universal axioms $\bigwedge x^\tau A_0(x)$ on $\underline{\chi}$, where $\tau \leq 2$*

plus axioms having the form

(2) *$\underline{\chi}^*$ s–maj $\underline{\chi}$ for $\underline{\chi}^* \in G_nR^\omega_{-}$,*

where (1),(2) are valid in the full type structure $\mathcal{S}^\omega$ under a suitable interpretation of $\underline{\chi}$ ($G_nR^\omega[\underline{\chi}]$ denotes the set of all closed terms of the extended theories).
In particular the bounds extracted in thm.2.2.2, 2.2.7 and cor.2.2.3 are still $\in G_nR^\omega_{-}[\Phi_1]$.

Proof: The theorem follows immediately from the proofs above (observing that also (2) is purely universal) if one extends the construction of t^* in the proof of prop.1.2.21 by the clause
'Replace all occurrences of χ_i in t by χ_i^*'. Since the majorizing terms χ_i^* are $\in \mathrm{G}_n\mathrm{R}^\omega_-$ this also holds for t^*.

Remark 2.2.9 *The reason for the restriction to deg(ρ_i) ≤ 1 in the theorem above is that the addition of symbols for higher type functionals χ in general destroys the possibility of elimination of extensionality since Ex(χ) may not be provable (and cannot be added simply as an axiom since it is not purely universal). Also* (2) *is no longer purely universal if deg(ρ_i) ≥ 2.*

By theorem 2.2.8 the extension by symbols for majorizable functions has no impact on the bounds extracted from a proof. This is the reason why in the following chapters at some places we will make free use of such extensions (e.g. we will add new function symbols for sin and cos in chapter 5) and will denote the resulting theories also by $\mathrm{G}_n\mathrm{A}^\omega$ etc.

By cor.2.1.3 and thm.2.2.2 we can extract realizing functionals respectively uniform bounds for $\bigwedge\bigvee A_0$–sentences (in the later case even for the more general sentences from thm.2.2.7). Since the theories $\mathrm{G}_n\mathrm{A}^\omega$ are based on classical logic it is in general not possible to extract computable realizations or bounds for $\bigwedge\bigvee\bigwedge A_0$–sentences: Let us consider e.g.

$$(+)\ \bigwedge x^0\bigvee y^0\bigwedge z^0(Pxy \vee \neg Pxz),$$

which holds by classical logic. If $Pxy :\equiv Txxy$, where T is the Kleene T–predicate, then any upper bound f on y, i.e.

$$\bigwedge x^0\bigvee y \leq_0 fx\bigwedge z^0(Pxy \vee \neg Pxz)$$

can be used to decide the halting–problem (and therefore must be ineffective): For h which is defined primitive recursively in f such that

$$hx := \begin{cases} 0, \text{ if } \bigvee y \leq fx(Txxy) \\ 1 \text{ otherwise} \end{cases}$$

one has $hx = 0 \leftrightarrow \bigvee yTxxy$ for all x.
T is elementary recursive and can therefore be defined already in $\mathrm{G}_3\mathrm{A}^\omega$.
If one generalizes (+) to tuples of number variables then – by Matijacevic's result on Hilbert's 1oth problem– there is a polynomial $P\underline{x}\,\underline{y}$ whith coefficients in $\mathbb{N}$ such that there is no tuple $t_1, \ldots, t_k$ of recursive functions (for $\underline{y} = y_1 \ldots y_k$) with

$$\bigwedge\underline{x}\bigvee y_1 \leq t_1\underline{x} \ldots \bigvee y_k \leq t_k\underline{x}\bigwedge\underline{z}(P\underline{x}\underline{y} = 0 \vee \neg P\underline{x}\underline{z} = 0).$$

Since $P \in \mathrm{G}_2\mathrm{R}^\omega$ and $\mathrm{G}_2\mathrm{R}^\omega$ allows the coding of finite tuples of natural numbers one can define already in $\mathrm{G}_2\mathrm{R}^\omega$ a predicate P such that there is no recursive bound on y in (+).

The use of non–constructive $\bigwedge\bigvee$–dependencies as in (+) is a characteristic feature of classical logic. If **intuitionistic** logic is used the situation changes completely: In chapter 8 we will show that even in the presence of a large class of non–constructive analytical axioms (including as a special case arbitrary $\bigwedge u^\delta \bigvee v \leq_\rho su \bigwedge w^\tau A_0$–sentences) one can extract uniform bounds $\Psi \in \mathrm{G}_n\mathrm{R}^\omega$ on z in sentences $\bigwedge x^1 \bigwedge y \leq_\gamma tx \bigvee z\ A(x,y,z)$, which are proved in $\mathrm{G}_n\mathrm{A}_i^\omega$ from such non–constructive axioms, where A **is an arbitrary formula** (containing only x, y, z free). This extraction is achieved by a new monotone version of modified realizability.

Although in the case of theories based on classical logic it is not always possible to extract effective bounds for $\bigwedge x \bigvee y A(x,y)$–sentences when A is not purely existential, one may obtain **relative bounds** :
By $\mathrm{AC}^{0,0}$–qf and classical logic

(1) $\bigwedge x^0 \bigvee y^0 \bigwedge z^0 (Pxy \vee \neg Pxz)$

is equivalent to

(2) $\bigwedge x, f^1 \bigvee y (Pxy \vee \neg Px(fy))$

and a bound on y in (2) is given by

$$\Psi xf := \max{}_0(0, f0) = f0$$

since[26]

$$(Px0 \vee \neg Px(f0)) \vee (Px(f0) \vee \neg Px(ff0)).$$

For a more complex situation let us consider

$$F :\equiv \Big(\bigwedge x^0 \bigvee y^0 \bigwedge z^0 A_0(x,y,z) \to \bigwedge u^0 \bigvee v^0 B_0(u,v) \Big),$$

which is –by $\mathrm{AC}^{0,0}$–$\bigwedge$ and prenexing– equivalent to

$$\tilde{F} :\equiv \bigwedge f^1, u \bigvee x, z, v \big(A_0(x, fx, z) \to B_0(u,v) \big).$$

$\tilde{F}$ is a $\bigwedge\bigvee F_0$–sentence. Thus v (and also x,z) can be bounded by a functional Ψuf in u, f with $\Psi \in \mathrm{G}_n\mathrm{R}^\omega$ if F is proved in $\mathrm{G}_n\mathrm{A}^\omega + \Delta$+AC–qf. Ψ is an effective bound **relatively to the oracle** f.
By raising the types one can replace $\tilde{F}$ by a different (and more complex) $\bigwedge\bigvee F_0$–sentence $\widehat{F}$ which is more closely related to F in that the equivalence of F and $\widehat{F}$ can be proved using only $\mathrm{AC}^{0,0}$–qf:

$$\begin{aligned} F &\leftrightarrow \Big(\bigvee \Phi^2 \bigwedge x^0, f^1 A_0(x, \Phi xf, f(\Phi xf)) \to \bigwedge u \bigvee v B_0(u,v) \Big) \\ &\leftrightarrow \bigwedge \Phi, u \bigvee x, f, v \big(A_0(x, \Phi xf, f(\Phi xf)) \to B_0(u,v) \big) \equiv: \widehat{F}. \end{aligned}$$

If $\widehat{F}$ is proved in $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf, then one can extract from this proof a term $t \in \mathrm{G}_n\mathrm{R}^\omega$ such that $t\Phi u$ realizes '$\bigvee v$'. If $\widehat{F}$ is proved in $\mathrm{G}_n\mathrm{A}^\omega + \Delta$+AC–qf one obtains (using monotone functional interpretation) a term $t^* \in \mathrm{G}_n\mathrm{R}^\omega$ such that for every Φ^* which majorizes Φ, $t^*\Phi^*u$ is a bound for v:

$$\Phi^* \text{ s–maj } \Phi \to \Big(\bigwedge x, f A_0(x, \Phi xf, f(\Phi xf)) \to \bigwedge u \bigvee v \leq t^*\Phi^* u\ B_0(u,v) \Big).$$

[26]More generally fz is an upper bound where z is a variable.

In this chapter we have determined the growth of uniform bounds $\Phi \underline{u}\,\underline{k}$ on $\bigvee w^\gamma$ (where $\gamma \le 2$) for sentences

$$(+)\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \le_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0$$ [27]

(and also the more general sentences from thm. 2.2.7) which are provable in $\mathrm{G}_n\mathrm{A}^\omega+$ axioms Δ having the form $\bigwedge x^\delta \bigvee y \le_\rho sx \bigwedge z^\eta B_0$.
In particular, for $\gamma = 0$ and $n \le 3$ we have bounds Φ such that

$\Phi \underline{u}\,\underline{k}$ is a linear function in $\underline{u}^M, \underline{k}$ (where $u_i^M := \lambda x^0.\max_0(u0, \ldots, ux)$), if $n = 1$,

$\Phi \underline{u}\,\underline{k}$ is a polynomial function in $\underline{u}^M, \underline{k}$ for which prop.1.2.30 applies, if $n = 2$,

$\Phi \underline{u}\,\underline{k}$ is an elementary recursive function in $\underline{u}^M$, $\underline{k}$, if $n = 3$.

These results will be used in the following chapters (besides other proof–theoretic methods) to determine the growth of extractable bounds from proofs which may use various genuine **analytical** theorems.

[27] Here $\underline{u}, \underline{k}$ denote tuple of variables of type $1, 0$.

3 Real numbers and continuous functions in $\mathbf{G}_2\mathbf{A}_i^\omega$: Enrichment of data

3.1 Representation of real numbers in $\mathbf{G}_2\mathbf{A}_i^\omega$

Suppose that a proposition $\bigwedge x \bigvee y A(x,y)$ is proved in one of the theories $\mathcal{T}^\omega$ from the previous chapters, where the variables x, y may range over $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or e.g. C[0,1] etc. What sort of numerical information on '$\bigvee y$' relatively to the 'input' x can be extracted from a given proof depends in particular on how x is represented, i.e. on the numerical data by which x is given:
Suppose e.g. x that is a variable on $\mathbb{R}$ and real numbers are represented by **arbitrary** Cauchy sequences of rational numbers x_n, i.e.

$$(1)\ \bigwedge k^0 \bigvee n^0 \bigwedge m, \tilde{m} \geq n \big(|x_m - x_{\tilde{m}}| \leq \frac{1}{k+1} \big).$$

Let us consider the (obviously true) proposition

$$(2)\ \bigwedge x \in \mathbb{R} \bigvee l \in \mathbb{N} (x \leq l).$$

Given x by a representative (x_n) in the sense of (1) it is not possible to compute an l which satisfies (2) on the basis of this representation, since this would involve the computation of a number n which fulfils a (in general undecidable) universal property like $\bigwedge m, \tilde{m} \geq n (|x_m - x_{\tilde{m}}| \leq 1)$: Define now $l := \lceil |x_n| \rceil + 1$.

If however real numbers are represented by Cauchy sequences with a **fixed Cauchy modulus**, e.g. $1/(k+1)$, i.e.

$$(3)\ \bigwedge m, \tilde{m} \geq k \big(|x_m - x_{\tilde{m}}| \leq \frac{1}{k+1} \big),$$

then the computation of l is trivial:

$$l := \Phi\left((x_n)\right) := \lceil |x_0| \rceil + 1.$$

Φ is not a function : $\mathbb{R} \to \mathbb{N}$ since it is not extensional: Different Cauchy sequences $(x_n), (\tilde{x}_n)$ which represent the same real number, i.e. $\lim_{n\to\infty}(x_n - \tilde{x}_n) = 0$, yield in general different numbers $\Phi((x_n)) \neq \Phi((\tilde{x}_n))$. Following E. Bishop [5] , [6] we call Φ an **operation** : $\mathbb{R} \to \mathbb{N}$. This phenomenon is a general one (and not caused by the special definition of Φ): The only computable operations $\mathbb{R} \to \mathbb{N}$, which are extensional, are operations which are constant, since the computability of Φ implies its continuity as a functional[28] : $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and therefore (if it is extensional w.r.t. $=_{\mathbb{R}}$) the continuity as a function $\mathbb{R} \to \mathbb{N}$.

The importance of the representation of complex objects as e.g. real numbers is also indicated by the fact that the logical form of properties of these objects depends essentially on the representation:
If $(x_n), (\tilde{x}_n)$ are arbitrary Cauchy sequences (in the sense of (1)) then the property that both sequences represent the same real number is expressed by the Π_3^0–formula

$$(4)\ \bigwedge k \bigvee n \bigwedge m, \tilde{m} \geq n \big(|x_m - \tilde{x}_m| \leq \frac{1}{k+1} \big).$$

[28] An operation $\Phi : \mathbb{R} \to \mathbb{N}$ is given by a functional : $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ (which is extensional w.r.t. $=_1$!) since sequences of rational numbers are coded as sequences of natural numbers.

For Cauchy sequences with fixed Cauchy modulus as in (2) this property can be expressed by the (logically much simpler) Π^0_1–formula

$$(5)\ \bigwedge k\big(|x_k - \tilde{x}_k| \le \frac{3}{k+1}\big).$$

For Cauchy sequences with modulus $1/(k+1)$ (4) and (5) are equivalent (provably in $\mathrm{G_2A}^\omega_i$). But for arbitrary Cauchy sequences (4) does not imply (5) in general.

If $(x_n) \subset \mathbb{Q}$ is an arbitrary Cauchy sequence then $\mathrm{AC}^{0,0}$ applied to

$$\bigwedge k \bigvee n \bigwedge m, \tilde{m} \ge n\big(|x_m - x_{\tilde{m}}| \le \frac{1}{k+1}\big)$$

yields the existence of a function f^1 such that

$$\bigwedge k \bigwedge m, \tilde{m} \ge fk\big(|x_m - x_{\tilde{m}}| \le \frac{1}{k+1}\big).$$

For $m, \tilde{m} \ge k$ this implies $|x_{fm} - x_{f\tilde{m}}| \le \frac{1}{k+1}$ (choose $k' \in \{m, \tilde{m}\}$ with $fk' \le fm, f\tilde{m}$ and apply the Cauchy property to $m' := fm, \tilde{m}' := f\tilde{m}$), i.e. the sequence $(x_{fn})_{n\in\mathbb{N}}$ is a Cauchy sequence with modulus $1/(k+1)$ which has the same limit as $(x_n)_{n\in\mathbb{N}}$.
Thus in the presence of $\mathrm{AC}^{0,0}$ (or more precisely the restriction $\mathrm{AC}^{0,0}$–$\bigwedge$ of $\mathrm{AC}^{0,0}$ to Π^0_1–formulas) both representations (1) and (2) equivalent. However $\mathrm{AC}^{0,0}$–$\bigwedge$ is not provable in any of our theories and the addition of this schema to the axioms would yield an explosion of the rate of growth of the provably recursive functions. In fact every $\alpha(<\varepsilon_0)$–recursive function is provably recursive in $\mathrm{G_2A}^\omega$+ $\mathrm{AC}^{0,0}$–$\bigwedge$. This follows from the fact that iterated use of $\mathrm{AC}^{0,0}$–$\bigwedge$ combined with classical logic yields full arithmetical comprehension

$$CA_{ar}\ :\ \bigvee f^1 \bigwedge x^0 (fx =_0 0 \leftrightarrow A(x)),$$

where A is an arithmetical formula, i.e. a formula containing only quantifiers of type 0. CA_{ar} applied to QF–IA proves the induction principle for every arithmetical formula. Hence full Peano–arithmetic PA is a subsystem of $\mathrm{G_2A}^\omega$+ $\mathrm{AC}^{0,0}$–$\bigwedge$.

As a consequence of this situation we have to specify the representation of real numbers we choose:

Definition 3.1.1 *A real number is given by a Cauchy sequence of rational numbers with modulus $1/(k+1)$.*

The reason for this representation is two–fold:

1) As we have seen already above any numerically interesting application of the extraction of a bound presupposes that the input is given as a numerical reasonable object. This is also the reason why in constructive analysis (in the sense of Bishop) as well as in complexity theory for analysis (in the sense of H. Friedman and K.–I. Ko, see [31]) real numbers are always endowed with a rate of convergence, continuous functions with a modulus of continuity and so on. Also in the work by H. Friedman, S. Simpson and others on the program of so–called 'reverse mathematics', real numbers are always given with a fixed rate of convergence.

2) For our representation of real numbers we can achieve that quantification over real numbers is nothing else then quantification over $\mathbb{N}^{\mathbb{N}}$, i.e. $\bigwedge x^1, \bigvee y^1$. Because of this many interesting theorems in analysis have the logical form $\bigwedge\bigvee F_0$ (see [39] for a discussion on that) so that our method of extracting feasible bounds applies.

1) and 2) are in fact closely related: If real numbers would be represented as arbitrary Cauchy sequences then a proposition $\bigwedge x \in \mathbb{R} \bigvee y \in \mathbb{N}\ A(x,y)$ would have the logical form

$$\bigwedge x^1 (\bigwedge k \bigvee n \bigwedge m F_0 \to \bigvee y^0 A),$$

where $(*)$ $\bigwedge k \bigvee n \bigwedge m F_0$ expresses the Cauchy property of the sequence of rational numbers coded by x^1. By our reasoning in chapter 2 we know that we can only obtain a bound on y which depends on x together with a Skolem function for $(*)$. But this just means that the computation of the bound requires that x is given with a Cauchy modulus.
As concerned with provability in our theories like $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf the representation with fixed modulus is no real restriction: In chapter 11 we will show in particular that the a proof of

$$\bigwedge (x_n)\big(\bigvee f^1 \bigwedge k \bigwedge m, \tilde{m} \geq fk(|x_m - \tilde{x}_m| \leq \frac{1}{k+1}) \to \bigvee y^0 A\big)$$

can be transformed into a proof of

$$\bigwedge (x_n)\big(\bigwedge k \bigvee n \bigwedge m, \tilde{m} \geq n(|x_m - \tilde{x}_m| \leq \frac{1}{k+1}) \to \bigvee y^0 A\big).$$

within the same theory (i.e. without any use of $\mathrm{AC}^{0,0}$) for a large class of formulas A.
In particular we show that for every definable Cauchy sequence the assertion of the existence of a Cauchy modulus is conservative (i.e. it does not cause any additional rate of growth).

The representation of $\mathbb{R}$ presupposes a **representation of** $\mathbb{Q}$: Rational numbers are represented as codes $j(n,m)$ of pairs (n,m) of natural numbers n,m. $j(n,m)$ represents

the rational number $\frac{\frac{n}{2}}{m+1}$, if n is even,
the negative rational $-\frac{\frac{n+1}{2}}{m+1}$ if n is odd.

By the surjectivity of our pairing function j from chapter 1 every natural number can be conceived as code of a uniquely determined rational number. On the codes of $\mathbb{Q}$, i.e. on $\mathbb{N}$, we define an equivalence relation by

$$n_1 =_{\mathbb{Q}} n_2 :\equiv \frac{\frac{j_1 n_1}{2}}{j_2 n_1 + 1} = \frac{\frac{j_1 n_2}{2}}{j_2 n_2 + 1} \text{ if } j_1 n_1, j_1 n_2 \text{ both are even}$$

and analgously in the remaining cases, where $\frac{a}{b} = \frac{c}{d}$ is defined to hold iff $ad =_0 cb$ (for $bd > 0$).
On $\mathbb{N}$ one easily defines functions $|\cdot|_{\mathbb{Q}}, +_{\mathbb{Q}}, -_{\mathbb{Q}}, \cdot_{\mathbb{Q}} :_{\mathbb{Q}}, \max_{\mathbb{Q}}, \min_{\mathbb{Q}} \in \mathrm{G}_2\mathrm{R}^\omega$ and (quantifier–free) relations) $<_{\mathbb{Q}}, \leq_{\mathbb{Q}}$ which represent the corresponding functions and relations on $\mathbb{Q}$. In the following we sometimes omit the index $\mathbb{Q}$ if this does not cause any confusion.

Notational convention: For better readability we often write e.g. $\frac{1}{k+1}$ instead of its code $j(2,k)$ in $\mathbb{N}$. So e.g. we write $x^0 \leq_{\mathbb{Q}} \frac{1}{k+1}$ for $x \leq_{\mathbb{Q}} j(2,k)$.

By the coding of rational numbers as natural numbers, **sequences of rationals** are just functions f^1 (and every function f^1 can be conceived as a sequence of rational numbers in a unique way). In particular representatives of real numbers are functions f^1 modulo this coding. We now show that **every** function can be conceived as an representative of a uniquely determined Cauchy sequence of rationals with modulus $1/(k+1)$ and therefore can be conceived as an representative of a uniquely determined real number.[29]
To achieve this we need the following functional

Definition 3.1.2 *The functional $\lambda f^1.\widehat{f} \in G_2R^\omega$ is defined such that*

$$\widehat{f}n = \begin{cases} fn, \text{ if } \bigwedge k, m, \tilde{m} \leq_0 n\big(m, \tilde{m} \geq_0 k \to |fm -_{\mathbb{Q}} f\tilde{m}| \leq_{\mathbb{Q}} \frac{1}{k+1}\big) \\ f(n_0 - 1) \text{ for } n_0 := \min l \leq_0 n[\bigvee k, m, \tilde{m} \leq_0 l\big(m, \tilde{m} \geq_0 k \wedge |fm -_{\mathbb{Q}} f\tilde{m}| >_{\mathbb{Q}} \frac{1}{k+1}\big)], \\ \text{otherwise.} \end{cases}$$

One easily verifies (within $\mathrm{G}_2\mathrm{A}_i^\omega$) that

1) if f^1 represents a Cauchy sequence of rational numbers with modulus $1/(k+1)$, then $\bigwedge n^0(fn =_0 \widehat{f}n)$,

2) for every f^1 the function $\widehat{f}$ represents a Cauchy sequence of rational numbers with modulus $1/(k+1)$.

Hence every function f gives a uniquely determined real number, namely that number which is represented by $\widehat{f}$. Quantification $\bigwedge x \in \mathbb{R}\, A(x)$ ($\bigvee x \in \mathbb{R}\, A(x)$) so reduces to the quantification $\bigwedge f^1 A(\widehat{f})$ ($\bigvee f^1 A(\widehat{f})$) for properties A which are extensional w.r.t. $=_{\mathbb{R}}$ below (i.e. which are really properties of real numbers). **Operations** $\Phi : \mathbb{R} \to \mathbb{R}$ are given by functionals $\Phi^{1(1)}$ (which are extensional w.r.t.$=_1$). A real function : $\mathbb{R} \to \mathbb{R}$ is given by a functional $\Phi^{1(1)}$ which (in addition) is extensional w.r.t. $=_{\mathbb{R}}$. Following the usual notation we write (x_n) instead of fn and $(\widehat{x}_n)$ instead of $\widehat{f}n$.
In the following we define various relations and operations on functions which correspond to the usual relations and operations on $\mathbb{R}$ for the real numbers represented by the respective functions:

Definition 3.1.3 *1)* $(x_n) =_{\mathbb{R}} (\tilde{x}_n) :\equiv \bigwedge k^0\big(|\widehat{x}_k -_{\mathbb{Q}} \widehat{\tilde{x}}_k| \leq_{\mathbb{Q}} \frac{3}{k+1}\big)$*;*

2) $(x_n) <_{\mathbb{R}} (\tilde{x}_n) :\equiv \bigvee k^0\big(\widehat{\tilde{x}}_k - \widehat{x}_k >_{\mathbb{Q}} \frac{3}{k+1}\big)$*;*

3) $(x_n) \leq_{\mathbb{R}} (\tilde{x}_n) :\equiv \neg(\widehat{\tilde{x}}_n) <_{\mathbb{Q}} (\widehat{x}_n)$*;*

4) $(x_n) +_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2n+1} +_{\mathbb{Q}} \widehat{\tilde{x}}_{2n+1})$*;*

5) $(x_n) -_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2n+1} -_{\mathbb{Q}} \widehat{\tilde{x}}_{2n+1})$*;*

6) $|(x_n)|_{\mathbb{R}} := (|\widehat{x}_n|_{\mathbb{Q}})$*;*

7) $(x_n) \cdot_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2(n+1)k} \cdot_{\mathbb{Q}} \widehat{\tilde{x}}_{2(n+1)k})$*, where* $k := \lceil \max_{\mathbb{Q}}(|x_0| + 1, |\tilde{x_0}| + 1) \rceil$*;*

[29] A related representation of real numbers is sketched in [3] .

8) For (x_n) and l^0 we define

$$(x_n)^{-1} := \begin{cases} (\max_{\mathbb{Q}}(\widehat{x}_{(n+1)(l+1)^2}, \frac{1}{l+1})^{-1}), & \textit{if } \widehat{x}_{2(l+1)} >_{\mathbb{Q}} 0 \\ (\min_{\mathbb{Q}}(\widehat{x}_{(n+1)(l+1)^2}, \frac{-1}{l+1})^{-1}), & \textit{otherwise;} \end{cases}$$

9) $\max_{\mathbb{R}}\big((x_n),(\tilde{x}_n)\big) := \big(\max_{\mathbb{Q}}(\widehat{x}_n, \widehat{\tilde{x}}_n)\big), \quad \min_{\mathbb{R}}\big((x_n),(\tilde{x}_n)\big) := \big(\min_{\mathbb{Q}}(\widehat{x}_n, \widehat{\tilde{x}}_n)\big).$

One easily verifies the following

Lemma: 3.1.4 *1) $(x_n) =_{\mathbb{R}} (\tilde{x}_n)$ resp. $(x_n) <_{\mathbb{R}} (\tilde{x}_n)$, $(x_n) \leq_{\mathbb{R}} (\tilde{x}_n)$ hold iff the correponding relations hold for those real numbers which are represented by $(x_n), (\tilde{x}_n)$.*

2) Provably in $G_2A^\omega_i$, $(x_n) +_{\mathbb{R}} (\tilde{x}_n)$, $(x_n) -_{\mathbb{R}} (\tilde{x}_n)$, $(x_n) \cdot_{\mathbb{R}} (\tilde{x}_n)$, $\max_{\mathbb{R}}\big((x_n),(\tilde{x}_n)\big)$, $\min_{\mathbb{R}}\big((x_n),(\tilde{x}_n)\big)$ and $|(x_n)|_{\mathbb{R}}$ also represent Cauchy sequences with modulus $1/(k+1)$ which represent the real number obtained by addition (subtraction,...) of those real numbers which are represented by $(x_n), (\tilde{x}_n)$. This also holds for $(x_n)^{-1}$ if $|(x_n)|_{\mathbb{R}} \geq_{\mathbb{R}} \frac{1}{l+1}$ for the number l used in the definition of $(x_n)^{-1}$. In particular the operations $+_{\mathbb{R}}, -_{\mathbb{R}}$ etc. are extensional w.r.t. to $=_{\mathbb{R}}$ and therefore represent functions[30]*.*

3) The functionals $+_{\mathbb{R}}, -_{\mathbb{R}}, \cdot_{\mathbb{R}}, \max_{\mathbb{R}}, \min_{\mathbb{R}}$ of type $1(1)(1)$, $|\cdot|_{\mathbb{R}}$ of type $1(1)$ and $()^{-1}$ of type $1(1)(0)$ are definable in G_2R^ω.

Proof: The lemma is easily proved using the following hints: Ad $=_{\mathbb{R}}$: If $\bigwedge k^0\big(|\widehat{x}_k -_{\mathbb{Q}} \widehat{\tilde{x}}_k| \leq_{\mathbb{Q}} \frac{3}{k+1}\big)$ then the Cauchy sequences $(x_n), (\tilde{x}_n)$ clearly have the same limit. If $\bigvee k^0\big(|\widehat{x}_k -_{\mathbb{Q}} \widehat{\tilde{x}}_k| >_{\mathbb{Q}} \frac{3}{k+1}\big)$ then $\bigwedge n \geq k\big(|\widehat{x}_n -_{\mathbb{Q}} \widehat{\tilde{x}}_n| >_{\mathbb{Q}} \frac{1}{k+1}\big)$ (since $(\widehat{x}_n), (\widehat{\tilde{x}}_n)$ have the Cauchy modulus $\frac{1}{n+1}$). Hence $(\widehat{x}_n), (\widehat{\tilde{x}}_n)$ have different limits.
Ad $\cdot_{\mathbb{R}}$: Because of $|ca - db| = |(c-d)a + (a-b)d| \leq |c-d|\cdot|a| + |a-b|\cdot|d|$ one has for $m, \tilde{m} \geq n$:
$|\widehat{x}_{2(m+1)} \cdot \widehat{\tilde{x}}_{2(m+1)k} -_{\mathbb{Q}} |\widehat{x}_{2(\tilde{m}+1)} \cdot \widehat{\tilde{x}}_{2(\tilde{m}+1)k}| \leq$
$|\widehat{x}_{2(m+1)k} -_{\mathbb{Q}} \widehat{x}_{2(\tilde{m}+1)k}| \cdot k + |\widehat{\tilde{x}}_{2(m+1)k} -_{\mathbb{Q}} \widehat{\tilde{x}}_{2(\tilde{m}+1)k}| \cdot k \leq$
$\frac{1}{2(n+1)k+1} \cdot k + \frac{1}{2(n+1)k+1} \cdot k < \frac{1}{n+1}$.
That the definition of $(x_n)^{-1}$ is correct is proved using
$|\frac{1}{q} - \frac{1}{p}| = \frac{1}{|pq|} \cdot |p-q|$ (for $p, q \neq 0$) and $|\max(p,r) - \max(q,r)| \leq |p-q|$.

Rational numbers q coded by r_q have as canonical representative in $\mathbb{R}$ (besides other representatives) the constant function $\lambda n^0.r_q$. One easily shows that

$$\bigwedge k\big(|(x_n) -_{\mathbb{R}} \lambda n.x_k| \leq_{\mathbb{R}} \frac{1}{k+1}\big)$$

for every function (x_n).
Notational convention: For notational simplicity we often omit the emmbedding $\mathbb{Q} \hookrightarrow \mathbb{R}$, e.g. $x^1 \leq_{\mathbb{R}} y^0$ stands for $x \leq_{\mathbb{R}} \lambda n.y^0$. From the type of the objects it will be always clear what is meant.

[30] The functional $()^{-1}$ is extensional for all l and $(x_n), (y_n)$ such that $|(x_n)|_{\mathbb{R}}, |(y_n)|_{\mathbb{R}} \geq \frac{1}{l+1}$.

If $(f_n)_{n\in\mathbb{N}}$ of type $1(0)$ represents a $\frac{1}{k+1}$–Cauchy sequence of **real** numbers, then $f(n) := \widehat{f}_{3(n+1)}(3(n+1))$ represents the limit of this sequence, i.e.

$$\bigwedge k\big(|f_k -_{\mathbb{R}} f| \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

(One only has to show that $\widehat{f} =_1 f$. But this follows from
$|\widehat{f}_{3(m+1)}(3(m+1)) -_{\mathbb{Q}} \widehat{f}_{3(\tilde{m}+1)}(3(\tilde{m}+1))| \leq$
$|\widehat{f}_{3(m+1)}(3(m+1)) -_{\mathbb{R}} f_{3(m+1)}| + |f_{3(m+1)} -_{\mathbb{R}} f_{3(\tilde{m}+1)}| + |f_{3(\tilde{m}+1)} -_{\mathbb{R}} \widehat{f}_{3(\tilde{m}+1)}(3(\tilde{m}+1))| \leq \frac{3}{3(n+1)}$
for $m, \tilde{m} \geq n$).

Representation of $\mathbb{R}^d$ in $\mathbf{G}_2\mathbf{A}^\omega_i$:

For every fixed d we represent $\mathbb{R}^d$ as follows: Elements of $\mathbb{R}^d$ are represented by functions f^1 in the following way: Using the construction $\widehat{f}$ from above, every f^1 can be conceived as a representative of such a d–tuple of Cauchy sequences of real numbers, namely the sequence which is represented by

$$\big(\widehat{\nu^d_1(f)}, \ldots, \widehat{\nu^d_d(f)}\big), \text{ where } \nu^d_i(f) := \lambda x^0.\nu^d_i(fx).$$

Since the $\widehat{\nu^d_i(f)}$ represent Cauchy sequences of rationals with Cauchy modulus $\frac{1}{k+1}$, elements of $\mathbb{R}^d$ are so represented as Cauchy sequences of elements in $\mathbb{Q}^d$ which have the Cauchy modulus $\frac{1}{k+1}$ w.r.t. the maximum norm $\|f^1\|_{\max} := \max_{\mathbb{R}}\big(|\nu^d_1(f)|_{\mathbb{R}}, \ldots, |\nu^d_d(f)|_{\mathbb{R}}\big)$.
Quantification $\bigwedge(x_1, \ldots, x_d) \in \mathbb{R}^d$ so reduces to $\bigwedge f^1 A(\widehat{\nu^d_1(f)}, \ldots, \widehat{\nu^d_d(f)})$ for $\mathbb{R}$–extensional properties A (likewise for $\bigvee$).
The operations $+_{\mathbb{R}^d}, -_{\mathbb{R}^d}, \ldots$ are defined via the corresponding operations on the components, e.g. $x^1 +_{\mathbb{R}^d} y^1 :\equiv \nu^d(\nu^d_1 x +_{\mathbb{R}} \nu^d_1 y, \ldots, \nu^d_d x +_{\mathbb{R}} \nu^d_d y)$.
Sequences of elements are represented by (f_n) of type $1(0)$.

Representation of [0,1]$\subset \mathbb{R}$ in $\mathbf{G}_2\mathbf{A}^\omega_i$

We now show that every element of $[0,1]$ can be represented already by a bounded function $f \in \{f : f \leq_1 M\}$, where M is a fixed function from $\mathrm{G}_2\mathrm{R}^\omega$ and that every function from this set can be conceived as an (representative of an) element in [0,1]: Firstly we define a function $q \in \mathrm{G}_2\mathrm{R}^\omega$ by

$$q(n) := \begin{cases} \min l \leq_0 n[l =_{\mathbb{Q}} n], \text{ if } 0 \leq_{\mathbb{Q}} n \leq_{\mathbb{Q}} 1 \\ 0^0, \text{ otherwise.} \end{cases}$$

It is clear that every rational number $\in [0,1] \cap \mathbb{Q}$ has a unique code by a number $\in q(\mathbb{N})$ and $\bigwedge n^0(q(q(n)) =_0 q(n))$. Also every such number codes an element of $\in [0,1] \cap \mathbb{Q}$. We may conceive every number n as a representative of a rational number $\in [0,1] \cap \mathbb{Q}$, namely of the rational coded by $q(n)$.
In contrast to $\mathbb{R}$ we can restrict the set of representing functions for [0,1] to the compact (in the sense of the Baire space) set $f \in \{f : f \leq_1 M\}$, where $M(n) := j(6(n+1), 3(n+1)-1)$:

Each fraction r having the form $\frac{i}{3(n+1)}$ (with $i \leq 3(n+1)$) is represented by a number $k \leq M(n)$, i.e. $k \leq M(n) \wedge q(k)$ codes r. Thus $\{k : k \leq M(n)\}$ contains (modulo this coding) an $\frac{1}{3(n+1)}$–net for [0,1].
We define a functional $\lambda f.\tilde{f} \in \mathrm{G_2R}^\omega$ such that

$$\tilde{f}(k) = q(i_0), \text{ where } i_0 = \mu i \leq_0 M(k)[\bigwedge j \leq_0 M(k)(|\widehat{f}(3(k+1)) -_{\mathbb{Q}} q(j)| \geq_{\mathbb{Q}} |\widehat{f}(3(k+1)) -_{\mathbb{Q}} q(i)|)].$$

$\tilde{f}$ has (provably in $\mathrm{G_2A}_i^\omega$) the following properties:

1) $\bigwedge f^1(\widehat{\tilde{f}} =_1 \tilde{f})$.

2) $\bigwedge f^1(0 \leq_{\mathbb{R}} \tilde{f} \leq_{\mathbb{R}} 1)$.

3) $\bigwedge f^1(0 \leq_{\mathbb{R}} f \leq_{\mathbb{R}} 1 \to f =_{\mathbb{R}} \tilde{f})$.

4) $\bigwedge f^1(\tilde{\tilde{f}} =_{\mathbb{R}} \tilde{f})$.

Proof: 1) $\bigwedge f^1(\tilde{\tilde{f}} =_1 \tilde{f})$ follows easily from the definition of $\tilde{f}$.
$\bigwedge f^1(\widehat{\tilde{f}} =_1 \tilde{f})$: Assume $m, \tilde{m} \geq_0 n$. $|\widehat{f}(3(m+1)) -_{\mathbb{Q}} \widehat{f}(3(\tilde{m}+1))| \leq \frac{1}{3(n+1)}$ and the fact that $\{q(i) : i \leq_0 M(n)\}$ contains a $\frac{1}{3(n+1)}$–net for $[0,1]$ imply that $|\tilde{f}(m) -_{\mathbb{Q}} \tilde{f}(\tilde{m})| \leq \frac{1}{n+1}$ (here one has to disinguish the cases $\widehat{f}(3(m+1)$ in $[0,1]$ or not in $[0,1]$), so $\tilde{f}$ has the appropriate Cauchy modulus.
2) follows again immediately from the definition of $\tilde{f}$.
3) follows from 1). 4) follows from 2) and 3).

By this construction quantification $\bigwedge x \in [0,1]\ A(x)$ and $\bigvee x \in [0,1]\ A(x)$ reduces to quantification having the form $\bigwedge f \leq_1 M\ A(\tilde{f})$ and $\bigvee f \leq M\ A(\tilde{f})$ for properties A which are $=_{\mathbb{R}}$–extensional (for f_1, f_2 such that $1 \leq_{\mathbb{R}} f_1, f_2 \leq_{\mathbb{R}} 1$), where $M \in \mathrm{G_2R}^\omega$. Similarly one can define a representation of $[a,b]$ for variable a^1, b^1 such that $a <_{\mathbb{R}} b$ by bounded functions $\{f^1 : f \leq_1 M(a,b)\}$. However by remark 3.1.5 below one can easily reduce the quantification over $[a,b]$ to quantification over $[0,1]$ so that we do not need this generalization. But on some occasions it is convenient to have an explicit representation for $[-k,k]$ for all natural numbers k. This representation is analogous to the representation of $[0,1]$ except that we now define $M_k(n) := j(6k(n+1), 3(n+1)-1)$ as the bounding function. The construction corresponding to $\lambda f.\tilde{f}$ is also denoted by $\tilde{f}$ since it will be always clear from the context what interval we have in mind.

Representation of $[0,1]^d$ in $\mathbf{G_2A}_i^\omega$

Using the construction $f \mapsto \tilde{f}$ from the representation of [0,1] we also can represent $[0,1]^d$ for every fixed number d by a bounded set $\{f^1 : f \leq_1 M_d\}$ of functions, where $M_d : \nu^d(M, \ldots, M) \in \mathrm{G_2R}^\omega$ for every fixed d:
$f(\leq M_d)$ represents the vector in $[0,1]^d$ which is represented by $(\widetilde{(\nu_1^d f)}, \ldots, \widetilde{(\nu_d^d f)})$. If (in the other direction) $f_1, \ldots, f_d$ represent real numbers $x_1, \ldots, x_d \in [0,1]$, then $f := \nu^d(\tilde{f}_1, \ldots, \tilde{f}_d) \leq_1 \nu^d(M, \ldots, M)$ represents $(x_1, \ldots, x_d) \in [0,1]^d$ in this sense.

Remark 3.1.5 *For $a, b \in \mathbb{R}$ with $a \leq_{\mathbb{R}} b$, quantification $\bigwedge x \in [a,b]\ A(x)$ ($\bigvee x \in [a,b]\ A(x)$) reduces to quantification over $[0,1]$ (and therefore –modulo our representation– over $\{f : f \leq_1 M\}$) by $\bigwedge \lambda \in [0,1]\ A(\lambda a+(1-\lambda)b)$ and analogously for $\bigvee x$. This transformation immediately generalizes to $[a_1,b_1] \times \cdots \times [a_d,b_d]$ using $\lambda_1,\ldots,\lambda_d$.*

3.2 Representation of continuous functions $f : [0,1]^d \to \mathbb{R}$ by number theoretic functions

Functions $f : [a,b] \to \mathbb{R}$ ($a, b \in \mathbb{R}, a < b$) are represented in $\mathrm{G}_n\mathrm{A}^\omega$ by functionals $\Phi^{1(1)}$ which are $=_{\mathbb{R}}$–extensional:

$$\bigwedge x^1, y^1 (a^1 \leq_{\mathbb{R}} x, y \leq_{\mathbb{R}} b^1 \wedge x =_{\mathbb{R}} y \to \Phi x =_{\mathbb{R}} \Phi y).$$

Let $f : [a,b] \to \mathbb{R}$ be a pointwise continuous function. Then (classically) f is uniformly continuous and possesses a modulus $\omega : \mathbb{N} \to \mathbb{N}$ of uniform continuity, i.e.

$$\bigwedge x, y \in [a,b], k \in \mathbb{N}(|x-y| \leq \frac{1}{\omega(k)+1} \to |fx - fy| \leq \frac{1}{k+1}).$$

In $\mathrm{G}_n\mathrm{A}^\omega$ this reads as follows

$$(+)\ \bigwedge x^1, y^1, k^0 (a^1 \leq_{\mathbb{R}} x, y \leq_{\mathbb{R}} b \wedge |x -_{\mathbb{R}} y| \leq_{\mathbb{R}} \frac{1}{\omega(k)+1} \to |\Phi x -_{\mathbb{R}} \Phi y| \leq_{\mathbb{R}} \frac{1}{k+1}).$$

Thus quantification over continuous functions : $[a,b] \to \mathbb{R}$ corresponds in $\mathrm{G}_n\mathrm{A}^\omega$ to quantification over all $\Phi^{1(1)}, \omega^1$ which fulfil (+).
In the following we show how this quantification over objects of type level 2 can be reduced to type–1–quantification and how the condition (+) can be eliminated so that quantification over continuous functions on $[a,b]$ corresponds exactly to (unrestricted) quantification over f^1. We do this first for $a = 0, b = 1$ and reduce the general case to this situation. Finally we generalize our treatment to functions on $[0,1]^d$ (and $[a_1,b_1] \times \ldots \times [a_d,b_d]$).

Let $f : [0,1] \to \mathbb{R}$ be a uniformly continuous function with modulus of uniform continuity ω_f.
f is already uniquely determined by its restriction to $[0,1] \cap \mathbb{Q}$. Thus continuous functions $f : [0,1] \to \mathbb{R}$ can be conceived as a pair (f_r, ω_f) of functions $f_r : [0,1] \cap \mathbb{Q} \to \mathbb{R}$, $\omega_f : \mathbb{N} \to \mathbb{N}$ which satisfy

$$(*)\ \bigwedge k \in \mathbb{N}, x, y \in [0,1] \cap \mathbb{Q}\big(|x-y| \leq \frac{1}{\omega_f(k)+1} \to |f_r x - f_r y| \leq \frac{1}{k+1}\big)$$

(See also [70] and [6]).

Remark 3.2.1 *To represent a continuous function $f \in C[0,1]$ as a pair including a modulus of uniform continuity is a numerical enrichment of the given data which we use here for reasons which are similar to the endowment of real numbers with a Cauchy modulus: As we will see below quantification over $C[0,1]$ so reduces to quantification over functions of type 1. Furthermore many functions on $C[0,1]$ as e.g. $\int_0^1 f(x)dx$ or $\sup\limits_{x\in[0,1]} f(x)$ are given by functionals $\in G_2R^\omega$ in these data (see paragraph 3 and 4 below). This has as a consequence that many important theorems on*

continous functions have the logical form of axioms Γ *or* Δ *in the theorems of chapter 2. Also many sentences* $\bigwedge f \in C[0,1] \bigwedge x \in \mathbb{R} \bigwedge y \in [0,1] \bigvee z \in \mathbb{N}\ A(f,x,y,z)$ *have the logical form* $\bigwedge f^1, x^1 \bigwedge y \leq_1 M \bigvee z^0\ \tilde{A}(f,x,y,z)$ *with* $\tilde{A} \in \Sigma^0_1$ *so that theorem 2.2.2 applies yielding bounds on* $\bigvee z$ *which depend only on* f, x *(if* f *is represented with a modulus of continuity).*
In chapter 7 we will extend $E\text{-}G_nA^\omega$ *by an axiom* F^- *having the form of the sentences* $\in \Delta$ *in thm. 2.2.2 (and therefore not contributing to the rate of growth) which implies that every pointwise continuous function* $f : [0,1] \to \mathbb{R}$ *is uniformly continuous and possesses a modulus of uniform continuity. Hence under* F^- *the enrichment by such a modulus does not imply a restriction on the class of functions. We also formulate a stonger axiom* F *of this type which even implies that* **every** *function* $f : [0,1] \to \mathbb{R}$ *which is given by a functional* $\Phi^{1(1)}$ *is uniformly continuous and possesses a modulus of uniform continuity. This is not contradictory to the existence of non–continuous functions since the proof of the existence of a functional* Φ *representing such a function would require higher comprehension which is not available in our theories.*

Modulo our representation of $\mathbb{Q}$ and $\mathbb{R}$, f_r is an object of type 1(0) (i.e. a sequence of number theoretic functions). Quantification over continuous functions on [0,1] reduces to quantification over all pairs $(f^{1(0)}, \omega^1)$ (and therefore by suitable coding to quantification over all functions of type 1) which satisfy $(*)$ by substituting $\lambda x^1.f(x)_{\mathbb{R}}$ for (f,ω) in the matrix where $f(x)_{\mathbb{R}} := \lim\limits_{k\to\infty} f(\tilde{x}(\omega(k)))$ $(\lambda k^0.f(\tilde{x}(\omega(k)))$ is a Cauchy sequence of real numbers with modulus $\frac{1}{k+1}$ and so its limit is definable in $\mathrm{G_2A}^\omega$).
For the program carried out in this paper it is of crucial importance to be able to eliminate the implicative premise $(*)$: Let us consider the theorem of the attainment of the maximum of a continuous function on [0,1]

$$\bigwedge f \in C[0,1] \bigvee x_0 \in [0,1] \bigwedge x \in [0,1] (f(x_0) \geq fx).$$

Without the need of the implicative premise $(*)$ on (f,ω) this theorem would have (using our representation) the logical form

$$\bigwedge f^1 \bigvee x_0 \leq_1 M \bigwedge x^1\ A(f, x_0, x),$$

where $A \in \Pi^0_1$, i.e. the logical form of an axiom Δ in the theorems 2.2.2 and 2.2.7 and corollary 2.2.3 from chapter 2. Similarly many other important non–constructive theorems would have the logical form of an axiom Δ and thus do not contribute to the rate of growth of the uniform bounds extracted from proofs which use these theorems.
In fact below we will show that the premise $(*)$ can be eliminated by constructing functionals $\tilde{\Psi}_1, \tilde{\Psi}_2 \in \mathrm{G_2R}^\omega$ such that the following holds

1) If $(f^{1(0)}, \omega^1)$ fulfils $(*)$, then $f =_{1(0)} \tilde{\Psi}_1 f\omega$ and $\tilde{\Psi}_2 f\omega$ is also a modulus of uniform continuity for f.

2) For every pair $(f^{1(0)}, \omega^1)$ the pair $(\tilde{\Psi}_1 f\omega, \tilde{\Psi}_2 f\omega)$ satisfies $(*)$.

By this construction the quantification

$$\bigwedge (f^{1(0)}, \omega^1)\big((*) \to A(f,\omega)\big)$$

reduces to

$$\bigwedge (f^{1(0)}, \omega^1)\ A(\tilde{\Psi}_1 f\omega, \tilde{\Psi}_2 f\omega)$$

(and likewise for $\bigvee$) for properties A which are extensional in the sense of $=_{C[0,1]}$.
In the following we write more suggestively f_ω, ω_f for $\tilde{\Psi}_1 f\omega, \tilde{\Psi}_2 f\omega$.

The underlying intuition for the following definition is roughly as follows: If f is uniformly continuous with modulus ω, then $f_\omega(n) := f(n)$. In the case that the continuity property is violated at the first time at a point n, then we define f_ω as a simple polygon using the f–values on the previous points:

Definition 3.2.2 *For $f^{1(0)}, \omega^1$ we define f_ω, ω_f as follows:*

$$f_\omega(n) :=_1 \begin{cases} f(n), \text{ if } A_0(f,\omega,n) :\equiv \bigwedge m, \tilde{m} \leq_0 \Phi_\omega(3n) \bigwedge k \leq_0 n^2 \\ \qquad\qquad \big(|q(m) -_{\mathbb{Q}} q(\tilde{m})| \leq \frac{1}{\tilde{\omega}(k)+1} \to |(f(\widehat{qm}))k -_{\mathbb{Q}} (f(\widehat{q\tilde{m}}))k| \leq \frac{3}{k+1}\big) \\ \\ p_{n_0,f}(n), \text{ for } n_0 \leq_0 n \text{ minimal such that } \neg A_0(f,\omega,n_0), \text{ otherwise}, \end{cases}$$

$$\overline{\omega}_f(n) :=_0 \begin{cases} \tilde{\omega}(3n), \text{ if } A_0(f,\omega,n) \\ \\ \max_0\bigg(\bigg(\max_0\bigg\{\bigg\lceil\bigg|\frac{\widehat{f(qi) -_{\mathbb{R}} f(qj)}}{qi -_{\mathbb{Q}} qj}\bigg|(1)\bigg\rceil + 1 : i,j \leq_0 \Phi_\omega(3n_0), q(i) \neq q(j)\bigg\}\bigg) \cdot (n+1), \tilde{\omega}(n)\bigg) \\ \qquad\qquad \text{for } n_0 \leq_0 n \text{ minimal such that } \neg A_0(f,\omega,n_0), \text{ otherwise}, \end{cases}$$

(here $|\ldots|(1)$ is the value of the sequence $|\ldots|$ at 1) where
$p_{n_0,f}$ is the polygon defined by $f(q0), \ldots, f(q(\Phi_\omega(3(n_0 \dot{-} 1))))$,
$\tilde{\omega}(k) :=_0 \max_0(k,1)^2 \cdot \big(\max_{i\leq k} \omega(i) + 1\big)$, $\omega_f(n) := \overline{\omega}_f(5(n+1))$ and
$\Phi_\omega(n) :=_0 j(2(\tilde{\omega}(n)+1), \tilde{\omega}(n)+1)$ (Note that $0,1$ are coded by $0, j(2,0) \leq_0 \Phi_\omega(3(n_0 \dot{-} 1))$).

Remark 3.2.3 *f_ω and ω_f are definable in G_2R^ω (as functionals in f,ω) since A_0 can be expressed quantifier–free and $p_{n_0,f}$ can be written as*

$$p_{n_0,f}(n) =_1 f(qi) +_{\mathbb{R}} \frac{f(qi) -_{\mathbb{R}} f(qj)}{qi -_{\mathbb{Q}} qj} \cdot_{\mathbb{R}} (qn -_{\mathbb{Q}} qi),$$

where $i,j \leq_0 \Phi_\omega(3(n_0 \dot{-} 1))$ are such that $qi \leq_{\mathbb{Q}} qn \wedge (|qi -_{\mathbb{Q}} qn|$ minimal$) \wedge qj >_{\mathbb{Q}} qn \wedge (|qj -_{\mathbb{Q}} qn|$ minimal$)$ (If $q(n) =_{\mathbb{Q}} 1$, then $p_{n_0,f}(n) =_1 f(q(n))$).

Lemma: 3.2.4 *1) $k_1 \geq_0 k_2 \to \tilde{\omega}(k_1) \geq_0 \tilde{\omega}(k_2)$.*

2) $\tilde{\omega}(k) \geq_0 k$ and $\tilde{\omega}(k) \geq_0 \omega(k)$.

3) $\tilde{\omega}(3 \cdot k) \geq_0 3 \cdot \tilde{\omega}(k) + 3$ for $k \geq 1$.

Proof: 1) and 2) follow immediately from the definition of $\tilde{\omega}$.

3) $\tilde{\omega}(3k) \overset{k\geq 1}{\geq} 9k^2 \cdot \big(\max\limits_{i\leq k} \omega(i) + 1\big) \geq 3k^2 \cdot \big(\max\limits_{i\leq k} \omega(i) + 1\big) + 6k^2$

$\qquad \overset{k\geq 1}{\geq} 3k^2\big(\max\limits_{i\leq k} \omega(i) + 1\big) + 3 = 3 \cdot \tilde{\omega}(k) + 3.$

Lemma: 3.2.5 *If $f^{1(0)}$ represents a uniformly continuous function $F : [0,1] \to \mathbb{R}$ with a modulus ω^1 of uniform continuity, i.e.*
$\bigwedge m, \tilde{m}, k\big(|qm -_{\mathbb{Q}} q\tilde{m}| \le_{\mathbb{Q}} \frac{1}{\omega(k)+1} \to |f(qm) -_{\mathbb{R}} f(q\tilde{m})| \le_{\mathbb{R}} \frac{1}{k+1}\big)$,
then $f_\omega =_{1(0)} f$ and ω_f is also a modulus of uniform continuity for F.

Proof: The first part of the lemma follows from the definition of f_ω observing that the case 'otherwise' never occurs because of the assumption, since

$$|qm -_{\mathbb{Q}} q\tilde{m}| \le \frac{1}{\tilde{\omega}(k)+1} \overset{l.3.2.4}{\le} \frac{1}{\omega(k)+1}$$

implies that

$$|(\widehat{f(qm)})k -_{\mathbb{Q}} (\widehat{f(q\tilde{m})})k| \le |f(qm) -_{\mathbb{R}} f(q\tilde{m})| + \frac{2}{k+1} \le \frac{3}{k+1}.$$

Furthermore $\overline{\omega}_f(n) = \tilde{\omega}(3n) \overset{l.3.2.4}{\ge_0} \omega(n)$. Hence together with ω also $\overline{\omega}_f$ and thus a fortiori ω_f is a modulus of uniform continuity.

Lemma: 3.2.6 *For every pair $(f^{1(0)}, \omega^1)$ the following holds:*
f_ω represents a uniformly continuous function : $[0,1] \cap \mathbb{Q} \to \mathbb{R}$ and ω_f is a modulus of uniform continuity for this function, i.e.

$$\bigwedge m, \tilde{m}, k\big(|qm -_{\mathbb{Q}} q\tilde{m}| \le \frac{1}{\omega_f(k)+1} \to |f_\omega(qm) -_{\mathbb{R}} f_\omega(q\tilde{m})| \le \frac{1}{k+1}\big).$$

Proof: Let $m, \tilde{m}, k \in \mathbb{N}$ be such that $|qm -_{\mathbb{Q}} q\tilde{m}| \le \frac{1}{\overline{\omega}_f(k)+1}$.
We may assume that $qm >_0 q\tilde{m}$.
<u>Case</u> 1: $A_0(f, \omega, qm)$. Then also $A_0(f, \omega, q\tilde{m})$ since the monotonicity of $\Phi_\omega(3n)$ and n^2 implies

$$n_1 \ge_0 n_2 \wedge A_0(f, \omega, n_1) \to A_0(f, \omega, n_2).$$

Hence $f(qm) =_{\mathbb{R}} f_\omega(qm)$ and $f(q\tilde{m}) =_{\mathbb{R}} f_\omega(q\tilde{m})$. By $\overline{\omega}_f(k) \ge_0 \tilde{\omega}(k), k$ the assumption on $m, \tilde{m}, k$ yields

$$(+)\ |qm -_{\mathbb{Q}} q\tilde{m}| \le \frac{1}{\tilde{\omega}(k)+1} \text{ and } (++)\ |qm -_{\mathbb{Q}} q\tilde{m}| \le \frac{1}{k+1}.$$

(++) implies that $k \le_0 (qm)^2$ (Because of $j_2(qm), j_2(q\tilde{m}) <_0 qm$, the (distinct) fractions coded by $qm, q\tilde{m}$ have denominaters $a, b \le_0 qm$. Thus $|\frac{i}{a} - \frac{j}{b}| \ge \frac{1}{ab} \ge \frac{1}{(qm)^2}$). Furthermore $qm, q\tilde{m} \le_0 \Phi_\omega(3(qm))$. Hence (+) and $A_0(f, \omega, qm)$ yield (using $\bigwedge x^0(q(qx) =_0 qx)$)

$$|(\widehat{f(qm)})k -_{\mathbb{Q}} (\widehat{f(q\tilde{m})})k| \le \frac{3}{k+1}$$

and therefore

$$|f_\omega(qm) -_{\mathbb{R}} f_\omega(q\tilde{m})| =_{\mathbb{R}} |f(qm) -_{\mathbb{R}} f(q\tilde{m})| \le \frac{5}{k+1}.$$

<u>Case</u> 2: $\neg A_0(f, \omega, qm)$.
2.1 $k \ge_0 n_0 := \min n \le_0 qm \neg A_0(f, \omega, n)$:

In this case we have $f_\omega(qm) =_{\mathrm{I\!R}} p_{n_0,f}(qm)$ and $f_\omega(q\tilde{m}) =_{\mathrm{I\!R}} p_{n_0,f}(q\tilde{m})$ (In the case $A_0(f,\omega,q\tilde{m})$ we have $q\tilde{m} < n_0 \leq \Phi_\omega(3(n_0-1))$ and so $f_\omega(q\tilde{m}) = f(q\tilde{m})$ is one of the f–values used in defining $p_{n_0,f}$). Since $\overline{\omega}_f$ is a modulus of uniform continuity for $p_{n_0,f}$ for $k \geq n_0$, the assumption on $m,\tilde{m}$ implies

$$|f_\omega(qm) -_{\mathrm{I\!R}} f_\omega(q\tilde{m})| \leq \frac{1}{k+1}.$$

2.2 $1 \leq_0 k <_0 n_0$: Then $A_0(f,\omega,k)$ and therefore $\overline{\omega}_f(k) = \tilde{\omega}(3k)$. Since all fractions $\frac{i}{\tilde{\omega}(3(n_0-1))+1}$ with $i \leq_0 \tilde{\omega}(3(n_0-1))+1$ have a code $\leq_0 \Phi_\omega(3(n_0-1))$, the maximal distance between two adjacent breaking points of $p_{n_0,f}$ is $\leq \frac{1}{\tilde{\omega}(3(n_0-1))+1}$. Hence there are $m^*,\tilde{m}^* \leq_0 \Phi_\omega(3(n_0-1))$ (i.e. 'breaking points' of the polygon $p_{n_0,f}$ next to $m,\tilde{m}$ satisfying (2) below) such that

$$(1)\ |qm^* -_{\mathbb{Q}} q\tilde{m}^*| \leq \frac{1}{\overline{\omega}_f(k)+1} + \frac{2}{\tilde{\omega}(3(n_0-1))+1} \overset{l.3.2.4}{\leq} \frac{3}{\tilde{\omega}(3k)+1} \overset{l.3.2.4}{\leq} \frac{3}{3\tilde{\omega}(k)+3+1} \leq \frac{1}{\tilde{\omega}(k)+1}$$

and

$$(2)\ |\underbrace{p_{n_0,f}(q\tilde{m}^*)}_{=_{\mathrm{I\!R}} f(q\tilde{m}^*)} -_{\mathrm{I\!R}} \underbrace{p_{n_0,f}(qm^*)}_{=_{\mathrm{I\!R}} f(qm^*)}| \geq_{\mathrm{I\!R}} |\underbrace{p_{n_0,f}(q\tilde{m})}_{=_{\mathrm{I\!R}} f_\omega(q\tilde{m})} -_{\mathrm{I\!R}} \underbrace{p_{n_0,f}(qm)}_{=_{\mathrm{I\!R}} f_\omega(qm)}|.$$

Since $A_0(f,\omega,n_0-1)$ and $k \leq_0 (n_0-1)^2$, (1) and (2) imply

$$|f_\omega(qm) -_{\mathrm{I\!R}} f_\omega(q\tilde{m})| \overset{(2)}{\leq} |f(qm^*) -_{\mathrm{I\!R}} f(q\tilde{m}^*)| \leq |(f(\widehat{qm^*}))k -_{\mathbb{Q}} (f(\widehat{q\tilde{m}^*}))k| + \tfrac{2}{k+1}$$
$$\overset{(1)}{\leq} \tfrac{3}{k+1} + \tfrac{2}{k+1} = \tfrac{5}{k+1}.$$

Put together we have shown that in both cases (for $k \geq 1$)

$$|qm -_{\mathbb{Q}} q\tilde{m}| \leq \frac{1}{\overline{\omega}_f(k)+1} \to |f_\omega(qm) -_{\mathrm{I\!R}} f_\omega(q\tilde{m})| \leq \frac{5}{k+1}.$$

Hence ω_f is a modulus of uniform continuity for f_ω.

Since every pair $(f^{1(0)},\omega^1)$ can be conceived now as a representation of a uniformly continuous function $[0,1]\cap\mathbb{Q} \to \mathrm{I\!R}$, namely that function which is represented by $(\tilde{\Psi}_1 f\omega, \tilde{\Psi}_2 f\omega)$ (where $\tilde{\Psi}_1 f\omega := f_\omega \circ q, \tilde{\Psi}_2 f\omega := \omega_f$).[31] And every function g^1 can be conceived as a pair (f,ω) by $g \mapsto (\lambda k^0, n^0.(j_1 g)(j(k,n)), j_2 g)$ (where $j_i g := \lambda x^0.j_i(gx)$), so g^1 represents the continuous function $(\Psi_1 g, \Psi_2 g)$, where $\Psi_1 g := \tilde{\Psi}_1(\lambda k^0, n^0.(j_1 g)(j(k,n)), j_2 g)$ and $\Psi_2 g := \tilde{\Psi}_2(\lambda k^0, n^0.(j_1 g)(j(k,n)), j_2 g)$. Since every pair (f,ω) can be coded by a function g, every uniformly continuous function $[0,1]\cap\mathbb{Q} \to \mathrm{I\!R}$ is represented by some function g. Together with $\tilde{\Psi}_i$ also the Ψ_i are in $\mathrm{G_2R^\omega}$.
Now we define the continuation on full $[0,1]$:

Definition 3.2.7 *The functional $\lambda g^1, x^1.g(x)_{\mathrm{I\!R}} \in G_2R^\omega$ is defined by $(g(x)_{\mathrm{I\!R}})(k^0) :=_0 \Psi_1 g(\tilde{x}(\widehat{\Psi_2 g(3}(k+1))))(3(k+1))$, $\tilde{x}$ is the construction used in our representation of $[0,1]$.*

[31] By switching from f_ω to $f_\omega \circ q$ we can formulate the continuity of $\tilde{\Psi}_1 f\omega$ now as $\bigwedge m,\tilde{m}\big(0 \leq_{\mathbb{Q}} m,\tilde{m} \leq_{\mathbb{Q}} 1 \wedge |m -_{\mathbb{Q}} \tilde{m}| \leq \frac{1}{\omega_f(k)+1} \to |(\tilde{\Psi}_1 f\omega)(m) -_{\mathrm{I\!R}} |(\tilde{\Psi}_1 f\omega)(\tilde{m})| \leq \frac{1}{k+1}\big)$, i.e. without mentioning q anymore.

Remark 3.2.8 *$g(x)_{\mathbb{R}}$ represents the value of the function $\in C[0,1]$, which is represented by g, applied to the real $\in [0,1]$, which is represented by x.*
Notation: If a function $\in C[0,1]$ is given as a pair $(f^{1(0)}, \omega^1)$ we also use the notation $f(x)_{\mathbb{R}}$ in order to avoid the need of spelling out the coding $(f,\omega) \mapsto g^1$.

Remark 3.2.9 *Quantification over $C[a,b]$ (where $a < b$) reduces to quantification over $C[0,1]$ by $f \in C[a,b] \mapsto g := \lambda x.f(a(1-x)+bx) \in C[0,1]$ and*
$g \in C[0,1] \mapsto f := \lambda x.g(\frac{x-a}{b-a}) \in C[a,b]$.

In [32] and [37] we used a different representation of the space $C[0,1]$ (following [8]) based on the Weierstraß approximation theorem: A function $f \in C[0,1]$ was represented as a Cauchy sequence w.r.t. $\|\cdot\|_\infty$ (with modulus $1/(k+1)$) of polynomials with rational coefficients. Then we applied a construction, similarly to $\widehat{f}$ used in our representation of $\mathbb{R}$ above, to ensure that every function f^1 could be conceived as such a Cauchy sequence.
However this representation is not convenient for our theory $\mathrm{G_2A}^\omega_i$ since the coding of an arbitrary sequence of polynomials requires the coding of finite sequences of natural numbers (the codes of the coefficients) of variable length which can be carried out in $\mathrm{G_3A}^\omega_i$ but not in $\mathrm{G_2A}^\omega_i$. Furthermore in practice the computation of an approximating sequence of polynomials to a given function is quite complicated (and even more when one deals with functions in several variables as we will do below) whereas for most functions occurring in mathematics a modulus of continuity can be written down directly. Hence it is much more useful to extract bounds which require as a function input only the function endowed with a modulus of uniform continuity than an approximating sequence of polynomials. In our applications to approximation theory we always obtained bounds in functions with a modulus of continuity. Because of this we conjectured in [37] that this will always hold for extractions of bounds from concrete proofs. By our new representation of $C[0,1]$ this conjecture is theoretically justified: From a proof of a sentence

$$\bigwedge f \in C[0,1] \bigvee y^0\ A(f,y),\ \text{where } A \in \Sigma^0_1$$

we obtain a bound on y in a representative of f in our sense, i.e. in f endowed with a modulus of uniform continuity.

The construction of f_ω, ω_f looks quite complicated. However if f is already given with a modulus ω (as in concrete applications) then f_ω does not change anything and $\omega_f(n)$ is just a slight modification of ω and the proof of this (3.2.5) is almost trivial. The complicated clause in the definition of f_ω, ω_f is needed only to ensure that an arbitrary given pair (f,ω) is transformed into a continuous function. The quite complicated proof of lemma 3.2.6 is not relevant for the extraction process since the statement of this lemma is a purely universal sentence and therefore an axiom of $\mathrm{G_2A}^\omega_i$.

For the construction of f_ω and ω_f we have made use of the fact that the values $fx, f\tilde{x}$ of f on two points $x < \tilde{x}$ can be connected by a line given by the simple function

$$(*)\ py := fx + \frac{f\tilde{x} - fx}{\tilde{x} - x}(y - x)$$

which extends f from $\{x, \tilde{x}\}$ to $[x, \tilde{x}]$. We have used the following properties of p:

1) $\min(fx, f\tilde{x}) \le py \le \max(fx, f\tilde{x})$ for $x, \tilde{x} \in [x, \tilde{x}]$.

2) $px = fx$ and $p\tilde{x} = f\tilde{x}$.

3) p has a (simple) modulus of uniform continuity $\in \mathrm{G}_2\mathrm{R}^\omega$ (in fact a Lipschitz constant) on $[x, \tilde{x}]$.

In the following we generalize this construction to the d–dimensional space $[0,1]^d$ and obtain (for every fixed d) a representation of $C([0,1]^d, \mathbb{R})$ by the functions of type 1.
$(*)$ can be written also in the following form:
Let $y \in [x, \tilde{x}]$. Then $y = (1-\lambda)x + \lambda\tilde{x}$ with $\lambda = \frac{y-x}{\tilde{x}-x} \in [0,1]$ and $py = (1-\lambda)fx + \lambda f\tilde{x}$.
This formulation of p easily generalizes to the dimension d:
Let us consider an d–dimensional rectangle (i.e. a regular parallel epipthed) in $[0,1]^d$ defined by

$$K_{x,\underline{n}} := \left\{ y \in [0,1]^d : \bigwedge_{i=1}^{d} (x_i \le y_i \le x_i + \frac{1}{n_i+1}) \right\},$$

where $x = (x_1, \ldots, x_d) \in [0,1]^d$ and $x_i \le 1 - \frac{1}{n_i+1}$ for $i = 1, \ldots, d$ and $\underline{n} := n_1, \ldots, n_d$.

$$V_{x,\underline{n}} := \left\{ (\overline{x}_1, \ldots, \overline{x}_d) \in [0,1]^d : \bigwedge_{i=1}^{d} (\overline{x}_i = x_i \vee \overline{x}_i = x_i + \frac{1}{n_i+1}) \right\} =: \{e_1, \ldots, e_{2^d}\},$$

denotes the set of vertices of $K_{x,\underline{n}}$.
We now define a construction by which a function f defined on $V_{x,\underline{n}}$ is continued on the whole rectangle $K_{x,\underline{n}}$:

$$A(x, \underline{n}, y) := \left\{ \overline{\lambda}_1 \cdot \ldots \cdot \overline{\lambda}_d \cdot f(\overline{x}_1, \ldots, \overline{x}_d) : \bigwedge_{i=1}^{d} [(\overline{\lambda}_i = 1 - \lambda_i \wedge \overline{x}_i = x_i) \vee (\overline{\lambda}_i = \lambda_i \wedge \overline{x}_i = x_i + \frac{1}{n_i+1})] \right\},$$

where $\lambda_i := \lambda(y_i) := (y_i - x_i)(n_i + 1)$. $\#A(x, \underline{n}, y) = 2^d$.

Definition 3.2.10 $p(x, \underline{n}, y) := \Sigma A(x, \underline{n}, y)$.

Remark 3.2.11 *For every fixed number d the function p is definable in G_2R^ω.*

Lemma: 3.2.12 *1) p equals f on the vertices of $K_{x,\underline{n}}$: $\bigwedge_{i=1}^{2^d} (p(x, \underline{n}, e_i) = fe_i)$.*

2) $\min(fe_1, \ldots, fe_{2^d}) \le p(x, \underline{n}, y) \le \max(fe_1, \ldots, fe_{2^d})$ for all $y \in K_{x,\underline{n}}$.

3) $p(x, \underline{n}, \cdot)$ is Lipschitz continuous on $K_{x,\underline{n}}$ w.r.t. $\|(y_1, \ldots, y_d)\|_{\max} := \max_{i=1,\ldots,d} |y_i|$ with Lipschitz constant $\lambda(x, \underline{n}) := \max([\max(fe_1, \ldots, fe_{2^d}) - \min(fe_1, \ldots, fe_{2^d})] \cdot (\max_0(n_1, \ldots n_d) + 1)d, 1)$.

Proof: 1) Let $(\overline{x}_1, \ldots, \overline{x}_d) \in K_{x,\underline{n}}$. Then

$$\overline{x}_i = x_i \to \overline{\lambda}(\overline{x}_i) = 1 - \lambda(\overline{x}_i) = 1 \wedge \lambda(\overline{x}_i) = 0$$

and

$$\overline{x}_i = x_i + \frac{1}{n_i+1} \to \overline{\lambda}(\overline{x}_i) = \lambda(\overline{x}_i) = 1 \wedge 1 - \lambda(\overline{x}_i) = 0.$$

Hence

$$f(\overline{x}_1,\dots,\overline{x}_d)=\overline{\lambda}(\overline{x}_1)\cdot\ldots\cdot\overline{\lambda}(\overline{x}_d)\cdot f(\overline{x}_1,\dots,\overline{x}_d)\in A(x,\underline{n},\overline{x}),\ \text{where}\ \overline{x}:=(\overline{x}_1,\dots,\overline{x}_d).$$

All other elements $\overline{\lambda}'_1\cdot\ldots\cdot\overline{\lambda}'_d\cdot f(\overline{x}'_1,\dots,\overline{x}'_d)\in A(x,\underline{n},\overline{x})$ are $=0$ since at least one factor $\overline{\lambda}'_i=0$. Hence $p(x,\underline{n},\overline{x})=f\overline{x}$.
2) Let $y\in K_{x,\underline{n}}$. Then $\lambda(y_i),1-\lambda(y_i)\in[0,1]$ for $i=1,\dots,d$. Define $M:=\max(fe_1,\dots,fe_{2^d})$. Then

$$\Sigma A(x,\underline{n},y)\le M\cdot\Sigma\left\{\overline{\lambda}(y_1)\cdot\ldots\cdot\overline{\lambda}(y_d):\bigwedge_{i=1}^{d}(\overline{\lambda}(y_i)=\lambda(y_i)\vee\overline{\lambda}(y_i)=1-\lambda(y_i))\right\}=M.$$

Analogously one shows the inequality $\min(fe_1,\dots,fe_{2^d})\le\Sigma A(x,\underline{n},y)$.
3) We show that for every $\varepsilon>0$ and every fixed i with $1\le i\le d$ and all $y_1,\dots,y_d,z\in[0,1]$

$$(+)\ \big(|y_i-z|\le\frac{\varepsilon}{\lambda(x,\underline{n})}\to|p(x,\underline{n},(y_1,\dots,y_d))-p(x,\underline{n},(y_1,\dots,y_{i-1},z,y_i,\dots,y_d))|\le\frac{\varepsilon}{d}\big).$$

This implies the claim of the lemma since for $(y_1,\dots,y_d),(\tilde{y}_1,\dots,\tilde{y}_d)\in[0,1]^d$

$$\|(y_1,\dots,y_d)-(\tilde{y}_1,\dots,\tilde{y}_d)\|_{\max}=\|(y_1-\tilde{y}_1),\dots,(y_d-\tilde{y}_d)\|_{\max}\le\frac{\varepsilon}{\lambda(x,\underline{n})}$$

implies

$$\bigwedge_{i=1}^{d}\big(|y_i-\tilde{y}_i|\le\frac{\varepsilon}{\lambda(x,\underline{n})}\big)$$

and therefore (by (+))

$$\begin{aligned}&|p(x,\underline{n},(y_1,\dots,y_d))-p(x,\underline{n},(\tilde{y}_1,\dots,\tilde{y}_d)|\\&\le\sum_{i=1}^{d}|p(x,\underline{n},(\tilde{y}_1,\dots,\tilde{y}_{i-1},y_i,y_{i+1},\dots,y_d))-p(x,\underline{n},(\tilde{y}_1,\dots,\tilde{y}_i,y_{i+1},\dots,y_d))|\\&\le d\cdot\big(\tfrac{\varepsilon}{d}\big)=\varepsilon.\end{aligned}$$

For notational simplicity we assume that $i=1$ (for an arbitrary $i=1,\dots,d$ the proof proceeds analogously):
$|y_1-z|\le\frac{\varepsilon}{\lambda(x,\underline{n})}$ implies for $\lambda(y_1)=(y_1-x_1)(n_1+1),\lambda(z)=(z-x_1)(n_1+1)$:

$$(0)\ \begin{cases}|\lambda(y_1)-\lambda(z)|\le(n_1+1)\cdot\frac{\varepsilon}{\lambda(x,\underline{n})}\le\varepsilon\cdot c,\\ \text{where } c:=\big(\max([\max(fe_1,\dots,fe_{2^d})-\min(fe_1,\dots,fe_{2^d})]\cdot d,1/(\max(n_i)+1))\big)^{-1}.\end{cases}$$

We may assume that $\lambda(y_1)\ge\lambda(z)$:
$\sum$:=
$\sum\left\{\overline{\lambda}_2\cdot\ldots\cdot\overline{\lambda}_d\cdot f(x_1,\overline{x}_2,\dots,\overline{x}_d):\bigwedge_{i=2}^{d}[(\overline{\lambda}_i=1-\lambda_i\wedge\overline{x}_i=x_i)\vee(\overline{\lambda}_i=\lambda_i\wedge\overline{x}_i=x_i+\frac{1}{n_i+1})]\right\}$,
$\overline{\sum}$:=

$$\sum\left\{\overline{\lambda}_2\cdot\ldots\cdot\overline{\lambda}_d\cdot f(\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_d):\bigwedge_{i=2}^{d}[(\overline{\lambda}_i=1-\lambda_i\wedge\overline{x}_i=x_i)\vee(\overline{\lambda}_i=\lambda_i\wedge\overline{x}_i=x_i+\tfrac{1}{n_i+1})]\right\},$$

where $\overline{x}_1=x_1+\frac{1}{n_1+1}$.

(1) $(1-\lambda(y_1))\cdot\sum-(1-\lambda(z))\cdot\sum\le-(\lambda(y_1)-\lambda(z))\cdot\min(fe_1,\ldots,fe_{2^d})$.

(2) $\lambda(y_1)\cdot\overline{\sum}-\lambda(z)\cdot\overline{\sum}\le(\lambda(y_1)-\lambda(z))\cdot\max(fe_1,\ldots,fe_{2^d})$

and Put together (0),(1) and (2) yield

$$\begin{aligned}
&\textstyle\sum A(x,\underline{n},y)-\sum A(x,\underline{n},(z,y_2,\ldots,y_d))\\
&=\lambda(y_1)\cdot\overline{\textstyle\sum}+(1-\lambda(y_1))\cdot\textstyle\sum-\big(\lambda(z)\cdot\overline{\textstyle\sum}-(1-\lambda(z))\cdot\textstyle\sum\big)\\
&=\big(\lambda(y_1)\cdot\overline{\textstyle\sum}-\lambda(z)\cdot\overline{\textstyle\sum}\big)+\big((1-\lambda(y_1))\cdot\textstyle\sum-(1-\lambda(z))\cdot\textstyle\sum\big)\\
&\le(\lambda(y_1)-\lambda(z))\cdot\big(\max(fe_1,\ldots,fe_{2^d})-\min(fe_1,\ldots,fe_{2^d})\big)\le\tfrac{\varepsilon}{d}.
\end{aligned}$$

Analogously: $\sum A(x,\underline{n},(z,y_2,\ldots,y_d))-\sum A(x,\underline{n},y)\le\frac{\varepsilon}{d}$.

Using p one can now define constructions $f_\omega^{1\overbrace{(0)\ldots(0)}^{d}}$ and ω_f^1 such that every pair $(f^{1(0)\ldots(0)},\omega^1)$ is transformed into a representative of a function $\in C([0,1]^d,\mathbb{R})$ together with a modulus of uniform continuity ω_f (w.r.t. to $\|\cdot\|_{\max}$). In the definition of $f_\omega(n_1,\ldots,n_d)$ we test whether the continuity property is satisfied for all $m_1,\tilde{m}_1,\ldots,m_d,\tilde{m}_d\le_0\Phi_\omega(3(\max(n_1,\ldots,n_d)))$ and $k\le_0\max_0(n_1,\ldots,n_d)^2$. Every 'lattice' in $[0,1]^d\cap\mathbb{Q}^d$ coded by $\left\{(m_1,\ldots,m_d):\bigwedge_{i=1}^{d}(m_i\le_0 k)\right\}$ defines a decomposition of $[0,1]^d$ into d–dimensional rectangles. Using our construction p we are able to continue the restriction of f on the vertices of each rectangle to a function on the whole rectangle. By carrying out this for every rectangle we obtain a function on the whole space $[0,1]^d$ which coincides with f on the 'lattice' points and is Lipschitz continuous with the maximum of the Lipschitz constants of the functions on all single rectangle (This follows from the fact that two functions corresponding to rectangles which have a face in common coincide on this face). Using this function instead of the polygon in definition 3.2.2 one obtains a representation analogously to f_ω,ω_f also for functions $\in C([0,1]^d,\mathbb{R})$ (together with a corresponding application $(\cdot)_{\mathbb{R}^d}$).

3.3 The functionals $\max_{\mathbb{R}},+_{\mathbb{R}}$ for sequences of variable length and $\sup\limits_{x\in[a,b]}fx$,$\int\limits_a^b f(x)dx$ in $\mathbf{G}_2\mathbf{A}_i^\omega$

For the computation of $\sup\limits_{x\in[a,b]}fx$ and $\int\limits_a^b f(x)dx$ for $f\in C[a,b]$ we need the maximum and the sum of a sequence of real numbers of variable length, i.e. $\max_{\mathbb{R}}\{f(r_i):i\le k\}$ and $f(r_0)+_{\mathbb{R}}\ldots+_{\mathbb{R}}f(r_k)$ for a sequence of rational numbers r_i. For the construction of such operations in $\mathrm{G}_2\mathrm{R}^\omega$ we need a special form of our representation of real numbers:
The computation of the addition of a sequence of x real numbers $a_0,\ldots,a_x$ requires the addition of corresponding sequences of the n–th rational approximations $\widehat{a}_0(n),\ldots,\widehat{a}_x(n)$ of these real numbers

(for all n). For this we need the computation of a common divisor of $\widehat{a}_0(n), \ldots, \widehat{a}_x(n)$. However the size of such a common divisor will (in general) have an exponential growth in x and therefore is not definable in $\mathrm{G}_2\mathrm{R}^\omega$ but only in $\mathrm{G}_3\mathrm{R}^\omega$. This difficulty is avoided by modifying representatives f of real numbers to representatives f' such that $f =_{\mathbb{R}} f'$ and the n–th rational approximation $f'n$ of f' is a (code of a) fraction with a fixed denominator. We choose $3(n+1)+1$ as this denominator in order to ensure the right rate of convergence such that $\widehat{f'} =_1 f'$. For the computation of $\max_{\mathbb{R}}(a_0, \ldots, a_x)$ this modification is (although not necessary) very convenient.

Definition 3.3.1

$$\check{f}n :=_0 \begin{cases} \min k \leq_0 j_1(\widehat{f}(3(n+1))) \cdot (3(n+1)+1) \left[\frac{\frac{k}{2}}{3(n+1)+1} \leq_{\mathbb{Q}} \widehat{f}(3(n+1)) <_{\mathbb{Q}} \frac{\frac{k}{2}+1}{3(n+1)+1}\right] \\ \qquad\qquad \textit{if it exists and } j_1(\widehat{f}(3(n+1))) \textit{ is even} \\ \\ \min k \leq_0 j_1(\widehat{f}(3(n+1))) \cdot (3(n+1)+1) \left[\frac{-\frac{k+1}{2}}{3(n+1)+1} \leq_{\mathbb{Q}} \widehat{f}(3(n+1)) <_{\mathbb{Q}} \frac{-\frac{k+1}{2}+1}{3(n+1)+1}\right] \\ \qquad\qquad \textit{if it exists and } j_1(\widehat{f}(3(n+1))) \textit{ is odd} \\ \\ 0^0, \quad \textit{otherwise.} \end{cases}$$

$f'(n) := j\big(\check{f}n, 3(n+1)\big)$.

Remark 3.3.2 *Together with $\lambda f.\widehat{f}$ also $\lambda f.\check{f}$ and therefore $\lambda f.f'$ are definable in G_2R^ω.*

Lemma: 3.3.3 $G_2A_i^\omega \vdash \bigwedge f^1(f' =_{\mathbb{R}} f)$.

Proof: The case 'otherwise' does not occur since by our coding of rational numbers[32]

$$-j_1\big(\widehat{f}(3(n+1))\big) - 1 \leq_{\mathbb{Q}} 2\widehat{f}(3(n+1))) \leq_{\mathbb{Q}} j_1\big(\widehat{f}(3(n+1))\big).$$

Hence

$$|f'(n) -_{\mathbb{Q}} \widehat{f}(3(n+1))| < \frac{1}{3(n+1)+1} \text{ for all } n \in \mathbb{N}.$$

It therefore suffices to show that f' has the right rate of convergence, i.e. $\widehat{f'} =_1 f'$: Assume $m, \tilde{m} \geq k$. Then

$$\left.\begin{array}{l} |f'm -_{\mathbb{Q}} \widehat{f}(3(m+1))| \leq \frac{1}{3(m+1)+1} \leq \frac{1}{3(k+1)+1} \\ |f'\tilde{m} -_{\mathbb{Q}} \widehat{f}(3(\tilde{m}+1))| \leq \frac{1}{3(\tilde{m}+1)+1} \leq \frac{1}{3(k+1)+1} \\ |\widehat{f}(3(m+1)) -_{\mathbb{Q}} \widehat{f}(3(\tilde{m}+1))| \leq \frac{1}{3(k+1)+1} \end{array}\right\} \Rightarrow |f'm -_{\mathbb{Q}} f'\tilde{m}| \leq \frac{1}{k+1}.$$

Definition 3.3.4 $\chi^1, \psi^{1(1)} \in G_2R^\omega$ *are defined such that (provably in $G_2A_i^\omega$)*

$$\chi n^0 =_0 \begin{cases} 1, & \textit{if } \bigvee m \leq_0 n(n =_0 2m) \\ 0, & \textit{otherwise.} \end{cases}$$

[32]Here we simply write $j_1(\widehat{f}(3(n+1)))$ instead of the code of this natural number as an element of $\mathbb{Q}$.

and

$$\psi g^1 k^0 =_0 \begin{cases} \max\limits_{i\le k}\big(g(i)\cdot\chi(gi)\big), & \textit{if } \bigvee i\le_0 k(\chi(gi)=_0 1) \\ \min\limits_{i\le k} g(i), & \textit{otherwise.}\end{cases}$$

Definition 3.3.5 $\Phi_{\max_{\rm I\!R}} \in G_2R^\omega$ *is defined by*
$\Phi_{\max_{\rm I\!R}} := \lambda f^{1(0)}, k^0, n^0.j\big(\psi(\lambda i^0.j_1((fi)'n),k), 3(n+1)\big)$.

Lemma: 3.3.6
$G_2A_i^\omega \vdash \bigwedge k^0, f^{1(0)}\big(\Phi_{\max_{\rm I\!R}} f0 =_{\rm I\!R} f0 \wedge \Phi_{\max_{\rm I\!R}} f(k+1) =_{\rm I\!R} \max_{\rm I\!R}(\Phi_{\max_{\rm I\!R}} fk, f(k+1))\big)$.

Proof: For notational simplicity we write Φ_m instead of $\Phi_{\max_{\rm I\!R}}$ in the following.
$(\Phi_m f0)(n) =_0 j(j_1((f0)'n), 3(n+1)) =_0 (f0)'n$ and therfore $\Phi_m f0 =_{\rm I\!R} (f0)' =_{\rm I\!R} f0$.
$k+1$: Case 1) $\bigvee i \le k+1\big(\chi(j_1((fi)'n)) = 1\big)$:
$j\big(\max\limits_{i\le k+1}\big(j_1((fi)'n)\cdot\chi(j_1((fi)'n))\big), 3(n+1)\big)$
$=_0 j\Big(\max_0\big(\max\limits_{i\le k}\big(j_1((fi)'n)\cdot\chi(j_1((fi)'n))\big), j_1((f(k+1))'n)\cdot\chi(j_1((f(k+1))'n))\big), 3(n+1)\Big)$
$=_{\mathbb{Q}} \max_{\mathbb{Q}}\big(j(\max\limits_{i\le k}(\ldots),3(n+1)), j\big(j_1((f(k+1))'n)\cdot\chi(j_1((f(k+1))'n))\big), 3(n+1))\big)$
$\overset{!}{=}_{\mathbb{Q}} \max_{\mathbb{Q}}\big((\Phi_m fk)n, (f(k+1))'n\big)$.
Hence $\Phi_m f(k+1) =_{\rm I\!R} \max_{\rm I\!R}(\Phi_m fk, f(k+1))$.
(Ad !: Case α) $\bigvee i \le k\big(\chi(j_1((fi)'n)) = 1\big)$:
$j\big(\max\limits_{i\le k}(\ldots), 3(n+1)\big) =_0 (\Phi_m fk)n$, hence $j\big(\max\limits_{i\le k}(\ldots), 3(n+1)\big) =_{\mathbb{Q}} (\Phi_m fk)n$ $\surd$.
Case β) $\bigwedge i \le k\big(\chi(j_1((fi)'n)) = 0\big)$:

(1) $(\Phi_m fk)n =_0 j\big(\min\limits_{i\le k} j_1((fi)'n), 3(n+1)\big) <_{\mathbb{Q}} 0$,

(2) $j\big(\max\limits_{i\le k}(\ldots), 3(n+1)\big) =_0 j(0, 3(n+1)) =_{\mathbb{Q}} 0$.

Since $j_1((f(k+1))'n)$ is even, it follows that

(3) $j\big(j_1((f(k+1))'n)\cdot\chi(j_1((f(k+1))'n)), 3(n+1)\big) =_0 j\big(j_1((f(k+1))'n), 3(n+1)\big) \ge_{\mathbb{Q}} 0$.

(1)–(3) imply '!').
Case 2) $\bigwedge i \le k+1\big(\chi(j_1((fi)'n)) = 0\big)$: Similarly!
Since the statement of lemma 3.3.6 is purely universal it is an axiom of $G_2A_i^\omega$.

Lemma: 3.3.7 *1)* $G_2A_i^\omega \vdash \bigwedge f^{1(0)}, m^0, \tilde m^0\big(m \ge_0 \tilde m \to \Phi_{\max_{\rm I\!R}} fm \ge_{\rm I\!R} \Phi_{\max_{\rm I\!R}} f\tilde m, f\tilde m\big)$.

2) $G_2A^\omega + AC^{0,0}\text{–}qf \vdash \bigwedge f^{1(0)}, m^0 \bigvee k \le_0 m(fk =_{\rm I\!R} \Phi_{\max_{\rm I\!R}} fm)$.

Proof: 1) follows by induction on m using lemma 3.3.6 . Since 1) is purely universal it is an axiom of $G_2A_i^\omega$.
2) Assume $\bigwedge k \le_0 m(fk <_{\rm I\!R} \Phi_{\max_{\rm I\!R}} fm)$. Then

$$\bigwedge k \le_0 m \bigvee l^0 (fk <_{\rm I\!R} \Phi_{\max_{\rm I\!R}} fm - \frac{1}{l+1}).$$

By $\mathrm{AC}^{0,0}$–qf one obtains a function χ^1 such that

$$\bigwedge k \leq_0 m(fk <_{\mathbb{R}} \Phi_{\max_{\mathbb{R}}} fm - \frac{1}{\chi k + 1}).$$

Put $l_0 := \max\limits_{k \leq m} \chi k$. Then $\bigwedge k \leq_0 m(fk <_{\mathbb{R}} \Phi_{\max_{\mathbb{R}}} fm - \frac{1}{l_0+1})$ and therefore

$$(1) \bigwedge k \leq_0 m\big((\widehat{fk})(6(l_0+1)) <_{\mathbb{Q}} (\widehat{\Phi_{\max_{\mathbb{R}}} fm})(6(l_0+1)) -_{\mathbb{Q}} \frac{2}{3(l_0+1)}\big).$$

One easily verifies that

$$(2)\ a^1, b^1 \leq_{\mathbb{R}} c^1 \rightarrow \max_{\mathbb{R}}(a,b) \leq_{\mathbb{R}} c.$$

Using this and the previous lemma one shows by induction on m that

$$(3) \bigwedge k \leq_0 m(fk \leq_{\mathbb{R}} c) \rightarrow \Phi_{\max_{\mathbb{R}}} fm \leq_{\mathbb{R}} c.$$

¿From this and the implication

$$(\widehat{fk})(6(l_0+1)) <_{\mathbb{Q}} (\widehat{\Phi_{\max_{\mathbb{R}}} fm})(6(l_0+1)) -_{\mathbb{Q}} \frac{2}{3(l_0+1)} \rightarrow fk <_{\mathbb{R}} \Phi_{\max_{\mathbb{R}}} fm -_{\mathbb{R}} \frac{1}{3(l_0+1)},$$

one concludes

$$(4) \left\{ \begin{array}{l} \bigwedge k \leq_0 m\big((\widehat{fk})(6(l_0+1)) <_{\mathbb{Q}} (\widehat{\Phi_{\max_{\mathbb{R}}} fm})(6(l_0+1)) -_{\mathbb{Q}} \frac{2}{3(l_0+1)}\big) \rightarrow \\ \qquad\qquad \Phi_{\max_{\mathbb{R}}} fm \leq_{\mathbb{R}} \Phi_{\max_{\mathbb{R}}} fm -_{\mathbb{R}} \frac{1}{3(l_0+1)}, \end{array} \right.$$

which is purely universal and hence an axiom of $\mathrm{G}_2\mathrm{A}^\omega$. (1) and (4) imply

$$\Phi_{\max_{\mathbb{R}}} fm \leq_{\mathbb{R}} \Phi_{\max_{\mathbb{R}}} fm - \frac{1}{3(l_0+1)},$$

which is a contradiction.

Remark 3.3.8 *1) The tedious proofs for the two lemmas above have no impact on the extraction of bounds: Lemma 3.3.6 and 3.3.7 1) are purely universal sentences. Since we have proved their truth they are treated as axioms. Lemma 3.3.7 2) (although not being universal) has the logical form* $\bigwedge x \bigvee y \leq sx \bigwedge z A_0$ *of an axiom* $\in \Delta$ *and therefore is treated as an axiom by our monotone (but not by the ususal) functional interpretation. The same is true for the next lemma.*

2) $\Phi_{\min_{\mathbb{R}}} fm$ *can be defined from* $\Phi_{\max_{\mathbb{R}}} fm$ *by* $:= -_{\mathbb{R}} \Phi_{\max_{\mathbb{R}}}\big(\lambda k.(-_{\mathbb{R}} fk), m\big)$.

Using $\Phi_{\max_{\mathbb{R}}}$ we are able to define $\sup\limits_{x \in [0,1]} f(x)$ for $f \in C[0,1]$:

Definition 3.3.9 $\Phi^{1(1)}_{\sup_{[0,1]}} \in G_2R^\omega$ *is defined as follows*

$$\Phi^{1(1)}_{\sup_{[0,1]}} := \lambda f^1, n^0.\Phi_{\max_{\mathbb{R}}}\big(\Psi_1 f, h(\Psi_2 f(3(n+1)))\big)(3(n+1)),$$

where $hn := j(2n,n)$ *and* $\Psi_1, \Psi_2 \in G_2R^\omega$ *are the functionals used in the representation of* $C[0,1]$.

Lemma: 3.3.10

$$G_2A_i^\omega \vdash \bigwedge f \in C[0,1](\bigwedge x \in [0,1]\big(\Phi_{\sup_{[0,1]}} f \geq_{\mathbb{R}} fx\big) \wedge \bigwedge k^0 \bigvee x \in [0,1]\big(\Phi_{\sup_{[0,1]}} f -_{\mathbb{R}} fx \leq \frac{1}{k+1}\big)).$$

Proof: In the following we write Φ_M instead of $\Phi_{\max_{\mathbb{R}}}$.

1) $\Phi f := \lambda k^0.\Phi_M(\Psi_1 f, h(\Psi_2 fk))$ is a Cauchy sequence of real numbers with Cauchy modulus $1/(k+1)$ (This implies that $\Phi^{1(1)}_{\sup_{[0,1]}} f$ represents the limit of this sequence):

Assume $m \geq_0 \tilde{m} \geq_0 k$: The monotonicity of $\Psi_2 f$ (see lemma 3.2.4) implies $\Psi_2 fm \geq_0 \Psi_2 f\tilde{m}$ and therefore (by 3.3.7 and the monotonicity of h) $\Phi fm \geq_{\mathbb{R}} \Phi f\tilde{m}$. By induction on l^0 we show

$$(*)\ \bigwedge l^0 \big(\Phi_M(\Psi_1 f, h(\Psi_2 f\tilde{m}) + l) \leq_{\mathbb{R}} \Phi f\tilde{m} + \frac{1}{k+1}\ :$$

The case $l = 0$ is trivial. $l+1$:

$$\Phi_M(\Psi_1 f, h(\Psi_2 f\tilde{m}) + l + 1) \overset{3.3.6}{=}_{\mathbb{R}} \max{}_{\mathbb{R}}(\underbrace{\Phi_M(\Psi_1 f, h(\Psi_2 f\tilde{m}) + l)}_{\overset{I.V.}{\leq}_{\mathbb{R}} \Phi f\tilde{m} + \frac{1}{k+1}}, \Psi_1 f(h(\Psi_2 f\tilde{m}) + l + 1))).$$

Thus it remains to show that

$$\Psi_1 f(h(\Psi_2 f\tilde{m}) + l + 1) \leq_{\mathbb{R}} \Phi f\tilde{m} + \frac{1}{k+1}.$$

¿From our represention of $[0,1] \cap \mathbb{Q}$ (which used the function q) it follows that there exists an $i \leq_0 h(\Psi_2 f\tilde{m})$ such that

$$|q(i) -_{\mathbb{Q}} q\big(h(\Psi_2 f\tilde{m}) + l + 1\big)| \leq_{\mathbb{Q}} \frac{1}{\Psi_2 f\tilde{m} + 1}$$

and therefore

$$\Psi_1 f(h(\Psi_2 f\tilde{m}) + l + 1) \leq_{\mathbb{R}} \Psi_1 fi + \frac{1}{\tilde{m}+1} \overset{3.3.7}{\leq}_{\mathbb{R}} \Phi f\tilde{m} + \frac{1}{\tilde{m}+1} \leq \Phi f\tilde{m} + \frac{1}{k+1},$$

which completes the proof of $(*)$.

Since $\Psi_2 fm \geq_0 \Psi_2 f\tilde{m}$ implies $h(\Psi_2 fm) \geq_0 h(\Psi_2 f\tilde{m})$, this yields

$$\Phi fm = \Phi_M(\Psi_1 f, h(\Psi_2 fm)) \leq_{\mathbb{R}} \Phi f\tilde{m} + \frac{1}{k+1},$$

which completes the proof of the Cauchy property.

2) $\Phi_{\sup_{[0,1]}} f \geq_{\mathbb{R}} f(x)_{\mathbb{R}}$ for all $x \in [0,1]$. We know that

$$(a)\ |f(x)_{\mathbb{R}} -_{\mathbb{R}} \Psi_1 f(\tilde{x}(\Psi_2 fk))| \leq \frac{1}{k+1},$$

where $\tilde{x}$ is the construction used in the representation of [0,1], and

$$(b)\ \bigvee i \leq_0 h\big(\Psi_2 f(3(k+1))\big) \left(|q(i) -_{\mathbb{Q}} q(\tilde{x}(\Psi_2 fk))| \leq_{\mathbb{Q}} \frac{1}{\Psi_2 f(3(k+1)) + 1}\right)$$

for all $k \in \mathbb{N}$.

$i \leq_0 h\big(\Psi_2 f(3(k+1))\big)$ implies $\Phi_M\big(\Psi_1 f, h(\Psi_2 f(3(k+1)))\big) \geq_{\mathbb{R}} \Psi_1 fi$ (see lemma 3.3.7). It follows that

$$\Phi_{\sup_{[0,1]}} fk \geq_{\mathbb{Q}} \Psi_1 fi - \frac{1}{3(k+1)+1}$$

and therefore

$$\begin{aligned}
&\Phi_{\sup_{[0,1]}} f \geq_{\mathbb{R}} \Psi_1 f i - \big(\tfrac{1}{3(k+1)+1} + \tfrac{1}{k+1}\big) \\
&\overset{(b)}{\geq_{\mathbb{R}}} \Psi_1 f(\tilde{x}(\Psi_2 f k)) - \big(\tfrac{2}{3(k+1)+1} + \tfrac{1}{k+1}\big) \\
&\overset{(a)}{\geq_{\mathbb{R}}} f(x)_{\mathbb{R}} - \big(\tfrac{2}{3(k+1)+1} + \tfrac{2}{k+1}\big)
\end{aligned}$$

for all $k \in \mathbb{N}$. Put together this yields our claim.
3) $\bigwedge n^0 \bigvee x \leq_1 M\big(\Phi_{\sup_{[0,1]}} f -_{\mathbb{R}} f(x)_{\mathbb{R}} \leq \frac{1}{n+1}\big)$, where M is the boundedness function from the representation of [0,1]:

$$\begin{aligned}
&i_0 := \min i \leq_0 h(\Psi_2 f(3(n+1))) \text{ such that} \\
&\bigwedge j \leq_0 h(\Psi_2 f(3(n+1))) \Big(\big(\widehat{\Psi_1 f i}\big)(3(n+1)) \geq_{\mathbb{Q}} \big(\widehat{\Psi_1 f j}\big)(3(n+1)) \Big).
\end{aligned}$$

We show by induction on k that

$$(+)\ \bigwedge k^0 \big(\Phi_M(\Psi_1 f, k) \leq_{\mathbb{R}} \Psi_1 f i_0 + \frac{1}{n+1}\big)$$

and therefore a fortiori

$$(++)\ \bigwedge k^0 \big(\Phi f k \leq_{\mathbb{R}} \Psi_1 f i_0 + \frac{1}{n+1}\big),$$

which implies –by 1)– that

$$(+++)\ \Phi_{\sup_{[0,1]}} f \leq_{\mathbb{R}} \Psi_1 f i_0 + \frac{1}{n+1}:$$

$k = 0$: $\Phi_M(\Psi_1 f, 0) =_{\mathbb{R}} \Psi_1 f 0 \leq_{\mathbb{R}} \Psi_1 f i_0 + \frac{2}{3(n+1)+1}$ by the definition of i_0. $k+1$:

$$\Phi_M(\Psi_1 f, k+1) \overset{3.3.7}{=_{\mathbb{R}}} \max\nolimits_{\mathbb{R}}(\underbrace{\Phi_M(\Psi_1 f, k)}_{\leq_{\mathbb{R}} \Psi_1 f i_0 + \frac{1}{n+1}}, \Psi_1 f(k+1)).$$

To show $\Psi_1 f(k+1) \leq_{\mathbb{R}} \Psi_1 f i_0 + \frac{1}{n+1}$:

$$\bigvee j_k \leq_0 h(\Psi_2 f(3(n+1))) \left(|q(k+1) -_{\mathbb{Q}} q(j_k)| \leq_{\mathbb{Q}} \frac{1}{\Psi_2 f(3(n+1)) + 1} \right).$$

Hence

$$|\Psi_1 f j_k -_{\mathbb{R}} \Psi_1 f(k+1)| \leq_{\mathbb{R}} \frac{1}{3(n+1)+1}.$$

Together with $\Psi_1 f j_k \leq_{\mathbb{R}} \Psi_1 f i_0 + \frac{2}{3(n+1)+1}$ we obtain $\Psi_1 f(k+1) \leq_{\mathbb{R}} \Psi_1 f i_0 + \frac{1}{n+1}$, which completes the proof of $(+)$ and so of $(+++)$.
Since $(+++)$ is purely universal its truth implies its provability in $G_2A_i^\omega$. Our claim follows immediately from $(+++)$ since the rational number $\in [0,1] \cap \mathbb{Q}$ which is coded by $q(i_0)$ has a representative x as a real number and so $x =_{\mathbb{R}} \tilde{x} \leq_1 M$.

In the following chapters we make liberal use of the usual mathematical expressions '$\sup\limits_{x\in[0,1]} fx$' and '$f \in C[0,1]$' and go back to the details of the actual representation of these notions in $\mathrm{G_2A}_i^\omega$ only when this is needed to determine the logical form of a sentence which involves these notions. For a function $f \in C[a,b]$ we can express $\sup\limits_{x\in[a,b]} fx$ as $\sup\limits_{x\in[0,1]} \tilde{f}x$, where $\tilde{f}x := f((1-x)a + xb)$.

For the definition of the sum of a sequence of real numbers of length x we need the following constructions.

Definition 3.3.11 *The functionals $\zeta, \overline{\zeta}, \xi \in G_2R^\omega$ are defined such that*

$$\zeta n^0 =_0 \begin{cases} n, & \text{if } \bigvee m \le n(n = 2m) \\ 0, & \text{otherwise.} \end{cases}$$

$$\overline{\zeta} n^0 =_0 \begin{cases} n+1, & \text{if } \bigvee m \le n(n = 2m+1) \\ 0, & \text{otherwise.} \end{cases}$$

$$\xi n^0 m^0 =_0 \begin{cases} n \dot{-} m, & \text{if } n \ge m \\ m \dot{-} n \dot{-} 1, & \text{otherwise.} \end{cases}$$

Using these functions we are now able to define a variable summation:

Definition 3.3.12 *$\Phi_{\Sigma_{\mathbb{R}}} \in G_2R^\omega$ is defined as*
$\Phi_{\Sigma_{\mathbb{R}}} := \lambda f^{1(0)}, k^0, n^0 . j\big(\xi\big(\sum\limits_{i=0}^{k} \zeta(j_1[(fi)'(\alpha(k,n))]), \sum\limits_{i=0}^{k} \overline{\zeta}(j_1[(fi)'(\alpha(k,n))])\big), 3(\alpha(k,n)+1)\big)$,
where $\alpha(k,n) := 2(k+1)(n+1)$.

Lemma: 3.3.13
$G_2A_i^\omega \vdash \bigwedge f^{1(0)}, k^0 \big(\Phi_{\Sigma_{\mathbb{R}}} f0 =_{\mathbb{R}} f0 \wedge \Phi_{\Sigma_{\mathbb{R}}} f(k+1) =_{\mathbb{R}} \Phi_{\Sigma_{\mathbb{R}}} fk +_{\mathbb{R}} f(k+1)\big)$.

Proof: We do not give a formalized proof in $\mathrm{G_2A}_i^\omega$ by induction on k but show informally that '$\Phi_{\Sigma_{\mathbb{R}}} fk = f0 +_{\mathbb{R}} \ldots +_{\mathbb{R}} fk$' (and hence the assertion of the lemma) is true. Since the lemma is a purely universal statement it therefore is an axiom of $\mathrm{G_2A}_i^\omega$. By the definition of the construction $f \mapsto f'$ and our coding of rational numbers we conclude:

$\frac{(\Sigma :=) \sum_{i=0}^{k} \zeta(j_1[(fi)'(2(k+1)(n+1))])}{3(2(k+1)(n+1)+1)+1} = 2\times$ the rational which is represented by $(fi_1)'(2(k+1)(n+1)) +_{\mathbb{Q}} \ldots +_{\mathbb{Q}} (fi_l)'(2(k+1)(n+1))$, where $\{i_1, \ldots, i_l\}$ ($\subset \{0, \ldots, k\}$) are the indices of the positive fractions.

Analogously, $-\frac{(\overline{\Sigma} :=) \sum_{i=0}^{k} \overline{\zeta}(j_1[(fi)'(2(k+1)(n+1))])}{3(2(k+1)(n+1)+1)+1} = 2\times$ the sum of the negative fractions among $(f0)'(2(k+1)(n+1)), \ldots, (fk)'(2(k+1)(n+1))$.

Case 1) $\Sigma \ge_0 \overline{\Sigma}$: Then $j(\xi(\Sigma, \overline{\Sigma}), 3(2(k+1)(n+1)+1))$ represents the fraction $\frac{\frac{\Sigma - \overline{\Sigma}}{2}}{3(2(k+1)(n+1)+1)+1}$, i.e. the fraction which is represented by
$(f0)'(2(k+1)(n+1)) +_{\mathbb{Q}} \ldots +_{\mathbb{Q}} (fk)'(2(k+1)(n+1))$.

Case 2) $\Sigma <_0 \overline{\Sigma}$: Analogously!
Since $(fi)'(2(k+1)(n+1))$ is a $1/(2(k+1)(n+1))$–approximation of fi, the rational $(f0)'(2(k+1)(n+1))+_{\mathbb{Q}}\ldots+_{\mathbb{Q}}(fk)'(2(k+1)(n+1))$ is a $1/2(n+1)$–approximation of $f0+_{\mathbb{R}}\ldots+_{\mathbb{R}}fk$. Hence $\Phi_{\Sigma_{\mathbb{R}}}fk$ has the Cauchy modulus $1/(n+1)$, i.e. $\widehat{\Phi_{\Sigma_{\mathbb{R}}}fk} =_1 \Phi_{\Sigma_{\mathbb{R}}}fk$, which concludes the proof of the lemma.

Using $\Phi_{\Sigma_{\mathbb{R}}}$ we now define the Riemann integral $\int_0^1 f(x)dx$ for $f \in C[0,1]$:

Let $S_n := \frac{1}{\omega_f(n)+1} \cdot \sum\limits_{i=0}^{\omega_f(n)} f\big(\frac{i}{\omega_f(n)+1}\big)$ denote the n–th Riemann sum (where ω_f is the modulus of uniform continuity from the representation of f). One easily follows from the usual proof of the convergence of the sequence of Riemann sums that $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with Cauchy modulus $2/(n+1)$ (which converges to $\int_0^1 f(x)dx$). Therefore we define:

Definition 3.3.14 *1) $\Phi_S \in G_2R^\omega$ is defined as*
$\Phi_S := \lambda f^1, n^0 . j(2, \Psi_2 fn) \cdot_{\mathbb{R}} \Phi_{\Sigma_{\mathbb{R}}}\big(\lambda i.(\Psi_1 f)(j(2i, \Psi_2 fn)), \Psi_2 fn\big)$.

2) $\Phi_I \in G_2R^\omega$ is defined as
$\Phi_I := \lambda f^1, n^0.[\Phi_S f(2(3(n+1))+1)](3(n+1))$.

Proposition: 3.3.15 *$\Phi_I f^1$ represents the real number $\int_0^1 F(x)dx$, where F is the function $\in C[0,1]$ which is represented by f.*

Proof: Since $j(2i, \Psi_2 fn)$ codes $\frac{i}{\Psi_2 fn+1}$ and Ψ_2 is a modulus of uniform continuity for the function $: [0,1]\cap\mathbb{Q} \to \mathbb{R}$ which is represented by Ψ_1, Φ_S is just the n–th Riemann sum for the function represented by f. As we have mentioned already above, these Riemann sums S_n form a Cauchy sequence with modulus $2/(n+1)$. Hence $(S_{2n+1})_{n\in\mathbb{N}}$ is a Cauchy sequence with modulus $1/(n+1)$. $\Phi_I f$ represents the limit of this sequence.

In the following we use the usual notation $\int_0^1 f(x)dx$ instead of Φ_I.

Proposition: 3.3.16 *The following properties of $\int_0^1$ are provable in $G_2A_i^\omega$ ($f, f_n, g \in C[0,1], \lambda \in \mathbb{R}$):*

1) $\int_0^1 (f+g)(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$.

2) $\int_0^1 (\lambda \cdot f)(x)dx = \lambda \int_0^1 f(x)dx$.

3) $f \le g \to \int_0^1 f(x)dx \le \int_0^1 g(x)dx$.

4) $\left|\int_0^1 f(x)dx\right| \le \int_0^1 |f|(x)dx \le \|f\|_\infty$.

5) $f_n \overset{\|\cdot\|_\infty}{\to} f \Rightarrow \int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$.

Proof: It is clear from the ususal proofs in analysis that 1)–5) are true. Since 1),2) and 4) are purely universal, they are axioms of $\mathrm{G_2A}_i^\omega$. 3) can be transformed into a purely universal sentence

$$3)' \quad \int_0^1 f(x)dx \le \int_0^1 \max(f,g)(x)dx.$$

The proof of the equivalence of 3) and 3)' uses the extensionality of $\int_0^1$, which follows immediately from 4) and thus is also provable in $\mathrm{G}_2\mathrm{A}_i^\omega$. 5) follows from 1),2) and 4).

Our definition of $\int_0^1$ easily generalizes to $\int_a^b F(x)dx$ for $F \in C[a,b]$ $(a < b)$. Let F be given as a pair $(\Psi^{1(1)}, \omega)$, where Ψ represents a function : $[a,b] \to \mathbb{R}$ which has the modulus of uniform continuity ω. Then a representative of $\int_a^b F(x)dx$ can be computed in Ψ, ω, a, b by a functional in $\mathrm{G}_2\mathrm{R}^\omega$. For this one has to replace the partition

$$\frac{0}{\omega(n)+1}, \ldots, \frac{\omega(n)+1}{\omega(n)+1}$$

of [0,1] by the partition

$$a_0, \ldots, a_{k(\omega(n)+1)}, \text{ where } a_i := a +_{\mathbb{R}} i(b-a) \cdot_{\mathbb{R}} \frac{1}{k(\omega(n)+1)} \text{ and } \mathbb{N} \ni k \geq b-a,$$

of $[a,b]$ which also has mesh $\leq 1/(\omega(n)+1)$.

We can define also a functional $\Phi_{I_a^x} \in \mathrm{G}_2\mathrm{R}^\omega$ such that $\Phi_{I_a^x}(x^1, a^1, \Psi^{1(1)}, \omega^1)$ represents the integral $\int_a^x \Psi x dx$ if Ψ represents a function $[a,b] \to \mathbb{R}$ $(a < b)$, which is uniformly continuous with modulus ω, and $x \in [a,b]$:

$$\Phi_{I_a^x}(x^1, a^1, \Psi^{1(1)}, \omega^1) := \lim_{n\to\infty} S_n(x, a, \Psi, \omega),$$

where

$$S_n (= S_n(x,a,\Psi,\omega)) := \frac{x -_{\mathbb{R}} a}{n+1} \cdot_{\mathbb{R}} \Phi_\Sigma(\lambda i.\Psi(a +_{\mathbb{R}} i(x -_{\mathbb{R}} a) \cdot_{\mathbb{R}} \frac{1}{n+1}), n+1).$$

¿From our reasoning above it is clear that (S_n) is a Cauchy sequence which converges to $\int_a^x \Psi x dx$. In order to be able to define $\lim_{n\to\infty} S_n$ in $\mathrm{G}_2\mathrm{R}^\omega$ we have to construct a Cauchy modulus for this sequence in $\mathrm{G}_2\mathrm{R}^\omega$. This however is possible since

$$|S_{k(\omega(n)+1)} - \int_a^x \Psi x dx| \leq \frac{k}{n+1},$$

where $k \in \mathbb{N}$ such that $k \geq x - a$.

The formula

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx \text{ for } a < c < b$$

is purely universal and hence an axiom of $\mathrm{G}_2\mathrm{A}_i^\omega$.

Summary of the main features of our representation of basic analytical notions

1) **Rational numbers** are coded by natural numbers with corresponding relations $=_{\mathbb{Q}}, \leq_{\mathbb{Q}}, <_{\mathbb{Q}}$ and operations $|\cdot|_{\mathbb{Q}}, +_{\mathbb{Q}}, -_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ on the codes.

2) **Sequences of rational numbers** are represented by number–theoretic functions.

3) **Real numbers** are given as Cauchy sequences of rational numbers with **fixed Cauchy modulus** $\frac{1}{k+1}$ and are therefore represented by functions f^1 with a corresponding equality relation $=_{\mathbb{R}}$.
Using the construction $f^1 \mapsto \widehat{f}^1$ **every** function can be conceived as a representative of a real number, namely the real number which is represented by $\widehat{f}$.
Using this construction we have relations $=_{\mathbb{R}}, \leq_{\mathbb{R}} \in \Pi^0_1, <_{\mathbb{R}} \in \Sigma^0_1$ and operations $+_{\mathbb{R}}, -_{\mathbb{R}}, \ldots \in \mathrm{G}_2\mathrm{R}^\omega$ on all functions f^1 which correspond to the usual relations and operations on $\mathbb{R}$.
Quantification over reals so reduces to $\bigwedge f^1 A(\widehat{f}), \bigvee f^1 A(\widehat{f})$ for $=_{\mathbb{R}}$–extensional properties A.

4) **Elements of** $\mathbb{R}^d$ are represented by functions f^1: f represents the d–tuple of real numbers which is represented by $(\widehat{\nu^d_1 f}, \ldots, \widehat{\nu^d_d f})$.

5) **The closed unit interval** $[0,1]$ is represented by $\{f^1 : f \leq_1 M\}$ (for a suitable $M \in \mathrm{G}_2\mathrm{R}^\omega$) using a construction $f \mapsto \tilde{f}$ such that $0 \leq_{\mathbb{R}} \tilde{f} \leq_{\mathbb{R}} 1$ and $0 \leq_{\mathbb{R}} f \leq_{\mathbb{R}} 1 \to f =_{\mathbb{R}} \tilde{f}$. Hence **quantification over** $[0,1]$ reduces to $\bigwedge f \leq_1 M\, A(\tilde{f}), \bigvee f \leq_1 M\, A(\tilde{f})$ for properties A which are $=_{\mathbb{R}}$–extensional. Similar for $[0,1]^d$.
Quantification over $[a,b]$ $([a_1,b_1] \times \ldots \times [a_d,b_d])$ is reduced to quantification over $[0,1]$ $([0,1]^d)$ by a convex transformation.

6) **Functions** $f : \mathbb{R} \to \mathbb{R}$ **(**$f : [a,b] \to \mathbb{R}$**)** are given by functionals $\Phi^{1(1)}$ which are $=_{\mathbb{R}}$–extensional.
Continuous functions $f : [0,1] \to \mathbb{R}$ **endowed with a modulus** ω^1 **of uniform continuity** can be represented as pairs of type–1–objects $(f_r^{1(0)}, \omega^1)$, where f_r represents the restriction of f on $[0,1] \cap \mathbb{Q}$. Using the functionals $\Psi_1, \Psi_2 \in \mathrm{G}_2\mathrm{R}^\omega$ **every** function f^1 represents such a pair $(\Psi_1 f, \Psi_2 f)$ and hence using the application $(\cdot)_{\mathbb{R}}$ a uniformly continuous function : $[0,1] \to \mathbb{R}$. Thus **quantification over** $C[0,1]$ reduces to $\bigwedge f^1 A(\lambda x^1.f(x)_{\mathbb{R}}, \Psi_2 f)$ for $=_{C[0,1]}$–extensional properties A. This generalizes to $C([0,1]^d)$.
Quantification over $C[a,b]$ $(C([a_1,b_1] \times \ldots \times [a_d,b_d]))$ is reduced to quantification over $C[0,1]$ $(C([0,1]^d))$.

7) **Maximum and sum for sequences of real numbers of variable length** are given by functionals $\Phi^{1(0)(1(0))}_{\max}, \Phi^{1(0)(1(0))}_{\Sigma} \in \mathrm{G}_2\mathrm{R}^\omega$.

8) $\sup\limits_{x \in [0,1]} fx, \int\limits_0^1 fxdx$ for $f \in C[0,1]$ are given by functionals $\Phi_{\sup}, \Phi_I \in \mathrm{G}_2\mathrm{R}^\omega$ in the representatives of f.
The definition of $\sup\limits_{x \in [a,b]} fx$ for $f \in C[a,b]$ reduces to $\sup\limits_{x \in [0,1]} \tilde{f}x$ for suitable $\tilde{f} \in C[0,1]$.
$\int\limits_a^x fxdx$ for $f \in C[a,b], x \in [a,b]$ is given by a functional $\Phi_{I^x_a} \in \mathrm{G}_2\mathrm{R}^\omega$ in $x^1, a^1, b^1, \Phi^{1(1)}_f, \omega^1$, where $\Phi^{1(1)}_f$ represents f and ω is a modulus of uniform continuity for this function.

The representation of all these notions can be carried out in $\mathrm{G}_n\mathrm{A}^\omega$ for $n \geq 2$. The basic properties of $\int_b^a fxdx$ and $\sup_{x\in[a,b]} fx$ (for $f \in C[a,b]$) and the variable maximum and sum for sequences of real numbers are expressible as purely universal sentences (or follow relatively to $\mathrm{G}_2\mathrm{A}^\omega$+$\mathrm{AC}^{0,0}$–qf easily from such sentences) and therefore contribute to the growth of bounds extractable from proofs which use these notions and their properties only by majorants $\in \mathrm{G}_2\mathrm{R}^\omega$ for the terms used in our representation. More general this holds for sentences having the form

$$(*) \bigwedge f \in C([0,1]^d), x \in [0,1], y \in \mathbb{R}, k \in \mathbb{N}\big(\varphi_1 fxyk \underset{<_{\mathbb{R}}}{\overset{\neq_{\mathbb{R}}}{}} \varphi_2 fxyk \to \varphi_3 fxyk \underset{\leq_{\mathbb{R}}}{\overset{=_{\mathbb{R}}}{}} \varphi_4 fxyk\big),$$

where the $\varphi_i \in G_2\mathrm{R}^\omega$ represent functionals $C([0,1]^d) \times [0,1] \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, since (modulo our representation) sentences $(*)$ are equivalent to purely universal sentences.
In particular, from a $\mathrm{G}_n\mathrm{A}^\omega$+AC-qf–proof of a sentence
$(+)\bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\rho t\underline{u}\,\underline{k} \bigvee w^0 A_0$ relatively to sentences $(*)$ which may be used as lemmas one can extract (using cor.2.2.3) a uniform bound $\bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\rho t\underline{u}\,\underline{k} \bigvee w \leq_0 \chi\underline{u}\,\underline{k}\, A_0$ such that

(i) χ is a polynomial in $\underline{u}^M, \underline{k}$ (where $u_i^M := \lambda x^0. \max_0(u0,\ldots,ux))$ for which prop. 1.2.30 applies, if $n=2$,

(ii) χ is elementary recursive in $\underline{u}^M, \underline{k}$, if $n=3$.

Using our representation many sentences in analysis have the form (+), in particular sentences of the form

$$(++) \bigwedge f \in C([0,1]^d), x \in \mathbb{R}, y \in [0,1]^m \bigvee k \in \mathbb{N}\, A(f,x,y,k),$$

where $A \in \Sigma_1^0$ and the bound χ only depends on (representatives of) f, x but not on y. In [37],[38] and [39] we study interesting examples of sentences (++).

4 Sequences and series in $G_2A_i^\omega$: Convergence with moduli involved

By our representation of real numbers by functions f^1 (see chapter 3), sequences of real numbers are given as functions $f^{1(0)}$ in $G_2A_i^\omega$. We will use the usual notation (a_n) instead of f. In this chapter we are concerned with the following properties of sequences of real numbers:

1) (a_n) is a Cauchy sequence, i.e.

$$\bigwedge k^0 \bigvee n^0 \bigwedge m, \tilde{m} \geq_0 n \big(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1} \big).$$

2) (a_n) is convergent, i.e.

$$\bigvee a^1 \bigwedge k^0 \bigvee n^0 \bigwedge m \geq_0 n \big(|a_m -_{\mathbb{R}} a| \leq_{\mathbb{R}} \frac{1}{k+1} \big).$$

3) (a_n) is convergent with a modulus of convergence, i.e.

$$\bigvee a^1, h^1 \bigwedge k^0 \bigwedge m \geq_0 hk (|a_m -_{\mathbb{R}} a| \leq_{\mathbb{R}} \frac{1}{k+1}).$$

4) (a_n) is a Cauchy sequence with a Cauchy modulus, i.e.

$$\bigvee h^1 \bigwedge k^0 \bigwedge m, \tilde{m} \geq_0 hk (|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1}).$$

One easily shows within $G_2A_i^\omega$ that

4) $\leftrightarrow$ 3) $\to$ 2) $\to$ 1).

Using

$$\mathrm{AC}^{0,0}\text{–}\bigwedge^0 \; : \; \bigwedge x^0 \bigvee y^0 \bigwedge z^0 A_0(x,y,z) \to \bigvee f^1 \bigwedge x^0, z^0 A_0(x, fx, z)$$

one can prove that 1) $\to$ 4) (and therefore 1) $\leftrightarrow$ 2) $\leftrightarrow$ 3) $\leftrightarrow$ 4)).

However, as we already have discussed in chapter 3, the addition of $\mathrm{AC}^{0,0}\text{–}\bigwedge^0$ to $G_2A_i^\omega$ would make all $\alpha(<\varepsilon_0)$–recursive functions provably recursive.
Thus since we are working in G_2A^ω we have to distinguish carefully between e.g. 1) and 4). In chapters 9–11 we will study the relationship between 1) and 4) in detail and show in particular that the use of sequences of single instances of 4) in proofs of $\bigwedge u^1 \bigwedge v \leq_\rho tu \bigvee w^2 A_0$–sentences relatively to e.g. $G_2A^\omega + \Delta + \mathrm{AC}$–qf (where Δ is defined as in thm.2.2.2) can be reduced the use of the same instances of 1).

For **monotone** sequences (a_n) the equivalence of 2) and 3) (and hence that of 2) and 4)) is already provable using only the **quantifier–free** choice $\mathrm{AC}^{0,0}$–qf:
Let (a_n) be say increasing, i.e.

(*i*) $\bigwedge n^0 (a_n \leq_{\mathbb{R}} a_{n+1})$, and a^1 be such that

$$(ii)\ \bigwedge k^0 \bigvee n^0 \bigwedge m \geq_0 n\big(|a_m - a| \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

$\mathrm{AC}^{0,0}$–qf applied to

$$\bigwedge k^0 \bigvee n^0 \big(\underbrace{|a_n - a| <_{\mathbb{R}} \frac{1}{k+1}}_{\in \Sigma^0_1} \big)$$

yields

$$\bigvee h^1 \bigwedge k^0 \big(|a_{hk} - a| <_{\mathbb{R}} \frac{1}{k+1}\big),$$

which gives

$$\bigvee h^1 \bigwedge k^0 \bigwedge m \geq_0 hk\big(|a_m - a| <_{\mathbb{R}} \frac{1}{k+1}\big),$$

since –by (i),(ii)– $a_{hk} \leq a_m \leq a$ for all $m \geq_0 hk$. (Here we use the fact that $\bigwedge n(a_n \leq_{\mathbb{R}} a_{n+1}) \rightarrow \bigwedge m, \tilde{m}(m \geq \tilde{m} \rightarrow a_{\tilde{m}} \leq_{\mathbb{R}} a_m)$. This follows in $\mathrm{G}_2\mathrm{A}^\omega$ from the universal sentence
(+) $\bigwedge a^{1(0)}_{(\cdot)}, n, l\big(\bigwedge k < n\big(\widehat{a}_k(l) \leq_{\mathbb{Q}} \widehat{a}_{k+1}(l) + \frac{3}{l+1}\big) \rightarrow \bigwedge m, \tilde{m} \leq n\big(m \geq \tilde{m} \rightarrow a_{\tilde{m}} \leq_{\mathbb{R}} a_m + \frac{5n}{l+1}\big)\big)$. (+) is true (and hence an axiom of $\mathrm{G}_2\mathrm{A}^\omega$) since $\widehat{a}_k(l) \leq_{\mathbb{Q}} \widehat{a}_{k+1}(l) + \frac{3}{l+1} \rightarrow a_k \leq_{\mathbb{R}} a_{k+1} + \frac{5}{l+1}$.)

If one of the properties 1), …,4) –say $i \in \{1, \ldots, 4\}$– is fulfilled for two sequences $(a_n), (b_n)$, then i) is also fulfilled (provably in $\mathrm{G}_2\mathrm{A}^\omega_i$) for $(a_n +_{\mathbb{R}} b_n), (a_n -_{\mathbb{R}} b_n), (a_n \cdot_{\mathbb{R}} b_n)$ and (if $b_n \neq 0$ and $b_n \rightarrow b \neq 0$) for $\big(\frac{a_n}{b_n}\big)$, where in the later case the modulus in 3),4) depends on an estimate $l \in \mathbb{N}$ such that $|b| \geq \frac{1}{l+1}$ (The construction of the moduli for $(a_n +_{\mathbb{R}} b_n), (a_n -_{\mathbb{R}} b_n), (a_n \cdot_{\mathbb{R}} b_n), \big(\frac{a_n}{b_n}\big)$ from the moduli for $(a_n), (b_n)$ (for i=3,4) is similar to our definition of $+_{\mathbb{R}}, -_{\mathbb{R}}, \cdot_{\mathbb{R}}, (\cdot)^{-1}$ given in chapter 3.
The most important property of bounded monotone sequences (a_n) of real numbers is their convergence. We call this fact 'principle of convergence for monotone sequences' (PCM). Because of the difference between 1) and 4) above we have in fact to consider two versions of this principle:

$$(\mathrm{PCM1}) : \begin{cases} \bigwedge a^{1(0)}_{(\cdot)}, c^1\big(\bigwedge n^0(c \leq_{\mathbb{R}} a_{n+1} \leq_{\mathbb{R}} a_n) \\ \qquad \rightarrow \bigwedge k^0 \bigvee n^0 \bigwedge m, \tilde{m} \geq_0 n(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1})\big), \end{cases}$$

$$(\mathrm{PCM2}) : \begin{cases} \bigwedge a^{1(0)}_{(\cdot)}, c^1\big(\bigwedge n^0(c \leq_{\mathbb{R}} a_{n+1} \leq_{\mathbb{R}} a_n) \\ \qquad \rightarrow \bigvee h^1 \bigwedge k^0 \bigwedge m, \tilde{m} \geq_0 hk(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1})\big), \end{cases}$$

Both principles cannot be derived in any of the theories $\mathrm{G}_n\mathrm{A}^\omega + \Delta$+AC–qf. They will be examined thoroughly in the chapters 9 and 11 below where the exact rate of growth of provable functionals is determined which may result from the use of PCM1 and PCM2 in proofs. In chapter 11 we will also study the rate of growth which is caused (potentionally) by the Bolzano–Weierstraß principle.

By lemma 3.3.13 there is a functional $\Phi^{1(0)(1(0))}_{\Sigma_{\mathbb{R}}} \in \mathrm{G}_2\mathrm{R}^\omega$ such that $\Phi_{\Sigma_{\mathbb{R}}}\big(a_{(\cdot)}\big)n$ is the partial sum

$\sum_{k=0}^{n} a_k$. Thus within $\mathrm{G_2R}^\omega$ we can form the sequence $s_n = \sum_{k=0}^{n} a_k$ of partial sums for the sequence (a_n).

Criteria for convergence of series in $\mathbf{G_2A}_i^\omega$

As far as the **Cauchy criterion** is concerned our remarks above on the relationship of 1) and 2) apply.
The **Leibniz criterion** is provable in $\mathrm{G_2A}_i^\omega$+$\mathrm{AC}^{0,0}$–qf in its strongest quantitative version:

$$(L) \begin{cases} \bigwedge(a_n) \subset \mathbb{R}\big(\bigwedge n^0(0 \le a_{n+1} \le a_n) \wedge \bigwedge k^0 \bigvee n^0 \bigwedge m \ge_0 n(a_m \le \frac{1}{k+1}) \\ \qquad \to \bigvee h^1 \bigwedge k^0 \bigwedge m, \tilde{m} \ge_0 hk(|\sum\limits_{i=0}^{m}(-1)^i a_i - \sum\limits_{i=0}^{\tilde{m}}(-1)^i a_i| \le \frac{1}{k+1})\big). \end{cases}$$

(L) follows from the simple observation that
$|\sum\limits_{i=0}^{n+k}(-1)^i a_i - \sum\limits_{i=0}^{n}(-1)^i a_i| = |a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \ldots| \le a_{n+1}$ and the above proof for the existence of a modulus of convergence for a convergent monotone sequence by AC–qf.

Remark 4.1 *1) $AC^{0,0}$–qf is needed only to prove the existence of a modulus h^1 such that $\bigwedge k^0 \bigwedge m \ge_0 hk(a_{hm} \le \frac{1}{k+1})$ (which can be done since (a_n) is decreasing to 0). If (a_n) is already given with such a modulus, then the proof of (L) needs no $AC^{0,0}$–qf.*

2) In various calculus textbooks the Leibniz criterion is proved as a consequence of PCM. However this proof (as it stands) does not provide any information on the rate of convergence of $\sum(-1)^i a_i$ relatively to the rate of the convergence $a_n \to 0$, since PCM is non–constructive.

The **comparison test** for series is also provable in a quantitative form within $\mathrm{G_2A}_i^\omega$+$\mathrm{AC}^{0,0}$–qf:
Let $(a_n), (c_n) \subset \mathbb{R}$ be such that $\bigwedge n^0(|a_n| \le c_n)$. If $\sum\limits_{i=0}^{\infty} c_i$ converges in the sense of 2) or 3) or 4), then $\sum\limits_{i=0}^{\infty} |a_n|$ (and a fortiori $\sum\limits_{i=0}^{\infty} a_n$) converges with a modulus of convergence, i.e. it converges in the sense of 3),4) and so a fortiori in the sense of 2). If $\sum\limits_{i=0}^{\infty} c_i$ converges in the sense of 1) (i.e. the sequence of its partial sums is a Cauchy sequence), then the same holds for $\sum\limits_{i=0}^{\infty} a_i$. All this follows immediately from the ususal proof of the comparison test and the fact that by $\mathrm{AC}^{0,0}$–qf one obtains a modulus of convergence for the monotone sequence of the partial sums of $\sum\limits_{i=0}^{\infty} c_i$ if this series fulfils 2). If $\sum\limits_{i=0}^{\infty} c_i$ satisfies 3) or 4) we do not need AC–qf.

In order to treat the **quotient criterion** we have to introduce the **geometric series** in $\mathrm{G_2A}_i^\omega$:

For this purpose we introduce (according to theorem 2.2.8) a new constant $P^{1(0)(0)}$ to $\mathrm{G_2A}_i^\omega$ (which is majorized by a suitable term $\in\mathrm{G_2A}^\omega$) whose intended meaning is that Px^0n^0 represents q^n (as a real number) for the rational number q which is coded by x, if $|q| \le 1$. The following purely universal sentences are true assertions about P (under this interpreation) and are therefore taken as axioms in $\mathrm{G_2A}_i^\omega \cup \{P\}$ (which we denote also by $\mathrm{G_2A}_i^\omega$ in the following):

1) $\bigwedge x^0, n^0(|x| \leq_{\mathbb{Q}} 1 \to |Pxn| \leq_{\mathbb{R}} 1)$.

2) $\bigwedge x^0, y^0, n^0(|x|, |y| \leq_{\mathbb{Q}} 1 \to |Pxn -_{\mathbb{R}} Pyn| \leq_{\mathbb{R}} n \cdot |x-y|)$[33].
   This is true since the absolute value of the derivative $n \cdot x^{n-1}$ of x^n is bounded by n on $[-1,1]$.

3) $\bigwedge x^0\big(Px0 =_{\mathbb{R}} 1 \wedge (|x| \leq_{\mathbb{Q}} 1 \to Px1 =_{\mathbb{R}} x)\big)$.

4) $\bigwedge x^0, n^0, m^0(|x| \leq_{\mathbb{Q}} 1 \to Px(n +_0 m) =_{\mathbb{R}} Pxn \cdot_{\mathbb{R}} Pxm)$.

By 2), using the application $(\cdot)_{\mathbb{R}}$, P extends to a continuous function $[-1,1] \times \mathbb{N} \to \mathbb{R}$, represented by a functional of type $1(0)(1)$, which we denote also by P. The axioms 3),4) imply

$$\bigwedge x^1\big(Px1 =_{\mathbb{R}} 1 \wedge (|x| \leq_{\mathbb{R}} 1 \to Px1 =_{\mathbb{R}} x)\big),$$

$$\bigwedge x^1, n^0, m^0\big(|x| \leq_{\mathbb{R}} 1 \to Px(n+m) =_{\mathbb{R}} Pxn \cdot_{\mathbb{R}} Pxm\big).$$

In contrast to 3),4) (or the case $|x| <_{\mathbb{R}} 1$) these propositions are not $\in \Pi^0_1$ but $\in \Pi^0_2$ and therefore cannot be treated directly as axioms.
Since we use Px^0n^0 only for $|x| \leq_{\mathbb{Q}} 1$, we are free to extend this function on $\mathbb{Q}$ by stipulating
5) $\bigwedge x^0(|x| >_{\mathbb{Q}} 1 \to Pxn =_{\mathbb{R}} 1)$.
Similar to our representation of $[0,1]$ where we used the construction $\tilde{f}$ such that $\tilde{f} =_{\mathbb{R}} f$ and $\tilde{f} \leq_1 M$, we can represent $[-1,1]$ with a corresponding construction $\tilde{f}$ and a function $M \in \mathrm{G}_2\mathrm{R}^\omega$. Hence we may assume that
6) $P \leq_{1(0)(0)} \lambda x^0, n^0.M$.
Because of 6) P can be majorized by a term $\in \mathrm{G}_2\mathrm{R}^\omega_{\preceq}$ (namely by any majorant $M^* \in \mathrm{G}_2\mathrm{R}^\omega_{\preceq}$ for M) so that theorem 2.2.8 applies.

Remark 4.2 *1) Within $G_2A^\omega + \Delta + AC$–qf one can not prove that*

$$(*)\ \bigwedge x^1, n^0(0 <_{\mathbb{R}} x \leq_{\mathbb{R}} 1 \to Pxn >_{\mathbb{R}} 0),$$

since this would yield e.g. for $x = \frac{1}{2}$ the exponential growth $\frac{1}{Pxn} \geq 2^n$ (hence contradicting cor.2.2.3). One easily verifies that $G_3A^\omega_i \vdash (*)$.

2) Within $G_3A^\omega_i$ one can define the x^n as a function in $x \in \mathbb{R}$ and $n \in \mathbb{N}$ on whole $\mathbb{R}$.

Using P we are now able to define the geometric series via the sum formula:

$$(1)\ \bigwedge x^1, n^0(|x| <_{\mathbb{R}} 1 \to \sum_{k=0}^{n} Pxk =_{\mathbb{R}} \frac{1 -_{\mathbb{R}} Px(n+1)}{1 -_{\mathbb{R}} x}).$$

Note that (1) can be transformed (by intuitionistic logic) into a purely universal sentence, i.e. an axiom of $\mathrm{G}_2\mathrm{A}^\omega_i$.
In order to obtain the convergence of $\sum\limits_{k=0}^{\infty} Pxk$ to $\frac{1}{1-x}$ we need the convergence $Pxn \overset{n\to\infty}{\to} 0$ which can be expressed in a quantitative form as a purely universal sentence:[34]

$$(2)\ \bigwedge x^1, n^0, k^0\big(|x| <_{\mathbb{R}} 1 \wedge n >_{\mathbb{R}} \frac{k+1}{1-x} \to |Pxn| \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

[33] Here and below we write simply n for the code $j(2n,0)$ of n as a rational number.
[34] See also [42] for a derivation of this modulus by a (variable) Herbrand disjunction for (PCM1).

Together with (1) this yields

$$(3)\ \bigwedge x^1, n^0, k^0 \big(|x| <_{\mathbb{R}} 1 \wedge n >_{\mathbb{R}} \frac{k+1}{1 -_{\mathbb{R}} x} \cdot \left\lceil \frac{1}{1-x} \right\rceil \to |\sum_{k=0}^{n} Pxk - \frac{1}{1-x}| \le_{\mathbb{R}} \frac{1}{k+1} \big).$$

The quotient criterion now follows as usual together with an explicit rate of convergence in $\theta \in (0,1)$ such that $|a_{n+1}/a_n| \le \theta$ (for all $n \in \mathbb{N}$).

$G_2A^\omega_i$ proves the divergence of the **harmonic series** $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ in the sense that the sequence of its partial sums $s_k := \sum\limits_{n=1}^{k} \frac{1}{n}$ is not a Cauchy sequence: This follows immediately from the universal axiom $|s_{2k} - s_k| \ge k \cdot \frac{1}{2k} = \frac{1}{2}$ (G_2A^ω(+Δ+AC–qf) does not prove that the harmonic series diverges to infinity: see chapter 9!).

In chapter 9 below we need the convergence of $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ in $G_2A^\omega_i$. This follows from the universal axiom $\sum\limits_{n=1}^{k} \frac{1}{n(n+1)} =_{\mathbb{R}} \frac{k}{k+1}$, which implies $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} =_{\mathbb{R}} 1$ with $hk := k$ as a modulus of convergence.

We have seen in this chapter that within G_nA^ω+AC–qf (for $n \ge 2$) one can treat infinite sequences and series of real numbers and establish the comparison test, the Leibniz criterion and the quotient criterion. The last two criteria can be proved even in a quantitative version, i.e. together with a modulus of convergence. This also holds for the comparison test, if the series of the majorizing sequence is given together with its limit. Thus the results on the growth of bounds extracted from proofs stated at the end of chapter 3 extend to proofs which use these principles for series.

Furthermore the function x^n in $x \in \mathbb{R}$ **and** $n \in \mathbb{N}$ can be introduced for $x \in [0,1]$ in G_2A^ω and for unrestricted x in G_3A^ω.
If a sequence (x_n) is definable in G_nA^ω together with a modulus of convergence for the sequence of its partial sums, then $\sum_{i=0}^{\infty} x_n$ is definable in G_nA^ω.
The principle of convergence for bounded monotone sequences of real numbers is not provable in G_nA^ω+AC–qf, not even in its weak form PCM1 which asserts the Cauchy property for such sequences. We will discuss PCM1 in chapter 9 where we determine its impact on the growth of bounds.
In chapter 11 we investigate the full principle of convergence of bounded monotone sequences PCM2 which asserts the existence of a limit (together with a modulus of convergence) and show that single arithmetical families of instances of PCM2 can be reduced to instances of PCM1 (this requires quite complicated proof–theoretic methods which are developed in chapter 10).

5 Trigonometric functions in $\mathrm{G}_2\mathrm{A}^\omega_i$: Moduli and universal properties

5.1 The functions sin, cos and tan in $\mathrm{G}_2\mathrm{A}^\omega_i$

In the following we introduce the functions **sin , cos** axiomatically by adding to $\mathrm{G}_2\mathrm{A}^\omega_i$ new function constants $\Phi_{\sin}, \Phi_{\cos}$ of type 1(0) which represent the restriction of sin and cos to $\mathbb{Q}$. Then the Lipschitz continuity of $\sin, \cos$ is used to continue these functions to $\mathbb{R}$ (If we would introduce $\sin, \cos$ directly as functions on $\mathbb{R}$, this would require new constants for **functionals** of type 1(1). In order to express their extensionality by universal axioms we also would have to make use of the Lipschitz continuity, since uniform continuity is just a uniform quantitative version of extensionality).

The following purely universal assertions on the function constants $\Phi_{\sin}, \Phi_{\cos}$ express true propositions on $\sin, \cos$ and are therefore taken as axioms in $\mathrm{G}_2\mathrm{A}^\omega_i \cup \{\Phi_{\sin}, \Phi_{\cos}\}$ (which we also denote by $\mathrm{G}_2\mathrm{A}^\omega_i$):

1) $\bigwedge x^0((\widehat{\Phi_{\sin}x}) =_1 \Phi_{\sin}x \leq_1 M \wedge (\widehat{\Phi_{\cos}x}) =_1 \Phi_{\cos}x \leq_1 M \wedge -1 \leq_{\mathbb{R}} \Phi_{\sin}x, \Phi_{\cos}x \leq_{\mathbb{R}} 1)$, where $M^1 \in \mathrm{G}_2\mathrm{R}^\omega$ is the boundedness function from the representation of $[-1,1]$ (one may take $M := \lambda n^0.j(6(n+1), 3(n+1)-1)$ see $[0,1]$).

2) $\bigwedge x^0, y^0, q^0\big(|x -_{\mathbb{Q}} y| \leq_{\mathbb{Q}} q \rightarrow |\Phi_{\sin}x -_{\mathbb{R}} \Phi_{\sin}y| \leq_{\mathbb{R}} q \wedge |\Phi_{\cos}x -_{\mathbb{R}} \Phi_{\cos}y| \leq_{\mathbb{R}} q\big)$.
(2) (together with 1)) asserts that $\Phi_{\sin}$ and $\Phi_{\cos}$ represent functions : $\mathbb{Q} \rightarrow [-1,1]$ which are Lipschitz continuous on $\mathbb{Q}$ with Lipschitz constant $\lambda = 1$).

3) $\bigwedge x^0\big(\Phi_{\sin}(-_{\mathbb{Q}}x) =_{\mathbb{R}} -_{\mathbb{R}}\Phi_{\sin}x \wedge \Phi_{\cos}(-_{\mathbb{Q}}x) =_{\mathbb{R}} \Phi_{\cos}x\big)$, $\Phi_{\cos}0 =_{\mathbb{R}} 1$.

4) $\bigwedge x^0, y^0\big(\Phi_{\sin}(x +_{\mathbb{Q}} y) =_{\mathbb{R}} (\Phi_{\sin}x) \cdot_{\mathbb{R}} (\Phi_{\cos}y) +_{\mathbb{R}} (\Phi_{\cos}x) \cdot_{\mathbb{R}} (\Phi_{\sin}y) \wedge$
$\Phi_{\cos}(x +_{\mathbb{Q}} y) =_{\mathbb{R}} (\Phi_{\cos}x) \cdot_{\mathbb{R}} (\Phi_{\cos}y) -_{\mathbb{R}} (\Phi_{\sin}x) \cdot_{\mathbb{R}} (\Phi_{\sin}y)\big)$.
$\bigwedge x^0, y^0\big(\Phi_{\sin}x -_{\mathbb{R}} \Phi_{\sin}y = 2 \cdot \Phi_{\cos}(\frac{x+_{\mathbb{Q}}y}{2}) \cdot_{\mathbb{R}} \Phi_{\sin}(\frac{x-_{\mathbb{Q}}y}{2}) \wedge$
$\Phi_{\cos}x -_{\mathbb{R}} \Phi_{\cos}y = -2 \cdot \Phi_{\sin}(\frac{x+_{\mathbb{Q}}y}{2}) \cdot_{\mathbb{R}} \Phi_{\sin}(\frac{x-_{\mathbb{Q}}y}{2})\big)$.

5) $\bigwedge x^0\big(0 <_{\mathbb{Q}} |x| \rightarrow \big|\frac{\Phi_{\sin}x}{x} -_{\mathbb{R}} 1\big| \leq_{\mathbb{R}} \frac{|x|^2}{6}\big)$.
This proposition on sin (which is proved e.g. in [15]) provides a quantitative version of the proposition $\frac{\sin x}{x} \overset{x\to 0}{\rightarrow} 1$. Only by this quantitative strengthening the proposition becomes purely universal (and therefore an axiom of $\mathrm{G}_2\mathrm{A}^\omega_i$).

Because of axiom 2) there are unique continuous extensions of the functions : $\mathbb{Q} \rightarrow \mathbb{R}$, which are represented by $\Phi_{\sin}, \Phi_{\cos}$, to the whole space $\mathbb{R}$. These extensions are represented by

$$\tilde{\Phi}^{1(1)}_{\sin}x^1 := \lambda k^0.\Phi_{\sin}(\widehat{x}(3(k+1)))(3(k+1)),$$

$$\tilde{\Phi}^{1(1)}_{\cos}x^1 := \lambda k^0.\Phi_{\cos}(\widehat{x}(3(k+1)))(3(k+1)).$$

Remark 5.1.1 *1) It is well–known that 2)–5) already characterize* $\sin, \cos$ *(see e.g. [24]).*

2) By the axioms 1) $\Phi_{\sin}$ *and* $\Phi_{\cos}$ *are majorizable by* $\lambda x^0, n^0.j(6(n+1), 3(n+1)-1) \in G_2R^\omega_{-}$. *Hence thm.2.2.8 applies.*

3) *In G_3A^ω we can* **define** *constants $\Phi'_{\sin}, \Phi'_{\cos}$ which satisfy (provable in $G_3A_i^\omega$) $-1 \le \Phi'_{\sin}x, \Phi'_{\cos}x \le 1$ and 2)–5) above using the usual definition via the Taylor expansion of* sin *and* cos. *If we now define $\Phi_{\sin}x := (\widetilde{\Phi'_{\sin}x})$ and $\Phi_{\cos}x := (\widetilde{\Phi'_{\cos}x})$ (where $\lambda y^1.\tilde{y} \in G_2R^\omega$ is the construction corresponding to our representation of $[-1,1]$ such that $\tilde{y} \le_1 M$, $y =_{\mathbb{R}} \tilde{y}$ if $-1 \le_{\mathbb{R}} y \le_{\mathbb{R}} 1$, and $-1 \le_{\mathbb{R}} \tilde{y} \le_{\mathbb{R}} 1$ for all y^1), then these functionals satisfy 1)–5).*

In the following we will write $\Phi_{\sin}$,$\Phi_{\cos}$ also for $\tilde{\Phi}_{\sin}, \tilde{\Phi}_{\cos}$ since from the type of the argument it will always be clear wether $\Phi_{\sin}$,$\Phi_{\cos}$ or their extensions $\tilde{\Phi}_{\sin}, \tilde{\Phi}_{\cos}$ are meant.

In the following we will introduce $\frac{\pi}{2}$ (and thus π) as the uniquely determined zero of the function cos on $[0, 2]$. This is possible since $\Phi_{\cos}0 =_{\mathbb{R}} 1$, $\Phi_{\cos}2 \le_{\mathbb{R}} -\frac{1}{3}$ and

$$(*)\ \bigwedge x^0, y^0 \big(0 \le_{\mathbb{Q}} y \le_{\mathbb{Q}} x \le_{\mathbb{Q}} 2 \to \Phi_{\cos}x -_{\mathbb{R}} \Phi_{\cos}y \le_{\mathbb{R}} -\frac{(x -_{\mathbb{Q}} y)^2}{18}\big)$$

are true purely universal assertions on cos (see below for the verfication of $(*)$) and hence axioms of $\mathrm{G_2A}_i^\omega$.
$(*)$ is a uniform quantitative version of the strict monotonicity of cos on $[0, 2]$. This strict monotonicity implies the uniqueness and hence (by a general meta–theorem from [37]) the effectivity of the uniquely determined zero of cos $[0, 2]$. This can be seen also directly as follows: The quantitative monotonicity $(*)$ immediately yields a modulus of uniqueness (in the sense of [37]) $\omega \in \mathrm{G_2R}^\omega$, namely $\omega(n) := \frac{1}{36(n+1)^2}$ and thus the computability of the zero of cos in $\mathrm{G_2R}^\omega \cup \Phi_{\cos}$:
Let $x_m, x_{\tilde{m}} \in [0, 2]$ be such that

$$|\cos x_m|, |\cos x_{\tilde{m}}| < \frac{1}{36(n+1)^2} \text{ and therefore } |\cos x_m - \cos x_{\tilde{m}}| < \frac{1}{18(n+1)^2}.$$

Then –by $(*)$– $|x_m - x_{\tilde{m}}| < \frac{1}{n+1}$, i.e. ω is a modulus of uniqueness. We define a partition of $[0, 2]$ by

$$x_i := \frac{i}{3 \cdot 36(n+1)^2} \text{ for } i = 0, \dots, 6 \cdot 36(n+1)^2$$

and compute for each i a rational $1/(6 \cdot 36(n+1)^2)$–approximation y_i of $|\cos x_i|$. Next we compute an i_n such that

$$|y_{i_n}| = \min\left\{|y_i| : i = 0, \dots, 6 \cdot 36(n+1)^2\right\}.$$

It follows

$$|\cos(x_{i_n})| \le \min_{i \le 6 \cdot 36(n+1)^2} |\cos x_i| + \frac{1}{3 \cdot 36(n+1)^2} \le \inf_{x \in [0,2]} |\cos x| + \frac{2}{3 \cdot 36(n+1)^2} < \frac{1}{36(n+1)^2}.$$

Hence (x_{i_n}) is a Cauchy sequence in $[0, 2]$ with Cauchy modulus $1/(n+1)$. (x_{i_n}) can be computed by a term t^1 in $\mathrm{G_2R}^\omega \cup \Phi_{\cos}$. Therefore we may define $\pi :=_1 2 \cdot_{\mathbb{R}} t$.

The following propositions on $\pi, \Phi_{\sin}, \Phi_{\cos}$ are purely universal and therefore axioms of $\mathrm{G_2A}_i^\omega$:

1) $2 \le_{\mathbb{R}} \pi \le_{\mathbb{R}} 4$, $\Phi_{\cos}(\frac{\pi}{2}) =_{\mathbb{R}} 0$.

2) $\bigwedge x^1 \big(\Phi_{\cos}(x +_{\mathrm{IR}} 2\pi) =_{\mathrm{IR}} \Phi_{\cos}x \wedge \Phi_{\sin}(x +_{\mathrm{IR}} 2\pi) =_{\mathrm{IR}} \Phi_{\sin}x \wedge$
$\Phi_{\cos}(x +_{\mathrm{IR}} \pi) =_{\mathrm{IR}} -\Phi_{\cos}x \wedge \Phi_{\sin}(x +_{\mathrm{IR}} \pi) =_{\mathrm{IR}} -\Phi_{\sin}x \wedge$
$\Phi_{\cos}x =_{\mathrm{IR}} \Phi_{\sin}(\frac{\pi}{2} -_{\mathrm{IR}} x) \wedge \Phi_{\sin}x =_{\mathrm{IR}} \Phi_{\cos}(\frac{\pi}{2} -_{\mathrm{IR}} x)\big)$.

3) Uniform quantitative strict monotonicity:
$\bigwedge x^0, y^0 \big((0 \le_{\mathbb{Q}} y \le_{\mathbb{Q}} x \le_{\mathbb{Q}} 4 \to \Phi_{\cos}(\tilde{x}) -_{\mathrm{IR}} \Phi_{\cos}(\tilde{y}) \le_{\mathrm{IR}} -\frac{(\tilde{x}-_{\mathrm{IR}}\tilde{y})^2}{18}\big) \wedge$
$\big(-2 \le_{\mathbb{Q}} y \le_{\mathbb{Q}} x \le_{\mathbb{Q}} 2 \to \Phi_{\sin}(\check{x}) -_{\mathrm{IR}} \Phi_{\sin}(\hat{y}) \ge_{\mathrm{IR}} \frac{(\check{x}-_{\mathrm{IR}}\hat{y})^2}{18}\big)\big)$,
where $\tilde{z} := \min_{\mathrm{IR}}(z,\pi)$, $\check{z} := \min_{\mathrm{IR}}(z,\pi/2)$ and $\widehat{z} := \max_{\mathrm{IR}}(z,-\pi/2)$.

3) implies (together with 1) and the continuity of $\cos,\sin$):

$$\text{3)'}\ \bigwedge x^1, y^1 \big((0 \le_{\mathrm{IR}} y \le_{\mathrm{IR}} x \le_{\mathrm{IR}} \pi \to \Phi_{\cos}(x) -_{\mathrm{IR}} \Phi_{\cos}(y) \le_{\mathrm{IR}} -\tfrac{(x-_{\mathrm{IR}}y)^2}{18}\big)\wedge$$
$$\big(-\tfrac{\pi}{2} \le_{\mathrm{IR}} y \le_{\mathrm{IR}} x \le_{\mathrm{IR}} \tfrac{\pi}{2} \to \Phi_{\sin}(x) -_{\mathrm{IR}} \Phi_{\sin}(y) \ge_{\mathrm{IR}} \tfrac{(x-_{\mathrm{IR}}y)^2}{18}\big)\big).$$

The reason for our somewhat complicated formulation 3) instead of 3)' is that 3) is in Π^0_1 (in contrast to 3)').
Proof of 3)' (and hence of 3) and $(*)$ above):
Since $\sin z \ge \frac{z}{3}$ for all $z \in [0,2]$ (see e.g. [15]), we obtain for all x,y such that $0 \le y \le x \le \frac{\pi}{2}$:

$$\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2}) \le -2(\frac{x+y}{6})(\frac{x-y}{6}) \le -\frac{(x-y)^2}{18}.$$

Because of $\cos x = -\cos(\pi - x)$, the claim follows for $x,y \in [0,\frac{\pi}{2}]$ and $x,y \in [\frac{\pi}{2},\pi]$. Now assume that $x \ge \frac{\pi}{2} \wedge y \le \frac{\pi}{2}$: Then
$\cos x - \cos y = \cos x - \cos\frac{\pi}{2} + \cos\frac{\pi}{2} - \cos y \le -2\big(\frac{x^2-y^2}{36}\big) \le -\frac{(x-y)^2}{18}$. Put together this yields the claim for $[0,\pi]$.
By $\sin x = -\cos(\frac{\pi}{2}+x)$ the corresponding claim for sin follows.

Remark 5.1.2 The proof of 3)' above can be conceived as an instance of cor.2.2.3 (of course a very simple one): When formalized within $\mathrm{G}_2\mathrm{A}^\omega$, the strict monotonicity of cos has (modulo a suitable prenexation) the logical form

$$(+)\ \bigwedge x,y \le_1 M_\pi, k^0 \bigvee n^0 \big(\underbrace{x \ge_{\mathrm{IR}} y + \frac{1}{k+1} \to \Phi_{\cos}x - \Phi_{\cos}y <_{\mathrm{IR}} -\frac{1}{n+1}}_{\equiv: A \in \Sigma^0_1 (\text{modulo prenexation})}\big).$$

Since (+) is provable in $\mathrm{G}_2\mathrm{A}^\omega$, cor.2.2.3 implies the extractability of a polynomial pk providing a bound on n which does not depend on x,y. Since A is monotone w.r.t. n, this bound in fact realizes '$\bigvee n$', i.e.

$$\mathrm{G}_2\mathrm{A}^\omega_i \vdash \bigwedge x,y \in [0,\pi], k^0 \big(x \ge_{\mathrm{IR}} y + \frac{1}{k+1} \to \Phi_{\cos}x - \Phi_{\cos}y <_{\mathrm{IR}} -\frac{1}{pk+1}\big).$$

Our proof of 3)' yields $pk := 18(k+1)^2$. The majorization used in this proof to eliminate the dependence on x,y is simply the inequality

$$(x+y)(x-y) \ge (x-y)^2 \ge \frac{1}{(k+1)^2} \text{ for } x \ge y + \frac{1}{k+1} \ge \frac{1}{k+1}.$$

The **tangent** function $\tan x := \frac{\sin x}{\cos x}$ is represented by a term $\Phi_{\tan}^{1(0)(1)} \in \mathrm{G_2R}^\omega \cup \{\Phi_{\sin}, \Phi_{\cos}\}$ such that

$$\bigwedge x^1, n^0 \big(-\frac{\pi}{2} + \frac{1}{n+1} \le_{\mathbb{R}} x \le_{\mathbb{R}} \frac{\pi}{2} - \frac{1}{n+1} \to \Phi_{\tan} xn =_{\mathbb{R}} \frac{\Phi_{\sin} x}{\Phi_{\cos} x}\big).$$

5.2 The functions arcsin, arccos and arctan in $\mathrm{G_2A}_i^\omega$

As we have seen above, $\sin x$ is strictly monotone on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with the 'modulus of uniform strict monotonicity' $\omega(\varepsilon) := \frac{\varepsilon^2}{18}$. Since $\sin x$ has the Lipschitz constant $\lambda = 1$,
$\bigwedge y \in [-1,1] \bigvee x \in [-\frac{\pi}{2}, \frac{\pi}{2}](\sin x = y)$ implies

$$(*)\ \bigwedge y \in [-1,1], n \in \mathbb{N} \bigvee r_n \in \{q_1, \ldots, q_{l_n}\}\big(|\sin r_n - y| \le \frac{1}{n+1}\big),$$

where $\{q_1, \ldots, q_{l_n}\} \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \cap \mathbb{Q}$ is a $1/(n+1)$–net for $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Similarly to the function M used in our representation of $[0,1]$ one constructs a function $M_\pi \in \mathrm{G_2R}^\omega$ such that $\{i : i \le_0 M_\pi n\}$ contains (modulo our coding of $\mathbb{Q}$) such a $1/(n+1)$–net (e.g. $M_\pi n := j(8(n+1), n)$). $(*)$ implies

$$\bigwedge y \le_1 M, n^0 \bigvee q \le_0 M_\pi n \big((\widehat{-\frac{\pi}{2}})(n) + \frac{1}{n+1} \le_{\mathbb{Q}} q \le_{\mathbb{Q}} (\widehat{\frac{\pi}{2}})(n) - \frac{1}{n+1} \wedge |\Phi_{\sin} q -_{\mathbb{R}} \tilde{y}| \le_{\mathbb{R}} \frac{3}{n+1}\big)^{35}$$

and therefore

$$\bigwedge y \le_1 M, n^0 \bigvee q \le_0 M_\pi n \big((\widehat{-\frac{\pi}{2}})(n) + \frac{1}{n+1} \le_{\mathbb{Q}} q \le_{\mathbb{Q}} (\widehat{\frac{\pi}{2}})(n) - \frac{1}{n+1} \wedge |(\Phi_{\sin} q)(n) -_{\mathbb{Q}} \tilde{y}(n)| \le_{\mathbb{Q}} \frac{5}{n+1}\big).$$

Bounded μ–search provides a functional $\tilde{\Psi}^{1(1)} \in \mathrm{G_2R}^\omega \cup \{\Phi_{\sin}\}$ such that

$$\bigwedge y \le_1 M, n^0 \big((\widehat{-\frac{\pi}{2}})(n) + \frac{1}{n+1} \le_{\mathbb{Q}} \tilde{\Psi} yn \le_{\mathbb{Q}} (\widehat{\frac{\pi}{2}})(n) - \frac{1}{n+1} \wedge |\Phi_{\sin}(\tilde{\Psi} yn))(n) -_{\mathbb{Q}} \tilde{y}(n)| \le_{\mathbb{Q}} \frac{5}{n+1}\big)$$

and therefore

$$\bigwedge y \le_1 M, n^0 \big((\widehat{-\frac{\pi}{2}})(n) + \frac{1}{n+1} \le_{\mathbb{Q}} \tilde{\Psi} yn \le_{\mathbb{Q}} (\widehat{\frac{\pi}{2}})(n) - \frac{1}{n+1} \wedge |\Phi_{\sin}(\tilde{\Psi} yn) -_{\mathbb{R}} \tilde{y}| \le_{\mathbb{R}} \frac{7}{n+1}\big)$$

Hence for $\Psi yn := \tilde{\Psi} y(7 \cdot 36(n+1)^2)$

$$\bigwedge y \in [-1,1], n \in \mathbb{N} \big(|\Phi_{\sin}(\Psi yn) -_{\mathbb{R}} \tilde{y}| < \frac{1}{36(n+1)^2}\big).$$

¿From the fact that $\omega(\varepsilon)$ is a modulus of strict monotonicity for sin we obtain that $(\Psi yn)_{n \in \mathbb{N}}$ is a Cauchy sequence in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with Cauchy modulus $1/(n+1)$: Suppose that $m, \tilde{m} \ge_0 n$, then

$$|\Phi_{\sin}(\Psi ym) - \Phi_{\sin}(\Psi y\tilde{m})| \le |\Phi_{\sin}(\Psi ym) - \tilde{y}| + |\tilde{y} - \Phi_{\sin}(\Psi y\tilde{m})| < \frac{1}{18(n+1)^2}$$

and therefore $|\Psi ym -_{\mathbb{Q}} \Psi y\tilde{m}| < \frac{1}{n+1}$.
Hence $\Phi_{\arcsin} y := \Psi \tilde{y}$ represents the inverse function of sin on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and is uniformly continuous on $[-1,1]$ with ω as a modulus of uniform continuity.
The inverse arccos of cos on $[0, \pi]$ is defined analogously.
Similarly to arcsin, arccos one can finally define arctan in $\mathrm{G_2A}_i^\omega$.

[35] Here again $\lambda y^1.\tilde{y} \in \mathrm{G_2R}^\omega$ is the construction corresponding to our representation of $[-1,1]$ such that $\tilde{y} \le_1 M$, $y =_{\mathbb{R}} \tilde{y}$ if $-1 \le_{\mathbb{R}} y \le_{\mathbb{R}} 1$, and $-1 \le_{\mathbb{R}} \tilde{y} \le_{\mathbb{R}} 1$ for all y^1.

5.3 The exponential functions $\exp_n$ and $\exp$ in $\mathbf{G_2A}_i^\omega$ and $\mathbf{G_3A}_i^\omega$

Since all terms $t^1 \in \mathrm{G_2R}^\omega$ are bounded by a polynomial (prop.1.2.28) it is clear that exp can neither be defined in $\mathrm{G_2A}_i^\omega$ nor can exp be represented by a new function constant which is majorized by a term from $\mathrm{G_2R}^\omega$. However for every **fixed** number $n \geq_0 1$ we can introduce the restriction of exp to $[-n,n](\subset \mathbb{R})$ by such a constant. This means that we can deal locally with exp in $\mathrm{G_2A}_i^\omega$ and e.g. may use exp for the solution of ordinary differential equations etc.

We add to $\mathrm{G_2A}_i^\omega$ a function constant $\Phi_{\exp_n}^{1(0)}$ which is intended to represent the restriction of exp on $[-n,n]\cap\mathbb{Q}$. Since exp is Lipschitz continuous on $[-n,n]$ with a Lipschitz constant e.g. $\lambda := 3^n$, we have the following universal axioms on $\Phi_{\exp_n}^{1(0)}$ in $\mathrm{G_2A}_i^{\omega}$[36]

$$(1)\ \bigwedge x^0\big(\widehat{\Phi_{\exp_n}}x =_1 \Phi_{\exp_n}x \leq_1 M_n \wedge 0 \leq_{\mathbb{R}} \Phi_{\exp_n}x \leq_{\mathbb{R}} 3^n\big),$$

where M_n is the boundedness function used in the representation of $[0,3^n]$ (e.g. $M_n(k) :=$ $j(6\cdot 3^n(k+1), 3(k+1)-1)$).[37]

$$(2)\ \bigwedge x^0,y^0,q^0\big(-n \leq_{\mathbb{Q}} x,y \leq_{\mathbb{Q}} n \wedge |x -_{\mathbb{Q}} y| \leq_{\mathbb{Q}} \frac{q}{3^n} \to |\Phi_{\exp_n}x -_{\mathbb{R}} \Phi_{\exp_n}y| \leq_{\mathbb{R}} q\big).$$

As in the case of $\Phi_{\sin}$, by (2) we can extend $\Phi_{\exp_n}$ to a constant $\tilde{\Phi}_{\exp_n}^{1(1)} \in \mathrm{G_2R}^\omega$ which represents the continuation of the function represented by $\Phi_{\exp_n}$ to $[-n,n]$. As for $\Phi_{\sin}$ we will denote this extension also by $\Phi_{\exp_n}$. The most important properties of exp (restricted on $[-n,n]$) can be expressed by purely universal sentences and thus are axioms of $\mathrm{G_2A}_i^\omega$:

$$(3)\ \bigwedge x^0,y^0\big(-n \leq_{\mathbb{Q}} y \leq_{\mathbb{Q}} x \leq_{\mathbb{Q}} n \to \int_y^x (\Phi_{\exp_n}t)dt =_{\mathbb{R}} \Phi_{\exp_n}x -_{\mathbb{R}} \Phi_{\exp_n}y\big),\ \Phi_{\exp_n}0 =_{\mathbb{R}} 1,$$

$$(4)\ \bigwedge x^0,y^0\big(-n \leq_{\mathbb{Q}} x,y,x +_{\mathbb{Q}} y \leq_{\mathbb{Q}} n \to \Phi_{\exp_n}(x +_{\mathbb{Q}} y) =_{\mathbb{R}} \Phi_{\exp_n}(x)\cdot_{\mathbb{R}} \Phi_{\exp_n}(y)\big).$$

By the continuity of $\Phi_{\exp_n}$, (3) and (4) immediately generalize to real arguments. Furthermore by the theorem that the derivative of $\int_0^x f(x)dx$ is f (which we will discuss in the next chapter in the context of $\mathrm{G_2A}_i^\omega$), (3) implies

$$(3)'\ \bigwedge x^1(-n \leq_{\mathbb{R}} x \leq_{\mathbb{R}} n \to \Phi'_{\exp_n}x =_{\mathbb{R}} \Phi_{\exp_n}x),\ \text{where } ' \text{ denotes the derivative.}$$

In contrast to $\mathrm{G_2A}_i^\omega$ we can define the unrestricted exponential function in $\mathrm{G_3A}_i^\omega$ as usual via the exponential series:[38] one easily defines the sequence of partial sums of this series for rational arguments. From the quotient criterion one gets the convergence of this series together with a modulus of convergence. By the continuity of this series in $x \in \mathbb{R}$ with the modulus $\omega(x,n) := 3^{\lceil|\widehat{x}(0)|+1\rceil}\cdot(n+1)$ we can continue it on $\mathbb{R}$.

[36] As in the case of $\Phi_{\sin}$ and $\Phi_{\cos}$ we denote (according to the discussion in connection with thm.2.2.8) $\mathrm{G_2A}_i^\omega \cup \{\Phi_{\exp_n}^{1(0)}\}$ also by $\mathrm{G_2A}_i^\omega$

[37] For notational simplicity we identify in the following the natural number n with its code $j(2n,0)$ as a rational number, e.g. we write $x^0 \leq_{\mathbb{Q}} n$ instead of $x^0 \leq_{\mathbb{Q}} j(2n,0)$ in order to express that the rational number which is coded by x is $\leq$ the natural number n.

[38] In particular we can define a term $\Phi_{\exp_n}$ in $\mathrm{G_3A}_i^\omega$ which satisfies (provably) (1)–(4).

Analogously to the definition of arcsin we can define the inverse function $\ln_n$ of $\exp_n$ using the fact that e.g. $\omega(\varepsilon) := \varepsilon \cdot 3^{-n}$ is a modulus of strict monotonicity for $\exp_n$ on $[-n, n]$.

In this chapter we have seen that $\sin, \cos$ can be introduced relatively to $\mathrm{G}_2\mathrm{A}^\omega$ via new constants $\Phi_{\sin}^{1(0)}, \Phi_{\cos}^{1(0)}$ and purely universal axioms which express the usual (characterizing) properties of $\sin, \cos$. tan and the inverse functions $\arcsin, \arccos, \arctan$ of $\sin, \cos, \tan$ as well as π can be defined in $\mathrm{G}_2\mathrm{A}^\omega$ using $\Phi_{\sin}, \Phi_{\cos}$. Furthermore for each **fixed** $n \in \mathbb{N}$ the restriction $\exp_n$ of the exponential function exp to $[-n, n]$ can be introduced relatively to $\mathrm{G}_2\mathrm{A}^\omega$ via a new constant $\Phi_{\exp_n}^{1(0)}$ and its characterizing properties can be expressed as universal axioms. Thus by theorem 2.2.8 the use of $\sin, \cos, \tan, \arcsin, \arccos, \arctan, \pi$ and the **local** use of exp only contributes to the growth of provably functionals by majorants $\in \mathrm{G}_2\mathrm{R}^\omega$ for the constants $\Phi_{\sin}^{1(0)}, \Phi_{\cos}^{1(0)}, \Phi_{\exp_n}^{1(0)}$ and the terms used in the formulation of their universal axioms and in the definition of $\pi, \arcsin, \arccos, \arctan$. Hence the results stated at the end of chapter 3 on polynomial growth of bounds extractable from proofs relatively to $\mathrm{G}_2\mathrm{A}^\omega$ (resp. finitely iterated exponential growth in case of $\mathrm{G}_3\mathrm{A}^\omega$) extend to proofs which use (besides the analytical tools discussed in chapter 3 and 4 above) also these trigonometric functions and $\exp_n$ and their usual properties. The result on finitely iterated exponential growth also applies in the presence of the unrestricted exponential function.

6 Analytical theorems which can be expressed as universal sentences in G_2A^ω or follow from $AC^{0,1}$–qf

In the previous chapters 3–5 we have seen that many basic special functions as e.g. sin, cos etc. and functionals as sup, $\int_a^b$ etc. can be introduced in G_2A^ω and their characteristic properties can be expressed as universal sentences (which are treated as axioms). As we have discussed in chapters 1,2 such universal axioms have a trivial functional interpretation and monotone functional interpretation. In particular their proofs are irrelevant and do not contribute neither to the extraction of bounds nor to the bounds itself (that is why we have taken universal sentences as axioms[39]). Only the terms (respectively their majorants) used to formulate these axioms may contribute to the growth of the bounds. Since we have used only terms which (are polynomials or) can be majorized by polynomials of degree ≤ 3, the order of growth which may result from the use of these function(al)s and their basic properties is quite low.[40]
In this chapter we show that the same holds for some basic analytical theorems by reducing them (in fact strengthening them) to universal sentences or a simple application of $AC^{0,1}$–qf. Since AC–qf also has a trivial functional interpretation and monotone functional interpretation this is as good as a reduction to a universal sentence.

6.1 Fundamental theorem of calculus

In this paragraph we consider the following theorem

Theorem 6.1.1 (Existence of a primitive function) *Let $f \in C[a,b]$, where $a < b$, and define $F(x) := \int\limits_a^x f(t)dt$ for $x \in [a,b]$. Then $F'(x) = f(x)$ on $[a,b]$, where F' denotes the derivative of F.*

We now verify that this theorem can be written as a purely universal sentence in $G_2A_i^\omega$ (and therefore is an axiom in $G_2A_i^\omega$):
Firstly we express the definition of the derivative as sequential limit

$$(1)\quad \lim_{\substack{h\to 0\\ |h|>0}} \frac{F(x+h)-F(x)}{h} = f(x)$$

in the form

$$(2)\ \bigwedge k \bigvee n \bigwedge y \in [a,b]\big(|x-y| \leq \frac{1}{n+1} \to |f(x)(x-y) - (F(x)-F(y))| \leq \frac{1}{k+1}\cdot |x-y|\big).$$

Remark 6.1.2 *(2) trivially implies (1) relatively to $G_2A_i^\omega$ whereas the proof of the implication '(1) $\to$ (2)' needs classical logic and $AC^{0,1}-$ qf (the proof is analogously to the proof of the equivalence of sequential continuity and ε–δ–continuity which we will discuss in detail in paragraph 3 of this chapter).*

[39] At least up to type–2–variables. However we use only such axioms where the types of the universal quantifiers are ≤ 1.

[40] Mainly we have used polynomials of degree 2 as $j(x,y)$ or the modulus of monotonicty $q^2/18$ for sin, cos or the functions from the representation of $C[0,1]$. Only in the representation of the inverse $1/x$ for a real number x with $|x| > 0$ we used a polynomial of degree 3.

In order to write (2) as a universal sentence we need a modulus of convergence. However the usual proof of (2) immediately yields such a modulus (in fact even a uniform one, i.e. a modulus which does not depend on x), namely any modulus ω_f of uniform continuity for f works:

$$(3)\ \bigwedge k \bigwedge y \in [a,b] \big(\underbrace{|x-y| < \frac{1}{\omega_f(k)+1} \to |f(x)(x-y) - (F(x) - F(y))| \leq \frac{1}{k+1} \cdot |x-y|}_{\equiv: A} \big).$$

Since $<_{\mathbb{R}} \in \Sigma^0_1$ and $\leq_{\mathbb{R}} \in \Pi^0_1$, the formula A is (when formalized in $\mathrm{G}_2\mathrm{A}^\omega_i$) equivalent to a purely universal formula. By our representation of $C[a,b]$ from chapter 3, quantification over $C[a,b]$ (and over $[a,b]$) reduces to quantification over f^1. Hence (3) can be expressed in $\mathrm{G}_2\mathrm{A}^\omega$ as a sentence $\bigwedge f^1 A$ with $A \in \Pi^0_1$.

Remark 6.1.3 *In constructive analysis the relation 'f is uniformly differentiable on $[a,b]$ with derivative f' ' is defined as*

$$\bigvee \alpha^1 \bigwedge k^0 \bigwedge x, y \in [a,b] \big(|x-y| \leq \frac{1}{\alpha(k)+1} \to |f'(x)(x-y) - (f(x) - f(y))| \leq \frac{1}{k+1} \cdot |x-y| \big)$$

(see e.g. [70]).
This is a uniform quantitative version of differentiation which classically is equivalent to the usual one but not constructively. From our treatment of the fundamental theorem of calculus we obtain in $G_2A^\omega_i$ as a corollary the differentiability in this strong sense of many basic functions. We illustrate this by a simple example:
The formalization of '$\bigwedge x \in [0,7] \big(\int_0^x \cos(t)dt = \sin(x) \big)$' in $G_2A^\omega_i$ is a purely universal sentence and hence taken as an axiom. Therefore $G_2A^\omega_i$ proves that sin *is uniformly differentiable (on $[0,7]$ and hence on $\mathbb{R}$ because* sin *is 2π–periodic) with derivative* cos *and the modulus $\alpha(k) := k$ (since* cos *is Lipschitz continuous with $\lambda := 1$).*

The theorem on the existence of a primitive function is sometimes called 'first part' of the fundamental theorem of calculus, where the 'second part' of this theorem refers to the proposition that every primitive function for $f \in C[a,b]$ differs from F only by an additive constant. This second part follows immediately from the mean value theorem of differentiation which will be discussed in the first paragraph of the next chapter.

Both parts of the fundamental theorem of calculus together yield $f(x) = f(a) + \int\limits_a^x f'(t)dt$ for all functions $f \in C[a,b]$ with derivative $f' \in C[a,b]$. From this one obtains in $\mathrm{G}_2\mathrm{A}^\omega_i$ for every **fixed** number k the **Taylor formula** for $k+1$–times continuously differentiable functions f with the integral form of the error term by the usual inductive procedure (In order to formulate this formula for variable k we need the functions $\lambda k.k!$ and $\lambda k.x^k$ which are definable in $\mathrm{G}_3\mathrm{A}^\omega_i$ but not in $\mathrm{G}_2\mathrm{A}^\omega_i$). The Taylor formula with the Lagrange error term follows from this as usual by the mean value theorem of integration which will be considered in chapter 7.

6.2 Uniform approximation of continuous functions by trigonometric polynomials

Let $f \in C[-\pi,\pi]$ be a continuous function with the modulus ω_f of uniform continuity and assume $f(\pi) =_{\mathbb{R}} f(-\pi)$. It is well–known that f can be approximated uniformly (i.e. w.r.t. the norm

$\|f\|_\infty := \sup\limits_{x\in[-\pi,\pi]} |fx|$) by trigonometric polynomials

$$\sum_{k=1}^{n}(a_k \cdot \cos(kx) + b_k \cdot \sin(kx)),$$

where $a_k, b_k \in \mathbb{R}$.
In order to express this theorem in $\mathrm{G_2A}_i^\omega$ as a universal sentence we have to define (within $\mathrm{G_2R}^\omega$) a sequence of approximating polynomials as a functional in f, ω_f together with a modulus of convergence. This can be achieved by a theorem due to Fejr (more precisely by the proof of this theorem as it is given e.g. in [53]):

Theorem 6.2.1 (Fejr) *Let* $\sigma_n(f,x) := \frac{1}{n} \cdot \sum\limits_{k=0}^{n-1} S(k,f,x)$, *where*
$S(n,f,x) := \frac{a_0}{2} + \sum\limits_{k=1}^{n} \big(a_k \cdot \cos(kx) + b_k \cdot \sin(kx)\big)$ *and*
$a_k := \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t)\cos(kt)dt,\ b_k := \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\sin(kt)dt.$
Then $\lim\limits_{n\to\infty} \sigma_n(f,x) = f(x)$ *uniformly on* $[-\pi,\pi]$.

Remark 6.2.2 *Usually theorem 6.2.1 is formulated for 2π–periodic continuous functions $f : \mathbb{R} \to \mathbb{R}$. This version follows immediately from our formulation (which is more suited to formalize this theorem as a universal sentence) by the fact that* $\sin, \cos$ *are periodic with period* 2π.
By the results from the chapters 3-5 on the definability of variable sums of real numbers, the Riemann integral and $\sin, \cos, \pi$ *in* $G_2R^\omega \cup \{\Phi_{\sin}, \Phi_{\cos}\}$ *and our representation of functions* $f \in C[-\pi,\pi]$ *and real numbers by functions* f^1, x^1 *we know that* $\sigma_n(f,x)$ *can be defined as a functional* $\Phi_\sigma \in G_2R^\omega \cup \{\Phi_{\sin}, \Phi_{\cos}\}$ *in* n^0, f^1, x^1

The **proof** of theorem 6.2.1 from [53] (pp.129–131) yields

$$\bigwedge x \in [-\pi,\pi]\big(|\sigma_{n_0}(f,x) - f(x)| \le \frac{1}{k+1}\big),$$

if n_0 is sufficiently large such that

$$\frac{2\cdot M_f}{\pi \cdot n_0\big(\sin\big(1/(\omega_f(2(k+1))+1)\big)\big)^2} \le \frac{1}{4(k+1)}, \text{i.e.}$$

$$n_0 \ge \frac{8(k+1)M_f}{\pi\cdot\big(\sin\big(1/(\omega_f(2(k+1))+1)\big)\big)^2},$$

where $M_f \ge \|f\|_\infty$ and $\omega_f : \mathbb{N} \to \mathbb{N}$ is a modulus of uniform continuity for f.
¿From the fact that $\omega(\varepsilon) := \frac{\varepsilon^2}{18}$ is a modulus of strict monotonicity for sin on $[-\frac{\pi}{2},\frac{\pi}{2}] \supset [0,1]$ (as we have proved in chapter 4) we obtain

$$\pi \cdot \sin\big(1/(\omega_f(2(k+1))+1)\big) \ge \frac{1}{6(\omega_f(2(k+1))+1)^2},$$

Hence for $\Psi \in \mathrm{G_2R}^\omega$ defined as

$$\Psi fk := 48(k+1)M_f \cdot (\omega_f(2(k+1))+1)^2,$$

we have

$$(+)\ \bigwedge f \in C[-\pi,\pi], m, n \in \mathbb{N}\big(m \geq_0 \Psi f n \wedge f(-\pi) =_{\mathbb{R}} f(\pi) \to \|\sigma_m - f\|_\infty \leq \frac{1}{n+1}\big).$$

$(+)$ is equivalent to

$$(+)'\ \bigwedge f \in C[-\pi,\pi], k, m, n\big(m \geq_0 \Psi \tilde{f} n \wedge |f(-\pi) - f(\pi)| <_{\mathbb{R}} \frac{1}{k+1} \to \|\sigma_m(\tilde{f}) - f\|_\infty \leq_{\mathbb{R}} \frac{1}{n+1} + \frac{1}{k+1}\big),$$

where $\tilde{f}x := fx + \left(\frac{x+\pi}{2\pi}\right)(f(-\pi) - f(\pi))$.
By our representation of $C[a,b]$ from chapter 3, '$\bigwedge f \in C[-\pi,\pi]$' reduces to '$\bigwedge f^1$'. Furthermore by the definability of $\|f\|_\infty$ in $\mathrm{G_2R^\omega}$ (and the computability of an upper bound $\mathbb{N} \ni M_f \geq \|f\|_\infty$; see also chapter 3) and the remark above, we conclude that $(+)'$ (in contrast to $(+)$) has the logical form $\bigwedge f^1, \underline{n}^0 A_0$ (when formalized in $\mathrm{G_2A}_i^\omega$) and thus is an axiom of $\mathrm{G_2A}_i^\omega$.

6.3 An application of $\mathrm{AC}^{0,1}$–qf

A function $\mathbb{R} \to \mathbb{R}$ in $\mathrm{G_2A}_i^\omega$ is given by a functional $F^{1(1)}$ which is extensional w.r.t. $=_{\mathbb{R}}$ (for short: $F : \mathbb{R} \to \mathbb{R}$), i.e.

$$\bigwedge x^1, y^1(x =_{\mathbb{R}} y \to Fx =_{\mathbb{R}} Fy).$$

As usual F is called **sequentially continuous** in x iff

$$\bigwedge x_{(\cdot)}^{1(0)}\big(\lim_{n\to\infty} x_n =_{\mathbb{R}} x \to \lim_{n\to\infty} F(x_n) =_{\mathbb{R}} F(x)\big),$$

where $(\lim\limits_{n\to\infty} x_n =_{\mathbb{R}} x) :\equiv \bigwedge k^0 \bigvee n^0 \bigwedge m \geq_0 n(|x_m -_{\mathbb{R}} x| \leq \frac{1}{k+1})$.
F is called ε–δ–continuous in x iff

$$\bigwedge k^0 \bigvee n^0 \bigwedge y^1\big(|x -_{\mathbb{R}} y| \leq \frac{1}{n+1} \to |F(x) -_{\mathbb{R}} F(y)| \leq \frac{1}{k+1}\big).$$

Proposition: 6.3.1 *The theory $G_2A^\omega + AC^{0,1}$–qf proves*

$$\bigwedge F^{1(1)} : \mathbb{R} \to \mathbb{R} \bigwedge x^1\big(F \textit{is sequentially continuous in } x \leftrightarrow F \textit{ is } \varepsilon\text{–}\delta\text{–continuous in } x\big).$$

Proof: '$\leftarrow$': Obvious!
'$\to$': Suppose that F is not ε–δ–continuous in x, i.e.

$$(*)\ \bigvee k^0 \bigwedge n^0 \bigvee y^1 \big(\underbrace{|x -_{\mathbb{R}} y| <_{\mathbb{R}} \frac{1}{n+1} \wedge |F(x) -_{\mathbb{R}} F(y)| >_{\mathbb{R}} \frac{1}{k+1}}_{\equiv: A \in \Sigma_1^0}\big).$$

By our coding of pairs of natural numbers and numbers into functions one can express $\bigvee y^1 A$ in the form $\bigvee y^1 A_0$. Hence $\mathrm{AC}^{0,1}$–qf applied to $(*)$ yields

$$\bigvee k^0, \xi^{1(0)} \bigwedge n^0\big(|x -_{\mathbb{R}} \xi n| <_{\mathbb{R}} \frac{1}{n+1} \wedge |F(x) -_{\mathbb{R}} F(\xi n)| >_{\mathbb{R}} \frac{1}{k+1}\big),$$

i.e. $(\xi n)_{n\in\mathbb{N}}$ represents a sequence of real numbers which converges to x. But
$\neg \lim\limits_{n\to\infty} F(\xi n) =_{\mathbb{R}} F(x)$ and thus F is not sequentially continuous in x.

Remark 6.3.2 *1) The the proof of the implication '$\leftarrow$' needs no $AC^{0,1}$–qf and can be carried out even in the intuitionistic theory $G_2A_i^\omega$. On the other hand it is known that the implication '$\rightarrow$' is not provable in elementary intuitionistic analysis: See [47] for details on this. The weaker 'global' implication 'F is sequentially continuous on $\mathbb{R} \to F$ is ε–δ–continuous on $\mathbb{R}$' can be proved in elementary intuitionistic analysis if a certain principle of local continuity is added which (although classically incorrect) is of interest in intuitionistic mathematics (see [70] ,[66] and also [48] for a discussion on this point).*

2) The use of $AC^{0,1}$–qf in the proof of

$$\bigwedge F : \mathbb{R} \to \mathbb{R} \bigwedge x \in \mathbb{R} \big(F \textit{ sequentially continuous in } x \to F \ \varepsilon\text{–}\delta\text{–}\textit{continuous in } x \big)$$

is unavoidable since this implication is known to be unprovable even in Zermelo–Fraenkel set theory ZF (and a fortiori in G_2A^ω): see [27] ,[23] and [12] (However the weaker global implication (see 1) can be proved without choice; see [70] (7.2.9)).

The results of this chapter imply that the statements on the growth of extractable bounds stated at the end of the previous chapter extend without any changes to proofs which may use in addition to the analytical principles studied in chapters 3–5 also

1) the fundamental theorem of calculus
2) Fejer's theorem on the uniform approximation of 2π–periodic continuous functions by trigonometric polynomials
3) the equivalence (local and global) of ε–δ –continuity and sequential continuity of $F : \mathbb{R} \to \mathbb{R}$ in $x \in \mathbb{R}$.

7 Axioms having the logical form $\bigwedge x^\delta \bigvee y \le_\rho sx \bigwedge z^\tau A_0(x,y,z)$

So far we have considered basic properties of special functions and functionals in analysis as well as analytical theorems which can be expressed as purely universal sentences or follow from such sentences by the use of AC–qf. These theorems therefore have both a trivial functional interpretation **and** a trivial monotone functional interpretation and contribute to the growth of bounds (if they contribute at all) only by their term structure but not by their proofs.
In this chapter we deal with sentences which have the much more general form

$$(*)\ \bigwedge x^\delta \bigvee y \le_\rho sx \bigwedge z^\tau A_0(x,y,z)$$

of the axioms Δ in the theorems 2.2.2, 2.2.8 and the cor.2.2.3 . Although all the sentences of this type which we consider in this chapter have no direct usual functional interpretation by computable functionals at all (and for the most interesting ones even their negative translations have no functional interpretation in Gödel's T) they have a simple direct (i.e. without negative translation) **monotone** functional interpretation by very simple functionals $\in \mathrm{G}_2\mathrm{R}^\omega_{\underline{\ }}$. In particular their proofs do not matter for the extraction of uniform bounds but only the growth of majorants for the terms needed to formulate these sentences which is very low (mainly polynomially of degree 2).

In §1 we show that the following theorems of analysis can be expressed as sentences $(*)$ with $s \in \mathrm{G}_2\mathrm{R}^\omega$ and $A_0 \in \mathcal{L}(\mathrm{G}_2\mathrm{A}^\omega)$[41]:

1) Attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$ on $[0,1]^d$.

2) Mean value theorem of integration.[42]

3) Cauchy–Peano existence theorem for ordinary differential equations.

4) Brouwer's fixed point theorem for continuous functions : $[0,1]^d \to [0,1]^d$.

In §2 we introduce new axioms F and F^- which both have the form $(*)$ and are true in the type structure of all strongly majorizable functionals (which was introduced in [4]) but are false in the full set– theoretic model. Thus, whereas F, F^- do not contribute to the construction of bounds extracted from a proof, the verification of these bounds so long uses these axioms. However F^- can be eliminated from the verification proof by further proof–theoretic transformations (which do not effect the bounds themselves) so that the bounds extracted can also be verified without F^-. For bounds of type ≤ 1 this is also possible for proofs using F. The importance of the F, F^- rests on the fact that they imply combined with $\mathrm{AC}^{1,0}$–qf (which also has a trivial monotone functional interpretation) relative to $\mathrm{G}_2\mathrm{A}^\omega$ many important analytical theorems in there direct formulation (i.e. without any special representation) which do not have the form $(*)$ by themselves: In §3 we show that F^- implies

5) Every pointwise continuous function $G : [0,1]^d \to \mathbb{R}$ is uniformly continuous on $[0,1]^d$ and possesses a modulus of uniform continuity.

6) $[0,1]^d \subset \mathbb{R}^d$ has the (sequential form of the) Heine–Borel covering property.

[41] These theorems which are formulated here only for $[0,1]^d$ generalize to variable rectangles $[a_1,b_1] \times \ldots \times [a_d,b_d]$, where $a_i < b_i$ for $i = 1,\ldots,d$.

[42] The mean value theorem of differentiation does not have this logical form but can easily be derived from 1) as in its usual proof.

7) Dini's theorem: Every sequence G_n of pointwise continuous functions $:[0,1]^d \to \mathbb{R}$ which increases pointwise to a pointwise continuous function $G:[0,1]^d \to \mathbb{R}$ converges uniformly on $[0,1]^d$ to G and there exists a modulus of uniform convergence.

8) Every strictly increasing pointwise continuous function $G:[0,1] \to \mathbb{R}$ possesses a uniformly continuous inverse function $G^{-1}:[G0, G1] \to [0,1]$ together with a modulus of uniform continuity.

As a consequence of this we obtain the following result: If $\bigwedge x^0 \bigwedge y \leq_\rho sx \bigvee z^0 A_0(x,y,z)$ is proved in $\mathrm{G}_2\mathrm{A}^\omega$ plus the theorems 1)–8), then one can extract from the proof a **polynomial** p which provides a uniform bound on '$\bigvee z$' (which does not depend on y), i.e. $\bigwedge x^0 \bigwedge y \leq_\rho sx \bigvee z \leq_0 p(x)\ A_0(x,y,z)$. If x has the type 1 one obtains a polynomial relatively to x (in the sense of prop.1.2.30).
By 5) our representation of $f \in C([0,1]^d, \mathbb{R})$ which presupposes that f is endowed with a modulus of uniform continuity does not impose any restriction on the domain of functions in the presence of F^- and $\mathrm{AC}^{1,0}$–qf.
It is well–known that the existence of a function $G:[0,1]^d \to \mathbb{R}$, represented by a functional $\Phi^{1(1)}$, which is not continuous can be proved only by an instance of arithmetical comprehension over functions

$$CA^1_{ar}\ :\ \bigvee \Phi^{0(1)} \bigwedge f^1 (\Phi f =_0 0 \leftrightarrow A(f)),\ \text{where } A \text{ is an arithmetical formula.}$$

Since CA^1_{ar} implies CA_{ar} it in particular makes all $\alpha(<\varepsilon_0)$–recursive functions provably recursive (when added to $\mathrm{G}_2\mathrm{A}^\omega$). Since we deal with theories which do not contain CA^1_{ar} it is consistent to assume as an axiom that all functions $G:[0,1]^d \to \mathbb{R}$, which are given explicitely by a functional $\Phi^{1(1)}$, are uniformly continuous.[43] This is achieved by the axiom F:

$\mathrm{G}_2\mathrm{A}^\omega + F + \mathrm{AC}^{1,0}$–qf proves: Every function $G:[0,1]^d \to \mathbb{R}$ is uniformly continuous and possesses a modulus of uniform continuity.

The use of F (which does not contribute to the bounds extracted) has the nice property that continuous functions $G:[0,1]^d \to \mathbb{R}$ are nothing else then functionals $\Phi^{1(1)}$ which are extensional w.r.t. $=_{[0,1]^d}$ and $=_\mathbb{R}$ (and thus represent a function : $[0,1]^d \to \mathbb{R}$). This simplifies the formalization of given proofs and thereby the extraction of bounds from these proofs. Moreover the proofs of 5)–8) (which now hold for arbitrary functions G, G_n) become more simple.

¿From the work on the program of so–called 'reverse mathematics' (see [16],[17],[60],[61], [56]) it is known that 1) and 3)–6) are provable in a subsystem RCA_0+WKL of second–order arithmetic which is based on the binary König's lemma and Σ^0_1–induction (see chapter 9)[44]. The provably recursive functions of RCA_0+WKL are just the primitive recursive ones. This was firstly proved by H. Friedman in 1979 (in an unpublished paper) using model–theoretic methods. Later on W.Sieg gave a proof–theoretic treatment of this result using cut–elimination (see [57]).[45] In [33] we proved

[43] The restriction 'given explicitely by a functional' is essential. Of course we can formulate functional dependencies in e.g. $\mathrm{G}_2\mathrm{A}^\omega$ which describe a discontinuous functional: E.g. we can prove $\bigwedge f^1 \bigvee !x^0 ([\bigvee y(fy=0) \to fx=0 \wedge \bigwedge \tilde{x} < x(f\tilde{x} \neq 0)] \wedge [\bigwedge y(fy \neq 0) \to x=0])$ and x does not depend continuously on f, but we cannot show the existence of a funtional $\Phi^{0(1)}$ which maps f to x.

[44] In fact this work shows that 1) and 3)–6) actually are equivalent to WKL over the base theory RCA_0. From this it follows that these theorems have no functional interpretation in Gödel's T.

[45] One should mention also [14] where the conservativity of a special version of WKL over a system of second–order arithmetic whose provably recursive functions are polynomial time computable is established by model–theoretic

the conservativity of WKL over the finite type theories PRA^ω and PA^ω even for higher type sentences $\bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee z^\tau A_0(x,y,z)$, where ρ, τ are arbitrary types. Moreover we gave a perspicuous method for the the extraction of bounds from proofs using WKL and arbitrary axioms $(*)$ by a new combination of functional interpretation with majorization which, in [39], was simplified even further to the monotone functional interpretation (see chapter 2). In [37], [38] this was applied to concrete proofs in best approximation theory yielding new numerical estimates which improved known estimates significantly (see [39] for a discussion of these results). In [37] we also gave a detailed representation of $\mathbb{R}$, $C[0,1]$ and more general complete separable metric spaces and showed that e.g. 1) (for d=1), 2) and 3) as well as some more specific theorems from approximation theory have the logical form $(*)$. However we did not determine the growth of the terms needed in the formalization of these theorems as axioms $(*)$. Only by our much more involved representation of $C[0,1]$ and its generalization to $C([0,1]^d, \mathbb{R})$ and the explicit definition of the basic function(al)s of analysis in the chapters 3 and 5 we are now able to show that 1)–4) can be expressed as axioms $(*)$ in $\mathbf{G}_2\mathbf{A}^\omega$.

Since 5)–8) do not have the logical form $(*)$ one has to consider their proofs. The proofs of 5) and 6) using WKL (relative to RCA_0) require a tedious coding technique. In particular pointwise continuous functions have to be coded as a complicated set of quadruples of rational numbers (see [60]). Although working in the more flexible language of finite types makes it much easier to speak about such functions (namely as functionals of type 1(1)) this does not help as long as one has to use WKL as the basic principle of proof. In fact even the formulation of WKL itself uses the coding of sequences of variable length and therefore cannot be carried out in $\mathrm{G}_2\mathrm{A}^\omega$. The motivation for our axioms F, F^- was to formulate a more general higher type version of WKL which can be formulated **and applied** without the need of coding up objects like functions $[0,1] \to \mathbb{R}$. This allows very short proofs for 5)–8) **in** $\mathbf{G}_2\mathbf{A}^\omega + F^- + \mathbf{AC}^{1,0}$**–qf**. In §2 we will study the relationship between F^- and (a generalization of) WKL (to sequences of trees) in great detail.

7.1 Examples of theorems in analysis which can be expressed as $\bigwedge x^1 \bigvee y \leq_1 sx \bigwedge z^{0/1} A_0$–sentences in $\mathbf{G}_2\mathbf{A}_i^\omega$

Example 1:

a) Attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$ **on** $[0,1]^d$**:**

$$(1)\ \bigwedge f \in C([0,1]^d, \mathbb{R}) \bigvee x_0 \in [0,1]^d \big(fx_0 = \sup_{x \in [0,1]^d} fx\big).$$

(1) is equivalent to

$$(2)\ \bigwedge f \in C([0,1]^d, \mathbb{R}) \bigvee x_0 \in [0,1]^d \bigwedge n^0 \big(fx_0 \geq fr_n\big),$$

where $(r_n)_{n\in\mathbb{N}}$ enumerates a dense subset of $[0,1]^d$ (e.g. $[0,1]^d \cap \mathbb{Q}^d$).
Modulo our representation of $C([0,1]^d, \mathbb{R})$, $[0,1]^d$ and $\mathbb{R}$, the formalization of (2) in $\mathrm{G}_2\mathrm{A}_i^\omega$ has the following logical form

$$\bigwedge f^1 \bigvee x_0 \leq_1 \nu^d(M, \ldots, M) \bigwedge n^0 \big(\underbrace{f(x_0)_{\mathbb{R}^d} \geq_{\mathbb{R}} \Psi_1 fn}_{\in \Pi^0_1}\big),$$

methods.

where $M := \lambda n.j(6(n+1), 3(n+1)-1)$ is the boundedness function from the representation of $[0,1]$, Ψ_1 is the functional used in the representation of $C([0,1]^d, \mathbb{R})$ and $(\cdot)_{\mathbb{R}^d}$ the corresponding application.
Hence (for each fixed number d) theorem (1) is an axiom $\in \Delta$ in theorems 2.2.2 ,2.2.7 ,2.2.8 and corollary 2.2.3 . This generalizes to functions $f \in C([a_1, b_1] \times \ldots \times [a_d, b_d], \mathbb{R})$, where $a_i < b_i$ for $i = 1, \ldots, d$.

b) Mean value theorem of differentiation: Assume $a < b$.

$$\bigwedge f \in C[a,b](f \text{ differentiable in } (a,b) \to \bigvee x_0 \in (a,b)\big(\frac{f(b)-f(a)}{b-a} = f'(x_0)\big).$$

This theorem does not have the logical form $\bigwedge x^1 \bigvee y \leq_1 sx \bigwedge z^0 A_0$ by itself since there are unbounded quantifiers hidden in $\bigvee x_0 \in (a,b)$ because

$$\bigvee x \in (a,b)\ A(x) \leftrightarrow \bigvee x \in [a,b](a < x < b \wedge A(x))$$

and $<_{\mathbb{R}} \in \Sigma^0_1$.
However the usual proof of the mean value theorem using the above theorem on the attainment of the maximum can easily be formalized in $\mathrm{G}_2\mathrm{A}^\omega$ so that the mean value theorem also does not go beyond polynomial growth.

Example 2: Mean value theorem of integration

$$\bigwedge f, \varphi \in C[0,1]\big(\varphi \geq 0 \to \bigvee x_0 \in [0,1]\big(\int_0^1 f(x)\varphi(x)dx = f(x_0) \cdot \int_0^1 \varphi(x)dx\big)\big).$$

Formalized in $\mathrm{G}_2\mathrm{A}^\omega_i$ this theorem has the following logical form:

$$\bigwedge f^1, \varphi^1 \bigvee x_0 \leq_1 M\big(\underbrace{\Phi_I(f \cdot \varphi^+) =_{\mathbb{R}} f(x_0)_{\mathbb{R}} \cdot_{\mathbb{R}} \Phi_I \varphi}_{\in \Pi^0_1}\big),$$

where φ^+ is the code of the pair $(\lambda x^0. \max_{\mathbb{Q}}(0, \Psi_1 \varphi x), \Psi_2 \varphi)$ into a single function (i.e. φ^+ represents the positive part of the function represented by φ) and $f \cdot \varphi^+$ is a representative of the product of the functions represented by f and φ^+. Again this generalizes to $[a,b]$ instead of $[0,1]$.

Example 3: Cauchy–Peano existence theorem

Let $F(x,y)$ be a continuous function on the rectangle $R := \{(x,y) : |x-\xi| \leq a, |y-\eta| \leq b\} \subset \mathbb{R}^2$, where $a, b \in \mathbb{R}_+ \setminus \{0\}, (\xi, \eta) \in \mathbb{R}^2$. Furthermore assume that $M_F := \sup\limits_{(x,y)\in R} |F(x,y)| > 0$ and define $\alpha := \min(a, \frac{b}{M_F})$. Then (one version of) the Cauchy–Peano existence theorem says (see e.g. [9])

$$(*) \left\{ \begin{array}{l} \text{There exists a continuously differentiable function } G : [\xi-\alpha, \xi+\alpha] \to [\eta-b, \eta+b] \text{ such that} \\ \bigwedge x \in [\xi-\alpha, \xi+\alpha]\big(G'(x) = F(x, G(x))\big) \wedge G(\xi) = \eta \end{array} \right. .$$

By the fundamental theorem of calculus, $(*)$ is implied by

$$(**)\begin{cases} \text{There exists a function } G \in C([\xi-\alpha,\xi+\alpha],[\eta-b,\eta+b]) \text{ such that} \\ \bigwedge x \in [\xi-\alpha,\xi+\alpha]\big(G(x) = \eta + \int\limits_{\xi}^{x} F(t,G(t))dt\big) \end{cases}.$$

$(**)$ immediately implies

$$(1)\ \bigwedge x \in [\xi-\alpha,\xi+\alpha]\big(|G(x)| \le b + |\eta|\big) \text{ and } (2)\ \bigwedge x,y \in [\xi-\alpha,\xi+\alpha]\big(|G(x)-G(y)| \le M_F \cdot |x-y|\big).$$

Hence we can $(**)$ write in the form

$$(***)\begin{cases} \bigwedge(\xi,\eta) \in \mathbb{R}^2, a,b \in \mathbb{R}, k \in \mathbb{N}\big(a,b > \frac{1}{k+1} \to \bigwedge F \in C(R,\mathbb{R})\big(M_F > \frac{1}{k+1} \to \\ \bigvee G : [\xi-\alpha,\xi+\alpha] \to [\eta-b,\eta+b] \bigwedge x,y \in [\xi-\alpha,\xi+\alpha] \\ \big(|G(x)-G(y)| \le M_F \cdot |x-y| \wedge G(x) = \eta + \int\limits_{\xi}^{x} F(t,Gt)dt\big)\big)\big). \end{cases}$$

$(***)$ can be formalized in $\mathrm{G_2A}_i^\omega$ in the following way

$$(+)\begin{cases} \bigwedge \xi^1,\eta^1,a^1,b^1,k^0\big(a,b >_{\mathbb{R}} \frac{1}{k+1} \to \bigwedge F^1\big(M_F >_{\mathbb{R}} \frac{1}{k+1} \to \bigvee G \le_{1(0)} \lambda k^0.M(\chi\eta b) \\ (\bigwedge x^0,y^0(0 \le_{\mathbb{Q}} x,y \le_{\mathbb{Q}} 1 \to (|Gx -_{\mathbb{R}} Gy| \le_{\mathbb{R}} 2\alpha M_F \cdot |x -_{\mathbb{Q}} y|) \wedge (\eta - b \le_{\mathbb{R}} Gx \le_{\mathbb{R}} \eta + b)) \\ \wedge \bigwedge z^1\big(G_{\xi,\alpha}(\breve{z})_{\mathbb{R}} =_{\mathbb{R}} \eta +_{\mathbb{R}} \Phi_{I_\xi^{\breve{z}}}(\lambda x^1.F(\nu^2(x,G_{\xi,\alpha}(x)_{\mathbb{R}}))_{\mathbb{R}^2})\big)\big)\big)\big),^{46} \end{cases}$$

where $\chi\eta b =_0 \lceil(|\eta| \widehat{+_{\mathbb{R}}} b)(1)\rceil + 1$[47], $Mk := \lambda n^0.j(6k(n+1), 3(n+1)-1)$ and $G_{\xi,\alpha}(x^1)_{\mathbb{R}} := G\big(\frac{x -_{\mathbb{R}} (\xi -_{\mathbb{R}} \alpha)}{2\alpha}\big)_{\mathbb{R}}$ and $\breve{z}^1 := \max_{\mathbb{R}}(\xi-\alpha, \min_{\mathbb{R}}(z,\xi+\alpha))$.

The fact that $(+)$ expresses $(***)$ in $\mathrm{G_2A}_i^\omega$ follows from our representation of $\mathbb{R}, \mathbb{R}^2, R, C(R,\mathbb{R}), \int_\xi^x$ and the following observations: Let $\mathcal{G}$ be a function $[\xi-\alpha,\xi+\alpha] \to [\eta-b,\eta+b]$ which is Lipschitz continuous with constant M_F. Then $\tilde{\mathcal{G}}x := \mathcal{G}((\xi-\alpha)(1-x)+(\xi+\alpha)x)$ is a function : $[0,1] \to [\eta-b,\eta+b]$ which is Lipschitz continuous with constant $2\alpha \cdot M_F$. Because of this continuity $\tilde{\mathcal{G}}$ is already determined by its restriction on $[0,1] \cap \mathbb{Q}$. Such a function is represented in $\mathrm{G_2A}_i^\omega$ by a function $G^{1(0)}$. Since $|\tilde{\mathcal{G}}(x)| \le |\eta| + b$ we may assume that $Gx^0 \le_1 M(\chi\eta b)$ (By our representation of $[-k,k](\subset \mathbb{R})$ there is a construction $x^1 \mapsto \tilde{x} \le_1 Mk$ such that $-k \le_{\mathbb{R}} \tilde{x} \le_{\mathbb{R}} k$ for all x^1 and $\tilde{x} =_{\mathbb{R}} x$ if $-k \le_{\mathbb{R}} x \le_{\mathbb{R}} k$.[48] Thus we can achieve that $Gx^0 \le_1 M(\chi\eta b)$ simply by switching to $\widetilde{Gx}$. Now $\lambda z^1.G_{\xi,\alpha}(z)_{\mathbb{R}}$ just represents the original function $\mathcal{G}$. In the other direction one only has to observe that any $G^{1(0)}$ such that

$$\bigwedge x^0,y^0(0 \le_{\mathbb{Q}} x,y \le_{\mathbb{Q}} 1 \to (|Gx -_{\mathbb{R}} Gy| \le_{\mathbb{R}} 2\alpha M_F \cdot |x -_{\mathbb{Q}} y|) \wedge (\eta - b \le_{\mathbb{R}} Gx \le_{\mathbb{R}} \eta + b))$$

represents a function $[0,1] \to [\eta-b,\eta+b]$ with Lipschitz constant $2\alpha \cdot M_F$ and hence that $G_{\xi,\alpha}$ represents a continuous function : $[\xi-\alpha,\xi+\alpha] \to [\eta-b,\eta+b]$.

[46] For notational simplicity we omit here the modulus of uniform continuity for $\lambda x.F(\nu^2(x,G_{\xi,\alpha}(x)_{\mathbb{R}}))$ which can be easily computed from the moduli of F, G.

[47] $\lceil x^0 \rceil$ denotes the least natural number which is an upper bound for the rational number coded by x (One easily shows that $\lceil\cdot\rceil \in \mathrm{G_2R}^\omega$). Hence $\chi\eta b$ is a natural number which is an upper bound of the real number (represented by) $|\eta| +_{\mathbb{R}} b$.

[48] Here we simply write k as representative of the natural number k in $\mathbb{R}$.

It is clear that (+) has (modulo a shift of the existential quantifiers hidden in $>_{\mathbb{R}}$ into the front of the implication) the form $\bigwedge \underline{x} \bigvee G \leq_{1(0)} s\underline{x} \bigwedge \underline{z} A_0$, where $\underline{x}, \underline{z}$ are tuples of variables whose types are ≤ 1 and $s \in \mathrm{G}_2\mathrm{R}^\omega$.

Example 4: Brouwer's fixed point theorem

Theorem 7.1.1 (Brouwer's fixed point theorem)
Every continuous function $F : [0,1]^d \to [0,1]^d$ has at least one fixed point, i.e. there exists an $x_0 \in [0,1]^d$ such that $F(x_0) = x_0$.

For every **fixed** number d continuous functions $F : [0,1]^d \to [0,1]^d$ can be represented in $\mathrm{G}_2\mathrm{A}_i^\omega$ as d–tuples of continuous functions $F_i : [0,1]^d \to [0,1]$ and therefore as d–tuples of number–theoretic functions f_i^1. Hence Brouwer's fixed point theorem has (formalized in $\mathrm{G}_2\mathrm{A}_i^\omega$) the logical form

$$\bigwedge f_1^1, \ldots, f_d^1 \bigvee x_0 \leq_1 \nu^d(M, \ldots, M) \underbrace{\bigwedge_{i=1}^{d} \big(\widetilde{f_i(x_0)}_{\mathbb{R}^d} =_{\mathbb{R}} (\widetilde{\nu_i^d x_0})\big)}_{\in \Pi_1^0},$$

where $M := \lambda n.j(6(n+1), 3(n+1)-1)$ is the boundedness function from our representation of $[0,1]$. This generalizes to any rectangle $[a_1,b_1] \times \ldots \times [a_d,b_d]$ (with variable $a_1, b_1, \ldots, a_d, b_d$ such that $a_i < b_i$ for $i = 1, \ldots, d$) instead of $[0,1]^d$.

7.2 The axiom F and the principle of uniform boundedness

In [39] we introduced the following axiom:[49]

$$\boldsymbol{F_0} :\equiv \bigwedge \Phi^2, y^1 \bigvee y_0 \leq_1 y \bigwedge z \leq_1 y (\Phi z \leq_0 \Phi y_0).$$

F_0 states that every functional Φ^2 assumes its maximum value on the fan $\{z^1 : z \leq_1 y\}$ for each y^1. This is an indirect way of expressing that Φ is bounded on $\{z^1 : z \leq_1 y\}$:

$$\boldsymbol{B_0} :\equiv \bigwedge \Phi^2, y^1 \bigvee x^0 \bigwedge z \leq_1 y (\Phi z \leq_0 x).$$

F_0 immediately implies B_0: Put $x := \Phi y_0$. The proof of the implication '$B_0 \to F_0$' uses the least number principle and classical logic:
If x is a bound for Φz on $\{z^1 : z \leq_1 y\}$ then there exists a minimal bound x_0 and therefore a z_0 such that $z_0 \leq_1 y \wedge \Phi z_0 =_0 x_0$ (since otherwise $\sup\limits_{\{z^1 : z \leq_1 y\}} \Phi z < x_0$, contradicting the minimality of n_0).

Our motivation for expressing B_0 via F_0 is that F_0 –in contrast to B_0– has (almost) the logical form $\bigwedge x \bigvee y \leq sx \bigwedge z A_0$ of an axiom $\in \Delta$ in theorems 2.2.2,2.2.7, 2.2.8 and cor.2.2.3 . This is the case because F_0 contains instead of the unbounded quantifier '$\bigvee x^0$' only the bounded quantifier '$\bigvee y_0 \leq_1 y$' (of higher type). The reservation 'almost' refers to the fact that there is still an

[49] In [39] this axiom is denoted by F instead of F_0. In this paper we reserve the name F for a generalization of this axiom which will be introduced below.

unbounded existential quantifier in F_0 hidden in the negative occurrence of '$z \leq_1 y$'. However this quantifier can be eliminated by the use of the extensionality axiom (E). By (E), F_0 is equivalent to

$$\tilde{F}_0 :\equiv \bigwedge \Phi^2, y^1 \bigvee y_0 \leq_1 y \bigwedge z^1 \big(\Phi(\min_1(z,y)) \leq_0 \Phi y_0\big) \text{ (see lemma 7.2.7 below).}$$

This use of extensionality does not cause problems for our monotone functional interpretation since the elimination of extensionality procedure applies: Because of the type–structure of F_0 the implication '$F_0 \to (F_0)_e$' is trivial.

F_0 is not true in the full type structure $\mathcal{S}^\omega$ of all set–theoretic functionals:

Definition 7.2.1

$$\begin{cases} \mathcal{S}_0 := \omega, \\ \mathcal{S}_{\tau(\rho)} := \{\textit{all set–theoretic functions } x : \mathcal{S}_\rho \to \mathcal{S}_\tau\}, \\ \mathcal{S}^\omega := \bigcup\limits_{\rho \in \mathbf{T}} \mathcal{S}_\rho, \end{cases}$$

where 'set–theoretic' is meant in the sense of ZFC.[50]

Proposition: 7.2.2 *$\mathcal{S}^\omega \not\models F_0$.*

Proof: Define

$$\Phi^2 y^1 := \begin{cases} \text{the least } n \text{ such that } yn =_0 0, \text{ if it exists} \\ 0^0, \text{ otherwise.} \end{cases}$$

Φ is not bounded on $\{z^1 : z \leq_1 \lambda x^0.1^0\}$ since $\Phi(\overline{1,x}) =_0 x$, where

$$(\overline{1,x})(k) := \begin{cases} 1^0, \text{ if } k <_0 x \\ 0^0, \text{ otherwise.} \end{cases}$$

On the other hand F_0 is true in the type structure $\mathcal{M}^\omega$ of all strongly majorizable set–theoretic functionals, which was introduced in [4] :

Definition 7.2.3

$$\mathcal{M}_0 := \omega,\ x^* \textit{ s–maj}_0\ x :\equiv x^*, x \in \omega \wedge x^* \geq x;$$
$$x^* \textit{ s–maj}_{\tau(\rho)}\ x :\equiv x^*, x \in \mathcal{M}_\tau^{\mathcal{M}_\rho} \wedge \bigwedge y^*, y \in \mathcal{M}_\rho (y^* \textit{ s–maj}_\rho\ y \to x^* y^* \textit{ s–maj}_\tau\ x^* y, xy),$$
$$\mathcal{M}_{\tau(\rho)} := \Big\{x \in \mathcal{M}_\tau^{\mathcal{M}_\rho} : \bigvee x^* \in \mathcal{M}_\tau^{\mathcal{M}_\rho} (x^* \textit{ s–maj}_{\tau(\rho)}\ x)\Big\};$$
$$\mathcal{M}^\omega := \bigcup_{\rho \in \mathbf{T}} \mathcal{M}_\rho$$

(Here $\mathcal{M}_\tau^{\mathcal{M}_\rho}$ denotes the set of all set–theoretic functions: $\mathcal{M}_\rho \to \mathcal{M}_\tau$).

Proposition: 7.2.4 *$\mathcal{M}^\omega \models F_0$.*

[50] The following proposition also holds if we omit the axiom of choice since only comprehension is used for the refutation of F_0.

Proof: It suffices to show that $\mathcal{M}^\omega \models B_0$: $\Phi \in \mathcal{M}_2$ implies the existence of a functional $\Phi^* \in \mathcal{M}_2$ such that Φ^* s–maj$_2$ Φ. Hence $\Phi^* y^M \geq_0 \Phi z$ for all y^1, z^1 such that $y \geq_1 z$ $(y^M x^0 := \max\limits_{i \leq x}(yi))$.

For our applications in this paper we also need a strengthening F of F_0, which generalizes F_0 to sequences of functionals and still holds in $\mathcal{M}^\omega$:

Definition 7.2.5

$\boldsymbol{F} :\equiv \bigwedge \Phi^{2(0)}, y^{1(0)} \bigvee y_0 \leq_{1(0)} y \bigwedge k^0 \bigwedge z \leq_1 yk \big(\Phi kz \leq_0 \Phi k(y_0 k)\big)$.

F implies the existence of a sequence of bounds for a sequence $\Phi^{2(0)}$ of type–2–functionals on a sequence of fan's:

Proposition: 7.2.6

$G_1A_i^\omega \vdash F \to \bigwedge \Phi^{2(0)}, y^{1(0)} \bigvee \chi^1 \bigwedge k^0 \bigwedge z \leq_1 yk(\Phi kz \leq_0 \chi k)$.

Proof: Put $\chi k := \Phi(y_0 k)k$ for y_0 from F.

Similarly to F_0 also F can be transformed into a sentence $\tilde{F}$ having the logical form $\bigwedge x \bigvee y \leq sx \bigwedge z\ A_0$:

Lemma: 7.2.7

$E\text{–}G_1A_i^\omega \vdash F \leftrightarrow \widetilde{\boldsymbol{F}} :\equiv \bigwedge \Phi^{2(0)}, y^{1(0)} \bigvee y_0 \leq_{1(0)} y \bigwedge k^0, z^1 \big(\Phi k(\min_1(z, yk)) \leq_0 \Phi k(y_0 k)\big)$.

Proof: '$\to$' is trivial. '$\leftarrow$' follows from $z \leq_1 yk \to \min_1(z, yk) =_1 z$ by the use of (E).

Because of this lemma we can treat F as an axiom $\in \Delta$ in the presence of (E). In order to apply our monotone functional interpretation we firstly have to eliminate (E) from the proof. This can be done as in cor.2.1.4 and remark 2.2.4 since $F \to (F)_e$.

Theorem 7.2.8 *Assume that $n \geq 1$. Let Δ be a set of sentences having the form $\bigwedge u^\gamma \bigvee v \leq_\delta tu \bigwedge w^\eta\ B_0$, where $t \in G_nR^\omega$ and $\gamma, \eta \leq 2$, $\delta \leq 1$ such that $\mathcal{S}^\omega \models \Delta$. Furthermore let $s \in G_nR^\omega$ and $A_0 \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula containing only x, y, z free and let $\alpha, \beta \in \mathbf{T}$ such that $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$, and $\tau \leq 2$. Then the following rule holds:*

$$\begin{cases} E\text{–}G_nA^\omega + F + \Delta + AC^{\alpha,\beta}\text{–}qf \vdash \bigwedge x^1 \bigwedge y \leq_1 sx \bigvee z^\tau\ A_0(x, y, z) \\ \Rightarrow \text{ by elimination of (E) and monotone functional interpretation } \exists \Psi \in G_nR^\omega_{-}[\Phi_1] : \\ G_nA_i^\omega + \tilde{F} + \Delta + b\text{–}AC \vdash \bigwedge x^1 \bigwedge y \leq_1 sx \bigvee z \leq_\tau \Psi x\ A_0(x, y, z) \\ \Rightarrow \mathcal{M}^\omega, \mathcal{S}^\omega \models \bigwedge x^1 \bigwedge y \leq_1 sx \bigvee z \leq_\tau \Psi x\ A_0(x, y, z).^{51} \end{cases}$$

Ψ is built up from $0^0, 1^0, \max_\rho, \Phi_1$ and majorizing terms[52] for the terms t occurring in the quantifier axioms $\bigwedge xGx \to Gt$ and $Gt \to \bigvee xGx$ which are used in the given proof by use of λ–abstraction and substitution.
If $\tau \leq 1$ then Ψ has the form $\Psi \equiv \lambda x^1.\Psi_0 x^M$, where $x^M := \Phi_1 x$ and Ψ_0 does not contain Φ_1 (An analogous result holds for E–PRA$^\omega$,E–PA$^\omega$).

[51] Note that the conclusion holds in $\mathcal{S}^\omega$ although $\mathcal{S}^\omega \not\models \tilde{F}$.

[52] Here $t^*[\underline{a}]$ is called a majorizing term if $\lambda \underline{a}.t^*$ s–maj $\lambda \underline{a}.t$, where $\underline{a}$ are all free variables of t.

Proof: By lemma 7.2.7 and elimination of extensionality the assumption yields

$$\mathrm{G}_n\mathrm{A}^\omega + \tilde{F} + \Delta + \mathrm{AC}^{\alpha,\beta}\text{–qf} \vdash \bigwedge x^1 \bigwedge y \leq_1 sx \bigvee z^\tau \ A_0(x,y,z).$$

By thm.2.2.2 there exists a $\Psi \in \mathrm{G}_n\mathrm{R}^\omega_-[\Phi_1]$ satifying the properties of the theorem such that

$$\mathrm{G}_n\mathrm{A}^\omega_i + \tilde{F} + \Delta + \text{b–AC} \vdash \bigwedge x^1 \bigwedge y \leq_1 sx \bigvee z \leq_\tau \Psi x \ A_0(x,y,z).$$

¿From [34] and the proof of prop.7.2.4 we know that $\mathcal{M}^\omega \models \mathrm{PA}^\omega + \tilde{F}$+b–AC and therefore $\mathcal{M}^\omega \models \mathrm{G}_n\mathrm{A}^\omega + \tilde{F}$+b–AC. Note that every $\mathcal{S}^\omega$–true universal sentence $\bigwedge x^\rho A_0(x)$ with $\deg(\rho \leq 2)$ as well as every sentence from Δ is also true in $\mathcal{M}^\omega$. This follows from $\mathcal{S}_0 = \mathcal{M}_0, \mathcal{S}_1 = \mathcal{M}_1$ and $\mathcal{S}_2 \supset \mathcal{M}_2$. Hence
$\mathcal{M}^\omega \models \mathrm{G}_n\mathrm{A}^\omega + \tilde{F} + \Delta$+b–AC
and therefore

$$\mathcal{M}^\omega \models \bigwedge x^1 \bigvee y \leq_1 sx \bigvee z \leq_\tau \Psi x A_0(x,y,z).$$

Since $\tau \leq 2$ this implies

$$\mathcal{S}^\omega \models \bigwedge x^1 \bigvee y \leq_1 sx \bigvee z \leq_\tau \Psi x A_0(x,y,z).$$

Remark 7.2.9 *It is the need of the (E)–elimination that prevents us from dealing with stronger forms of F, where y_0 may be given as a functional in Φ and y, since for such a strengthened version the interpretation $(F)_e$ would not follow from F (without using (E) already). The same obstacle arises when F is generalized to higher types $\rho > 1$:*

$$F_\rho :\equiv \bigwedge \Phi^{0\rho 0}, y^{\rho 0} \bigvee y_0 \leq_{\rho 0} y \bigwedge k^0 \bigwedge z \leq_\rho yk\big(\Phi kz \leq_0 \Phi k(y_0 k)\big).$$

F_ρ, which still is true in $\mathcal{M}^\omega$, will be used in the intuitionistic context studied in chapter 8 below.

In our applications of F we actually make use of the following consequence of F+$\mathrm{AC}^{1,0}$–qf:

Definition 7.2.10 *The schema of* **uniform Σ^0_1–boundednes** *is defined as*

$$\mathbf{\Sigma^0_1\text{–UB}}: \begin{cases} \bigwedge y^{1(0)}\big(\bigwedge k^0 \bigwedge x \leq_1 yk \bigvee z^0 \ A(x,y,k,z) \\ \qquad\qquad \to \bigvee \chi^1 \bigwedge k^0 \bigwedge x \leq_1 yk \bigvee z \leq_0 \chi k \ A(x,y,k,z)\big), \end{cases}$$

where $A \equiv \bigvee \underline{l} A_0(\underline{l})$ and $\underline{l}$ is a tuple of variables of type 0 and A_0 is a quantifier–free formula (which may contain parameters of arbitrary types).

Proposition: 7.2.11 *Assume that $n \geq 2$.*
G_nA^ω+$AC^{1,0}$–qf $\vdash F \to \Sigma^0_1$–UB.

Proof: $\bigwedge k^0 \bigwedge x^1 \leq_1 yk \bigvee z^0 \ A(x,y,k,z)$ implies
$\bigwedge k^0 \bigwedge x^1 \bigvee z^0, v^0\big(xv \leq_0 ykv \to A(x,y,k,z)\big)$. Thus using the fact that k, x as well as $z, v, \underline{l}$ can be coded together in $\mathrm{G}_2\mathrm{A}^\omega$, one obtains by $\mathrm{AC}^{1,0}$–qf the existence of a functional $\Phi^{2(0)}$ such that $\bigwedge k^0 \bigwedge x \leq_1 yk \ A(x,y,k,\Phi kx)$. Proposition 7.2.6 yields
$\bigvee \chi^1 \bigwedge k^0 \bigwedge x \leq_1 yk(\chi k \geq_0 \Phi kx)$.

Remark 7.2.12 *In the proof above we have made use of classical logic for the shift of the quantifier on v as an existential quantifier in front of the implication. Nevertheless we will make use of the principle of uniform boundedness (and even generalizations of this principle) in the intuitionistic context studied in chapter 8. This is possible since instead of classical logic we could have used also (E) to derive $\bigwedge k, x \bigvee z\, A(\min_1(x, yk), y, k, z)$ and (E) does not cause any problems intuitionistically.*

Σ^0_1–UB together with classical logic implies the existence of a modulus of uniform continuity for each extensional $\Phi^{1(1)}$ on $\{z^1 : z \leq_1 y\}$ (where 'continuity' refers to the usual metric on the Baire space $\mathbb{N}^{\mathbb{N}}$):

Proposition: 7.2.13 *For $n \geq 2$ the following holds*

$$G_nA^\omega + \Sigma^0_1\text{–}UB \vdash$$
$$\bigwedge \Phi^{1(1)}, y^1\big(ext(\Phi) \rightarrow \bigvee \chi^1 \bigwedge k^0 \bigwedge z_1, z_2 \leq_1 y\big(\bigwedge_{i \leq_0 \chi k} (z_1 i =_0 z_2 i) \rightarrow \bigwedge_{j \leq_0 k} (\Phi z_1 j =_0 \Phi z_2 j)\big)\big),$$

where $ext(\Phi) :\equiv \bigwedge z^1_1, z^1_2(z_1 =_1 z_2 \rightarrow \Phi z_1 =_1 \Phi z_2)$.

Proof: $\bigwedge z_1, z_2 \leq_1 y(z_1 =_1 z_2 \rightarrow \Phi z_1 =_1 \Phi z_2)$ implies

$$\bigwedge z_1, z_2 \leq_1 y \bigwedge k^0 \bigvee n^0\big(\bigwedge_{i \leq_0 n} (z_1 i =_0 z_2 i) \rightarrow \bigwedge_{j \leq_0 k} (\Phi z_1 j =_0 \Phi z_2 j)\big).$$

By Σ^0_1–UB (using the coding of z_1, z_2 into a single variable) we conclude

$$\bigvee \chi^1 \bigwedge k^0 \bigwedge z_1, z_2 \leq_1 y\big(\bigwedge_{i \leq_0 \chi k} (z_1 i =_0 z_2 i) \rightarrow \bigwedge_{j \leq_0 k} (\Phi z_1 j =_0 \Phi z_2 j)\big).$$

Remark 7.2.14 *The weaker axiom F_0 instead of F proves Σ^0_1–UB only in a weaker version which asserts instead of the bounding function χ^1 only the existence of a bound n^0 for every k^0. This is sufficient to prove that every $\Phi^{1(1)}$ is uniformly continuous but not to show the existence of a modulus of uniform continuity.*

For many applications a weaker version F^- of F is sufficient which we will study now for the following reasons:

1) F^- has already the logical form $\bigwedge x \bigvee y \leq sx \bigwedge z A_0$ of an axiom $\in \Delta$ and needs (in contrast to F) no further transformation. This simplifies the extraction of bounds and allows the generalization to higher types (see thm.7.2.20 below).

2) F^- can be eliminated from the proof for the verification of the bound extracted in a simple purely syntactical way (see thm.7.2.20) yielding a verification in $\mathrm{G}_{\max(3,n)}\mathrm{A}^\omega_i$. In particular no relativation to $\mathcal{M}^\omega$ is needed. For F such an elimination uses much more complicated tools and gives a verification only in HA^ω and only for $\tau \leq 1$ in thm.7.2.8 (see [39]).

Definition 7.2.15
$F^- :\equiv \bigwedge \Phi^{2(0)}, y^{1(0)} \bigvee y_0 \leq_{1(0)} y \bigwedge k^0, z^1, n^0\big(\bigwedge_{i <_0 n} (zi \leq_0 yki) \rightarrow \Phi k(\overline{z, n}) \leq_0 \Phi k(y_0 k)\big)$, *where, for $z^{\rho 0}$, $(\overline{z, n})(k^0) :=_\rho zk$, if $k <_0 n$ and $:= 0^\rho$, otherwise (It is clear that $\lambda z, n.(\overline{z, n}) \in G_2R^\omega$).*

Remark 7.2.16 *Since F^- is a weakening of F (to 'finite' sequences) it is also true in $\mathcal{M}^\omega$. By the proof of prop.7.2.2 F^- does not hold in $\mathcal{S}^\omega$.*

Lemma: 7.2.17
$G_1A_i^\omega \vdash F^- \to \bigwedge \Phi^{2(0)}, y^{1(0)} \bigvee \chi^{1(0)} \bigwedge k^0, z^1, n^0 \big(\bigwedge\limits_{i<_0 n} (zi \leq_0 yki) \to \Phi k(\overline{z,n}) \leq_0 \chi k \big).$

Definition 7.2.18 *The schema Σ_1^0–UB$^-$ is defined as the following weakening of Σ_1^0–UB:*

$$\mathbf{\Sigma_1^0\text{–}UB^-} : \begin{cases} \bigwedge y^{1(0)} \big(\bigwedge k^0 \bigwedge x \leq_1 yk \bigvee z^0 \ A(x,y,k,z) \to \bigvee \chi^1 \bigwedge k^0, x^1, n^0 \\ \qquad \big(\bigwedge\limits_{i<_0 n} (xi \leq_0 yki) \to \bigvee z \leq_0 \chi k \ A((\overline{x,n}), y, k, z) \big) \big), \end{cases}$$

where $A \in \Sigma_1^0$.

Proposition: 7.2.19 *For each $n \geq 2$ we have*
$G_nA^\omega + AC^{1,0}$–qf $\vdash F^- \to \Sigma_1^0$–UB$^-$.

Proof: Analogously to the proof of prop.7.2.11 using lemma 7.2.17 instead of prop.7.2.6.

Theorem 7.2.20 *Assume $n \geq 1,\ \tau \leq 2,\ s \in G_nR^\omega$. Let $A_0(x,y,z) \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula containing only x,y,z as free variables. Then the following rule holds:*

$$\begin{cases} G_nA^\omega \oplus AC\text{–}qf \oplus F^- \vdash \bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee z^\tau \ A_0(x,y,z) \\ \Rightarrow \quad \text{by monotone functional interpretation } \exists \Psi \in G_nR^\omega_-[\Phi_1] \text{ such that} \\ G_{\max(3,n)}A_i^\omega \vdash \bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee z \leq_\tau \Psi x \ A_0(x,y,z). \end{cases}$$

Ψ is built up from $0^0, 1^0, \max_\rho, \Phi_1$ and majorizing terms for the terms t occurring in the quantifier axioms $\bigwedge xGx \to Gt$ and $Gt \to \bigvee xGx$ which are used in the given proof by use of λ–abstraction and substitution.[53]
If $\tau \leq 1$ then Ψ has the form $\Psi \equiv \lambda x^1.\Psi_0 x^M$, where $x^M := \Phi_1 x$ and Ψ_0 does not contain Φ_1.
For $\rho \leq 1$, $G_nA^\omega \oplus AC$–qf$\oplus F^-$ can be replaced by E–$G_nA^\omega + AC^{\alpha,\beta}$–qf$+F^-$, where α, β are as in thm.7.2.8 . A remark analogous to 2.2.4 applies. Furthermore on may add axioms Δ (having the form as in thm. 2.2.2) to $G_nA^\omega \oplus AC$–qf$\oplus F^-$. Then the conclusion holds in $G_{\max(3,n)}A_i^\omega + \Delta + b$–AC.
An analogous result holds for PRA^ω and PA^ω with $\Psi \in \widehat{PR}^\omega$ resp. $\in T$.

Proof: The assumption implies

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC\text{–}qf} \vdash \big(\bigvee Y \leq \lambda \Phi^{2(0)}, y^{1(0)}.y \bigwedge \Phi, \tilde{y}^{1(0)}, k^0, \tilde{z}^1, n^0 \\ \big(\bigwedge\limits_{i<n} (\tilde{z}i \leq \tilde{y}ki) \to \Phi k(\overline{\tilde{z},n}) \leq_0 \Phi k(Y\Phi\tilde{y}k) \big) \to \bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee z^\tau A_0(x,y,z) \big),$$

and therefore

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC\text{–}qf} \vdash \bigwedge Y \leq \lambda \Phi, y.y \bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee \Phi, \tilde{y}, k, \tilde{z}, n, z(\ldots).$$

[53]Here $\oplus$ means that F^- and AC–qf must not be used in the proof of the premise of an application of the quantifier–free rule of extensionality QF–ER. $\mathrm{G}_n\mathrm{A}^\omega$ satisfies the deduction theorem w.r.t $\oplus$ but not w.r.t $+$.

By theorem 2.2.2 and a remark on it we can extract $\Psi_1, \Psi_2 \in \mathrm{G}_n\mathrm{R}^\omega_-[\Phi_1]$ such that

$$\mathrm{G}_n\mathrm{A}^\omega_i \vdash \bigwedge Y \leq \lambda\Phi, y.y \bigwedge x^1 \bigwedge y \leq_\rho sx \bigvee \Phi, \tilde{y}, k, \tilde{z} \bigvee n \leq_0 \Psi_1 x \bigvee z \leq_\tau \Psi_2 x(\ldots).$$

Hence

$$\begin{aligned} &\mathrm{G}_n\mathrm{A}^\omega_i \vdash \bigwedge x\big(\bigvee Y \leq \lambda\Phi^{2(0)}, y^{1(0)}.y \bigwedge \Phi, \tilde{y}^{1(0)}, k^0, \tilde{z}^1 \bigwedge n \leq_0 \Psi_1 x \\ &\quad\big(\bigwedge_{i<n} (\tilde{z}i \leq \tilde{y}ki) \to \Phi k(\overline{\tilde{z}, n}) \leq \Phi k(Y\Phi\tilde{y}k)\big) \to \bigwedge y \leq_\rho sx \bigvee z \leq_\tau \Psi_2 x A_0(x, y, z)\big). \end{aligned}$$

It remains to show that

$$\begin{aligned} &\mathrm{G}_3\mathrm{A}^\omega_i \vdash \bigwedge n_0 \bigvee Y \leq \lambda\Phi^{2(0)}, y^{1(0)}.y \bigwedge \Phi, \tilde{y}^{1(0)}, k^0, \tilde{z}^1 \bigwedge n \leq_0 n_0 \\ &\qquad\qquad\big(\bigwedge_{i<n} (\tilde{z}i \leq \tilde{y}ki) \to \Phi k(\overline{\tilde{z}, n}) \leq \Phi k(Y\Phi\tilde{y}k)\big): \end{aligned}$$

Define

$$\tilde{Y} := \lambda\Phi, \tilde{y}, k, n_0. \max_{j \leq_0 (\overline{\tilde{y}k})n_0} \Phi k\big(\overline{(\min_1(\lambda i.(j)_i, \tilde{y}k), n_0}\big).$$

One easily shows (using the fact that $\Phi_{\langle\cdot\rangle} \in \mathrm{G}_3\mathrm{R}^\omega$) that $\tilde{Y}$ is definable in $\mathrm{G}_3\mathrm{A}^\omega_i$. In the same way we can define (using μ_b)

$$\widehat{Y} := \lambda\Phi, \tilde{y}, k, n_0. \min_{j \leq_0 (\overline{\tilde{y}k})n_0} \Big[\Phi k\big(\overline{(\min_1(\lambda i.(j)_i, \tilde{y}k), n_0}\big) =_0 \tilde{Y}\Phi\tilde{y}kn_0\Big].$$

For every n_0 we now put

$$Y := \lambda\Phi, \tilde{y}, k.\big(\overline{\min_1\big(\lambda i.(\widehat{Y}\Phi\tilde{y}kn_0)_i, \tilde{y}k), n_0}\big).$$

We now show that F^- implies (relatively to $\mathrm{G}_1\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$–qf) a generalization of the binary ('weak') König's lemma WKL:

Definition 7.2.21 (Troelstra(74))
WKL:$\equiv \bigwedge f^1\big(T(f) \wedge \bigwedge x^0 \bigvee n^0(lth\ n =_0 x \wedge fn =_0 0) \to \bigvee b \leq_1 \lambda k.1 \bigwedge x^0(f(\bar{b}x) =_0 0)\big)$,
where $Tf :\equiv \bigwedge n^0, m^0(f(n * m) =_0 0 \to fn =_0 0) \wedge \bigwedge n^0, x^0(f(n * \langle x\rangle) =_0 0 \to x \leq_0 1)$ *(i.e.* $T(f)$ *asserts that* f *represents a 0,1–tree).*

In the following we generalize WKL to a sequential version WKL_{seq} which states that for every sequence of infinite 0,1–trees there exists a sequence of infinite branches:

Definition 7.2.22

$$WKL_{seq} :\equiv \begin{cases} \bigwedge f^{1(0)}\big(\bigwedge k^0(T(fk) \wedge \bigwedge x^0 \bigvee n^0(lth\ n =_0 x \wedge fkn =_0 0)) \\ \qquad\qquad \to \bigvee b \leq_{1(0)} \lambda k^0, i^0.1 \bigwedge k^0, x^0(fk((\overline{bk})x) =_0 0)\big). \end{cases}$$

This formulation of WKL and WKL_{seq} (which is used e.g. in [68] and [57],[59] and in a similar way in the system RCA_0 considered in the context of 'reverse mathematics' with set variables instead of function variables) uses the functional $\Phi_{\langle\cdot\rangle}bx = \bar{b}x$ which is definable in $\mathrm{G}_n\mathrm{A}^\omega_i$ only for $n \geq 3$ and causes exponential growth. Since we are mostly interested in polynomial growth and therefore in systems based on G_2A^ω we need a different formulation WKL^2_{seq} of WKL_{seq} which avoids the coding of finite sequences (of variable length) as numbers and can be used in G_2A^ω and is equivalent to WKL_{seq} in the presence of the functional $\Phi_{\langle\cdot\rangle}$. This is achieved by expressing trees as higher type (≥ 2) functionals which are available in our finite type theories:

Definition 7.2.23

$$WKL^2_{seq} :\equiv \begin{cases} \bigwedge \Phi^{0010}\big(\bigwedge k^0, x^0 \bigvee b \leq_1 \lambda n^0.1^0 \bigwedge\limits_{i=0}^{x} (\Phi k(\overline{b,i})i =_0 0) \\ \qquad\qquad \to \bigvee b \leq_{1(0)} \lambda k^0, n^0.1 \bigwedge k^0, x^0 (\Phi k(\overline{bk,x})x =_0 0)\big). \end{cases}$$

Proposition: 7.2.24
$G_3A^\omega_i \vdash WKL^2_{seq} \leftrightarrow WKL_{seq}$.

Proof: '$\to$': Define $\Phi k^0 b^1 x^0 := fk(\bar{b}x)$ and assume $\bigwedge k^0 T(fk)$ and (+) $\bigwedge k, x \bigvee n(lth\ n = x \wedge fkn = 0)$. It follows that

$$\bigwedge k, x \bigvee b \leq \lambda n.1 \bigwedge_{i=0}^{x} (\Phi k(\overline{b,i})i =_0 0)$$

(Put $b := \lambda i.(n)_i$ for n as in (+)).
Hence WKL^2_{seq} yields

$$\bigvee b \leq \lambda k, n.1 \bigwedge k, x(\Phi k(\overline{bk,x})x =_0 0),$$

i.e.

$$\bigvee b \leq \lambda k, n.1 \bigwedge k, x\big(fk((\overline{bk})x) =_0 0\big).$$

'$\leftarrow$': Define

$$fkn := \begin{cases} \Phi k(\lambda i.(n)_i)(lth\ n), \text{ if } \bigwedge j \leq lth\ n\big(\big(\Phi k(\overline{\lambda i.(n)_i, j})j =_0 0\big) \wedge (n)_j \leq 1\big) \\ 1^0, \quad \text{otherwise.} \end{cases}$$

The assumption $\bigwedge k, x \bigvee b \leq_1 \lambda n^0.1^0 \bigwedge\limits_{i=0}^{x} \big(\Phi k(\overline{b,i})i =_0 0\big)$ implies $\bigwedge k, x \bigvee n(lth\ n = x \wedge fkn = 0)$. Since furthermore $T(fk)$ for all k (by f–definition), WKL_{seq} yields

$$\bigvee b \leq_{1(0)} \lambda k, n.1 \bigwedge k^0, x^0 \big(fk((\overline{bk})x) =_0 0\big),$$

i.e.

$$\bigvee b \leq \lambda k, n.1 \bigwedge k, x(\Phi k(\overline{bk,x})x =_0 0).$$

Theorem 7.2.25
$G_2A^\omega + AC^{0,1}\text{–}qf \vdash \Sigma^0_1\text{–}UB^- \to WKL^2_{seq}$.

Proof: Assume that

$$\bigwedge b \leq_{1(0)} \lambda k^0, i^0.1 \bigvee k^0, x^0 \big(\Phi k(\overline{bk,x})x \neq_0 0\big).$$

By Σ^0_1–UB$^-$ it follows that (since the type 1(0) can be coded in type 1):

$$(*)\ \bigvee x_0 \bigwedge b \leq_{1(0)} \lambda k, i.1 \bigvee k, x \leq_0 x_0 \big(\Phi k \big(\underbrace{\overline{(\overline{bk, x_0}), x}}_{=_1 \overline{bk, x}} \big) x \neq_0 0 \big).$$

Assume $\bigwedge k^0, x^0 \bigvee b^1 \big(\bigwedge\limits_{i=0}^{x} (bi \leq_0 1 \wedge \Phi k(\overline{b, i}) i =_0 0) \big)$. AC^{0,1}–qf yields

$$\bigwedge x^0 \bigvee b^{1(0)} \bigwedge k^0 \big(\bigwedge_{i=0}^{x} (bki \leq_0 1 \wedge \Phi k(\overline{bk, i}) i =_0 0) \big)$$

Since $\overline{bk, i} =_1 \overline{(\overline{bk, x}), i}$ for $i \leq x$ and $\overline{bk, x} \leq_1 \lambda i.1$ if $\bigwedge\limits_{i=0}^{x} (bki \leq_0 1)$ this implies

$$\bigwedge x^0 \bigvee b \leq_{1(0)} \lambda k, i.1 \bigwedge k \bigwedge_{i=0}^{x} \big(\Phi k(\overline{bk, i}) i = 0 \big),$$

which contradicts $(*)$.

Together with prop.7.2.19 this theorem implies the following

Corollary 7.2.26 *Let $n \geq 2$. Then*

$$G_n A^\omega \oplus AC^{1,0}\text{–}qf \oplus AC^{0,1}\text{–}qf \oplus F^- \vdash WKL^2_{seq}.$$

Hence theorem 7.2.8 and theorem 7.2.20 capture proofs using WKL^2_{seq}. In particular (combined with cor.2.2.3) we have the following rule

$$\begin{cases} E\text{–}G_2A^\omega + AC^{\alpha,\beta}\text{–}qf + WKL^2_{seq} \vdash \bigwedge x^0 \bigwedge y \leq_1 sx \bigvee z^0 A_0(x, y, z) \\ \Rightarrow \exists (eff.) k, c_1, c_2 \in \mathbb{N} \text{ such that} \\ G_3A^\omega_i \vdash \bigwedge x^0 \bigwedge y \leq_1 sx \bigvee z \leq_0 c_1 x^k + c_2 \ A_0(x, y, z), \end{cases}$$

where $s \in G_2R^\omega$ and A_0 is a quantifier–free formula of G_2A^ω which contains only x, y, z as free variables and $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$.

Remark 7.2.27 *WKL^2_{seq} does not imply F^- since $\mathcal{S}^\omega \models WKL^2_{seq}$, but $\mathcal{S}^\omega \not\models F^-$.*

7.3 Applications of F+AC^{1,0} (resp. F^-+AC^{1,0}) relatively to G$_2$A$^\omega$

Application 1:

Proposition: 7.3.1 *For every* **fixed** *number d the following holds:*

1) *G_2A^ω+$AC^{1,0}$ + F proves:*
   Every function $F : [0, 1]^d \to \mathbb{R}$ is uniformly continuous and possesses a modulus of uniform continuity.

2) *G_2A^ω+$AC^{1,0}$ + F^- proves:*
   Every pointwise continuous function $F : [0, 1]^d \to \mathbb{R}$ is uniformly continuous and possesses a modulus of uniform continuity.

Proof: 1) Formulated in G_2A^ω the assertion above reads as follows:
If $\Phi^{1(1)}$ (note that we do not need the complicated representation of $C([0,1]^d,\mathbb{R})$ from chapter 3) represents a function $[0,1]^d \to \mathbb{R}$, i.e.
$\bigwedge x_1^1, x_2^1 (\bigwedge_{i=1}^{d} (0 \leq_{\mathbb{R}} \nu_i^d(x_1), \nu_i^d(x_2) \leq_{\mathbb{R}} 1 \wedge \nu_i^d(x_1) =_{\mathbb{R}} \nu_i^d(x_2)) \to \Phi x_1 =_{\mathbb{R}} \Phi_2)$, then Φ is uniformly continuous on $[0,1]^d$ and possesses a modulus of uniform continuity.
By the representation of $[0,1]$ from chapter 3 we can restrict ourselves to representatives x^1 of elements of $[0,1]^d$ which satisfy $\nu_i^d(x) \leq_1 M$ for $i=1,\ldots,d$ (where $M := \lambda n.j(6(n+1), 3(n+1)-1)$).

$$\bigwedge x_1, x_2 \leq_1 \nu^d(M,\ldots,M)(\bigwedge_{i=1}^{d} (\widetilde{\nu_i^d(x_1)} =_{\mathbb{R}} \widetilde{\nu_i^d(x_2)}) \to \Phi\tilde{x}_1 =_{\mathbb{R}} \Phi\tilde{x}_2)$$

is equivalent to[54]

$$\bigwedge x_1, x_2 \leq_1 \nu^d(M,\ldots,M)\bigwedge k^0 \bigvee n^0 \underbrace{\big(\|\tilde{x}_1 -_{\mathbb{R}^d} \tilde{x}_2\|_{\max} \leq_{\mathbb{R}} \frac{1}{n+1} \to |\Phi\tilde{x}_1 -_{\mathbb{R}} \Phi\tilde{x}_2|_{\mathbb{R}} <_{\mathbb{R}} \frac{1}{k+1} \big)}_{\equiv: A \in \Sigma_1^0},$$

where $\|\cdot\|_{\max}$ denotes the maximum metric[55] on $\mathbb{R}^d$.
Since x_1, x_2 can be coded together, Σ_1^0–UB (which is derivable by prop.7.2.11) yields (using the monotonicity of A w.r.t. n)

$$\bigvee \chi^1 \bigwedge x_1, x_2 \leq_1 \nu^d(M,\ldots,M) \bigwedge k^0 \big(\|\tilde{x}_1 -_{\mathbb{R}^d} \tilde{x}_2\|_{\max} \leq \frac{1}{\chi k+1} \to |\Phi\tilde{x}_1 -_{\mathbb{R}} \Phi\tilde{x}_2|_{\mathbb{R}} < \frac{1}{k+1} \big).$$

2) Using Σ_1^0–UB$^-$ instead of Σ_1^0–UB in the proof of 1) one obtains

$$\bigvee \chi^1 \bigwedge x_1, x_2 \leq_1 \nu^d(M,\ldots,M) \bigwedge l^0, k^0 \big(\|\widetilde{(\overline{x_1,l})} -_{\mathbb{R}^d} \widetilde{(\overline{x_2,l})}\|_{\max} \leq \frac{1}{\chi k+1} \to |\Phi\widetilde{(\overline{x_1,l})} -_{\mathbb{R}} \Phi\widetilde{(\overline{x_2,l})}|_{\mathbb{R}} < \frac{1}{k+1} \big).$$

Since $\|\widetilde{(\overline{x,l})} -_{\mathbb{R}^d} \tilde{x}\|_{\max} \leq \frac{2}{k+1}$ for $l > 3(k+1)$, this together with the pointwise continuity of Φ implies the claim .

This result generalizes also to variable rectangles $[a_1,b_1] \times \ldots \times [a_d,b_d]$ instead of $[0,1]^d$ (where $a_i < b_i$ for $i = 1,\ldots,d$).

Remark 7.3.2 (to the proof prop.7.3.1) *In the proof above we actually used only Σ_1^0–UB (resp. Σ_1^0–UB$^-$) and classical logic (more precisely Markov's principle).*

Corollary 7.3.3 *G_2A^ω+$AC^{1,0}$–qf+F proves: Every $\Phi^{1(1)}$ which represents an unrestricted function $\mathbb{R}^d \to \mathbb{R}$ is pointwise continuous on $\mathbb{R}^d$ and possesses a modulus of pointwise continuity operation.*

---

[54]Here $\tilde{x}$ is a shortage for $\nu^d(\widetilde{\nu_1^d(x_1)},\ldots,\widetilde{\nu_d^d(x_d)})$.

[55]Instead of $\|\cdot\|_{\max}$ we can also use e.g. the euclidean metric on $\mathbb{R}^d$ thereby obtaining a modulus of continuity w.r.t. this metric. However, since both norms on $\mathbb{R}^d$ are contructively equivalent, a modulus of uniform continuity w.r.t. one norm can be easily transformed into a modulus for the other norm.

Proof: ¿From the proof of 1) above we obtain a function $\chi^{1(0)}$ such that $\chi(m)$ is a modulus of uniform continuity for Φ on $[-m,m]^d$ by applying Σ^0_1–UB to

$$\bigwedge m^0 \bigwedge x_1, x_2 \leq_1 \nu^d(M(m),\ldots,M(m)) \bigwedge k^0 \bigvee n^0 \big(\underbrace{\|x_1 -_{\mathbb{R}^d} x_2\|_{\max} \leq \frac{1}{n+1} \to |\Phi x_1 -_{\mathbb{R}} \Phi x_2|_{\mathbb{R}} < \frac{1}{k+1}}_{\in \Sigma^0_1} \big),$$

where $M(m) := \lambda n.j(6m(n+1), 3(n+1)-1)$ is the boundedness function from our representation of $[-m,m]$.

Now define $\xi^{0(1)} \in \mathrm{G_2R}^\omega$ by $\xi(x^1) := \max_0(\lceil(\widehat{\nu^d_1(x)})(1)\rceil + 2, \ldots, \lceil(\widehat{\nu^d_d(x)})(1)\rceil + 2)$. The natural number $\xi(x^1)$ is an upper bound for $\|x^1\|_{\max} + 1$. Hence $\omega x^1 := \lambda k^0.\chi(\xi(x),k)$ is a modulus of pointwise continuity in x, since $\|x-y\|_{\max} \leq \frac{1}{\omega(x,k)+1}$ implies that $\|x\|_{\max}, \|y\|_{\max} \leq \xi(x)$.

Remark 7.3.4 *The modulus of pointwise continuity $\omega(x^1,k^0)$ is only an operation (see chapter 3) and not a function of x as an element of $\mathbb{R}^d$ (but a function of $x \in \mathbb{N}^{\mathbb{N}}$ as an representative of such an element) since it is not extensional w.r.t. $=_{\mathbb{R}^d}$.*

Application 2: Sequential form of the Heine–Borel covering property of $[0,1]^d$ and other compact spaces

Let $B_\varepsilon(x_0) := \{y \in \mathbb{R}^d : \|x_0 - y\|_E < \varepsilon\}$ denote the open ball with center $x_0 \in \mathbb{R}^d$ and radius ε (w.r.t. the euclidean norm).

Proposition: 7.3.5 *$G_2A^\omega_i$ + Σ^0_1–UB^- (and therefore G_2A^ω + F^-+$AC^{1,0}$–qf) proves that every sequence of open balls which cover $[0,1]^d$ contains a finite subcover.*

Proof: We have to show

$$(1)\ \bigwedge f : \mathbb{N} \to \mathbb{R}_+ \setminus \{0\} \bigwedge g : \mathbb{N} \to [0,1]^d \big(\bigwedge x \in [0,1]^d \bigvee k \in \mathbb{N}(x \in B_{fk}(gk)) \\ \to \bigvee k_0 \bigwedge x \in [0,1]^d \bigvee k \leq k_0 (x \in B_{fk}(gk))\big).$$

When formalized in $\mathrm{G_2A}^\omega_i$ (1) has the form (compare application 1 above)

$$(2)\quad \bigwedge f^{1(0)}, g^{1(0)} \big(\bigwedge l^0 (fl >_{\mathbb{R}} 0) \wedge \bigwedge x \leq_1 \nu^d(M,\ldots,M) \bigvee k^0 (\|\tilde{x} -_{\mathbb{R}^d} gk\|_E <_{\mathbb{R}} fk) \\ \to \bigvee k^0_0 \bigwedge x \leq_1 \nu^d(M,\ldots,M) \bigvee k \leq_0 k_0 (\|\tilde{x} -_{\mathbb{R}^d} gk\|_E <_{\mathbb{R}} fk)\big).$$

Using Σ^0_1–UB$^-$ and the fact that $<_{\mathbb{R}} \in \Sigma^0_1$ we obtain

$$(3)\quad \bigwedge f^{1(0)}, g^{1(0)} \big(\bigwedge l^0 (fl >_{\mathbb{R}} 0) \wedge \bigwedge x \leq_1 \nu^d(M,\ldots,M) \bigvee k^0 (\|\tilde{x} -_{\mathbb{R}^d} gk\|_E <_{\mathbb{R}} fk) \\ \to \bigvee k^0_0 \bigwedge x \leq_1 \nu^d(M,\ldots,M) \bigwedge n^0 \bigvee k, m \leq_0 k_0 (\|(\widetilde{x,n}) -_{\mathbb{R}^d} gk\|_E <_{\mathbb{R}} fk - \tfrac{1}{m+1})\big).$$

Since $\|(\widetilde{x,n}) -_{\mathbb{R}^d} \tilde{x}\|_{\max} \leq \frac{2}{k+1}$ for $n > 3(k+1)$, (3) implies (2) which concludes the proof.

Similarly one shows this result for $[a_1,b_1] \times \ldots \times [a_d,b_d]$ and also for other compact spaces as e.g. $K_{c,\lambda} := \{f \in C[0,1] : \|f\|_\infty \leq c \wedge f$ has Lipschitz constant $\lambda\}$: We have already verified in our treatment of the Cauchy–Peano existence theorem in example 3 above that $K_{c,\lambda}$ can be represented by a bounded set of functions f^1 so that Σ^0_1–UB applies.

Application 3: Attainment of the maximum value for $f \in C([0,1]^d, \mathbb{R})$

Proposition: 7.3.6 *1) $G_2A^\omega + \Sigma^0_1$–UB+$AC^{0,0}$–qf (and therefore $G_2A^\omega + F$+$AC^{1,0}$–qf) proves: Every function $F : [0,1]^d \to \mathbb{R}$ attains it maximum value on $[0,1]^d$.*

2) $G_2A^\omega + \Sigma^0_1$–UB$^-$+$AC^{0,0}$–qf (and therefore $G_2A^\omega + F^-$+$AC^{1,0}$–qf) proves: Every pointwise continuous function $F : [0,1]^d \to \mathbb{R}$ attains it maximum value on $[0,1]^d$.

Proof: In view of prop.7.3.1 and the remark to its proof we only have to show 2). Assume

$$(1)\ \bigvee\Phi : [0,1]^d \to \mathbb{R} \bigwedge x \in [0,1]^d \bigvee r \in [0,1]^d \cap \mathbb{Q}^d (\Phi x < \Phi r)\big).$$

The proposition $\bigwedge x \in [0,1]^d \bigvee r \in [0,1]^d \cap \mathbb{Q}^d (\Phi x < \Phi r)$ has the following logical form

$$(2)\ \bigwedge x \leq_1 \nu^d(M,\ldots,M) \underbrace{\bigvee n^0(\Phi\tilde{x} <_{\mathbb{R}} \Phi(\lambda k^0.q(n)))}_{\in\Sigma^0_1},$$

where $q \in \mathrm{G_2R}^\omega$ is an enumeration of $[0,1]^d \cap \mathbb{Q}^d$.
Σ^0_1–UB$^-$ applied to (2) yields

$$\bigvee n_0 \bigwedge x \leq_1 \nu^d(M,\ldots,M) \bigwedge l^0 \bigvee n \leq_0 n_0(\Phi(\widetilde{\overline{x,l}}) <_{\mathbb{R}} \Phi(\lambda k^0.q(n))).$$

By lemma 3.3.7 2) there exists an $n_1 \leq n_0$ be such that

$$\Phi(\lambda k^0.q(n_1)) =_{\mathbb{R}} \max_{\mathbb{R}}(\Phi(\lambda k^0.q(0)),\ldots,\Phi(\lambda k^0.q(n_0))).$$

Since there exist x^1, l^0 such that $x \leq_1 \nu^d(M,\ldots,M)$ and $(\widetilde{\overline{x,l}}) =_{\mathbb{R}} \lambda k^0.q(n_1)$ we obtain a contradiction to (1). Hence

$$(3)\ \bigwedge\Phi : [0,1]^d \to \mathbb{R} \bigvee x \in [0,1]^d \bigwedge r \in [0,1]^d \cap \mathbb{Q}^d (\Phi x \geq \Phi r)\big),$$

which implies

$$\bigwedge\Phi : [0,1]^d \to \mathbb{R}(\Phi \text{ pointwise continuous } \to \bigvee x \in [0,1]^d \bigwedge y \in [0,1]^d (\Phi x \geq \Phi y)\big).$$

Application 4: Dini's theorem

Proposition: 7.3.7 *1) $G_2A^\omega_i + \Sigma^0_1$–UB (and therefore $G_2A^\omega + F$+ $AC^{1,0}$–qf) proves: Every sequence Φ_n of functions : $[0,1]^d \to \mathbb{R}$ which increases pointwise to a function $\Phi : [0,1]^d \to \mathbb{R}$ converges uniformly on $[0,1]^d$ to Φ, and there exists a modulus of uniform convergence.*

2) $G_2A^\omega_i + \Sigma^0_1$–UB$^-$ (and therefore $G_2A^\omega + F^-$+$AC^{1,0}$–qf) proves: Every sequence Φ_n of pointwise continuous functions : $[0,1]^d \to \mathbb{R}$ which increases pointwise to a pointwise continuous function $\Phi : [0,1]^d \to \mathbb{R}$ converges uniformly on $[0,1]^d$ to Φ, and there exists a modulus of uniform convergence.

Proof: By the assumption we have

$$\bigwedge k^0 \bigwedge x \in [0,1]^d \bigvee n^0 (\Phi x - \Phi_n x <_{\mathbb{R}} \frac{1}{k+1}).$$

Similarly to the proof of prop.7.3.6 one obtains using Σ_1^0–UB

$$\bigvee \chi^1 \bigwedge k^0 \bigwedge x \in [0,1]^d \bigvee n \leq_0 \chi(k) (\Phi x - \Phi_n x <_{\mathbb{R}} \frac{1}{k+1}).$$

Since $(\Phi_n)_{n \in \mathbb{N}}$ is increasing this implies

$$(*) \ \bigvee \chi^1 \bigwedge k^0 \bigwedge x \in [0,1]^d \bigwedge n \geq_0 \chi(k) (\Phi x - \Phi_n x <_{\mathbb{R}} \frac{1}{k+1}),$$

which concludes the proof of 1).
By Σ_1^0–UB$^-$ we obtain $(*)$ only for a dense subset of $[0,1]^d$. However this implies $(*)$ if Φ_n, Φ are assumed to be pointwise continuous on $[0,1]^d$.

Application 5: Existence of the inverse function of a strictly monotone function

Proposition: 7.3.8 *1) $G_2A^\omega + \Sigma_1^0$–UB (and therefore $G_2A^\omega + F$+$AC^{1,0}$–qf) proves: Every strictly increasing function $\Phi : [0,1] \to \mathbb{R}$ possesses a strictly increasing inverse function $\Phi^{-1} : [\Phi 0, \Phi 1] \to [0,1]$ which is uniformly continuous on $[\Phi 0, \Phi 1]$ and has a modulus of uniform continuity.*

2) $G_2A^\omega + \Sigma_1^0$–UB$^-$ (and therefore $G_2A^\omega + F^-$+$AC^{1,0}$–qf) proves: Every strictly increasing pointwise continuous function $\Phi : [0,1] \to \mathbb{R}$ possesses a strictly increasing inverse function $\Phi^{-1} : [\Phi 0, \Phi 1] \to [0,1]$ which is uniformly continuous on $[\Phi 0, \Phi 1]$ and has a modulus of uniform continuity.

Proof: The strict monotonicity of Φ implies

$$(1) \ \bigwedge x, y \in [0,1] \bigwedge k^0 \bigvee n^0 (x \geq y + \frac{1}{k+1} \to \Phi x > \Phi y + \frac{1}{n+1}).$$

Modulo our representation of $[0,1]$, Φ and $\geq_{\mathbb{R}}, >_{\mathbb{R}}$ (1) has the logical form

$$\bigwedge x, y \leq_1 M \bigwedge k^0 \bigvee n^0 (\underbrace{\tilde{x} \geq_{\mathbb{R}} \tilde{y} +_{\mathbb{R}} \frac{1}{k+1} \to \Phi\tilde{x} >_{\mathbb{R}} \Phi\tilde{y} + \frac{1}{n+1}}_{\equiv: A \in \Sigma_1^0}).$$

By Σ_1^0–UB we obtain (using the monotonicity of A w.r.t. n) a modulus of uniform strict monotonicity, i.e.

$$(2) \ \bigvee \chi^1 \bigwedge x, y \leq_1 M \bigwedge k^0 (\tilde{x} \geq_{\mathbb{R}} \tilde{y} +_{\mathbb{R}} \frac{1}{k+1} \to \Phi\tilde{x} >_{\mathbb{R}} \Phi\tilde{y} + \frac{1}{\chi k+1})$$

(If we use Σ_1^0–UB$^-$ only instead of Σ_1^0–UB we obtain the restriction of (2) to a dense subset of $[0,1]$ which implies (2) if Φ is assumed to be pointwise continuous).
Analogously to our definition of the inverse functions of $\sin, \cos$ in chapter 5 (where we used the modulus ω of uniform strict monotonicity) one now shows the existence of the inverse function Φ^{-1} and the fact that χ is a modulus of uniform continuity for Φ^{-1} on $[\Phi 0, \Phi 1]$. That Φ^{-1} again is strictly increasing is clear.

Remark 7.3.9 *¿From the proofs of applications 1)–5) it is clear that the propositions can be proved already in* $G_2A^\omega \oplus F \oplus AC^{1,0}$*–qf (resp.* $G_2A^\omega \oplus F^- \oplus AC^{1,0}$*–qf) instead of* $G_2A^\omega + F + AC^{1,0}$*–qf* ($G_2A^\omega + F^- + AC^{1,0}$*–qf).*

The applications 1–5 show that F^- combined with $\mathrm{AC}^{1,0}$–qf allows to give very short proofs for important theorems in analysis. In these proofs one can treat continuous functions $\Phi : [0,1]^d \to \mathbb{R}$ simply as functionals of type 1(1) (which are $=_{[0,1]^d}, =_{\mathbb{R}}$–extensional) without the need of the quite complicated representation of $C([0,1]^d, \mathbb{R})$ from chapter 3.[56] Moreover the applications 1–4 generalize to other compact spaces K instead of $[0,1]^d$ as long as the elements of K can be represented by $\{f^1 : f \leq_1 t\}$ for a suitable term t.

Since the formulation of the examples 1–4 uses only terms which are majorizable in $\mathrm{G}_2\mathrm{A}^\omega$ and the applications 1–5 (for continuous functions) can be carried out in $\mathrm{G}_n\mathrm{A}^\omega + F^- + \mathrm{AC}^{1,0}$–qf for all $n \geq 2$ we can conclude (using the results obtained so far):
If a sentence $(+) \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\rho t\underline{uk} \bigvee w^0 A_0$ is proved in $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf plus the analytical tools developed in chapters 3–6 plus

1) Attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$ on $[0,1]^d$
2) Mean value theorem of integration
3) The mean value theorem of differentiation
4) Cauchy–Peano existence theorem for ordinary differential equations
5) Brouwer's fixed point theorem for continuous functions $f : [0,1]^d \to [0,1]^d$
6) Every pointwise continuous function $G : [0,1]^d \to \mathbb{R}$ is uniformly continuous on $[0,1]^d$ and possesses a modulus of uniform continuity
7) $[0,1]^d \subset \mathbb{R}^d$ has the (sequential form of the) Heine–Borel covering property
8) Dini's theorem: Every sequence G_n of pointwise continuous functions $:[0,1]^d \to \mathbb{R}$ which increases pointwise to a pointwise continuous function $G : [0,1]^d \to \mathbb{R}$ converges uniformly on $[0,1]^d$ to G and there exists a modulus of uniform convergence
9) Every strictly increasing pointwise continuous function $G : [0,1] \to \mathbb{R}$ possesses a uniformly continuous strictly increasing inverse function $G^{-1} : [G0, G1] \to [0,1]$ together with a modulus of uniform continuity

as lemmas one can extract a uniform bound $\bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\rho t\underline{uk} \bigvee w \leq_0 \chi\underline{uk}\, A_0$ such that

(i) χ is a polynomial in $\underline{u}^M, \underline{k}$ (where $u_i^M := \lambda x^0.\max_0(u0, \ldots, ux)$) for which prop. 1.2.30 applies, if $n = 2$,

(ii) χ is elementary recursive in $\underline{u}^M, \underline{k}$, if $n = 3$.

[56]Because of this, application 3 is usefull although this theorem can be treated directly as an axiom when one uses our representation of $C([0,1]^d, \mathbb{R})$.

8 Relative constructivity

In the previous chapters we studied various analytical principles in the context of theories G_nA^ω+AC–qf (mainly for $n = 2$) whose underlying logic is the usual classical logic and applied the meta–theorems from chapter 2 to determine the growth of provably recursive functionals. As we already have discussed at the end of chapter 2, the use of classical logic has the consequence that the extractability of an effective (and for $n = 2$ polynomial) bound from a proof of an $\bigwedge\bigvee A$–sentence is (in general) guaranteed only if A is quantifier–free. In this chapter we study proofs which may use **non–constructive analytical principles** as e.g. Brouwer's fixed point theorem, Cauchy–Peano existence theorem, attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$ and so on, but apply these principles only **in the context of the intuitionistic theories (E)–$\mathbf{G}_n\mathbf{A}^\omega_i$**. The restriction to intuitionistic logic guarantees the extractability of (uniform) effective bounds ($\in G_nR^\omega$) for arbitrary $\bigwedge\bigvee A$–sentences. Furthermore instead of analytical axioms Δ having the form $\bigwedge x^\delta \bigvee y \leq_\rho sx \bigwedge z^\tau A_0(x,y,z)$ we may use more general sentences as axioms, e.g. arbitrary sentences having the form $(*)\ \bigwedge x^\delta(A \to \bigvee y \leq_\rho sx \neg B)$, where A, B are arbitrary formulas (such that $(*)$ is closed). The methods by which such extractions are achieved are monotone versions of the so–called 'modified realizability' interpretations mr and mrt. Modified realizability was introduced in [41] and is studied in great detail in [67] and [69] (to which we refer).[57] In [67],[69] these interpretations are developed for theories like E-HA$^\omega$. However both interpretations immediately apply also to our theories E–$G_nA^\omega_i$:
The interpretation of the logical part can be carried out using only $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}, \overline{sg}, 0^0$ and definition by cases which is available in E–$G_nA^\omega_i$. The non–logical axioms can be expressed (using μ_b and $\min(x,y) = 0 \leftrightarrow x = 0 \vee y = 0$) as purely universal sentences (without $\vee$) which are trivially interpreted (with the empty tuple of realizing terms).

Whereas the usual modified realizability interpretation extracts tuples of closed terms $\underline{t} = t_1, \ldots, t_k$ such that $\underline{t}\ mr\ A$ (where A is a closed formula, the types of t_i and the length k of the tuple depends only on the logical form of A, and '$\underline{x}\ mr\ A$' (in words '$\underline{x}$ (modified) realizes A') is a formula defined by induction on A) we are interested in majorants of such realizing terms, i.e. $t_1^*, \ldots, t_k^*$ such that

$$(+)\ \bigvee x_1, \ldots, x_k \bigwedge_{i=1}^{k} (t_i^* \text{ s–maj } x_i \wedge \underline{x}\ mr\ A).$$

By saying that '$\underline{t}^*$ fulfils the monotone mr–interpretation of A' we simply mean that '$\underline{t}^*$ fulfils (+)' (analogously for the 'modified realizability with truth' variant mrt of mr).[58] For E–$G_nA^\omega_i$ such terms $\underline{t}^*$ can be obtained by applying at first the usual mr–interpretation and subsequent construction of majorants for the resulting terms by proposition 1.2.21. As in the case of functional interpretation it is also possible to extract such majorizing terms directly from a given proof (i.e. without extracting $\underline{t}$ at first). However the simplification achieved in this way is not as significant as for the functional interpretation since no decision of prime formulas is needed for the mr–interpretation (in contrast to usual functional interpretation, where this is avoided only by our monotone variant) and it will be therefore not studied further.
The monotone mr–interpretation is closed under deduction as the usual mr–interpretation. Hence in order to treat the extension of E–$G_nA^\omega_i$ by new axioms, we only have to consider what terms

[57] In [70] 'mrt' is denoted by 'mq'.
[58] This variant has the property that $\underline{x}\ mrt\ A$ implies A; see [70], [69] for information on this.

are needed to fulfil their monotone mr–interpretation (and what principles are necessary to verify them). We will show that for an axiom $(*)$ any majorant s^* for s satisfies its monotone mr–interpretation (provably in E–$\mathrm{G}_n\mathrm{A}^\omega_i$ + $(*)$+b-AC), whereas such axioms in general do not have a usual mr-interpretation by computable functionals at all. So sentences $(*)$ contribute to extractable bounds only by majorants for the terms occuring in their formulation but not by their proofs. That is why we conceive them as axioms (if they are true in $\mathcal{S}^\omega$ or –as F– in $\mathcal{M}^\omega$).

Definition 8.1 ([67]) *The independence–of–premise schema $IP_\neg$ for negated formulas is defined as*[59]

$$IP_\neg \ : \ (\neg A \to \bigvee y^\rho B) \to \bigvee y^\rho(\neg A \to B),$$

where y is not free in A.

Notational convention 8.2 *In the theorems of this chapter we consider* **always closed formulas**, *i.e. e.g. in the theorem below A, B, C resp. D contain (at most) x, (x,y), (u,v) resp. (u,v,w) as free variables.*

Theorem 8.3 *Let s,t be $\in$ G_nR^ω, $A,B,C,D \in \mathcal{L}(E\text{–}G_nA^\omega_i)$. Then the following holds:*

$$\begin{cases} E\text{–}G_nA^\omega_i + \bigwedge x^\delta(A \to \bigvee y \le_\rho sx\neg B)(+AC+IP_\neg) \vdash \bigwedge u^1 \bigwedge v \le_\gamma tu(\neg C \to \bigvee w^2 D) \\ \Rightarrow \exists \text{ (eff.) } \Psi \in G_nR^\omega_{\_}[\Phi_1] \text{ such that} \\ E\text{–}G_nA^\omega_i + \bigvee Y \le_{\rho\delta} s\bigwedge x(A \to \neg B(x,Yx))(+AC+IP_\neg) \vdash \bigwedge u^1\bigwedge v \le_\gamma tu\bigvee w \le_2 \Psi u(\neg C \to D) \\ \Rightarrow E\text{–}G_nA^\omega + b\text{-}AC + \bigwedge x^\delta(A \to \bigvee y \le_\rho sx\neg B)(+AC) \vdash \bigwedge u^1\bigwedge v \le_\gamma tu\bigvee w \le_2 \Psi u(\neg C \to D). \end{cases}$$

An analogous result holds for $E\text{–}PRA^\omega_i$,$\widehat{PR}^\omega$ and $E\text{–}PA^\omega_i$, T instead of $E\text{–}G_nA^\omega_i$, $G_nR^\omega_{\_}[\Phi_1]$.

Proof: By intuitionistic logic one shows

$$\bigvee Y \neg\neg\big(Y \le s \wedge \bigwedge x(A \to \neg B(x,Yx))\big) \leftrightarrow \bigvee Y\big(Y \le s \wedge \bigwedge x(A \to \neg B(x,Yx))\big)$$

and

$$\bigvee Y\big(Y \le s \wedge \bigwedge x(A \to \neg B(x,Yx))\big) \to \bigwedge x(A \to \bigvee y \le sx \neg B(x,y)).$$

Hence the assumption gives

$$\mathrm{E–G}_n\mathrm{A}^\omega_i + \bigvee Y \neg\neg(Y \le s \wedge \bigwedge x(A \to \neg B(x,Yx)))(+\mathrm{AC}+\mathrm{IP}_\neg) \vdash \bigwedge u^1 \bigwedge v \le_\gamma tu(\neg C \to \bigvee w D).$$

By prop.1.2.21 we can construct a term $s^* \in \mathrm{G}_n\mathrm{R}^\omega_{\_}$ such that E–$\mathrm{G}_n\mathrm{A}^\omega_i \vdash s^*$ s–maj s.
$\mathcal{T}$:=E–$\mathrm{G}_n\mathrm{A}^\omega_i + \bigvee Y \le s \bigwedge x(A \to \neg B(x,Yx))$ proves

$$(+) \ \bigvee u\big(s^* \text{ s–maj } u \wedge u \ mrt\big(\bigvee \tilde{Y} \neg\neg(\tilde{Y} \le s \wedge \bigwedge x(A \to \neg B(x,\tilde{Y}x)))\big)\big) :$$

By the definition of mrt and the easy fact that $(\underline{x} \ mrt \ \neg F) \leftrightarrow \neg F$ (and $\underline{x}$ is the empty sequence) for negated formulas one shows

$$u \ mrt \ \big(\bigvee \tilde{Y} \neg\neg(\tilde{Y} \le s \wedge \bigwedge x(A \to \neg B(x,\tilde{Y}x)))\big) \leftrightarrow \neg\neg\big(u \le s \wedge \bigwedge x(A \to \neg B(x,ux))\big).$$

[59]In [67] $\mathrm{IP}_\neg$ is denoted by IP^ω.

(+) now follows by taking $u := Y$ since s^* s–maj $s \wedge s \geq Y$ implies s^* s–maj Y (see lemma 1.2.11). Thus $\mathcal{T}$(+AC+IP$_\neg$) has a monotone mrt–interpretation in itself by terms $\in$G$_n$R$^\omega_{\underline{\ }}$. In particular (by the assumption) one can extract $\underline{\Psi} = \Psi_1, \ldots, \Psi_k \in$ G$_n$R$^\omega_{\underline{\ }}$ such that[60]

$$\mathcal{T}(\text{+AC+IP}_\neg) \vdash \bigvee \underline{\chi}\big(\underline{\Psi} \text{ s–maj } \underline{\chi} \wedge \underline{\chi}\ mrt\ \big(\bigwedge u \bigwedge v \leq tu(\neg C \to \bigvee w^2 D(w))\big)\big).$$

Let $t^* \in$G$_n$R$^\omega_{\underline{\ }}$ be such that E–G$_n$A$^\omega_i \vdash t^*$ s–maj t.
The following implications hold in E–G$_n$A$^\omega_i$:

$$\begin{aligned}
&\underline{\chi}\ mrt\ \big(\bigwedge u \bigwedge v \leq tu(\neg C \to \bigvee w^2 D(w))\big) \ \to \\
&\bigwedge u \bigwedge v(v \leq tu \wedge \neg C \to \chi_2 uv \ldots \chi_k uv\ mrt\ D(\chi_1 uv)) \ \to \ (\text{because } \underline{x}\ mrt\ D \to D) \\
&\bigwedge u, v(v \leq tu \wedge \neg C \to D(\chi_1 uv)) \overset{\Psi_1 \text{ s–maj } \chi_1}{\to} \\
&\bigwedge u \bigwedge v \leq tu\big(\underbrace{\lambda y^1.\Psi_1 u^M(t^* u^M) y^M}_{\Psi u :=} \geq_2 \chi_1 uv \wedge (\neg C \to D(\chi_1 uv))\big) \ \to \\
&\bigwedge u \bigwedge v \leq tu \bigvee w \leq_2 \Psi u(\neg C \to D(w)).
\end{aligned}$$

It remains to show that

$$\text{E–G}_n\text{A}^\omega \text{ +(b-AC) } \vdash \bigwedge x(A \to \bigvee y \leq sx \neg B) \to \bigvee Y \leq s \bigwedge x(A \to \neg B(x, Yx)) :$$

$$\begin{aligned}
\bigwedge x(A \to \bigvee y \leq sx \neg B) &\overset{(E)}{\to} \bigwedge x(A \to \bigvee y \neg B(x, \min\nolimits_\rho(y, sx))) \\
&\overset{\text{class.logic}}{\to} \bigwedge x \bigvee y(A \to \neg B(\min\nolimits_\rho(y, sx))) \\
&\to \bigwedge x \bigvee y \leq sx(A \to \neg B(x, y)) \\
&\overset{\text{(b–AC)}}{\to} \bigvee Y \leq s \bigwedge x(A \to \neg B(x, Yx)).
\end{aligned}$$

Corollary 8.4 (to the proof) *1) If $A \equiv \neg \tilde{A}$ is a negated formula, then the conclusion can be proved in E–$G_nA^\omega_i$+b-AC+$\bigwedge x(A \to \bigvee y \leq sx \neg B)$+IP$_\neg$(+AC).*

2) If the variable x is not present (i.e. if we only have closed axioms $A \to \bigvee y \leq s \neg B(y)$, then the conclusion can be proved without b-AC.

3) Instead of a single axiom $\bigwedge x(A \to \bigvee y \leq sx \neg B)$ we may also use a finite set of such axioms.

Definition 8.5 ([67]) *A formula $A \in \mathcal{L}(E\text{–}G_nA^\omega_i)$ is called $\bigvee$–free (or 'negative') if A is built up from quantifier–free formulas by means of $\wedge, \to, \neg, \bigwedge$ (i.e. A does not contain $\bigvee$ and contains $\vee$ only within quantifier–free subformulas[61]).*

Definition 8.6 ([67]) *The subset Γ_1 of formulas $\in \mathcal{L}(E\text{–}G_nA^\omega_i)$ is defined inductively by*

1) Quantifier–free formulas are in Γ_1.

[60]Here $\underline{\Psi}$ s-maj $\underline{\chi}$ means $\bigwedge_{i=1}^{k} (\Psi_i \text{ s–maj } \chi_i)$.

[61]Troelstra distinguishes between negative formulas which are built up from the double negation $\neg\neg P$ of prime formulas (instead of the arbitrary quantifier–free formulas in our definition) and $\bigvee$–free formulas where P instead of $\neg\neg P$ may be used. Since our theories have only decidable prime formulas both notions coincide with our definition.

2) $A, B \in \Gamma_1 \Rightarrow A \wedge B, A \vee B, \bigwedge x\, A, \bigvee x\, A \in \Gamma_1$.

3) If A *is* $\bigvee$*–free and* $B \in \Gamma_1$*, then* $(\bigvee \underline{x} A \to B) \in \Gamma_1$.

Definition 8.7 ([67]) *The independence–of–premise schema for* $\bigvee$*–free formulas is defined as*

$$IP_{\vee f}\ :\ (A \to \bigvee y^\rho\ B) \to \bigvee y^\rho (A \to B),$$

where A *is* $\bigvee$*–free and does not contain* y *as a free variable.*

Theorem 8.8 *Let* A, D *be* $\in \Gamma_1$ *and* B, C *denote* $\bigvee$*–free formulas;* $s, t \in G_n R^\omega$*. Then the following rule holds*

$$\begin{cases} E\text{–}G_nA_i^\omega + \bigwedge x^\delta (A \to \bigvee y \leq_\rho sx\ B) + AC\text{+}IP_{\vee f} \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu(C \to \bigvee w^2 D(w)) \\ \Rightarrow \exists \text{ (eff.) } \Psi \in G_n R^\omega_-[\Phi_1] \text{ such that} \\ E\text{–}G_nA_i^\omega + \bigvee Y \leq_{\rho\delta} s \bigwedge x(A \to B(x, Yx)) \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w \leq_2 \Psi u(C \to D(w)) \\ \Rightarrow E\text{–}G_nA^\omega + b\text{-}AC + \bigwedge x^\delta (A \to \bigvee y \leq_\rho sx\ B) \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu \bigvee w \leq_2 \Psi u(C \to D(w)). \end{cases}$$

An analogous result holds for $E\text{–}PRA_i^\omega, \widehat{PR}^\omega$ *and* $E\text{–}PA_i^\omega$*,* T *instead of* $E\text{–}G_nA_i^\omega$*,* $G_nR^\omega_-[\Phi_1]$*.*

Proof: Since quantifier–free formulas can be transformed into formulas $t\underline{x} =_0 0$, we may assume that the $\bigvee$–free formulas B, C do not contain $\vee$. The assumption of the theorem implies

$$(*)\ \mathcal{T} := \text{E–G}_n\text{A}_i^\omega + \bigvee Y \leq s \bigwedge x^\delta (A \to B(x, Yx)) + \text{AC+IP}_{\vee f} \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu(C \to \bigvee w^2 D(w)).$$

We now show that $\mathcal{T}$ has a monotone mr–interpretation in $\mathcal{T}^- := \mathcal{T} \setminus \{\text{AC,IP}_{\vee f}\}$ by terms $\in \text{G}_n\text{R}^\omega_-$. For $\text{E–G}_n\text{A}_i^\omega + \text{AC+IP}_{\vee f}$ this follows from the proof of the fact that $\text{E–HA}^\omega + \text{AC+IP}_{\vee f}$ has a mr–interpretation in E–HA^ω (see [69]) combined with our remarks in the introduction of this chapter and prop.1.2.21 (The mr–interpretation of $\text{AC+IP}_{\vee f}$ requires only terms built up from Π, Σ). Next we show that

$$\mathcal{T}^- \vdash \bigvee u \big(s^* \text{ s–maj } u \wedge u\ mr\ (\bigvee Y \leq s \bigwedge x(A \to B(x, Yx)))\big):$$

Since for $\bigvee$–free formulas $(\underline{x}\ mr\ B) \equiv B$ ($\underline{x}$ being the empty sequence) the mr–definition yields

$$u\ mr\ \big(\bigvee Y \leq s \bigwedge x(A \to B(x, Yx))\big) \leftrightarrow u \leq s \wedge \bigwedge x\big(\bigvee \underline{v}(\underline{v}\ mr\ A) \to B(x, ux)\big).$$

The right side of this equivalence is fulfilled by taking $u := Y$ since $\bigvee \underline{v}(\underline{v}\ mr\ A) \to A$ (because of the assumption $A \in \Gamma_1$). Hence $\mathcal{T}$ has a monotone mr–interpretation in $\mathcal{T}^-$ by terms $\in \text{G}_n\text{R}^\omega_-$. Therefore $(*)$ implies the extractability of terms $\underline{\Psi} = \Psi_1, \ldots, \Psi_k \in \text{G}_n\text{R}^\omega_-$ such that

$$\bigvee \underline{\chi}\big(\underline{\Psi} \text{ s–maj } \underline{\chi} \wedge \underline{\chi}\ mr\ (\bigwedge u \bigwedge v \leq tu(C \to \bigvee w D(w)))\big).$$

The following chain of implications holds in $\text{E–G}_n\text{A}_i^\omega$:

$$\begin{aligned} &\underline{\chi}\ mr\ \big(\bigwedge u \bigwedge v \leq tu(C \to \bigvee w\, D(w))\big) \overset{C\ \vee\text{–free}}{\to} \\ &\bigwedge u, v(v \leq tu \wedge C \to \chi_2 uv \ldots \chi_k uv\ mr\ D(\chi_1 uv)) \overset{D \in \Gamma_1}{\to} \\ &\bigwedge u, v, (v \leq tu \wedge C \to D(\chi_1 uv)) \overset{\Psi_1 \text{ s–maj } \chi_1}{\to} \\ &\bigwedge u \bigwedge v \leq tu(\lambda y^1.\Psi_1 u^M (t^* u^M) y^M \geq_2 \chi_1 uv \wedge (C \to D(\chi_1 uv)) \to \\ &\bigwedge u \bigwedge v \leq tu \bigvee w \leq_2 \Psi u(C \to D(w)), \end{aligned}$$

where $t^* \in \mathrm{G}_n\mathrm{R}^\omega_-$ such that $\mathrm{E\text{–}G}_n\mathrm{A}^\omega_i \vdash t^*$ s–maj t and
$\Psi := \lambda u, y.\Psi_1 u^M (t^* u^M) y^M \in \mathrm{G}_n\mathrm{R}^\omega_-[\Phi_1]$.
As in the proof of the previous theorem one shows

$$\mathrm{E\text{–}G}_n\mathrm{A}^\omega\text{+(b-AC)} \vdash \bigwedge x(A \to \bigvee y \le sx\, B) \to \bigvee Y \le s \bigwedge x(A \to B).$$

Corollary 8.9 (to the proof) *1) If $A \equiv \neg\tilde{A}$ is a negated (resp. $\bigvee$–free) formula, then the conclusion can be proved in E–$G_nA^\omega_i$+$IP_\neg$ + (b-AC) + $\bigwedge x(A \to \bigvee y \le sx\, B)$ (resp. E–$G_nA^\omega_i$+$IP_{\vee f}$ + (b-AC) + $\bigwedge x(A \to \bigvee y \le sx\, B)$).*

2) If the variable x is not present, i.e. if only axioms $A \to \bigvee y \le sx\, B(y)$ are used ($A \in \Gamma_1, B$ $\bigvee$–free), then the conclusion can be proved without b-AC.

3) Instead of a single axiom $\bigwedge x(A \to \bigvee y \le sx\, B(y))$ we may also use a finite set of such axioms.

Remark 8.10 *For every $\bigvee$–free formula A of our theories the equivalence $A \leftrightarrow \neg\neg A$ holds intuitionistically (since the prime formulas are stable). So the allowed axioms in thm.8.3 include the axioms allowed in thm.8.8.*

Although theorem 8.8 is weaker than theorem 8.3 in **some** respects (e.g. A, D have to be in Γ_1) it is of interest for the following reason:
Despite the fact that the schema AC of full choice may be used in the proof of the assumption, the proof of the conclusion uses only b-AC instead of AC. This has the consequence that the conclusion is valid in the model $\mathcal{M}^\omega$, if $\bigwedge x(A \to \bigvee y \le sx\, B)$ holds in $\mathcal{M}^\omega$ (although $\mathcal{M}^\omega \not\models$AC, see [34]). Let us e.g. consider the theory $\mathrm{E\text{–}G}_n\mathrm{A}^\omega_i + F$+AC, where F is the axiom studied in chapter 7 §2. Since F has the form $\bigwedge x(A \to \bigvee y \le sx\, B)$ (with $A(:\equiv 0 = 0) \in \Gamma_1$ and B $\bigvee$–free) of an allowed axiom in thm.8.8 (and a fortiori in thm.8.3) we can apply thm. 8.8 and obtain the following rule

$$\begin{cases} \mathrm{E\text{–}G}_n\mathrm{A}^\omega_i + F\text{+AC} \vdash \bigwedge u^1 \bigwedge v \le_1 tu(C \to \bigvee w^2 D(w)) \\ \Rightarrow \exists \text{ (eff.) } \Psi \in \mathrm{G}_n\mathrm{R}^\omega_-[\Phi_1] \text{ such that} \\ \mathrm{E\text{–}G}_n\mathrm{A}^\omega_i + F\text{+(b-AC)} \vdash \bigwedge u^1 \bigwedge v \le_1 tu \bigvee w \le_2 \Psi u(C \to D(w)). \end{cases}$$

The conlusion of this rule implies (see the proof of thm.7.2.8)

$$\mathcal{M}^\omega \models \bigwedge u^1 \bigwedge v \le_1 tu \bigvee w \le_2 \Psi u(C \to D(w)).$$

If all positively occuring $\bigwedge x^\rho$–quantifiers and all negatively occuring $\bigvee x^\rho$–quantifiers in this formula have types $\rho \le 1$ and if all other quantifiers have types ≤ 2, then we can conclude (since $\mathcal{M}_1 = \mathcal{S}_1$ and $\mathcal{M}_2 \subset \mathcal{S}_2$)

$$\mathcal{S}^\omega \models \bigwedge u^1 \bigwedge v \le_1 tu \bigvee w \le_2 \Psi u(C \to D(w)).$$

Hence the bound Ψ is classically valid although it has been extracted from a proof in a theory which classically is inconsistent:

Claim: $\mathrm{E\text{–}G}_n\mathrm{A}^\omega + F\text{+AC} \vdash 0 = 1$.

Proof of the claim: Consider

$$\bigwedge f \leq_1 \lambda x.1 \bigvee n^0 (\bigvee k^0 (fk = 0) \to fn = 0),$$

which holds by classical logic. AC yields the existence of a functional $\Psi^{0(1)}$ such that

$$\bigwedge f \leq_1 \lambda x.1 (\bigvee k^0 (fk = 0) \to f(\Psi f) = 0).$$

F applied to Ψ implies (prop.7.2.6)

$$\bigvee n_0 \bigwedge f \leq_1 \lambda x.1 \bigvee n \leq_0 n_0 (\bigvee k^0 (fk = 0) \to fn = 0),$$

which –of course– is wrong.

The (intuitionistically consistent) combination of F and AC (instead of AC–qf only, which we have used in the classical setting of chapter 7) can be used to prove strengthened versions of various classical theorems which may have non–constructive counterexamples, but no constructive ones. These proofs rely on the fact that F and AC prove a very general principle of uniform boundedness for **arbitrary formulas**:

Proposition: 8.11

$$E\text{–}G_nA_i^\omega + F + \ AC \ \vdash$$
$$\bigwedge y^{1(0)} \big(\bigwedge k^0 \bigwedge x \leq_1 yk \bigvee z^0 A(x,y,k,z) \to \bigvee \chi^1 \bigwedge k^0 \bigwedge x \leq_1 yk \bigvee z \leq_0 \chi k \, A(x,y,k,z) \big),$$

where A is an arbitrary formula of $\mathcal{L}(E\text{–}G_nA^\omega)$.

Proof: Similarly to the proof of prop.7.2.11 using remark 7.2.12.

Example 1: Pointwise convergence implies uniform convergence or 'Dini's theorem without monotonicity assumption'[62]

$$\text{E–G}_2\text{A}_i^\omega + F\text{+AC} \ \vdash \bigwedge \Phi_n, \Phi : [0,1]^d \to \mathbb{R} (\Phi_n \text{ converges pointwise to } \Phi \to$$
$$\Phi_n \text{ converges uniformly on } [0,1]^d \text{ to } \Phi \text{ and there exists a modulus of convergence}).$$

Proof: By the assumption we have

$$\bigwedge k^0 \bigwedge x \in [0,1]^d \bigvee n^0 \bigwedge l \geq_0 n \big(|\Phi x - \Phi_l x| \leq \frac{1}{k+1} \big).$$

By prop.8.11 and the fact that '$\bigwedge x \in [0,1]^d$' has the form '$\bigwedge x \leq_1 M$' in our representation of $[0,1]^d$ one obtains

$$\bigvee \chi^1 \bigwedge k^0 \bigwedge x \in [0,1]^d \bigvee n \leq_0 \chi k \bigwedge l \geq_0 n \big(|\Phi x - \Phi_l x| \leq \frac{1}{k+1} \big)$$

and therefore

$$\bigvee \chi^1 \bigwedge k^0 \bigwedge x \in [0,1]^d \bigwedge l \geq_0 \chi k \big(|\Phi x - \Phi_l x| \leq \frac{1}{k+1} \big).$$

[62] This principle has been studied in [2] in a purely intuitionistic context, i.e. without our (in general non–constructive) axioms $\bigwedge x(A \to \bigvee y \leq sx \neg B), \bigwedge x(C \to \bigvee y \leq sx \, D)$.

Remark 8.12 *1) The usual counterexamples to the theorem above do not occur in* $E\text{–}G_nA_i^\omega$ *since they use classical logic to verify the assumption of pointwise convergence: E.g. consider the well–known example* $\Phi_n(x) := \max(n - n^2|x - \frac{1}{n}|, 0)$ $(n \geq 1)$*. The proof that* Φ_n *converges pointwise to* 0 *requires the instance '*$\bigwedge x \in [0,1](x = 0 \vee x > 0)$*' of the tertium–non–datur schema, which cannot be proved in* $E\text{–}G_nA_i^\omega$*.*

2) Note that the monotonicity assumption of Dini's theorem has been used in our treatment in chapter 7 §3 just to eliminate the universal quantifier '$\bigwedge l \geq_0 n$*' which reduces the application of the general principle of uniform boundedness to an application of its restriction* Σ_1^0*–UB to* Σ_1^0*–formulas (since* $\leq$ *can be replaced by* $<$*).*

Example 2: Heine–Borel property for $[0,1]^d$ **and sequences of arbitrary (not necessarily open) balls**

$$\begin{aligned}
&\text{E–G}_2\text{A}_i^\omega\text{+AC } + F \vdash \bigwedge f : \mathbb{N} \to \mathbb{R}_+ \bigwedge g : \mathbb{N} \to [0,1]^d \bigwedge h^1 \\
&\quad \big(\bigwedge x \in [0,1]^d \bigvee k^0 \big((hk = 0 \wedge \|x - gk\|_E < fk) \vee (hk \neq 0 \wedge \|x - gk\|_E \leq fk)\big) \to \\
&\quad \bigvee k_0 \bigwedge x \in [0,1]^d \bigvee k \leq_0 k_0 \big((hk = 0 \wedge \|x - gk\|_E < fk) \vee (hk \neq 0 \wedge \|x - gk\|_E \leq fk)\big)\big).
\end{aligned}$$

Proof: Similarly to the proof of the Heine–Borel property in chapter 7 §3, but note that now Σ_1^0–UB would not suffice since there is a universal quantifier hidden in '$\|x - gk\|_E \leq fk$'.

Examples of sentences having (in $\mathbf{E\text{–}G_2A_i^\omega}$**) the form** $G \equiv \bigwedge x(A \to \bigvee y \leq sx \neg B)$ **or** $H \equiv \bigwedge x(C \to \bigvee y \leq sx\, D)$ **where** D **is** $\bigvee$**–free and** $C \in \Gamma_1$**:**

1) All sentences having the form $\bigwedge x^\delta \bigvee y \leq_\rho sx \bigwedge z^\tau A_0(x, y, z)$ are axioms G, H, in particular the examples 1)–4) from chapter 7 §1: Attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$, mean value theorem of integration, Cauchy–Peano existence theorem, Brouwer's fixed point theorem.

2) The generalization of the axiom F to arbitrary types ρ:

$$F_\rho :\equiv \bigwedge \Phi^{0\rho 0}, y^{\rho 0} \bigvee y_0 \leq_{\rho 0} y \bigwedge k^0 \bigwedge z \leq_\rho yk \big(\Phi kz \leq_0 \Phi k(y_0 k)\big)$$

has the form of an axiom H (and so a fortiori of G) since '$\bigwedge k^0 \bigwedge z \leq_\rho yk\big(\Phi kz \leq_0 \Phi k(y_0 k)\big)$' is $\bigvee$–free.

3) Our generalization WKL^2_{seq} of the binary König's lemma WKL has the form H (and therefore G) since its implicative premise '$\bigwedge k^0, x^0 \bigvee b \leq_1 \lambda n^0.1^0 \bigwedge\limits_{i=0}^{x} (\Phi k(\overline{b,i})i =_0 0)$' is in Γ_1.

4) The 'double negation shift' DNS : $\bigwedge x \neg\neg A \to \neg\neg \bigwedge x\, A$ has the form G and –if A is $\bigvee$–free– the form H.

5) The 'lesser limited principle of omniscience' is defined as:[63]

$$\mathrm{LLPO}\ :\ \bigwedge f^1 \bigvee k \leq_0 1([k=0 \to \bigwedge n(f'(2n)=0)] \wedge [k=1 \to \bigwedge n(f'(2n+1)=0)]),$$

where

$$f'n := \begin{cases} 1, \text{ if } fn = 1 \wedge \bigwedge k < n(fk \neq 1) \\ 0, \text{ otherwise.} \end{cases}$$

LLPO can be formulated also in the following equivalent form

$$\bigwedge x^1, y^1 \bigvee k \leq_0 1([k=0 \to x \leq_{\mathbb{R}} y] \wedge [k=1 \to y \leq_{\mathbb{R}} x]).$$

LLPO has the form of an axiom G, H (see [7] for a discussion of LLPO).

6) Comprehension for negated (resp. $\bigvee$–free) formulas:

$$CA^\rho_\neg :\ \bigvee \Phi \leq_{0\rho} \lambda x^\rho.1^0 \bigwedge y^\rho \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big),\ \text{where } A \text{ is arbitrary,}$$

$$CA^\rho_{\vee f} :\ \bigvee \Phi \leq_{0\rho} \lambda x^\rho.1^0 \bigwedge y^\rho \big(\Phi y =_0 0 \leftrightarrow A(y)\big),\ \text{where } A \text{ is } \bigvee\text{–free.}$$

By intuitionistic logic we have

$$\neg\neg \bigwedge y^\rho \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big) \leftrightarrow \bigwedge y^\rho \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big).$$

Hence $CA^\rho_\neg$ is (equivalent to) an axiom G.
$CA^\rho_{\vee f}$ is an axiom H since together with A also $\bigwedge y^\rho \big(\Phi y =_0 0 \leftrightarrow A(y)\big)$ is $\bigvee$–free.

Remark 8.13 *1) In order to express the examples 1)–4) from chapter 7 §1 as axioms G, H we do not have to use the quite complicated representation of $f \in C([0,1]^d, \mathbb{R})$ from chapter 3: Since an implicative premise $A \in \Gamma_1$ is now allowed (in contrast to the axioms $\in \Delta$ in the classical setting), the (purely universal) implicative assumption $(*)$ expressing that ω is a modulus of uniform continuity for f (which had to be eliminated by the constructions Ψ_1, Ψ_2 in chapter 3) does not cause any problems.*

2) WKL^2_{seq} does not have the form of an axiom $\in \Delta$ and therefore has to be derived from F and AC–qf in the classical context of chapter 7. In E–$G_nA^\omega_i$ it can be treated directly as an axiom.

3) DNS and LLPO follow of course from classical logic but are not derivable in E-$G_nA^\omega_i$.

4) F_ρ and AC prove a principle of uniform boundedness for the type ρ:

$$UB_\rho :\ \bigwedge y^{\rho 0} \big(\bigwedge k^0 \bigwedge x \leq_\rho yk \bigvee z^0 A(x,y,k,z) \to \bigvee \chi^1 \bigwedge k^0 \bigwedge x \leq_\rho yk \bigvee z \leq_0 \chi k\, A(x,y,k,z)\big).$$

[63] Usually one quantifies over all functions ≤ 1 which are $=1$ in at most one point. This is achieved by our transformation $f \mapsto f'$.

5) One easily shows that LLPO is implied by $CA^1_{\vee f}$.

6) $CA^0_{\neg}$ added to $E\text{–}G_nA^\omega_i$ yields the axiom schema of induction for arbitrary negated formulas

$$IA_{\neg}: \ \neg A(0) \wedge \bigwedge x^0(\neg A(x) \to \neg A(x+1)) \to \bigwedge x^0 \neg A(x):$$

Apply (QF–IA) to the characteristic function of $\neg A(x^0)$ which exists by $CA^0_{\neg}$.
Likewise $E\text{–}G_nA^\omega_i + CA^0_{\vee f}$ proves induction for arbitrary $\bigvee$–free formulas ($IA_{\vee f}$) . Whereas in the classical theories $E\text{–}G_nA^\omega$ the restricted schemas $IA_{\neg}$ and $IA_{\vee f}$ are equivalent to the unrestricted schema of induction, which (for $n \geq 2$) makes every $\alpha(<\varepsilon_0)$–recursive function provably recursive, $IA_{\neg}$ and $IA_{\vee f}$ do not cause any growth of provable functionals when added to the **intuitionistic** *theories $E\text{–}G_nA^\omega_i$.*

One real limitation for applications of the theorems 8.3 and 8.8 is due to the fact that the Markov principle

$$M^\omega: \ \bigwedge x(A \vee \neg A) \wedge \neg\neg \bigvee x\, A \to \bigvee x\, A$$

is not an allowed axiom, not even in its weak form

$$M_{pr}: \ \neg\neg \bigvee x^0 A_0(x) \to \bigvee x^0 A_0(x),$$

where A_0 is a quantifier–free formula.
In fact the addition of M_{pr} would make the theory $\text{E–G}_n\text{A}^\omega_i$+AC+$F$+$\text{IP}_{\neg}$ inconsistent:

$$\text{E–G}_n\text{A}^\omega_i + M_{pr} + \text{IP}_{\neg} \vdash \bigwedge f \leq_1 \lambda x.1 \bigvee k^0(\neg\neg \bigvee n(fn=0) \to fk=0).$$

Together with AC and F this gives a contradiction.

As we have discussed in [39] many $\bigwedge\bigvee$–sentences in classical analysis come from sentences

$$(1)\ \bigwedge x \in X(Fx =_{\mathbb{R}} 0 \to Gx =_{\mathbb{R}} 0)$$

by prenexation to

$$(2)\ \bigwedge x \in X \bigwedge k^0 \bigvee n^0 \big(|Fx| \leq \frac{1}{n+1} \to |Gx| < \frac{1}{k+1}\big),$$

what intuitionistically just needs M_{pr} (Here X is a complete separable metric space and $F, G: X \to \mathbb{R}$ are constructive functions).
We now prove a theorem which covers M^ω but still allows the extraction of bounds for arbitrary $\bigwedge\bigvee$–sentences. The price we have to pay for this is that the allowed axioms have to be restricted to the class Δ from the theorems in chapter 2 (and that we can use only the quantifier–free rule of extensionality instead of (E)).

Definition 8.14 ([67])

$IP^\omega_0: \ \bigwedge x(A \vee \neg A) \wedge (\bigwedge x\, A \to \bigvee y\, B) \to \bigvee y(\bigwedge x\, A \to B),$
where y is not free in A.

Theorem 8.15 *Let $s,t \in G_nR^\omega$, A_0, B_0 be quantifier–free and C be an arbitrary formula (respecting the convention made before thm.8.3). Then*

$$\begin{cases} G_nA_i^\omega\text{+}AC\text{+}IP_0^\omega + M^\omega + \bigwedge x^\delta \bigvee y \le_\rho sx \bigwedge z^\gamma A_0 \vdash \bigwedge u^1 \bigwedge v \le_\tau tu(\bigwedge a^\eta B_0 \to \bigvee w^2 C) \\ \Rightarrow \text{ by monotone functional interpretation one can extract } \Psi \in G_nR^\omega_{\_}[\Phi_1] \text{ such that} \\ G_nA_i^\omega\text{+}AC\text{+}IP_0^\omega + M^\omega + \bigwedge x^\delta \bigvee y \le_\rho sx \bigwedge z^\gamma A_0 \vdash \bigwedge u^1 \bigwedge v \le_\tau tu \bigvee w \le_2 \Psi u(\bigwedge a^\eta B_0 \to C(w)). \end{cases}$$

An analogous result holds for PRA_i^ω,$\widehat{PR}^\omega$ and PA_i^ω, T instead of $G_nA_i^\omega$, $G_nR^\omega_{\_}[\Phi_1]$.

Proof: As an abbreviation we define $\mathcal{T}$:=$\mathrm{G}_n\mathrm{A}_i^\omega$+AC+$\mathrm{IP}_0^\omega + M^\omega + \bigwedge x^\delta \bigvee y \le_\rho sx \bigwedge z^\gamma A_0$. By the assumption and IP_0^ω we obtain

$$\mathcal{T} \vdash \bigwedge u, v \bigvee w(v \le tu \wedge \bigwedge a\, B_0 \to C(w)).$$

Monotone functional interpretation extracts a term $\tilde{\Psi} \in \mathrm{G}_n\mathrm{R}^\omega_{\_}$ such that

$$\begin{aligned} \tilde{\mathcal{T}} := \mathcal{T} + \bigvee Y \le s \bigwedge x, z\, A_0(x, Yx, z) \vdash \\ \bigvee \chi\big(\tilde{\Psi} \text{ s–maj } \chi \wedge \bigwedge u \bigwedge v(v \le tu \wedge \bigwedge a\, B_0 \to C(\chi uv))^D\big). \end{aligned}$$

By [67] (3.5.10) we have $\mathcal{T} \vdash A^D \leftrightarrow A$ for all formulas A. Hence

$$\tilde{\mathcal{T}} \vdash \bigvee \chi \bigwedge u \bigwedge v \le tu\big(\underbrace{\lambda y^1.\tilde{\Psi} u^M(t^* u^M) y^M}_{\Psi u :=} \ge_2 \chi uv \wedge (\bigwedge a\, B_0 \to C(\chi uv))\big),$$

and thus

$$\tilde{\mathcal{T}} \vdash \bigwedge u \bigwedge v \le tu \bigvee w \le_2 \Psi u\big(\bigwedge a\, B_0 \to C(w)\big).$$

Since AC implies

$$\bigwedge x^\delta \bigvee y \le_\rho sx \bigwedge z^\gamma A_0 \to \bigvee Y \le_{\rho\delta} s \bigwedge x^\delta, z^\gamma A_0(x, Yx, z),$$

the proof is finished.

Let us summarize now the main consequences of the results obtained in this chapter on the growth of uniform bounds which are extractable from proofs in **classical analysis**:

If a proof of a sentence

(1) $\bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \le_\rho t\underline{u}\,\underline{k} \bigvee w^0 A$

uses in the **intuitionistic** context of E–$\mathrm{G}_n\mathrm{A}_i^\omega$+AC only the analytical tools developed in chapters 3–6 (except the equivalence between ε–δ–continuity and sequential continuity) plus the (non–constructive!) principles

1) attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$ on $[0,1]^d$
2) mean value theorem of integration

3) Cauchy–Peano existence theorem for ordinary differential equations

4) Brouwer's fixed point theorem for continuous functions : $[0,1]^d \to [0,1]^d$

5) the schema of comprehension for negated formulas $\mathrm{CA}_\neg$,

then one can extract from this proof (using thm.8.3, prop.1.2.22) a bound

$$(2)\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\rho t\underline{u}\,\underline{k} \bigvee w \leq_0 \chi \underline{u}\,\underline{k}\, A$$

such that (2) is true in the full type structure $\mathcal{S}^\omega$ and

(i) χ is a polynomial in $\underline{u}^M, \underline{k}$ (where $u_i^M := \lambda x^0. \max_0(u0, \ldots, ux)$) for which prop. 1.2.30 applies, if $n = 2$,

(ii) χ is elementary recursive in $\underline{u}^M, \underline{k}$, if $n = 3$.

The most important feature of this result is that the restriction to the intuitionistic theory $\mathrm{G}_n\mathrm{A}_i^\omega$ instead of $\mathrm{G}_n\mathrm{A}^\omega$ ensures (even relatively to the non–constructive theorems 1)–5) above) the extractability of such bounds for **arbitrary** formulas A (instead of quantifier–free ones only).

For $A \in \Gamma_1$ such that all positively occurring $\bigwedge x^\rho$ (resp. negatively occuring $\bigvee x^\rho$) in A have types ≤ 1 and all other quantifiers in A have types ≤ 2, $\rho \leq 1$ and 5) replaced by the schema of comprehension for $\bigvee$-free formulas we may use even the axiom F from chapter 7 in the proof of (1) and still obtain (using thm.8.8) a χ with the properties above. This covers proofs using the uniform continuity of every pointwise continuous function : $[0,1]^d \to \mathbb{R}$ and Dini's theorem and the (sequential) Heine–Borel property for $[0,1]^d$ (the last two principles even in strengthened versions which can be refuted in the presence of full classical logic).

9 Applications of logically complex induction in analysis and their impact on the growth of provably recursive function(al)s

By logically complex induction we mean instances of induction (or closely related schemas as bounded collection, see chapter 11 below) which go beyond quantifier–free induction QF–IA. One of the weakest induction principles which is stronger than QF–IA is the **rule** of Σ^0_1–induction:

$$\Sigma^0_1\text{–IR}\ :\ \frac{\bigvee y^0_0 A_0(0,y_0)\ ,\ \bigwedge x^0\big(\bigvee y^0_1 A_0(x,y_1)\to\bigvee y^0_2 A_0(x',y_2)\big)}{\bigwedge x^0\bigvee y^0 A_0(x,y)},$$

where A_0 is a quantifier–free formula.

Assume now that

$$\mathrm{G}_n\mathrm{A}^\omega(+\Delta+\text{AC–qf})\vdash\bigvee y^0_0 A_0(0,y_0)\wedge\bigwedge x^0\big(\bigvee y^0_1 A_0(x,y_1)\to\bigvee y^0_2 A_0(x',y_2)\big),$$

and therefore

$$\mathrm{G}_n\mathrm{A}^\omega(+\Delta+\text{AC–qf})\vdash\bigvee y^0_0 A_0(0,y_0)\wedge\bigwedge x^0,y^0_1\bigvee y^0_2\big(A_0(x,y_1)\to A_0(x',y_2)\big),$$

where Δ as in 2.2.2. By functional interpretation (or by monotone functional interpretation and bounded search) one can extract terms $s,t\in\mathrm{G}_n\mathrm{R}^\omega$ such that $\mathrm{G}_n\mathrm{A}^\omega$(+$\Delta$+b–AC) proves

$$(*)\ A_0(0,s\underline{a})\wedge\bigwedge x,y_1(A_0(x,y_1)\to A_0(x',t\underline{a}xy_1)),$$

where $\underline{a}$ are the parameters of A_0. A realizing term for the conclusion $\bigwedge x\bigvee yA_0(x,y)$ of Σ^0_1–IR is constructed by an iteration of t:

$$(**)\ \begin{cases}\tilde{t}\underline{a}0:=s\underline{a}\\ \tilde{t}\underline{a}x':=t\underline{a}x(\tilde{t}\underline{a}x).\end{cases}$$

One easily verifies (using only QF–IA) that $\bigwedge x\ A_0(x,\tilde{t}\underline{a}x)$.
In general $\tilde{t}\notin\mathrm{G}_n\mathrm{R}^\omega$: E.g. if $n\geq 2$ and $taxy:=A_n(a,y),sa:=1$ (where A_n is the function from def. 1.2.1). Then $\tilde{t}ax=A_{n+1}(a,x)$ but $A_{n+1}\notin\mathrm{G}_n\mathrm{R}^\omega$ by prop.1.2.28 and the well–known fact that $A_{n+1}\notin\mathcal{E}^n$. On the other hand we have $\tilde{t}\in\widehat{PR}^\omega$ ($\in$T) if $s,t\in\widehat{PR}^\omega$ ($s,t\in$T), since Φ_{it} can be defined in $\widehat{PR}^\omega$ and T.
If Σ^0_1–IR is restricted to formulas A which contain only number parameters $\underline{a}$ (i.e. free variables of type 0), then $\tilde{t}$ can be defined in $\mathrm{G}_{n+1}\mathrm{R}^\omega$ if $s,t\in\mathrm{G}_n\mathrm{R}^\omega$. Let Σ^0_1–IR$^-$ denote this restriction.
If the upper formulas of Σ^0_1–IR$^-$ are provable in $\mathrm{G}_n\mathrm{A}^\omega$(+$\Delta$+AC–qf) for $n\geq 2$, then the conclusion is provable in $\mathrm{G}_{n+1}\mathrm{A}^\omega$(+$\Delta$+b–AC) together with a term $\in\mathrm{G}_{n+1}\mathrm{R}^\omega$ which realizes $\bigwedge x\bigvee y\ A_0$.

We now give a (very simple) example of an application of Σ^0_1–IR$^-$ in analysis, where such a speed up of growth (in our example from $\mathrm{G}_2\mathrm{R}^\omega$ to $\mathrm{G}_3\mathrm{R}^\omega$) actually happens:

Claim:

$$\mathrm{G}_2\mathrm{A}^\omega+\Sigma^0_1\text{–IR}^-\vdash\sum_{k=1}^{\infty}\tfrac{1}{k}=\infty,\ \text{i.e.}$$
$$\mathrm{G}_2\mathrm{A}^\omega+\Sigma^0_1\text{–IR}^-\vdash\bigwedge n^0\bigvee m^0\big(s_m:=\sum_{k=1}^{m}\tfrac{1}{k}>_{\mathrm{I\!R}}n\big).$$

Proof: $n := 0$: Put $m := 1$. $n \mapsto n+1$: Assume $s_m > n$. Because of $s_{2k} - s_k \geq \frac{1}{2}$ we obtain $s_{4m} = (s_{4m} - s_{2m}) + (s_{2m} - s_m) + s_m > n+1$. Σ^0_1–IR$^-$ now yields $\bigwedge n \bigvee m(s_m >_{\mathbb{R}} n)$.

In this example s is 1 and $tm := 4m$. Hence $\tilde{t}n = 4^n$.
¿From that well–known fact that $\lim\limits_{n\to\infty}(\sum_{i=1}^n \frac{1}{i} - ln(n)) = C$ (where $C = 0,57721\ldots$ is the Euler–Mascheroni constant) it is clear that any function f which realizes (or –what is equivalent– is a bound for) $\bigwedge n \bigvee m(s_m > n)$ has to have exponential growth.

We now come back to the principle (PCM1)

$'$Every decreasing sequence $(a_n) \subset \mathbb{R}$ which is bounded from below is a Cauchy sequence$'$,

which we mentioned already in chapter 4. We show that, relatively to $\mathrm{G_2A^\omega}$, the principle (PCM1) implies the axiom of Σ^0_1–induction (and is implied by this axiom relatively to $\mathrm{G_3A^\omega}$)

$$\Sigma^0_1\text{–IA} \;:\; \bigwedge g^{000}\big(\bigvee y^0(g0y =_0 0) \wedge \bigwedge x^0\big(\bigvee y^0(gxy =_0 0) \to \bigvee y^0(gx'y =_0 0)\big) \to \bigwedge x^0 \bigvee y^0(gxy =_0 0)\big).$$

Remark 9.1 *This axiom is (relative to $G_nA^\omega_i$) equivalent to the schema of induction for all Σ^0_1–formulas in $\mathcal{L}(G_nA^\omega)$: Let $\bigvee y^0 A_0(\underline{x}, y)$ be a Σ^0_1–formula (containing only $\underline{x}$ as free variables). Then by prop. 1.2.6 there exists a term $t_{A_0} \in G_nR^\omega$ such that*

$$G_nA^\omega_i \vdash \bigwedge \underline{x}(\bigvee y^0 A_0(\underline{x}, y) \leftrightarrow \bigvee y^0(t\underline{x}y =_0 0)).$$

Proposition: 9.2 *One can construct functionals $\Psi_1, \Psi_2 \in G_2R^\omega$ such that:*
1) *G_3A^ω proves*

$$\begin{aligned}&\bigwedge a^{1(0)}\Big(\bigwedge k^0\big[\bigvee y^0(\Psi_1 ak0y =_0 0) \wedge \bigwedge x^0\big(\bigvee y^0(\Psi_1 akxy =_0 0) \to \bigvee y^0(\Psi_1 akx'y =_0 0)\big) \to \\ &\qquad \bigwedge x^0 \bigvee y^0(\Psi_1 akxy =_0 0)\big] \to \big[\bigwedge n^0(0 \leq_{\mathbb{R}} a(n+1) \leq_{\mathbb{R}} an) \\ &\qquad \to \bigwedge k^0 \bigvee n^0 \bigwedge m >_0 n\big(|am -_{\mathbb{R}} an| \leq_{\mathbb{R}} \tfrac{1}{k+1}\big)\big]\Big).\end{aligned}$$

2) *G_2A^ω proves*

$$\begin{aligned}&\bigwedge g^{000}\Big(\big[\bigwedge n^0(0 \leq_{\mathbb{R}} \Psi_2 g(n+1) \leq_{\mathbb{R}} \Psi_2 gn \leq_{\mathbb{R}} 1) \to \bigwedge k^0 \bigvee n^0 \bigwedge m \geq_0 n(|\Psi_2 gm -_{\mathbb{R}} \Psi_2 gn| \leq_{\mathbb{R}} \tfrac{1}{k+1})\big] \\ &\qquad \to \big[\bigvee y^0(g0y =_0 0) \wedge \bigwedge x^0\big(\bigvee y^0(gxy =_0 0) \to \bigvee y^0(gx'y =_0 0)\big) \to \bigwedge x^0 \bigvee y^0(gxy =_0 0)\big]\Big).\end{aligned}$$

Proof: 1) Assume that $\bigwedge n^0(0 \leq_{\mathbb{R}} a(n+1) \leq_{\mathbb{R}} an)$ and $\bigvee k \bigwedge n \bigvee m > n\big(|am -_{\mathbb{R}} an| >_{\mathbb{R}} \frac{1}{k+1}\big)$. By Σ^0_1–IA one proves that

$$(+)\; \bigwedge n^0 \bigvee i^0 \bigwedge j <_0 n\big((i)_j < (i)_{j+1} \wedge (a((i)_j) \widehat{-_{\mathbb{R}}} a((i)_{j+1}))(3(k+1)) >_{\mathbb{Q}} \frac{2}{3(k+1)}\big).$$

Let $C \in \mathbb{N}$ be such that $C \geq a_0$. For $n := 3C(k+1)$, (+) yields an $i \in \mathbb{N}$ such that

$$\bigwedge j < 3C(k+1)\big(a((i)_j) -_{\mathbb{R}} a((i)_{j+1}) >_{\mathbb{R}} \frac{1}{3(k+1)}\big).$$

Hence $a((i)_0) \geq \sum\limits_{j=0}^{3C(k+1)-1} \big(a((i)_j) -_{\mathbb{R}} a((i)_{j+1})\big) >_{\mathbb{R}} C$ which contradicts the assumption $\bigwedge n(a_n \leq C)$. Define

$$\Psi_1 akni :=_0 \begin{cases} 0, \text{ if } \bigwedge j <_0 n\big((i)_j < (i)_{j+1} \wedge (a((i)_j) \widehat{-_{\mathbb{R}}} a((i)_{j+1}))(3(k+1)) >_{\mathbb{Q}} \frac{2}{3(k+1)}\big) \\ 1, \text{otherwise.} \end{cases}$$

2) Define $\Psi_2 \in \mathrm{G}_2\mathrm{R}^\omega$ such that $\Psi_2 gn =_{\mathbb{R}} 1 -_{\mathbb{R}} \sum\limits_{i=1}^{n} \frac{\chi gni}{i(i+1)}$, where $\chi \in \mathrm{G}_2\mathrm{R}^\omega$ such that

$$\chi gni =_0 \begin{cases} 1, \text{ if } \bigvee l \leq_0 n(gil =_0 0) \\ 0, \text{ otherwise.} \end{cases}$$

¿From $\sum\limits_{i=1}^{\infty} \frac{1}{i(i+1)} = 1$ (which is provable in $\mathrm{G}_2\mathrm{A}^\omega_i$ as we have seen in chapter 4) it follows that

$$\bigwedge n^0 (0 \leq_{\mathbb{R}} \Psi_2 g(n+1) \leq_{\mathbb{R}} \Psi_2 gn \leq_{\mathbb{R}} 1).$$

By the assumption there exists an n_x for every $x > 0$ such that

$$\bigwedge m, \tilde{m} \geq n_x \big(|\Psi_2 gm -_{\mathbb{R}} \Psi_2 g\tilde{m}| < \frac{1}{x(x+1)}\big).$$

Claim: $\bigwedge \tilde{x}\big(0 < \tilde{x} \leq_0 x \to \big(\bigvee y(g\tilde{x}y = 0) \leftrightarrow \bigvee y \leq n_x(g\tilde{x}y = 0)\big)\big)$:
Assume that $\bigvee l^0(g\tilde{x}l = 0) \wedge \bigwedge l \leq n_x(g\tilde{x}l \neq 0)$ for some $\tilde{x} > 0$ with $\tilde{x} \leq x$.
Subclaim: Let l_0 be minimal such that $g\tilde{x}l_0 = 0$. Then $l_0 > n_x$ and

$$\Psi_2 g(\max(l_0, \tilde{x})) \leq_{\mathbb{R}} \Psi_2 g(\max(l_0, \tilde{x}) - 1) -_{\mathbb{R}} \frac{1}{\tilde{x}(\tilde{x}+1)}.$$

Proof of the subclaim: i) $\sum\limits_{i=1}^{\max(l_0,\tilde{x})} \frac{\chi g(\max(l_0,\tilde{x}))i}{i(i+1)}$ contains $\frac{1}{\tilde{x}(\tilde{x}+1)}$ as an element of the sum, since $g\tilde{x}l_0 = 0$ and therefore $\chi g(\max(l_0, \tilde{x}))\tilde{x} = 1$.
ii) $\sum\limits_{i=1}^{\max(l_0,\tilde{x})-1} \frac{\chi g(\max(l_0,\tilde{x})-1)i}{i(i+1)}$ does not contain $\frac{1}{\tilde{x}(\tilde{x}+1)}$ as an element of the sum:
Case 1. $\tilde{x} \geq l_0$: Then $\max(l_0, \tilde{x}) - 1 = \tilde{x} - 1 < \tilde{x}$.
Case 2. $l_0 > \tilde{x}$: Then $\max(l_0, \tilde{x}) - 1 = l_0 - 1$. Since l_0 is the minimal l such that $g\tilde{x}l = 0$, it follows that

$$\bigwedge i \leq \max(l_0, \tilde{x}) - 1(g\tilde{x}i \neq 0) \text{ and thus } \chi g(\max(l_0, \tilde{x}) - 1)\tilde{x} = 0,$$

which finishes case 2.
Because of

$$\chi g(\max(l_0, \tilde{x}) - 1)i \neq 0 \to \chi g(\max(l_0, \tilde{x}))i \neq 0$$

, i) and ii) yield

$$\sum_{i=1}^{\max(l_0,\tilde{x})} \frac{\chi g(\max(l_0, \tilde{x}))i}{i(i+1)} \geq \sum_{i=1}^{\max(l_0,\tilde{x})-1} \frac{\chi g(\max(l_0, \tilde{x}) - 1)i}{i(i+1)} + \frac{1}{\tilde{x}(\tilde{x}+1)},$$

which concludes the proof of the subclaim (Note that the assertion of the subclaim is purely universal. Hence its provability in $\mathrm{G_2A}^\omega_i$ is also clear now).
The subclaim implies

$$\max(l_0,\tilde{x})-1\geq n_x\wedge|\Psi_2 g(\max(l_0,\tilde{x}))-_{\mathbb{R}}\Psi_2 g(\max(l_0,\tilde{x})-1)|\geq\frac{1}{x(x+1)}.$$

However this contradicts the construction of n_x and therefore concludes the proof of the claim.
Assume

$$(a)\ \bigvee y_0(g0y_0=0).$$

Define $\Phi\in\mathrm{G_2R}^\omega$ such that

$$\Phi g\tilde{x}y=\begin{cases}\min\tilde{y}\leq_0 y[g\tilde{x}\tilde{y}=_0 0],\ \text{if}\ \bigvee\tilde{y}\leq_0 y(g\tilde{x}\tilde{y}=_0 0)\\ 0^0,\text{otherwise.}\end{cases}$$

By the claim above and (a) we obtain for $y:=\max(n_x,y_0)$:

$$(b)\ \bigwedge\tilde{x}\leq_0 x\big(\bigvee\tilde{y}(g\tilde{x}\tilde{y}=_0 0)\leftrightarrow g\tilde{x}(\Phi g\tilde{x}y)=_0 0\big).$$

QF–IA applied to $A_0(x):\equiv\big(gx(\Phi gxy)=_0 0\big)$ yields

$$g0(\Phi g0y)=0)\wedge\bigwedge\tilde{x}<x\big((g\tilde{x}(\Phi g\tilde{x}y)=0\to g\tilde{x}'(\Phi g\tilde{x}'y)=0\big)\to gx(\Phi gxy)=0.$$

¿From this and $(a),(b)$ we obtain

$$\bigvee y_0(g0y_0=0)\wedge\bigwedge\tilde{x}<x\big(\bigvee\tilde{y}(g\tilde{x}\tilde{y}=0)\to\bigvee\tilde{y}(g\tilde{x}'\tilde{y}=0)\big)\to\bigvee\tilde{y}(gx\tilde{y}=0).$$

Corollary 9.3
$G_3A^\omega\vdash\Sigma^0_1$*–IA* $\leftrightarrow(PCM1)$.

Remark 9.4 *1) ¿From the proof of prop.9.2 it follows that 2) is already provable in the intuitionistic theory* $G_2A^\omega_i$. *In particular*

$$G_2A^\omega_i\vdash(PCM1)\to\Sigma^0_1\text{–}IA.$$

The other implication Σ^0_1*–IA* $\to(PCM1)$ *cannot be proved intuitionistically since* $(PCM1)$ *implies the non–constructive so–called 'limited principle of omniscience' (see [45] for a discussion on this).*

2) Prop.9.2 provides much more information than cor.9.3 . In particular one can compute (in $G_2A^\omega_i$*) uniformly in* g *a decreasing sequence of positive real numbers such that the Cauchy property of this sequence implies induction for the* Σ^0_1*–formula* $A(x):\equiv\bigvee y(gxy=0)$. *The convers is not so explicit (due to the non–constructivity of this implication) but* Ψ_1 *provides an* **arithmetical family** $A_k(x):\equiv\bigvee y(\Psi_1akxy=0)$ *of* Σ^0_1*–formulas such that the induction principle for these formulas classically implies the Cauchy property of the decreasing sequence of positive reals* a.

We now determine the rate growth of uniform bounds for provably recursive functionals which may be caused by the use of (PCM1) in proofs:
Using the construction $\tilde{a}(n) := \max_{\mathbb{R}}(0, \min\limits_{i\leq n}(a(i)))$ we can express (PCM1) in the following logically more simple form[64]

$$(1)\ \bigwedge a^{1(0)} \bigwedge k^0 \bigvee n^0 \bigwedge m >_0 n\big(\tilde{a}(n) -_{\mathbb{R}} \tilde{a}(m) \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

(If $a^{1(0)}$ fulfils $\bigwedge n(0 \leq_{\mathbb{R}} a(n+1) \leq_{\mathbb{R}} a(n))$, then $\bigwedge n(\tilde{a}(n) =_{\mathbb{R}} a(n))$. Furthermore $\bigwedge n(0 \leq_{\mathbb{R}} \tilde{a}(n+1) \leq_{\mathbb{R}} \tilde{a}(n))$ for all $a^{1(0)}$. Thus by the transformation $a \mapsto \tilde{a}$, quantification over all decreasing sequences $\subset \mathbb{R}_+$ reduces to quantification over all $a^{1(0)}$).
By $\mathrm{AC}^{0,0}$–qf (1) is equivalent to

$$(2)\ \bigwedge a^{1(0)}, k^0, g^1 \bigvee n^0 \big(gn >_0 n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

We now construct a functional Ψ which provides a bound for $\bigvee n$, i.e.

$$(3)\ \bigwedge a^{1(0)}, k^0, g^1 \bigvee n \leq_0 \Psi akg\big(gn >_0 n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

Let $C(a) \in \mathbb{N}$ be an upper bound for the real number represented by $\tilde{a}(0)$, e.g. $C(a) := (a(0))(0)+1$.
We show that
$\Psi akg := \max\limits_{i<C(a)k'} \big(\Phi_{it} i0g\big)(= \max\limits_{i<C(a)k'} \big(g^i(0)\big)$ satisfies (3) (provably in PRA^ω):
Claim: $\bigvee i < C(a)k'\big(g(g^i0) > g^i0 \to \tilde{a}(g^i0) -_{\mathbb{R}} \tilde{a}(g(g^i0)) \leq_{\mathbb{R}} \frac{1}{k+1}\big)$.
Case 1: $\bigvee i < C(a)k'(g(g^i0) \leq g^i0)$: Obvious!
Case 2: $\bigwedge i < C(a)k'(g(g^i0) > g^i0)$: Assume $\bigwedge i < C(a)k'\big(\tilde{a}(g^i0) -_{\mathbb{R}} \tilde{a}(g(g^i0)) >_{\mathbb{R}} \frac{1}{k+1}\big)$.
Then $\tilde{a}(0) -_{\mathbb{R}} \tilde{a}(g^{C(a)k'}0) > C(a)$, contradicting $\tilde{a}(n) \in [0, C(a)]$.
In contrast to (2) the bounded proposition (3) has the form of an axiom Δ in thm.2.2.2 ,2.2.7 and cor.2.2.3. Hence the monotone functional interpretation of (3) requires just a majorant for Ψ. In particular we may use $\Psi \in \widehat{PR}^\omega$ itself since Ψ s–maj Ψ.
Thus from a proof of e.g. a sentence $\bigwedge x^0 \bigwedge y \leq_\rho sx \bigvee z^0 A_0(x,y,z)$ in $\mathrm{G}_n\mathrm{A}^\omega + \Delta + (PCM1)$+AC–qf we can (in general) extract only a bound t for z (i.e. $\bigwedge x \bigwedge y \leq sx \bigvee z \leq tx\ A_0(x,y,z)$) which is defined in $\widehat{PR}^\omega$ since the definition of Ψ uses the functional Φ_{it} which is not definable in $\mathrm{G}_\infty\mathrm{R}^\omega$ (see chapter 1). If however the proof uses (3) above only for functions g which can be bounded by terms in $\mathrm{G}_k\mathrm{R}^\omega$, then we can extract a $t \in \mathrm{G}_{\max(k+1,n)}\mathrm{R}^\omega$ since the iteration of a function $\in \mathrm{G}_k\mathrm{R}^\omega$ is definable in $\mathrm{G}_{k+1}\mathrm{R}^\omega$ (for $k \geq 2$).

The monotone functional interpretation of the negative translation of (1) requires (taking the quantifier hidden in $\leq_{\mathbb{R}}$ into account) a majorant for a functional Φ which bounds '$\bigvee n$' in

$$(3)'\ \bigwedge a^{1(0)}, k^0, g^1, h^1 \bigvee n\big(gn > n \to \widehat{\tilde{a}(n)}(hn) -_{\mathbb{Q}} \widehat{\tilde{a}(gn)}(hn) \leq \frac{1}{k+1} + \frac{3}{h(n)+1}\big).$$

[64]Here we use that $\bigwedge n^0\big(a(n+1) \leq_{\mathbb{R}} an\big) \to \bigwedge n^0\big(\Phi_{\min_{\mathbb{R}}}(a,n) =_{\mathbb{R}} an\big)$. This follows in $\mathrm{G}_2\mathrm{A}^\omega$ from the purely universal sentence
(+) $\bigwedge a^{1(0)}, n, k\Big(\bigwedge l < n\big(\big(\widehat{a(l+1)}\big)(k) \leq_{\mathbb{Q}} (\widehat{al})(k) + \frac{3}{k+1}\big) \to |\Phi_{\min_{\mathbb{R}}}(a,n) -_{\mathbb{R}} an| \leq_{\mathbb{R}} \frac{5n}{k+1}\Big)$. (+) is true (and hence an axiom of $\mathrm{G}_2\mathrm{A}^\omega$) since $\big(\widehat{a(l+1)}\big)(k) \leq_{\mathbb{Q}} (\widehat{al})(k) + \frac{3}{k+1} \to a(l+1) \leq_{\mathbb{R}} al + \frac{5}{k+1}$.

However every Φ which provides a bound for (2) a fortiori yields a bound for $(3)'$ (which does not depend on h). Hence Ψ satisfies (provably in PRA^ω_i) the monotone functional interpretation of the negative translation of (1).

In this chapter we have considered principles which may have a significant impact on the rate of growth of extractable bounds:
An instance of the Σ^0_1–induction **rule** (without function parameters) may increase the growth of a bound by one level in the $\mathrm{G}_n\mathrm{R}^\omega$–hierarchy (for $n \geq 2$) by a (single) iteration process. In particular if the upper formulas of this rule are derivable in $\mathrm{G}_2\mathrm{A}^\omega$ the growth of any bound for the conclusion may be exponential. This has been demonstrated using the example $\sum\limits_{n=1}^{l} \frac{1}{n} \overset{l\to\infty}{\to} \infty$.

The **axiom** of Σ^0_1–induction may contribute to the growth of bounds by the iteration functional Φ_{it}. Relatively to $\mathrm{G}_3\mathrm{A}^\omega$, Σ^0_1–IA is equivalent to the Cauchy property of bounded monotone sequences in $\mathbb{R}$ $(PCM1)$ which contributes to the growth by a term which fulfils the monotone functional interpretation of its negative translation, namely $\Psi := \lambda a^{1(0)}, k^0, g^1. \max\limits_{i<C(a)k'} \big(\Phi_{it} i 0 g\big)$, where $\mathbb{N} \ni C(a) \geq_{\mathbb{R}} a(0)$.
In the important special case where (3) above is applied only to $g := S$ in a given proof, one has $\Psi a k S \leq C(a) \cdot k'$ and the results on polynomial growth stated at the end of chapter 7 apply.
In general only the existence of a primitive recursive bound is guaranteed (this is unavoidable since Σ^0_1–IA suffices to introduce all primitive recursive functions when added to $\mathrm{G}_2\mathrm{A}^\omega$).

10 Elimination of Skolem functions of type $0(0)\ldots(0)$ in higher type theories for monotone formulas: no additional growth

There are central theorems in analysis whose proofs use arithmetical instances of AC, i.e. instances of

$$\mathrm{AC}_{ar}: \ \bigwedge x^0 \bigvee y^0 A(x,y) \to \bigvee f^1 \bigwedge x^0 A(x,fx),$$

where $A \in \Pi^0_\infty$ is not quantifier–free. Examples are the following theorems

1) Every bounded monotone sequence of real numbers has a limit (or equivalently –as we have seen in chapter 4– every bounded monotone sequence of reals has a Cauchy modulus: PCM2).

2) For every sequence of real numbers which is bounded from above there exists a least upper bound.

3) The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$ (for every fixed d).

4) The Arzelà–Ascoli lemma.

We will investigate these theorems (w.r.t. to their contribution to the rate of growth of uniform bounds extractable from proofs which use them) in chapter 11 below and discuss now only (PCM2) in order to motivate the results of the present chapter:

$$(\mathrm{PCM2}) \ : \left\{ \begin{array}{l} \bigwedge a^{1(0)}_{(\cdot)}, c^1 \big(\bigwedge n^0 (c \le_{\mathbb{R}} a_{n+1} \le_{\mathbb{R}} a_n) \\ \qquad\qquad \to \bigvee h^1 \bigwedge k^0 \bigwedge m, \tilde{m} \ge_0 hk (|a_m -_{\mathbb{R}} a_{\tilde{m}}| \le_{\mathbb{R}} \frac{1}{k+1})\big) \end{array} \right.$$

follows immediately from

$$(\mathrm{PCM1}) \ : \left\{ \begin{array}{l} \bigwedge a^{1(0)}_{(\cdot)}, c^1 \big(\bigwedge n^0 (c \le_{\mathbb{R}} a_{n+1} \le_{\mathbb{R}} a_n) \\ \qquad\qquad \to \bigwedge k^0 \bigvee n^0 \bigwedge m, \tilde{m} \ge_0 n (|a_m -_{\mathbb{R}} a_{\tilde{m}}| \le_{\mathbb{R}} \frac{1}{k+1})\big) \end{array} \right.$$

by an application of AC_{ar} to

$$A :\equiv \bigwedge m, \tilde{m} \ge n (|a_m -_{\mathbb{R}} a_{\tilde{m}}| \le_{\mathbb{R}} \frac{1}{k+1}) \in \Pi^0_1.$$

It is well–known that a constructive functional interpretation of the negative translation of AC_{ar} requires so–called bar recursion and cannot be caried out e.g. in Gödel's term calculus T (see [64] and [43]). In fact AC_{ar} is (using classically logic) equivalent to CA_{ar}+$\mathrm{AC}^{0,0}$–qf and therefore causes an immense rate of growth (when added to e.g. $\mathrm{G}_2\mathrm{A}^\omega$) as we have already discussed in chapter 3 §1. ¿From the work in the context of 'reverse mathematics' (see e.g. [61]) it is known that 1)–4) imply CA_{ar} relatively to a (second order version) of $\widehat{PA}^\omega\!\upharpoonright$+$\mathrm{AC}^{0,0}$–qf. In the next chapter we show that this holds even relative to $\mathrm{G}_2\mathrm{A}^\omega$.

In contrast to these general facts we prove in this chapter a meta–theorem which in particular

implies that if (PCM2) is applied in a proof only to sequences (a_n) which are given explicitely in the parameters of the proposition (which is proved) then this proof can be (effectively) transformed (without causing new growth) into a proof of the same conclusion which uses only (PCM1) for these sequences. By this transformation the use of AC_{ar} is eliminated and the determination of the growth caused (potentially by (PCM2)) reduces to the determination of the growth caused by (PCM1) which we have already carried out in chapter 9.
More precisely our meta–theorem has the following consequence:

Let $\mathcal{T}^\omega :=\mathrm{G}_n\mathrm{A}^\omega+\Delta$, where Δ is the set of axioms from thm.2.2.2 and cor.2.2.3. Then the following rule holds

$$(1)\left\{\begin{array}{l}\mathcal{T}^\omega + \text{AC–qf} \vdash \bigwedge u^1 \bigwedge v \leq_\rho tu\big(\bigvee h^1 \bigwedge k^0 \bigwedge m,\tilde{m} \geq_0 hk\big(|(\widetilde{\chi uv})_m -_{\mathrm{IR}} (\widetilde{\chi uv})_{\tilde{m}}| \leq \frac{1}{k+1}\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \to \bigvee w^\tau A_0(u,v,w)\big) \\ \Rightarrow \text{ there exists a } \Phi \in \mathrm{G}_n\mathrm{A}^\omega \text{ such that} \\ \mathcal{T}^\omega_i + \text{b-AC} \vdash \Big(\bigwedge u^1 \bigwedge v \leq_\rho tu\big(\bigwedge k^0 \bigvee n^0 \bigwedge m,\tilde{m} \geq_0 n\big(|(\widetilde{\chi uv})_m -_{\mathrm{IR}} (\widetilde{\chi uv})_{\tilde{m}}| \leq \frac{1}{k+1}\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \to \bigvee w^\tau A_0(u,v,w)\big)\Big) \\ \qquad\qquad \wedge\Phi \text{ fulfils the monotone funct. interpr. of the negative trans. of } \Big(\ldots\Big).\end{array}\right.$$

In contrast to (PCM2) the (negative translation of the) principle (PCM1) has a simple constructive monotone functional interpretation which is fulfilled by the functional Ψ defined at the end of chapter 9. Because of the nice behaviour of the monotone functional interpretation with respect to the modus ponens one obtains (by applying Φ to Ψ) a monotone functional interpretation of

$$\bigwedge u^1 \bigwedge v \leq_\rho tu \bigvee w^\tau A_0(u,v,w)$$

and so (if $\tau \leq 2$) a uniform bound ξ for $\bigvee w$, i.e.

$$\bigwedge u^1 \bigwedge v \leq_\rho tu \bigvee w \leq_\tau \xi u A_0(u,v,w).$$

If $\Delta=\emptyset$ then no b–AC is needed.
Let us assume now for simplicity that $\Delta=\emptyset$ and consider the following general situation:
For

$$F :\equiv \bigwedge x_1^0 \bigvee y_1^0 \ldots \bigwedge x_k^0 \bigvee y_k^0 F_0(x_1,y_1,\ldots,x_k,y_k,\underline{a}),$$

where $\underline{x},\underline{y},\underline{a}$ are all free variables of F, we define the **Skolem normal form** F^S of F by

$$F^S :\equiv \bigvee f_1,\ldots,f_k \bigwedge x_1^0,\ldots,x_k^0\ F_0(x_1,f_1x_1,\ldots,x_k,f_kx_1\ldots x_k,\underline{a}).$$

If we could prove that

$$(2)\left\{\begin{array}{l}\mathcal{T}^\omega(+\text{AC–qf}) \vdash \bigwedge u^1 \bigwedge v \leq_\rho tu\big(F^S(u,v) \to \bigvee w^\tau A_0(u,v,w,\underline{a})\big) \Rightarrow \\ \mathcal{T}^\omega \vdash \bigwedge u^1 \bigwedge v \leq_\rho tu\big(F(u,v) \to \bigvee w^\tau A_0(u,v,w,\underline{a})\big),\end{array}\right.$$

then (for $\Delta=\emptyset$) (1) would follow as a special case.
(2) in turn is implied by

$$(3)\ \mathcal{T}^\omega(+\text{AC–qf}) \vdash G^H \Rightarrow \mathcal{T}^\omega \vdash G,$$

where

$$G^H :\equiv \bigwedge u^1 \bigwedge v \leq_\rho tu \bigwedge h_1, \ldots, h_k \bigvee y_1^0, \ldots, y_k^0, w^\tau G_0(u, v, y_1, h_1 y_1, y_2, h_2 y_1 y_2, \ldots, y_k, h_k y_1 \ldots y_k, w)$$

is the (generalized)[65] **Herbrand normal form** of

$$G :\equiv \bigwedge u^1 \bigwedge v \leq_\rho tu \bigvee y_1^0 \bigwedge x_1^0 \ldots \bigvee y_k^0 \bigwedge x_k^0 \bigvee w^\tau G_0(u, v, y_1, x_1, \ldots, y_k, x_k, w).$$

Since $\bigwedge u^1 \bigwedge v \leq_\rho tu\big(F(u,v) \to \bigvee w^\tau A_0\big)$ can be transformed into a prenex normal form G whose Herbrand normal form is logically equivalent to $\bigwedge u \bigwedge v \leq tu\big(F^S(u,v) \to \bigvee w A_0\big)$, (2) is a special case of (3).

Unfortunately (3) is wrong (even without AC–qf) for $\mathcal{T}^\omega = \mathrm{G}_n\mathrm{A}^\omega$, PRA^ω and much weaker theories. In fact it is false already for the first order fragments of these theories augmented by function variables. For (the second order fragment of) $\mathrm{PRA}^\omega + \Sigma_1^0$–IA this was proved firstly in [35] (thereby detecting a false argument in the literature). Below we will prove a result which implies this as a special case and refutes (3) also for $\mathrm{G}_n\mathrm{A}^\omega$ (and their second order fragments).
On the other hand (3) is true for $\mathcal{T}^\omega = \mathrm{G}_n\mathrm{A}^\omega$ (but remains false for $\mathcal{T}^\omega = \mathrm{PRA}^\omega$) if G satisfies a certain monotonicity condition (see def.10.6 below) which is fulfilled e.g. in (1). This result will be used in the next chapter to determine the growth caused by instances of

1) Principle of convergence of bounded monotone sequences (PCM2).
2) Least upper bound for bounded sequences of real numbers.
3) Bolzano–Weierstraß principle for bounded sequences in $\mathbb{R}^d$.
4) Arzelà–Ascoli lemma.
5) The existence of lim sup and lim inf for bounded sequences in $\mathbb{R}$.
6) The restriction of AC_{ar} and CA_{ar} to Π_1^0 formulas: Π_1^0–AC, Π_1^0–CA.

We now prove a result which in particular refutes (3) (even without AC–qf) for $\mathrm{G}_n\mathrm{A}^\omega$ (with $n \geq 2$), $\mathrm{G}_\infty\mathrm{A}^\omega$ and PRA^ω:

Let $\mathrm{G}_2\mathrm{A}^+$ be the first–order part of $\mathrm{G}_2\mathrm{A}^\omega$ augmented by function variables and a substitution rule

$$SUB : \frac{A(f)}{A(g)}.$$

$\mathrm{G}_2\mathrm{A}^+$ contains the schema of quantifier–free induction **with function parameters** .

Proposition: 10.1 *Let $A \in \Pi_\infty^0$ be a theorem of (first order) Peano arithmetic PA. Then one can construct a sentence $\tilde{A} \in \Pi_\infty^0$ such that*

$$G_2A^+ \vdash \tilde{A}^H \quad \textit{and} \quad G_2A \vdash A \leftrightarrow \tilde{A}.$$

[65]The Hebrand normal form is usually defined only for arithmetical formulas, i.e. if u, v, w are not present. In this case it coincides with our definition. In $\mathrm{G}_2\mathrm{A}^+$ below, u, v, v do not occur and the h_i are free function variables.

Proof: If PA $\vdash A$, then there are arithmetical instances (without function parameters) of the induction schema such that for their universal closure $\tilde{F}_1, \ldots, \tilde{F}_k$

$$G_2A^+ \vdash \bigwedge_{i=1}^{k} \tilde{F}_i \to A,$$

since PA $\subset G_2A^+ + \Pi^0_\infty$–IA$^-$, where Π^0_∞–IA$^-$ is the induction schema for all arithmetical formulas without function variables.

Let B be any prenex normal form of $\big(\bigwedge_{i=1}^{k}(y_i =_0 0 \leftrightarrow F_i(x_i)) \to A\big)$, where F_i denotes the induction formula of $\tilde{F}_i$, then

$$\tilde{A} :\equiv \bigvee \underline{a}, x_1, \ldots, x_k \bigwedge y_1, \ldots, y_k\, B(x_1, \ldots, x_k, y_1, \ldots, y_k)$$

is a prenex normal form of

$$\bigwedge \underline{a}, x_1, \ldots, x_k \bigvee y_1, \ldots, y_k \bigwedge_{i=1}^{k} (y_i = 0 \leftrightarrow F_i(x_i)) \to A,$$

where $\underline{a}$ are the (number) parameters of the induction formulas F_i. Because of

$$G_2A \vdash \bigwedge \underline{a}, x_1, \ldots, x_k \bigvee y_1, \ldots, y_k \bigwedge_{i=1}^{k} (y_i = 0 \leftrightarrow F_i(x_i)),$$

we obtain

$$G_2A \vdash A \leftrightarrow \tilde{A}.$$

Since $\tilde{A}^H$ is logically implied by

$$C :\equiv \bigvee \underline{a}, x_1, \ldots, x_k\, B(x_1, \ldots, x_k, f_1\underline{a}x_1 \ldots x_k, \ldots, f_k\underline{a}x_1 \ldots x_k),$$

it remains to show that $G_2A^+ \vdash C$:

Assume $\bigwedge \underline{a}, x_1, \ldots, x_k \bigwedge_{i=1}^{k} \big(f_i\underline{a}x_1 \ldots x_k = 0 \leftrightarrow F_i(x_i)\big)$. Quantifier–free induction applied to $A_0(x_i) :\equiv f_i(\underline{a}, 0, \ldots, 0, x_i, 0, \ldots 0) = 0$ yields $\tilde{F}_i$. Hence

$$G_2A^+ \vdash \bigwedge \underline{a}, x_1, \ldots, x_k \bigwedge_{i=1}^{k} \big(f_i\underline{a}x_1 \ldots x_k = 0 \leftrightarrow F_i(x_i)\big) \to A,$$

i.e. $G_2A^+ \vdash C$.

Corollary 10.2 (to the proof) *Let G_2A be the first order fragment of G_2A^+ (i.e. G_2A^+ without function variables and the rule SUB) and let $G_2A[f_1, \ldots, f_k]$ denote the extension of G_2A which is obtained by adding new function symbols $f_1, \ldots, f_k$ which may occur in instances of QF–IA. Then $G_2A[f_1, \ldots, f_k] \vdash \tilde{A}^H$ and $G_2A \vdash A \leftrightarrow \tilde{A}$ (with $A, \tilde{A}$ as in the proof above), where $f_1, \ldots, f_k$ are the function symbols used in the definition of $\tilde{A}^H$.*

Corollary 10.3 *1) For each $n \in \mathbb{N}$ one can construct a sentence $A \in \Pi^0_\infty$ such that*

$$G_2A^\omega \vdash A^H,\ \ but\ \ G_\infty A^\omega + \Sigma^0_n\text{–}IA \subset PRA^\omega + \Sigma^0_n\text{–}IA \nvdash A.$$

2) For each $n \in \mathbb{N}$ one can construct sentences $A \in \Pi^0_\infty$ and $\bigwedge x^0 \bigvee y^0 B_0(x,y) \in \Pi^0_2$ such that

$$G_2A^\omega \vdash A^H,\ \ but\ \ G_2A^\omega + A \vdash \bigwedge x^0 \bigvee y^0 B_0(x,y),$$

where $fx := \min y[B_0(x,y)]$ is not ω_n–recursive.

Proof: 1) Let $A \in \mathcal{L}(\mathrm{PA})$ be an instance of Σ^0_{n+1}–IA which is not provable in $\mathrm{PRA}^\omega + \Sigma^0_n$–IA (such an instance exists since every ω_{n+1}–recursive function is provably recursive in $\mathrm{G_2A}^\omega + \Sigma^0_{n+1}$–IA, but in $\mathrm{PRA}^\omega + \Sigma^0_n$–IA only ω_n–recursive functions are provably recursive and there are ω_{n+1}–recursive functions which are not ω_n–recursive). Construct now $\tilde{A}$ as in prop.10.1 . It follows that $\mathrm{G_2A}^\omega \vdash \tilde{A}^H$, but $\mathrm{PRA}^\omega + \Sigma^0_n$–IA $\nvdash \tilde{A}$.
2) follows from prop.10.1 and the fact that every $\alpha(<\varepsilon_0)$–recursive function is provably recursive in PA.

The reason for the provability of $\tilde{A}^H$ in prop.10.1 is that the schema of quantifier–free induction is applicable to the index functions used in defining $\tilde{A}^H$. This always is the case in the presence of the substitution rule SUB or $\bigwedge^1$–elimination in theories like $\mathrm{G_2A}^\omega$ where quantification over functions is possible.

In the following we show that the same phenomenon occurs if QF–IA is restricted to formulas **without function variables** (we call this restricted system $\mathrm{G_2A}$) but instead of this new functional symbols $\Phi_{\max,n}$ are added (for each number $n \in \mathbb{N}$) together with the axioms

$$(\max,n):\ \bigwedge_{i=1}^{n}(y_i \le_0 x_i) \to f\underline{y} \le_0 \Phi_{\max,n} f\underline{x},$$

where f is an n–ary function variable.

$$(\max)\ := \cup_n(\max,n).$$

Remark 10.4 $(\max,1)$ *is fulfilled by the functional $\Phi_1 fx = \max(f0,\ldots,fx)$ from G_nA^ω. By λ–abstraction and finite iteration of Φ_1 one can easily define a functional satisfying* $(\max,n)$ *(Hence G_nA+*$(\max)$ *is a subsystem of G_nA^ω). This is the reason for calling this axiom* $(\max)$*. Of course instead of Φ_1 one could also use $\Phi_2 fx = \sum_{i=0}^{x} fi$,$\Phi_3 fx = \prod_{i=0}^{x} fi$ or $\Phi_{\langle\rangle} fx := \overline{f}x$.*

Proposition: 10.5 *Let $A \in \Pi^0_\infty$ be a theorem of PA. Then one can construct a sentence $\tilde{A} \in \Pi^0_\infty$ such that*

$$G_2A + (\max) \vdash \tilde{A}^H\ \ and\ \ G_2A\ \vdash A \leftrightarrow \tilde{A}.$$

Proof: Since PA $\vdash A$ there are arithmetical instances (without function parameters) of the induction schema such that for their universal closure $\tilde{F}_1,\dots,\tilde{F}_k$

$$\mathrm{G}_2\mathrm{A} \vdash \bigwedge_{i=1}^{k} \tilde{F}_i \to A.$$

Lets consider now the so–called collection principle

$$\mathbf{CP}\ :\ \bigwedge \tilde{x}^0 \big(\bigwedge x <_0 \tilde{x} \bigvee y^0 F(x,y,\underline{a}) \to \bigvee z \bigwedge x <_0 \tilde{x} \bigvee y <_0 z\, F(x,y,\underline{a})\big),$$

where $x, y, \underline{a}$ are all free variables of F. This principle has been studied proof–theoretically in [51] and also in [57]. By [57] (prop.4.1 (iv)) one can construct for every instance $\tilde{F}$ of Σ^0_n–IA instances F_i of Σ^0_{n+1}–CP (i.e. CP restricted to Σ^0_{n+1}–formulas) such that $\bigwedge\limits_i F_i \to \tilde{F}$. From the proof in [57] (which uses only QF–IA and the function $\dot{-}$) it follows that $\mathrm{G}_2\mathrm{A} \vdash \bigwedge\limits_i F_i \to \tilde{F}$. Let $F_1,\dots,F_l$ denote such instances of CP whose universal closures imply $\tilde{F}_1,\dots,\tilde{F}_k$. F_i has the form

$$F_i :\equiv \big(\bigwedge x <_0 \tilde{x} \bigvee y^0 G_i(x,y,\underline{a}) \to \bigvee z \bigwedge x < \tilde{x} \bigvee y < z\, G_i(x,y,\underline{a})\big).$$

Thus

$$(1)\ \mathrm{G}_2\mathrm{A}\ \vdash \bigwedge \underline{a},\tilde{x} \bigwedge_{i=1}^{l} \big(\bigwedge x_i <_0 \tilde{x} \bigvee y_i^0 G_i(x_i,y_i,\underline{a}) \to \bigvee z_i \bigwedge x_i < \tilde{x} \bigvee y_i < z_i\, G_i(x_i,y_i,\underline{a})\big) \to A.$$

Consider now

$$B :\equiv \Big(\bigwedge \underline{a},\tilde{x},x_1,\dots,x_l \bigvee y_1,\dots,y_l \bigwedge_{i=1}^{l} \big(\bigwedge u_i < \tilde{x} \bigvee w_i G_i(u_i,w_i,\underline{a}) \to (x_i < \tilde{x} \to G_i(x_i,y_i,\underline{a}))\big) \to A\Big)$$

and

$$C :\equiv \Big(\bigwedge_{i=1}^{l} \big(\bigwedge u_i < \tilde{x} \bigvee w_i G_i(u_i,w_i,\underline{a}) \to (x_i < \tilde{x} \to G_i(x_i,y_i,\underline{a}))\big) \to A\Big).$$

Let C^{pr} be an (arbitrary) prenex normal form of C. Then

$$B^{pr} :\equiv \bigvee \underline{a},\tilde{x},x_1,\dots,x_l \bigwedge y_1,\dots,y_l C^{pr}(\tilde{x},x_1,\dots,x_l,y_1,\dots,y_l,\underline{a})$$

is a prenex normal form of B.
We now show i) $\mathrm{G}_2\mathrm{A}$+(max) $\vdash (B^{pr})^H$ and ii) $\mathrm{G}_2\mathrm{A} \vdash B^{pr} \leftrightarrow A$.
i) Define

$$\widehat{B} :\equiv \bigvee \underline{a},\tilde{x},x_1,\dots,x_l C^{pr}(\tilde{x},x_1,\dots,x_l,f_1\underline{a}\tilde{x}x_1\dots x_l,\dots,f_l\underline{a}\tilde{x}x_1\dots x_l,\underline{a}).$$

The implication $\widehat{B} \to (B^{pr})^H$ holds logically. Hence we have to show that $\mathrm{G}_2\mathrm{A}$+(max) $\vdash \widehat{B}$: $\widehat{B}$ is logically equivalent to

$$(2)\ \bigwedge \underline{a},\tilde{x} \bigwedge_{i=1}^{l} \big(\underbrace{\bigwedge u_i < \tilde{x} \bigvee w_i G_i(u_i,w_i,\underline{a}) \to \bigwedge x_i (x_i < \tilde{x} \to G_i(x_i,f_i\underline{a}\tilde{x}x_1\dots x_l,\underline{a}))}_{H_i :\equiv}\big) \to A.$$

By (max) applied to f_i, $\bigwedge x_i(x_i < \tilde{x} \to G_i(x_i, f_i\underline{a}\tilde{x}x_1 \dots x_l, \underline{a}))$ implies $\bigvee z_i \bigwedge x_i < \tilde{x} \bigvee y_i < z_i G_i(x_i, y_i, \underline{a})$. Thus

$$\mathrm{G_2A} + (\max) \vdash H_i \to F_i \text{ for } i = 1, \dots l.$$

By (1),(2) this yields $\mathrm{G_2A}$+(max) $\vdash \widehat{B}$. ii) We have to show that $\mathrm{G_2A} \vdash B \leftrightarrow A$. This follows immediately from the fact that

$$\bigwedge \underline{a}, \tilde{x}, x_1, \dots, x_l \bigvee y_1, \dots, y_l \bigwedge_{i=1}^{l} \left(\bigwedge u_i < \tilde{x} \bigvee w_i G_i(u_i, w_i, \underline{a}) \to (x_i < \tilde{x} \to G_i(x_i, y_i, \underline{a}))\right)$$

holds logically.

Prop.10.1 and prop.10.5 show that for theories like $\mathrm{G}_n\mathrm{A}^\omega$ the Herbrand normal form A^H of a formula A is in general much weaker than A with respect to provability in $\mathrm{G}_n\mathrm{A}^\omega$ (compare cor.10.3). This phenomenon does not occur if A satisfies the following monotonicity condition:

Definition 10.6 *Let $A \in \mathcal{L}(G_nA^\omega)$ be a formula having the form*

$$A \equiv \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee y_1^0 \bigwedge x_1^0 \dots \bigvee y_k^0 \bigwedge x_k^0 \bigvee w^\gamma A_0(u, v, y_1, x_1, \dots, y_k, x_k, w),$$

where A_0 is quantifier–free and contains only $u, v, \underline{y}, \underline{x}, w$ free. Furthermore let t be $\in G_nR^\omega$ and τ, γ are arbitrary finite types.

1) A is called (arithmetically) **monotone** *if*

$$Mon(A) :\equiv \begin{cases} \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge x_1, \tilde{x}_1, \dots, x_k, \tilde{x_k}, y_1, \tilde{y}_1, \dots y_k, \tilde{y}_k \\ \qquad \Big(\bigwedge_{i=1}^{k} (\tilde{x}_i \leq_0 x_i \wedge \tilde{y}_i \geq_0 y_i) \wedge \bigvee w^\gamma A_0(u, v, y_1, x_1, \dots, y_k, x_k, w) \\ \qquad\qquad \to \bigvee w^\gamma A_0(u, v, \tilde{y}_1, \tilde{x}_1, \dots, \tilde{y}_k, \tilde{x}_k, w)\Big). \end{cases}$$

2) The **Herbrand normal form** *A^H of A is defined to be*

$$A^H :\equiv \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge h_1^{\rho_1}, \dots, h_k^{\rho_k} \bigvee y_1^0, \dots, y_k^0, w^\gamma \underbrace{A_0(u, v, y_1, h_1y_1, \dots, y_k, h_ky_1 \dots y_k, w)}_{A_0^H :\equiv}, \text{ where } \rho_i = 0 \underbrace{(0) \dots (0)}_{i}.$$

Theorem 10.7 *Let $\Psi_1, \dots, \Psi_k \in G_nR^\omega$. Then*

$$G_nA^\omega + Mon(A) \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge h_1, \dots, h_k \Big(\bigwedge_{i=1}^{k} (h_i \text{ monotone}) \to \bigvee y_1 \leq_0 \Psi_1 u\underline{h} \dots \bigvee y_k \leq_0 \Psi_k u\underline{h} \bigvee w^\gamma A_0^H\Big) \to A,$$

where $(h_i \text{ monotone}) :\equiv \bigwedge x_1, \dots, x_i, y_1, \dots, y_i\big(\bigwedge_{j=1}^{i} (x_i \geq_0 y_i) \to h_i\underline{x} \geq_0 h_i\underline{y}\big)$.

Theorem 10.8 *Let A be as in thm.10.7 and Δ be as in thm.2.2.2 and let A' denote the negative translation of A*[66]*. Then the following rule holds:*

$$\begin{cases} G_nA^\omega + AC\text{–}qf + \Delta \vdash A^H \wedge Mon(A) \Rightarrow \\ G_nA^\omega + \Delta + b\text{-}AC \vdash A \textit{ and} \\ \textit{by monotone functional interpretation one can extract a tuple } \underline{\Psi} \in G_nR^\omega \textit{ such that} \\ G_nA_i^\omega + b\text{–}AC + \Delta \vdash \underline{\Psi} \textit{ satisfies the monotone functional interpretation of } A'. \end{cases}$$

Proof of theorem 10.7 : We assume that

$$\bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge h_1, \ldots, h_k (\bigwedge_{i=1}^{k} (h_i \text{ monotone}) \to \bigvee y_1, \ldots, y_k \leq_0 \Psi u\underline{h} \bigvee w^\gamma A_0^H)$$

(This assumption follows from the implicative premise in the theorem by taking $\Psi u\underline{h} := \max_0(\Psi_1 u\underline{h}, \ldots, \Psi_k u\underline{h})$). By cor.1.2.24 and the corollary to the proof of prop.1.2.21 one can construct a term $\Psi^*[u, \underline{h}]$ such that

1) $\Psi^*[u, \underline{h}]$ is built up from $u, \underline{h}, A_0, \ldots, A_n, S^1, 0^0, \max_0$ only (by substitution).

2) $\lambda u, \underline{h}.\Psi^*[u, \underline{h}]$ s–maj Ψ.

1) in particular implies
1*) Every occurrence of an $h_j \in \{h_1, \ldots, h_k\}$ in $\Psi^*[u, \underline{h}]$ has the form $h_j(r_{n_1}, \ldots, r_{n_j})$, i.e. h_j occurs only with a full stock of arguments but not as a function argument in the form $s(h_j r_{n_1} \ldots r_{n_l})$ for some $l < j$.
By 2), $\bigwedge u^1(u^M$ s–maj $u)$ and (h_i monotone $\to h_i$ s–maj h_i) we have
2*) $\mathrm{G_nA^\omega} \vdash \bigwedge u \bigwedge h_1, \ldots, h_k (\bigwedge_{i=1}^{k} (h_i \text{ monotone}) \to \Psi^*[u^M, \underline{h}] \geq_0 \Psi u\underline{h})$.
(Note the the replacement of h_i by $h_i^M := \lambda x_1, \ldots, x_i. \max\limits_{\substack{\tilde{x}_1 \leq x_1 \\ \vdots \\ \tilde{x}_i \leq x_i}} h(\tilde{x}_1, \ldots, \tilde{x}_i)$, which would make the monotonicity assumption on h_i superfluous, would destroy property 1*) on which the proof below is based. This is the reason why we have to assume h_i to be monotone. In order to overcome this assumption we will use essentially the monotonicity of A).
Let $r_1, \ldots, r_l$ be all subterms of $\Psi^*[u^M, \underline{h}]$ which occur as an argument of a function $\in \{h_1, \ldots, h_k\}$ in $\Psi^*[u^M, \underline{h}]$ plus the term $\Psi^*[u^M, \underline{h}]$ itself.
Let $\widehat{r_j}[a_1, \ldots, a_{q_j}]$ be the term which results from r_j if every occurrence of a maximal $h_1, \ldots, h_k$–subterm (i.e. a maximal subterm which has the form $h_i(s_1, \ldots, s_i)$ for an $i = 1, \ldots, k$) is replaced by a new variable and let $a_1, \ldots, a_{q_j}$ denote these variables. We now define

$$\tilde{r}_j a_1 \ldots a_{q_j} := \max\Big(\max_{\substack{\tilde{a}_1 \leq a_1 \\ \vdots \\ \tilde{a}_{q_j} \leq a_{q_j}}} \widehat{r_j}[\tilde{a}_1, \ldots, \tilde{a}_{q_j}], a_1, \ldots, a_{q_j}\Big).$$

[66] Here we can use any of the various negative translations. For a systematical treatment of negative translations see [43].

($\tilde{r}_j$ can be defined in $\mathrm{G}_n\mathrm{R}^\omega$ from $\widehat{r}_j$ by successive use of Φ_1).
By the construction of $\tilde{r}_j$ we get

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \big(\lambda\underline{a}.\tilde{r}_j\underline{a} \text{ s–maj } \lambda\underline{a}.\widehat{r}_j[a_1,\ldots,a_{q_j}]\big) \wedge \bigwedge\underline{a}(\tilde{r}_j\underline{a} \geq_0 a_1,\ldots,a_{q_j}).$$

Since $\Psi^*[u^M,\underline{h}]$ is built up from $\widehat{r}_j,\underline{h}$ and u^M only (by substitution) and (h_i monotone $\to h_i$ s–maj h_i), u^M s–maj u, this implies

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \bigwedge u,h_1,\ldots,h_k\big(\bigwedge_{i=1}^{k}(h_i \text{ monotone}) \to \overline{\Psi}[u^M,\underline{h}] \geq_0 \Psi^*[u^M,\underline{h}] \geq_0 \Psi u\underline{h}\big),$$

where $\overline{\Psi}[u^M,\underline{h}]$ is built up as $\Psi^*[u^M,\underline{h}]$ but with $\tilde{r}_j a_1\ldots a_{q_j}$ instead of $\widehat{r}_j[a_1,\ldots,a_{q_j}]$.
Summarizing the situation achieved so far we have obtained a term $\overline{\Psi}[u^M,\underline{h}]$ such that

(α) $\bigwedge u^1\bigwedge v \leq_\tau tu\bigwedge\underline{h}\big(\underline{h} \text{ monotone} \to \bigvee y_1,\ldots,y_k \leq_0 \overline{\Psi}[u^M,\underline{h}]\bigvee w^\gamma A_0^H\big)$.

(β) $h_1,\ldots,h_k$ occur in $\overline{\Psi}[u^M,\underline{h}]$ only as in 1*), i.e. with all places for arguments filled and not as function arguments themselves.

(γ) For $\overline{\Psi}[u^M,\underline{h}]$ and all subterms s which occur as an argument of a function $h_1,\ldots,h_k$ in $\overline{\Psi}[u^M,\underline{h}]$ we have $\widehat{s}[a_1,\ldots,a_q] \geq_0 a_1,\ldots,a_q$, where $\widehat{s}$ results by replacing every occurrence of a maximal $h_1,\ldots,h_k$–subterm in s by a new variable a_l.

In the following we only use (α)–(γ) and $Mon(A)$.
¿From now on let $r_1,\ldots,r_l$ denote all subterms of $\overline{\Psi}[u^M,\underline{h}]$ which occur as an argument of a function $\in \{h_1,\ldots,h_k\}$ in $\overline{\Psi}[u^M,\underline{h}]$ plus $\overline{\Psi}[u^M,\underline{h}]$ itself. $M := \{r_1,\ldots,r_l\}$ (This set formation is meant w.r.t. identity $\equiv$ of terms and not $=_0$, i.e. '$s \in M$' means '$s \equiv r_1 \vee \ldots \vee s \equiv r_l$').
We now show that we can reduce '$\bigvee y_1,\ldots,y_k \leq \overline{\Psi}[u^M,\underline{h}]$' in ($\alpha$) to a disjunction with fixed length, namely to the disjunction over M:

$$(1)\ \begin{cases} \bigwedge u^1\bigwedge v \leq_\tau tu\bigwedge\underline{h}\big(\underline{h} \text{ monotone on } M \to \bigvee s_1,\ldots,s_k \in M\bigvee w^\gamma \\ \qquad\qquad A_0(u,v,s_1,h_1s_1,\ldots,s_k,h_ks_1\ldots s_k,w)\big). \end{cases}$$

Proof of (1): Let $h_1,\ldots,h_k$ be monotone on M. We order the terms r_i w.r.t. $\leq_0$. The resulting ordered tuple depends of course on $u,h_1,\ldots,h_k$. For notational simplicity we assume that $r_1 \leq_0 \ldots \leq_0 r_l$. We now define (again depending on $u,\underline{h}$) functions $\tilde{h}_1,\ldots,\tilde{h}_k$ by

$$\tilde{h}_i y_1^0\ldots y_i^0 := h_i(r_{j_{y_1}},\ldots,r_{j_{y_i}}), \text{where}$$

$$j_{y_q} := \begin{cases} 1, \text{ if } y_q \leq_0 r_1 \\ j+1, \text{ if } r_j <_0 y_q \leq_0 r_{j+1} \\ l, \text{ if } r_l <_0 y_q. \end{cases}$$

Since l (and therefore the number of cases in this definition of $\tilde{h}_i$) is a (from outside) fixed number depending only on the term structure of $\overline{\Psi}[u^M,\underline{h}]$ but not on $u,\underline{h}$, the functions $\tilde{h}_i$ can be defined

uniformly in $u, \underline{h}$ within $\mathrm{G}_n\mathrm{A}^\omega$. On M, $\tilde{h}_i$ equals h_i.
By the definition of $\tilde{h}_i$ and the assumption that $h_1, \ldots, h_k$ are monotone on M we conclude

(a) $\tilde{h}_1, \ldots, \tilde{h}_k$ are monotone everywhere.

By (β) we know that $h_1, \ldots, h_k$ occur in $\overline{\Psi}[u^M, \underline{h}]$ only in the form $h_i s_1 \ldots s_i$ for certain terms $s_1, \ldots, s_i \in M$. Hence we can define the h–**depth** of a term $s \in M$ as the maximal number of nested occurrences of $h_1, \ldots, h_k$ in s and show by induction on this rank (on the meta–level):

$$(b) \begin{cases} \bigwedge\limits_{i=1}^{l} (r_i =_0 \tilde{r}_i), \text{ where } \tilde{r}_i \text{ results if in } r_i \in M \text{ the functions } h_1, \ldots, h_k \\ \text{are replaced by } \tilde{h}_1, \ldots, \tilde{h}_k \text{ everywhere. In particular } \overline{\Psi}[u^M, \underline{\tilde{h}}] =_0 \overline{\Psi}[u^M, \underline{h}]. \end{cases}$$

By (α), (a) and (b) it follows (for all $u^1, v \leq tu$ and all $\underline{h}$ which are monotone on M) that

(c) $\bigvee y_1, \ldots, y_k \leq_0 \overline{\Psi}[u^M, \underline{h}] \bigvee w^\gamma A_0(u, v, y_1, \tilde{h}_1 y_1, \ldots, y_k, \tilde{h}_k y_1 \ldots y_k, w)$.

Let $y_1, \ldots, y_k \leq_0 \overline{\Psi}[u^M, \underline{h}]$ be such that (c) is fulfilled. Because of $\tilde{h}_i y_1 \ldots y_i = h_i(r_{j_{y_1}}, \ldots, r_{j_{y_i}})$ this implies

(d) $\bigvee w^\gamma A_0(u, v, y_1, h_1 r_{j_{y_1}}, \ldots, y_k, h_k r_{j_{y_1}}, \ldots, r_{j_{y_k}}, w)$.

With $y_q \leq r_{j_{y_q}}$ for $q = 1, \ldots .k$ (since $y_q \leq \overline{\Psi}[u^M, \underline{h}] \leq r_l$ –because of $\overline{\Psi}[u^M, \underline{h}] \in M$ and the y_q–assumption– the case '$y_q > r_l$' does not occur) and $Mon(A)$ we conclude

$$\bigvee w^\gamma A_0(u, v, r_{j_{y_1}}, h_1 r_{j_{y_1}}, \ldots, r_{j_{y_k}}, h_k r_{j_{y_1}}, \ldots, r_{j_{y_k}}, w)$$

and therefore

(e) $\bigvee s_1, \ldots, s_k \in M \bigvee w^\gamma A_0(u, v, s_1, h_1 s_1, \ldots, s_k, h_k s_1 \ldots s_k, w)$.

This concludes the proof of (1) (from $(\alpha), (\beta)$) which can easily be carried out in $\mathrm{G}_n\mathrm{A}^\omega$, i.e.

$\mathrm{G}_n\mathrm{A}^\omega \vdash Mon(A) \wedge (\alpha) \rightarrow (1)$.

We now define $N := \bigcup\limits_{i=1}^{k} N_i$, where $N_i := \{h_i(s_1, \ldots, s_i) : s_1, \ldots, s_i \in M\}$ (Again this set is meant w.r.t. identity $\equiv$ between terms). With the terms in N we associate new number variables according to their $\underline{h}$–depth as follows: Let p the maximal $\underline{h}$–depth of all terms $\in N$.

1. Let $t \in N$ be a term with $\underline{h}$–depth$(t) = p$. Then $t \mapsto y_i^1$, if $t \in N_i$.

2. Let $t \in N$ be a term with $\underline{h}$–depth$(t) = p - 1$. Then $t \mapsto y_i^2$, if $t \in N_i$.

   $\vdots$

p. Let $t \in N$ be a term with $\underline{h}$–depth$(t) = 1$. Then $t \mapsto y_i^p$, if $t \in N_i$.

This association of variables to the terms in N has the following properties:

(i) Terms $s_1, s_2 \in N$ with different $\underline{h}$–depth have different variables associated with.

(ii) If $s_1, s_2 \in N$ have the same $\underline{h}$–depth, then the variables associated with s_1 and s_2 are equal iff $s_1, s_2 \in N_i$ for an $i = 1, \ldots, k$.

For $r \in M \cup N$ we define $\widehat{r}$ as the term which results if every maximal $\underline{h}$–subterm occurring in r is replaced by its associated variable. Thus $\widehat{r}$ does not contain $h_1, \ldots, h_k$. For $r \in N$, $\widehat{r}$ is just the variable associated with r. $\widehat{M} := \{\widehat{r} : r \in M\}$.
We now show that (1) implies a certain index function–free (i.e. $h_1, \ldots, h_k$–free) disjunction ((2) below):
For q with $2 \leq q \leq p$ let $\widehat{r}^q_1, \ldots, \widehat{r}^q_{n_q}$ be all terms $\in \widehat{M}$ whose smallest upper index i of a variable y^i_j occurring in them equals q (i.e. there occurs a variable y^q_j in the term and for all variables y^i_m occurring in the term, $i \geq q$ holds). Since for $r \in M$ the $\underline{h}$–depth of $h_1(r) \in N$ is strictly greater than those of subterms of r, there are no terms $\widehat{r} \in \widehat{M}$ containing a variable y^1_j. $\widehat{r}^{p+1}_1, \ldots, \widehat{r}^{p+1}_{n_{p+1}}$ denote those terms $\in \widehat{M}$ which do not contain any variable y^i_j at all.
We now show that (1) implies (for all u and for all $v \leq tu$)

$$(2) \quad \begin{cases} \bigwedge y^1_1, \ldots, y^1_k; \ldots; y^p_1, \ldots, y^p_k \Big(\bigwedge\limits_{\substack{q=1,\ldots,p-1 \\ l=1,\ldots,p-q}} (y^q_1, \ldots, y^q_k > \widehat{r}^{q+l}_1, \ldots \widehat{r}^{q+l}_{n_{q+l}}, \widehat{r}^{p+1}_1, \ldots, \widehat{r}^{p+1}_{n_{p+1}}, y^{q+l}_1, \ldots, y^{q+l}_k) \\ \qquad \to \bigvee\limits_{\widehat{s}_1, \ldots, \widehat{s}_k \in \widehat{M}} \bigvee w^\gamma A_0(u, v, \widehat{s}_1, \widehat{h_1 s_1}, \ldots, \widehat{s}_k, h_k \widehat{s_1 \ldots} s_k, w) \Big). \end{cases}$$

Assume that there are values $y^1_1, \ldots, y^1_k; \ldots; y^p_1, \ldots, y^p_k$ such that

$$(+) \quad \bigwedge_{\substack{q=1,\ldots,p-1 \\ l=1,\ldots,p-q}} \big(y^q_1, \ldots, y^q_k > \widehat{r}^{q+l}_1, \ldots \widehat{r}^{q+l}_{n_{q+l}}, \widehat{r}^{p+1}_1, \ldots, \widehat{r}^{p+1}_{n_{p+1}}, y^{q+l}_1, \ldots, y^{q+l}_k \big)$$

and

$$\bigwedge_{\widehat{s}_1, \ldots, \widehat{s}_k \in \widehat{M}} \neg \bigvee w^\gamma A_0(u, v, \widehat{s}_1, \widehat{h_1 s_1}, \ldots, \widehat{s}_k, h_k \widehat{s_1 \ldots} s_k, w).$$

We construct (working in $\mathrm{G}_n\mathrm{A}^\omega$) functions $h_1, \ldots, h_k$ which are monotone on M and satisfy

$$\bigwedge s_1, \ldots, s_k \in M \neg \bigvee w\, A_0(u, v, s_1, h_1 s_1, \ldots, s_k, h_k s_1 \ldots s_k, w)$$

yielding a contradiction to (1): Define for $i = 1, \ldots, k$

$$h_i(x_1, \ldots, x_i) := \begin{cases} y_i^{\min_{1 \leq l \leq i}(q_l) - 1}, \text{ if } \bigvee \widehat{r}^{q_1}_{j_1}, \ldots, \widehat{r}^{q_i}_{j_i} \in \widehat{M} \big((x_1, \ldots, x_i) =_0 (\widehat{r}^{q_1}_{j_1}, \ldots, \widehat{r}^{q_i}_{j_i}) \big) \\ 0^0, \text{ otherwise.}^{67} \end{cases}$$

We have to show:

(i) The h_i are well–defined functions : $\underbrace{\mathbb{N} \times \ldots \times \mathbb{N}}_{i} \to \mathbb{N}$ and the definition above can be carried out in $\mathrm{G}_n\mathrm{A}^\omega$.

(ii) $\widehat{r} =_0 r$ for all $r \in M \cup N$ (for these $h_1, \ldots, h_k$).

[67] For $\widehat{r}^{q_i}_{j_i} \in \widehat{M}$ we have $q_i \geq 2$ since e.g. $h_1 r_{j_i} (\in N)$ has an $\underline{h}$–depth which is strictly greater than those of subterms in r_{j_i}.

(iii) $h_1,\ldots,h_k$ are monotone on $\widehat{M}$ (and hence –by (ii)– on M).

Ad (i): Consider $(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i})$ and $(\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i})$. We show that $y_i^{\min_{1\le l\le i}(q_l)-1} \neq y_i^{\min_{1\le l\le i}(\tilde{q}_l)-1}$ implies $(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i}) \neq (\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i})$:
We may assume $\min\limits_{1\le l\le i}(q_l) < \min\limits_{1\le l\le i}(\tilde{q}_l)$. Let l_0 be such that $q_{l_0} = \min\limits_{1\le l\le i}(q_l) \wedge 1 \le l_0 \le i$. $\widehat{r}^{q_{l_0}}_{j_{l_0}}$ contains a variable $y_d^{q_{l_0}}$ for some $d=1,\ldots,k$. By the property (γ) of $\overline{\Psi}[u^M,\underline{h}]$ this implies

$$\widehat{r}^{q_{l_0}}_{j_{l_0}} \ge y_d^{q_{l_0}} \overset{(+),q_{l_0}<\tilde{q}_{l_0}}{>} \widehat{r}^{\tilde{q}_{l_0}}_{\tilde{j}_{l_0}} \text{ and thus } (\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i}) \neq (\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i}).$$

Hence h_i can be defined in $\mathrm{G}_n\mathrm{A}^\omega$ by a definition by cases which are pairwise exclusive.
Ad (ii): (ii) follows from the definition of $h_1,\ldots,h_k$ by induction on the $\underline{h}$–depth of r.
Ad (iii): Assume $\bigwedge\limits_{l=1}^{i}(\widehat{r}^{q_l}_{j_l} \le_0 \widehat{r}^{\tilde{q}_l}_{\tilde{j}_l})$. Let l_0 $(1\le l_0\le i)$ be such that $q_{l_0} = \min\limits_{1\le l\le i}(q_l)$. By contraposition of the implication established in the proof of (i) one has: $\min\limits_{1\le l\le i}(q_l) \ge \min\limits_{1\le l\le i}(\tilde{q}_l)$.
Case 1: $\min\limits_{1\le l\le i}(q_l) = \min\limits_{1\le l\le i}(\tilde{q}_l)$. Then (by h_i–definition)

$$h_i(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i}) = y_i^{\min(q_l)-1} = y_i^{\min(\tilde{q}_l)-1} = h_i(\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i}).$$

Case 2: $q_{l_0} = \min\limits_{1\le l\le i}(q_l) > \min\limits_{1\le l\le i}(\tilde{q}_l) = q_{\tilde{l}_0}$ (where $1\le l_0,\tilde{l}_0\le i$). Then

$$h_i(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i}) = y_i^{q_{l_0}-1} \overset{(+)}{<} y_i^{\tilde{q}_{\tilde{l}_0}-1} = h_i(\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i}).$$

Hence $h_1,\ldots,h_k$ are monotone on $\widehat{M}$ and therefore (by (ii)) on M, which concludes the proof of (2) from $(1)\wedge(\alpha)\wedge(\beta)\wedge(\gamma)$. Since (1) follows (in $\mathrm{G}_n\mathrm{A}^\omega$) from $Mon(A)\wedge(\alpha)\wedge(\beta)$, and

$$F :\equiv \bigwedge u^1 \bigwedge v \le_\tau tu \bigwedge \underline{h}\big(\underline{h} \text{ monotone } \to \bigvee y_1,\ldots,y_k \le_0 \Psi u\underline{h} \bigvee w^\gamma A_0^H\big)$$

implies (in $\mathrm{G}_n\mathrm{A}^\omega$) (α)–(γ), we have shown altogether

$$(3)\ \mathrm{G}_n\mathrm{A}^\omega + Mon(A) \vdash$$
$$\begin{cases} F \to \Big[v \le tu \wedge \bigwedge\limits_{\substack{q=1,\ldots,p-1\\ l=1,\ldots,p-q}} \big(y_1^q,\ldots,y_k^q > \widehat{r}_1^{q+l},\ldots\widehat{r}^{q+l}_{n_{q+l}},\widehat{r}_1^{p+1},\ldots,\widehat{r}^{p+1}_{n_{p+1}},y_1^{q+l},\ldots,y_k^{q+l}\big) \\ \qquad \to \bigvee\limits_{\widehat{s}_1,\ldots,\widehat{s}_k\in\widehat{M}} \bigvee w^\gamma A_0(u,v,\widehat{s}_1,\widehat{h_1 s_1},\ldots,\widehat{s}_k,h_k\widehat{s_1\ldots}s_k,w)\Big]. \end{cases}$$

It remains to show that (3) implies

$$(4)\ \mathrm{G}_n\mathrm{A}^\omega + Mon(A) \vdash F \to A.$$

We prove this by a suitable application of quantifier introduction rules: We start with the variables with smallest upper index, i.e. $y_1^1,\ldots,y_k^1$. Under these variables we first take those of maximal lower index, i.e. with y_k^1: We split the assumption

$$(+)\ \bigwedge_{\substack{q=1,\ldots,p-1\\ l=1,\ldots,p-q}} \big(y_1^q,\ldots,y_k^q > \widehat{r}_1^{q+l},\ldots\widehat{r}^{q+l}_{n_{q+l}},\widehat{r}_1^{p+1},\ldots,\widehat{r}^{p+1}_{n_{p+1}},y_1^{q+l},\ldots,y_k^{q+l}\big)$$

as well as the disjunction

$$A^d :\equiv \bigvee_{\widehat{s}_1,\ldots,\widehat{s}_k\in\widehat{M}} \bigvee w^\gamma A_0(u,v,\widehat{s}_1,\widehat{h_1 s_1},\ldots,\widehat{s}_k,\widehat{h_k s_1\ldots s_k},w)$$

into the part in which y_k^1 occurs and into its y_k^1–free part:

$$(5)\begin{cases} F\to\Big[v\le tu\wedge \bigwedge\limits_{l=1,\ldots,p-1}\big(y_k^1>\widehat{r}_1^{1+l},\ldots,\widehat{r}_{n_{1+l}}^{1+l},\widehat{r}_1^{p+1},\ldots,\widehat{r}_{n_{p+1}}^{p+1},y_1^{1+l},\ldots,y_k^{1+l}\big) \\ \qquad\wedge\quad \underbrace{\bigwedge{}'(\ldots)}_{y_k^1\text{–free part of }(+)} \quad\to\bigvee\limits_j\bigvee w^\gamma A_0(u,v,\widehat{s}_1^j,\widehat{h_1 s_1^j},\ldots,\widehat{s}_k^j,y_k^1,w)\vee \underbrace{\bigvee\limits_{j'}(\ldots)}_{y_k^1\text{–free part of }A^d}\quad\Big]. \end{cases}$$

y_k^1 does not occur at any place other than indicated. Hence $\bigwedge$–introduction applied to y_k^1 yields:

$$(6)\ F\to\bigwedge y_k^1\big[v\le tu\wedge\bigwedge_l(y_k^1>\ldots)\wedge\bigwedge{}'(\ldots)\to\bigvee_j\bigvee w^\gamma A_0(\ldots,y_k^1,w)\vee\bigvee_{j'}(\ldots)\big].$$

Using $Mon(A)$ this implies

$$(7)\ F\to\big[v\le tu\wedge\bigwedge{}'(\ldots)\to\bigwedge y_k^1\bigvee_j\bigvee w^\gamma A_0(\ldots,y_k^1,w)\vee\bigvee_{j'}(\ldots)\big].$$

(Proof: In (6) put $\tilde{y}_k^1:=\max\limits_{1\le l\le p-1}\big(y_k^1,\widehat{r}_1^{1+l},\ldots\widehat{r}_{n_{1+l}}^{1+l},\widehat{r}_1^{p+1},\ldots,\widehat{r}_{n_{p+1}}^{p+1},y_1^{1+l},\ldots,y_k^{1+l}\big)+1$ for y_k^1.
(6) then gives

$$F\to\big[v\le tu\wedge\bigwedge{}'(\ldots)\to\bigvee_j\bigvee w^\gamma A_0(\ldots,\tilde{y}_k^1,w)\vee\bigvee_{j'}(\ldots)\big].$$

$Mon(A)$ and $\bigvee\limits_j\bigvee w^\gamma A_0(\ldots,\tilde{y}_k^1,w)$ imply $\bigvee\limits_j\bigvee w^\gamma A_0(\ldots,y_k^1,w)$, since $\tilde{y}_k^1\ge y_k^1$. Now $\bigwedge$–introduction applied to y_k^1 and shifting $\bigwedge y_k^1$ in front of $\bigvee\limits_j$, which is possible since y_k^1 occurs only in this disjunction, proves (7)).
Again by $Mon(A)$ we obtain $\bigvee\limits_j\bigwedge y_k^1\bigvee w^\gamma A_0(\ldots,y_k^1,w)$ from $\bigwedge y_k^1\bigvee\limits_j\bigvee w^\gamma A_0(\ldots,y_k^1,w)$:
Assume $\bigwedge\limits_j\bigvee y_k^1\bigwedge w^\gamma\neg A_0(\ldots,y_k^1,w)$. Then $\bigvee y\bigwedge\limits_j\bigvee y_k^1\le_0 y\bigwedge w^\gamma\neg A_0(\ldots,y_k^1,w)$. Using $Mon(A)$ this implies $\bigvee y\bigwedge\limits_j\bigwedge w^\gamma\neg A_0(\ldots,y,w)$.
Hence (7) implies

$$(8)\begin{cases} F\to\big[v\le tu\wedge\bigwedge{}'(\ldots)\to\bigvee\limits_j\bigvee x\bigwedge y\bigvee wA_0(u,v,\widehat{s}_1^j,\widehat{h_1 s_1^j},\ldots,h_{k-1}\widehat{s_1^j\ldots} s_{k-1}^j,x,y,w) \\ \qquad\qquad\qquad\qquad\vee\bigvee\limits_{j'}(\ldots)\big]. \end{cases}$$

Next we apply the same procedure to the variable y_{k-1}^1 and then to y_{k-2}^1 and so on until all $y_1^1,\ldots,y_k^1$ are bounded. We then continue with y_k^2, y_{k-1}^2 and so on. This corresponds to the sequence of

quantifications used in the usual proofs of Herbrand's theorem in order to show that there is a direct proof from the Herbrand disjunction of a first order formula to this formula itself: By taking always variables of minimal upper index it is ensured that any variable to which the $\bigwedge$–introduction rule is applied occurs in the disjunction $\bigvee A_0$ only at places where it is universal quantified in the original formula A. By quantifying under these variables firstly the one with maximal lower index one ensures that a universal quantifier is introduced only if the quantifiers which stand behind this one in A have already been introduced. In addition to these two reasons for the special sequence of quantifications there is in our situation another (essentially used) property which is fulfilled only if variables which have minimal lower index are quantified first: If y^i_j has minimal index i (under all variables which still have to be quantified), then y^i_j occurs in the implicative assumption (+) only in the form '$y^i_j >$ (...y^i_j–free...)'. So we are in the situation at the begining for y^1_k and are able to eliminate this part of (+) which is connected with y^i_j altogether using $Mon(A)$ (as we have shown for y^1_k).
Finally we have derived

$$(9)\ F \to \big[v \le tu \to \bigvee \bigvee x^0_1 \bigwedge y^0_1 \ldots \bigvee x^0_k \bigwedge y^0_k \bigvee w^\gamma A_0(u, v, x_1, y_1, \ldots, x_k, y_k, w)$$

and therefore (by contraction of $\bigvee$)

$$(10)\ F \to \big[v \le tu \to \bigvee x^0_1 \bigwedge y^0_1 \ldots \bigvee x^0_k \bigwedge y^0_k \bigvee w^\gamma A_0(u, v, x_1, y_1, \ldots, x_k, y_k, w)$$

which (by $\bigwedge$–introduction applied to u, v) yields

$$(11)\ F \to A.$$

Remark 10.9 The proof of thm.10.7 also works for various other systems $\mathcal{T}$ and domains of terms S than $\mathrm{G}_n\mathrm{A}^\omega$ and $\mathrm{G}_n\mathrm{R}^\omega$. What actually is used in the proof is:

1) Every term $\Psi^\rho \in S$ with $\deg(\rho) \le 2$ has a majorant $\Psi^*[\underline{h}^1]$ such that
   (i) $\mathcal{T} \vdash \lambda \underline{h}.\Psi^*[\underline{h}]$ s–maj Ψ,
   (ii) $\Psi^*[\underline{h}]$ is built up only from $\underline{h}$ and terms $\in S$ of type level ≤ 1 (by substitution).
2) S is (provably in $\mathcal{T}$) closed under definition by cases, λ –abstraction and contains the variable maximum–functional Φ_1.

Condition 1) is a sort of an upper bound for the complexity of $\mathcal{T}, S$. E.g. 1) is not satisfied if S contains the iteration functional Φ_{it}. In the next chapter we will show that thm.10.7 becomes false if $\mathrm{G}_n\mathrm{R}^\omega$ is replaced by $\widehat{PR}^\omega$ (see also remark 10.12). Since Φ_{it} is on some sense the simplest functional for which 1) fails, this shows that the upper bound provided by 1) is quite sharp. 1) essentially says that Ψ^{001} can be majorized by a term $\Psi^*[x^0, h^1]$ which uses $\underline{h}$ only at a fixed number of arguments, i.e. there exists a fixed number n (which depends only on the structure of Ψ^* but not on x, h) such that for all h, x the value of of $\Psi^*[x, h]$ only depends on (at most) n–many h–values. Let us illustrate this by an example: $\Phi_1 hx = \max(h0, \ldots, hx)$ depends on $x + 1$–many h–values but is majorized by $\Phi^* hx := hx$ for monotone h which for every x depends only on one h–value, namely on hx. If a term Ψ has a majorant which satisfies 1) we say that Ψ **is majorizable with finite support**. One easily convinces oneself that Φ_{it} is not majorizable with finite support.
2) is a lower bound on the complexity of $\mathcal{T}, S$, which also is essential. E.g. take $\mathcal{T} := \mathcal{L}^2$ and

$S := \{0^0\}$, where $\mathcal{L}^2$ is first–order logic with $=_0, \leq_0$ extended by quantification over functions and two constants $0^0, 1^0$. Consider now

$$G :\equiv \bigvee x^0 \bigwedge y^0 \bigvee z^0, f^1 (F_0(f,z) \rightarrow A_0(x,y)),$$

where $F_0(f,z) :\equiv (fz = 0 \wedge 0 \neq 1)$ and $A_0(x,y) :\equiv (y \neq 0 \wedge x = x \rightarrow \perp)$. Then

$$\mathcal{L}^2 \vdash \bigwedge g^1 \bigvee x, z \leq_0 0 \bigvee f (F_0(f,z) \rightarrow A_0(x, gx)) \wedge Mon(G), \text{ but } \mathcal{L}^2 \nvdash G,$$

i.e. thm.10.7 fails for $\mathcal{L}^2, S$. If however $\mathcal{L}^2$ is extended by λ–abstraction, then G becomes derivable since we can form $f := \lambda x^0.1^0$.

Corollary 10.10 *Let A be as in def.10.6 and thm.10.7, $n \geq 2$. Then*

1) $$G_n A^\omega \oplus F^- \oplus AC\text{–}qf \vdash A^H \Rightarrow G_{\max(n,3)} A^\omega + Mon(A) \vdash A.$$

In particular

$$G_n A^\omega \oplus F^- \oplus AC\text{–}qf \vdash A \Rightarrow G_{\max(n,3)} A^\omega + Mon(A) \vdash A.$$

2) $$G_n A^\omega \oplus WKL^2_{seq} \oplus AC\text{–}qf \vdash A^H \Rightarrow G_{\max(n,3)} A^\omega + Mon(A) \vdash A.$$

In particular

$$G_n A^\omega \oplus WKL^2_{seq} \oplus AC\text{–}qf \vdash A \Rightarrow G_{\max(n,3)} A^\omega + Mon(A) \vdash A.$$

If $\tau \leq 1$ (in A) then $G_n A^\omega \oplus F^- \oplus$ AC–qf can be replaced by E–$G_n A^\omega + F^- + AC^{\alpha,\beta}$–qf (with $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$).

Proof: 1) By thm.7.2.20 $\mathrm{G}_n\mathrm{A}^\omega \oplus F^- \oplus$ AC–qf $\vdash A^H$ implies the extractability of a $\Psi \in \mathrm{G}_n\mathrm{R}^\omega$ such that

$$\mathrm{G}_{\max(n,3)}\mathrm{A}^\omega \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \underline{h} \bigvee y_1, \ldots, y_k \leq_0 \Psi u \underline{h}\, A_0^H.$$

Theorem 10.7 now yields $\mathrm{G}_{\max(n,3)}\mathrm{A}^\omega + Mon(A) \vdash A$.
2) follows from 1) by cor.7.2.26.

Proof of theorem 10.8 :

$\mathrm{G}_n\mathrm{A}^\omega + \Delta +$AC–qf $\vdash A^H$ implies (by thm.2.2.2) the extractability (by monotone functional interpretation and the remarks after 2.2.6) of terms $\underline{\Psi} := \Psi_1, \ldots, \Psi_l \in \mathrm{G}_n\mathrm{R}^\omega$ such that $\underline{\Psi}$ satisfies the monotone functional interpretation of $(A^H)'$ provably in $\mathrm{G}_n\mathrm{A}_i^\omega + \tilde{\Delta}$, where
$\tilde{\Delta} := \{\bigvee Y \leq_{\rho\delta} s \bigwedge x^\delta, z^\eta F_0(x, Yx, z) : \bigwedge x^\delta \bigvee y \leq_\rho sx \bigwedge z^\eta F_0(x,y,z) \in \Delta\}$. From these terms Ψ_i one constructs (as in the proof of thm.2.2.2) uniform bounds $\tilde{\Psi}_i \in \mathrm{G}_n\mathrm{R}^\omega$ on $\bigvee y_i$ which depend only on u and $\underline{h}$:

$$(1)\ \mathrm{G}_n\mathrm{A}_i^\omega + \tilde{\Delta} \vdash \underbrace{\bigwedge u \bigwedge v \leq tu \bigwedge \underline{h} \bigvee y_1 \leq_0 \tilde{\Psi}_1 u \underline{h} \ldots \bigvee y_k \leq_0 \tilde{\Psi}_k u \underline{h} \bigvee w\, A_0^H}_{A^{H,B} :\equiv}.$$

The assumption assumption $\mathrm{G}_n\mathrm{A}^\omega + \Delta$+AC–qf $\vdash Mon(A)$ implies (by monotone functional interpretation, since $Mon(A)$ is implied by the monotone functional interpretation of its negative translation) that

(2) $\mathrm{G}_n\mathrm{A}^\omega_i + \tilde{\Delta} \ \vdash Mon(A)$.

Theorem 10.7 combined with (1) and (2) yields (using that each sentence $\in \tilde{\Delta}$ follows from the corresponding sentence in Δ by b-AC)

$\mathrm{G}_n\mathrm{A}^\omega + \Delta +$ b-AC $\vdash A$.

Again by thm.10.7 and the assumption $\mathrm{G}_n\mathrm{A}^\omega + \Delta$+AC–qf $\vdash Mon(A)$ we have

$\mathrm{G}_n\mathrm{A}^\omega + \Delta +$ AC–qf $\vdash A^{H,B} \to A$

and therefore using (1)

$\mathrm{G}_n\mathrm{A}^\omega + \tilde{\Delta}$+AC–qf $\vdash A$.

The second part of the theorem now follows by monotone functional interpretation, since $\tilde{\Delta}$ also is a set of allowed axioms Δ in thm.2.2.2.

For our applications in the next chapter we need the following corollary of theorem 10.8:

Corollary 10.11 *Let $\bigwedge x^0 \bigvee y^0 \bigwedge z^0 A_0(u^1, v^\tau, x, y, z) \in \mathcal{L}(G_nA^\omega)$ be a formula which contains only u, v as free variables and satisfies provably in $G_nA^\omega+\Delta+AC$–qf the following monotonicity property:*

$$(*)\ \bigwedge u, v, x, \tilde{x}, y, \tilde{y}(\tilde{x} \leq_0 x \wedge \tilde{y} \geq_0 y \wedge \bigwedge z^0 A_0(u, v, x, y, z) \to \bigwedge z^0 A_0(u, v, \tilde{x}, \tilde{y}, z)),$$

(i.e. $Mon(\bigvee x \bigwedge y \bigvee z \neg A_0)$). Furthermore let $B_0(u, v, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ be a formula which contains only u, v, w as free variables and $\gamma \leq 2$. Then from a proof

$$G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(\bigvee f^1 \bigwedge x, z\, A_0(u, v, x, fx, z) \to \bigvee w^\gamma B_0(u, v, w)\big) \wedge (*)$$

one can extract a term $\chi \in G_nR^\omega$ such that

$$G_nA^\omega_i + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \Psi^*\big((\Psi^* \text{ satisfies the mon.funct.interpr. of } \bigwedge x^0, g^1 \bigvee y^0 A_0(u, v, x, y, gy)) \to \bigvee w \leq_\gamma \chi u \Psi^*\, B_0(u, v, w)\big)^{68}.$$

Proof: We may assume that $\gamma = 2$. The property $Mon(F)$ for

$F :\equiv \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee x^0 \bigwedge y^0 \bigvee z^0, w^2\big(A_0(u, v, x, y, z) \to B_0(u, v, w)\big)$

follows logically from the monotonicity assumption $(*)$. By the assumption of the corollary we have

$\mathrm{G}_n\mathrm{A}^\omega + \Delta +$ AC–qf $\vdash F^H + Mon(F)$.

[68] 'Ψ^* satisfies the mon. funct.interpr. of $\bigwedge x, g \bigvee y A_0(u, v, x, y, gy)$' is meant here for fixed u, v (and not uniformly as a functional in u, v), i.e. $\bigvee \Psi\big(\Psi^* \text{ s–maj } \Psi \wedge \bigwedge x, g\, A_0(u, v, x, \Psi xg, g(\Psi xg))\big)$.

¿From this we conclude by thm.10.8 that

$$\mathrm{G}_n\mathrm{A}^\omega_i + \Delta + \text{b-AC} \vdash \underline{\tilde{\chi}} \text{ satisfies the monotone funct.interpr. of } F',$$

for a suitable tuple $\underline{\tilde{\chi}}$ of terms $\in \mathrm{G}_n\mathrm{R}^\omega$ which can be extracted from the proof.
F' is intuitionistically equivalent to

$$\bigwedge u \bigwedge v \leq tu \neg\neg \bigvee x^0 \bigwedge y^0 \neg\neg \bigvee z, w(A_0 \to B_0)$$

of F (This follows immediately if one uses the negative translation which is denoted by $*$ in [43]).
By intuitionistic logic the following implication holds

$$F' \to \bigwedge u \bigwedge v \leq_\tau tu\big(\bigwedge x \neg\neg \bigvee y \bigwedge z\, A_0(u,v,x,y,z) \to \neg\neg \bigvee w\, B_0(u,v,w)\big).$$

Hence from $\underline{\tilde{\chi}}$ we obtain a term which satisfies the monotone functional interpretation of the right side of this implication. In particular we obtain a term $\widehat{\chi} \in \mathrm{G}_n\mathrm{R}^\omega$ such that

$$\begin{aligned}\mathrm{G}_n\mathrm{A}^\omega_i + \Delta + \text{b-AC} \vdash \bigvee W\big(\widehat{\chi} \text{ s--maj } W \wedge \bigwedge u \bigwedge v \leq tu \bigwedge \Psi \\ \big(\bigwedge x, g\, A_0(u,v,x,\Psi xg, g(\Psi xg)) \to B_0(u,v,Wuv\Psi)\big)\big).\end{aligned}$$

Define $\chi \in \mathrm{G}_n\mathrm{R}^\omega$ by $\chi := \lambda u^1, \Psi, y^1.\widehat{\chi} u^M (t^* u^M)\Psi y^M$, where $t^* \in \mathrm{G}_n\mathrm{R}^\omega$ is such that $\mathrm{G}_n\mathrm{A}^\omega_i \vdash t^*$ s–maj t. Then

$$\begin{aligned}\bigwedge u \bigwedge v \leq tu \bigwedge \Psi^*\big(\bigvee \Psi\big(\Psi^* \text{ s--maj } \Psi \wedge \bigwedge x, g\, A_0(u,v,x,\Psi xg, g(\Psi xg))\big) \\ \to \bigvee w \leq_2 \chi u \Psi^*\, B_0(u,v,w)\big),\end{aligned}$$

since $\widehat{\chi}$ s–maj W and Ψ^* s–maj Ψ imply $\bigwedge u \bigwedge v \leq tu(\chi u \Psi^* \geq_2 Wuv\Psi)$.

Remark 10.12 *In §3 of the next chapter we will show that cor.10.11 does not hold for $PRA^\omega, \widehat{PR}^\omega, PRA^\omega_i$ (or $G_nA^\omega + \Sigma^0_1$–IA, $\widehat{PR}^\omega$, $G_nA^\omega_i + \Sigma^0_1$–IA) instead of G_nA^ω ,G_nR^ω, $G_nA^\omega_i$ (even for $\Delta = \emptyset$).*
Since the proof of cor.10.11 from thm.10.8 as well as the proof of thm.10.8 from thm.10.7 extends to these theories it follows that also the theorems 10.7 and 10.8 do not hold for them. The proof of thm.10.7 fails for $\Psi \in \widehat{PR}^\omega$ since $\widehat{PR}^\omega$ contains functionals like Φ_{it} which are not majorizable with finite support (see also remark 10.9). The proof of thm.10.8 fails for $PRA^\omega + \Sigma^0_1$–IA since the (monotone) functional interpretation of Σ^0_1–IA requires Φ_{it} and thus thm. 10.7 is not applicable.

The mathematical significance of corollary 10.11 for the growth of bounds extractable from given proofs rests on the following fact: Direct monotone functional interpretation of

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \text{AC--qf} \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(\bigvee f^1 \bigwedge x, z\, A_0(u,v,x,fx,z) \to \bigvee w^\gamma B_0(u,v,w)\big)$$

yields only a bound on $\bigvee w$ which depends on a functional which satisfies the monotone functional interpretation of (1) $\bigvee f \bigwedge x, z\, A_0$ or if we let remain the double negation in front of $\bigvee$ (which comes from the negative translation) (2) $\neg\neg \bigvee f \bigwedge x, z\, A_0$. However in our applications the monotone functional interpretation of (1) would require non–computable functionals (since f is not recursive) and the monotone functional interpretation of (2) can be carried out only using bar recursive

functionals. In contrast to this the bound χ only depends on a functional which satisfies the monotone functional interpretation of $\bigwedge x \bigvee y \bigwedge z\, A_0(x,y,z)$: In our applications such a functional can be constructed in $\widehat{PR}^{\omega}$.
In particular the use of the **analytical** premise

$$\bigvee f^1 \bigwedge x, z A_0$$

has been reduced to the **arithmetical** premise

$$\bigwedge x^0 \bigvee y^0 \bigwedge z^0 A_0.$$

11 The rate of growth caused by sequences of instances of analytical principles whose proofs rely on arithmetical comprehension

In this chapter we apply the results from the previous chapter in order to determine the impact on the rate of growth of uniform bounds for provably $\bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w^\gamma A_0$–sentences which may result from the use of sequences (which however may depend on the parameters of the proposition to be proved) of instances of:

1) $(PCM2)$ and the convergence of bounded monotone sequences of real numbers.
2) The existence of a greatest lower bound for every sequence of real numbers which is bounded from below.
3) Π^0_1–CA and Π^0_1–AC.
4) The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$ (for every fixed d).
5) The Arzelà–Ascoli lemma.
6) The existence of lim sup and lim inf for bounded sequences in $\mathbb{R}$.

11.1 $(PCM2)$ and the convergence of bounded monotone sequences of real numbers

Let $a^{1(0)}$ be such that $\bigwedge n^0(0 \leq_{\mathbb{R}} a(n+1) \leq_{\mathbb{R}} an)$[69]
$(PCM2)$ implies

$$\bigvee h^1 \bigwedge k^0, m^0 (m \geq_0 hk \to a(hk) -_{\mathbb{R}} a(m) \leq_{\mathbb{R}} \frac{1}{k+1}).$$

$\big(a(hk)\big)_k$ is a Cauchy sequence with modulus $\frac{1}{k+1}$ whose limit equals the limit of $(a(m))_{n\in\mathbb{N}}$. The existence of a limit a_0 of $(a(m))_m$ now follows from the remarks below lemma 3.1.4 : $a_0 k := (a(h(\widehat{3(k+1)})))(3(k+1))$. Thus we only have to consider $(PCM2)$. In order to simplify the logical form of $(PCM2)$ we use the construction $\tilde{a}(n) := \max_{\mathbb{R}}(0, \min\limits_{i\leq n}(a(i))$ from chapter 9 (recall that this construction ensures that $\tilde{a}$ is monotone decreasing and bounded from below by 0. If a already fulfils these properties nothing is changed by the passage from a to $\tilde{a}$).

$$(PCM2)(a^{1(0)}) :\equiv \bigvee h^1 \bigwedge k^0, m^0 \big(m \geq_0 hk \to \tilde{a}(hk) -_{\mathbb{R}} \tilde{a}(m) \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

We now show that the contribution of single instances $(PCM2)(a)$ of $(PCM2)$ to the growth of uniform bounds is (at most) given by the functional $\Psi akg := \max\limits_{i<C(a)k'} \big(\Phi_{it} i0g\big)$ (where $\mathbb{N} \ni C(a) \geq \tilde{a}(0))$ from chapter 9:

[69] The restriction to the lower bound 0 is (convenient but) not essential: If $\bigwedge n^0(c \leq_{\mathbb{R}} a(n+1) \leq_{\mathbb{R}} an)$ we may define $a'(n) := a(n) -_{\mathbb{R}} c$. $(PCM2)$ applied to a' implies $(PCM2)$ for a. Everything holds analogously for increasing sequences which are bounded from above.

Proposition: 11.1.1 *Let $n \geq 2$ and $B_0(u^1, v^\tau, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula which contains only u^1, v^τ, w^γ free, where $\gamma \leq 2$. Furthermore let $\xi, t \in G_nR^\omega$ and Δ be as in thm.2.2.2. Then the following rule holds*

$$
\begin{cases}
G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big((PCM2)(\xi uv) \to \bigvee w^\gamma B_0(u,v,w)\big) \\
\Rightarrow \exists (eff.) \chi, \tilde{\chi} \in G_nR^\omega \textit{ such that} \\
G_nA_i^\omega + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \tilde{\Psi}^*\big((\tilde{\Psi}^* \textit{ satifies the mon.funct.interpr. of} \\
\qquad \bigwedge k^0, g^1 \bigvee n^0 (gn > n \to (\widetilde{\xi uv})(n) -_{\mathbb{R}} (\widetilde{\xi uv})(gn) \leq_{\mathbb{R}} \frac{1}{k+1})) \to \bigvee w \leq_\gamma \tilde{\chi} u \tilde{\Psi}^* B_0(u,v,w)\big) \\
\textit{and} \\
G_nA_i^\omega + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \Psi^*\big((\Psi^* \textit{ satifies the mon. funct.interpr. of} \\
\qquad \bigwedge a^{1(0)}, k^0, g^1 \bigvee n^0 (gn > n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1})) \to \bigvee w \leq_\gamma \chi u \Psi^*\, B_0(u,v,w)\big) \\
\textit{and therefore} \\
PRA_i^\omega + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w \leq_\gamma \chi u \Psi\, B_0(u,v,w),
\end{cases}
$$

where $\Psi := \lambda a, k, g. \max\limits_{i < C(a)k'} \big(\Phi_{it} i0g\big)$ and $C(a) := (a(0))(0) + 1$.

If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\bigvee$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta + AC$–$qf$ may be replaced by E–$G_nA^\omega + \Delta + AC^{\alpha,\beta}$–$qf$, where α, β are as in cor.10.10.

Proof: The existence of $\tilde{\chi}$ follows from cor.10.11 since

$$
\begin{aligned}
\mathrm{G_2A^\omega} \vdash \bigwedge a^{1(0)} \bigwedge k, \tilde{k}, n, \tilde{n}\big(\tilde{k} \leq_0 k \wedge \tilde{n} \geq_0 n \wedge \bigwedge m \geq_0 n(\tilde{a}(n) -_{\mathbb{R}} \tilde{a}(m) \leq_{\mathbb{R}} \tfrac{1}{k+1}) \\
\to \bigwedge m \geq_0 \tilde{n}(\tilde{a}(\tilde{n}) -_{\mathbb{R}} \tilde{a}(m) \leq_{\mathbb{R}} \tfrac{1}{\tilde{k}+1})\big).
\end{aligned}
$$

Ψ fulfils the monotone functional interpretation of
$\bigwedge a^{1(0)}, k^0, g^1 \bigvee n^0 (gn > n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1})$ (see the end of chapter 9) and hence $\Psi(\xi^*(u^M, t^*u^M))$ satisfies the monotone functional interpretation of

$$\bigwedge k^0, g^1 \bigvee n^0 (gn > n \to (\widetilde{\xi uv})(n) -_{\mathbb{R}} (\widetilde{\xi uv})(gn) \leq_{\mathbb{R}} \frac{1}{k+1}), \text{ where } \xi^* \text{ s–maj } \xi \wedge t^* \text{ s–maj } t.$$

χ is defined by $\chi := \lambda u, \Psi^*. \tilde{\chi} u\big(\Psi^*(\xi^*(u^M, t^*u^M))\big)$.

Remark 11.1.2 *1) The computation of the bound $\tilde{\chi}$ in the proposition above needs only a functional $\tilde{\Psi}^*$ which satifies the monotone functional interpretation of*

$$(+)\ \bigwedge k^0, g^1 \bigvee n^0 (gn > n \to (\widetilde{\xi uv})(n) -_{\mathbb{R}} (\widetilde{\xi uv})(gn) \leq_{\mathbb{R}} \frac{1}{k+1}).$$

For special ξ such a functional may be constructable without the use of Φ_{it}. Furthermore for fixed u the number of iterations of g only depends on the k–instances of $(+)$ which are used in the proof.

2) If the given proof of the assumption of this proposition applies Ψ only to functions g of low growth, then also the bound $\chi u \Psi$ is of low growth: e.g. if only $g := S$ is used and type/$w = 0$, then $\chi u \Psi$ is a polynomial in u^M.

Corollary to the proof of prop.11.1.1:
The rule

$$\begin{cases} \mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC\text{–}qf} \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(\bigvee f^0 \bigwedge k \bigwedge m, \tilde{m} > fk(|(\xi uv)(\tilde{m}) -_{\mathbb{R}} (\xi uv)(m)| \leq \frac{1}{k+1}) \to \\ \qquad\qquad\qquad\qquad \bigvee w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \\ \mathrm{G}_n\mathrm{A}^\omega_i + \Delta + \mathrm{b\text{-}AC} \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(\bigwedge k \bigvee n \bigwedge m, \tilde{m} > n(|(\xi uv)(\tilde{m}) -_{\mathbb{R}} (\xi uv)(m)| \leq \frac{1}{k+1}) \to \\ \qquad\qquad\qquad\qquad \bigvee w^\gamma B_0(u,v,w)\big) \end{cases}$$

holds for arbitrary sequences $(\xi uv)^{1(0)}$ of real numbers. The restriction to bounded monotone sequences $\widetilde{\xi uv}$ is used only to ensure the existence of a functional Ψ which satisfies the monotone functional interpretation of $(+)$ above.

We now consider a generalization $(PCM2^*)(a^{1(0)(0)}_{(\cdot)})$ of $(PCM2)(a^{1(0)})$ which asserts the existence of a sequence of Cauchy moduli for a sequence $\widetilde{a_l}$ of bounded monotone sequences:

$$(PCM2^*)(a^{1(0)(0)}_{(\cdot)}) :\equiv \bigvee h^{1(0)} \bigwedge l^0, k^0 \bigwedge m \geq_0 hkl\big(\widetilde{(a_l)}(hkl) -_{\mathbb{R}} \widetilde{(a_l)}(m) \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

Proposition: 11.1.3 *Let $n, B_0(u,v,w), t, \Delta$ be as in prop.11.1.1. $t, \xi \in G_nR^\omega$. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big((PCM2^*)(\xi uv) \to \bigvee w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.)\chi \in G_nR^\omega \text{ such that} \\ G_nA^\omega_i + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \Psi^*\big((\Psi^* \text{ satifies the mon. funct.interpr. of} \\ \bigwedge a^{1(0)(0)}, k^0, g^1 \bigvee n^0 (gn > n \to \bigwedge l \leq k(\widetilde{(a_l)}(n) -_{\mathbb{R}} \widetilde{(a_l)}(gn) \leq_{\mathbb{R}} \frac{1}{k+1}))) \to \bigvee w \leq_\gamma \chi u \Psi^* \, B_0(u,v,w)\big) \\ \Rightarrow PRA^\omega_i + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w \leq_\gamma \chi u \Psi' \, B_0(u,v,w), \end{cases}$$

where $\Psi' := \lambda a, k, g. \max\limits_{i < C(a,k)(k+1)^2} \big(\Phi_{it} i 0 g\big)$ *and* $\mathbb{N} \ni C(a,k) \geq \max_{\mathbb{R}}(\widetilde{(a_0)}(0), \ldots, \widetilde{(a_k)}(0))$.
If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\bigvee$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta + AC$–qf may be replaced by E–$G_nA^\omega + \Delta + AC^{\alpha,\beta}$–qf, where α, β are as in cor.10.10.
As in prop.11.1.1 we also have a term $\tilde{\chi}$ which needs only a $\tilde{\Psi}^*$ for the instance $a := \xi uv$.

Proof: The first part of the proposition follows from cor.10.11 since $(PCM2^*)(a)$ is implied by

$$\bigvee h^1 \bigwedge k^0 \bigwedge m \geq_0 hk \bigwedge l \leq_0 k\big(\widetilde{(a_l)}(hk) -_{\mathbb{R}} \widetilde{(a_l)}(m) \leq_{\mathbb{R}} \frac{1}{k+1}\big)$$

and

$$\begin{aligned} \mathrm{G}_2\mathrm{A}^\omega \vdash \bigwedge a^{1(0)(0)}_{(\cdot)} \bigwedge k, \tilde{k}, n, \tilde{n}\big(&\tilde{k} \leq_0 k \wedge \tilde{n} \geq_0 n \wedge \bigwedge m \geq_0 n \bigwedge l \leq_0 k(\widetilde{(a_l)}(n) -_{\mathbb{R}} \widetilde{(a_l)}(m) \leq_{\mathbb{R}} \tfrac{1}{k+1}) \\ &\to \bigwedge m \geq_0 \tilde{n} \bigwedge l \leq_0 \tilde{k}(\widetilde{(a_l)}(\tilde{n}) -_{\mathbb{R}} \widetilde{(a_l)}(m) \leq_{\mathbb{R}} \tfrac{1}{\tilde{k}+1})\big). \end{aligned}$$

It remains to show that Ψ' satisfies the monotone functional interpretation of $\bigwedge a^{1(0)(0)}, k^0, g^1 \bigvee n^0 \big(gn > n \to \bigwedge l \leq k\big(\widetilde{(a_l)}(n) - \widetilde{(a_l)}(gn) \leq \frac{1}{k+1}\big)\big)$:
Assume

$$\bigwedge i < C(a,k)(k+1)^2\big(g(g^i 0) > g^i 0 \wedge \bigvee l \leq k\big(\widetilde{(a_l)}(g^i 0) - \widetilde{(a_l)}(g(g^i 0)) > \frac{1}{k+1}\big)\big).$$

Then

$$\begin{array}{l} \bigwedge i < C(a,k)(k+1)^2\big(g(g^i 0) > g^i 0\big) \text{ and} \\ \bigvee l \leq k \bigvee j\big(\bigwedge i < C(a,k)(k+1) \dot{-} 1\big((j)_i < (j)_{i+1} < C(a,k)(k+1)^2\big) \wedge \\ \qquad\qquad \bigwedge i < C(a,k)(k+1)\big(\widetilde{(a_l)}(g^{(j)_i} 0) - \widetilde{(a_l)}(g(g^{(j)_i} 0)) > \frac{1}{k+1}\big)\big) \end{array}$$

and therefore

$$\begin{array}{l} \bigvee l \leq k \bigvee j\Big(\bigwedge i < C(a,k)(k+1) \dot{-} 1\big(g^{(j)_{i+1}} 0 > g^{(j)_i} 0 \wedge \widetilde{(a_l)}(g^{(j)_i} 0) - \widetilde{(a_l)}(g^{(j)_{i+1}} 0) > \frac{1}{k+1}\big) \\ \qquad\qquad \wedge g(g^{(j)_{C(a,k)(k+1) \dot{-} 1}}(0)) > g^{(j)_{C(a,k)(k+1) \dot{-} 1}}(0) \\ \qquad\qquad \wedge \widetilde{(a_l)}(g^{(j)_{C(a,k)(k+1) \dot{-} 1}}(0)) - \widetilde{(a_l)}(g(g^{(j)_{C(a,k)(k+1) \dot{-} 1}}(0))) > \frac{1}{k+1}\Big). \end{array}$$

Hence

$$\bigvee l \leq k \bigvee j \bigwedge i < C(a,k)(k+1)\big(g^{(j)_{i+1}} 0 > g^{(j)_i} 0 \wedge \widetilde{(a_l)}(g^{(j)_i} 0) - \widetilde{(a_l)}(g^{(j)_{i+1}} 0) > \frac{1}{k+1}\big),$$

which contradicts $\widetilde{(a_l)} \subset [0, C(a,k)]$.

11.2 The principle (GLB) 'every sequence of real numbers in $\mathbb{R}_+$ has a greatest lower bound'

This principle can be easily reduced to $(PCM2)$ (provably in $\mathrm{G_2A}^\omega$):
Let $a^{1(0)}$ be such that $\bigwedge n^0 (0 \leq_{\mathbb{R}} an)$. Then $(PCM2)(a)$ implies that the decreasing sequence $(\tilde{a}(n))_n \subset \mathbb{R}_+$ has a limit $\tilde{a}_0^1$. It is clear that $\tilde{a}_0$ is the greatest lower bound of $(a(n))_n \subset \mathbb{R}_+$. Thus we have shown

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \bigwedge a^{1(0)}\big((PCM2)(a) \to (GLB)(a)\big).$$

By this reduction we may replace $(PCM2)(\xi uv)$ by $(GLB)(\xi uv)$ in the assumption of prop.11.1.1. There is nothing lost (w.r.t to the rate of growth) in this reduction since in the other direction we have

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{0,0}\text{–qf} \vdash \bigwedge a^{1(0)}\big((GLB)(a) \to (PCM2)(a)\big):$$

Let $a^{1(0)}$ be as above and a_0 its greatest lower bound. Then $a_0 = \lim\limits_{n\to\infty} \tilde{a}_n$. Using $\mathrm{AC}^{0,0}$–qf one obtains (see chapter 4) a modulus of convergence and so a Cauchy modulus for $(\tilde{a}(n))_n$.

11.3 Π^0_1–CA and Π^0_1–AC

$$\Pi^0_1\text{–CA}(f^{1(0)}) :\equiv \bigvee g^1 \bigwedge x^0 \big(gx =_0 0 \leftrightarrow \bigwedge y^0 (fxy =_0 0)\big).$$

(Note that iteration of $\bigwedge f^{1(0)}(\Pi^0_1\text{–CA}(f))$ yields CA_{ar}).

Π^0_1–CA can also be reduced to $(PCM2)(a)$:

Proposition: 11.3.1

$$G_2A^\omega \vdash \bigwedge f^{1(0)} \big((PCM2)(\lambda n^0.\Psi_2 f'n) \to \Pi^0_1\text{–}CA(f)\big),$$

where $\Psi_2 \in G_2R^\omega$ is the functional from prop. 9.2.2) such that $\Psi_2 fn =_{\mathrm{IR}} 1 -_{\mathrm{IR}} \sum_{i=1}^{n} \frac{\chi fni}{i(i+1)}$ and $\chi \in G_2R^\omega$ such that

$$\chi fni =_0 \begin{cases} 1^0, \textit{if } \bigvee l \leq_0 n(fil =_0 0) \\ 0^0, \textit{otherwise}, \quad \textit{and} \end{cases}$$

$$f' := \lambda x, y.\overline{sg}(fxy).$$

Proof: ¿From the proof of prop.9.2.2) we know

$$(1)\ \bigwedge n^0 (0 \leq_{\mathrm{IR}} \Psi_2 f'(n+1) \leq_{\mathrm{IR}} \Psi_2 f'n)$$

and

$$(2)\ \begin{cases} \bigwedge x, n >_0 0\Big(\big(\bigwedge m, \tilde{m} \geq n \to |\Psi_2 f'm -_{\mathrm{IR}} \Psi_2 f'\tilde{m}| <_{\mathrm{IR}} \frac{1}{x(x+1)}\big) \to \\ \bigwedge \tilde{x}\big(0 <_0 \tilde{x} \leq_0 x \to \big(\bigvee y(f'\tilde{x}y = 0) \leftrightarrow \bigvee y \leq_0 n(f'\tilde{x}y = 0)\big)\big)\Big) \end{cases}$$

By (1) and $(PCM2)(\lambda n^0.\Psi_2 f'n)$ there exists a function h^1 such that

$$\bigwedge x >_0 0 \bigwedge m, \tilde{m} \geq_0 hx\big(|\Psi_2 f'm -_{\mathrm{IR}} \Psi_2 f'\tilde{m}| <_{\mathrm{IR}} \frac{1}{x(x+1)}\big).$$

Hence by (2)

$$\bigwedge x >_0 0\big(\bigvee y(f'xy = 0) \leftrightarrow \bigvee y \leq_0 hx(f'xy = 0)\big).$$

Furthermore, classical logic yields $\bigvee z_0(z_0 =_0 0 \leftrightarrow \bigwedge y(f0y = 0))$. Define

$$gx := \begin{cases} z_0, \text{ if } x = 0 \\ \varphi hf'x, \text{ otherwise,} \end{cases}$$

where

$$\varphi hfx := \begin{cases} 1^0, \text{ if } \bigvee y \leq hx(fxy = 0) \\ 0^0, \text{ otherwise.} \end{cases}$$

It follows that $\bigwedge x^0(gx = 0 \leftrightarrow \bigwedge y(fxy = 0))$, i.e. Π^0_1–CA(g).

Remark 11.3.2 *Proposition 11.3.1 in particular implies that relatively to G_2A^ω the principle $(PCM2)$ implies CA_{ar}. For a second order version RCA_0 of $\widehat{PA}^\omega\!\upharpoonright + AC^{0,0}$–qf (instead of G_2A^ω) this implication is stated in [17]. A proof (which is different to our proof) can be found in [61].*

Prop.11.1.1 combined with prop.11.3.1 yields

Proposition: 11.3.3 *Let $n, B_0(u,v,w), \xi, t, \Delta$ be as in prop.11.1.1. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(\Pi^0_1\text{–}CA(\xi uv) \to \bigvee w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.)\chi \in G_nR^\omega \text{ such that} \\ G_nA^\omega_i + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \Psi^*\big((\Psi^* \text{ satifies the mon. funct.interpr. of} \\ \qquad \bigwedge a^{1(0)}, k^0, g^1 \bigvee n^0(gn > n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1})) \to \bigvee w \leq_\gamma \chi u \Psi^* \, B_0(u,v,w)\big) \\ \Rightarrow PRA^\omega_i + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w \leq_\gamma \chi u \Psi \, B_0(u,v,w), \end{cases}$$

where $\Psi := \lambda a, k, g. \max\limits_{i<C(a)k'} \big(\Phi_{it} i0g\big).$

If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\bigvee$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta + AC$–qf may be replaced by E–$G_nA^\omega + \Delta + AC^{\alpha,\beta}$–qf, where α, β are as in cor.10.10.

As in prop.11.1.1 we also have a term $\tilde{\chi}$ which needs only a $\tilde{\Psi}^*$ for an instance $a := \zeta uv$ (where ζ is a suitable term in G_2R^ω).[70]

We now consider Π^0_1–instances of AC$_{ar}$:

$$\Pi^0_1\text{–AC}(f^{1(0)(0)(0)}) :\equiv \bigwedge l^0\big(\bigwedge x^0 \bigvee y^0 \bigwedge z^0 (flxyz =_0 0) \to \bigvee g^1 \bigwedge x^0, z^0 (flx(gx)z =_0 0)\big).$$

Π^0_1–AC(f) can be reduced to Π^0_1–CA(g) by

Proposition: 11.3.4

$G_2A^\omega + AC^{0,0}$–qf $\vdash \bigwedge f^{1(0)(0)(0)}\big(\Pi^0_1\text{–}CA(f') \to \Pi^0_1\text{–}AC(f)\big)$,

where $f' := \lambda v^0, z^0. f(\nu^3_1(v), \nu^3_2(v), \nu^3_3(v), z)$.

Proof: By Π^0_1–CA(f') there exists a function h^1 such that

$$\bigwedge v^0 (hv = 0 \leftrightarrow \bigwedge z (f'vz = 0)).$$

$\tilde{h}lxy := h(\nu^3(l,x,y))$. Then

$$\bigwedge l, x, y(\tilde{h}lxy = 0 \leftrightarrow \bigwedge z(flxyz = 0)).$$

AC^{0,0}–qf applied to $\bigwedge x \bigvee y(\tilde{h}lxy = 0)$ yields $\bigvee g \bigwedge x, z(flx(gx)z = 0)$.

As a corollary of prop.11.3.3 and prop.11.3.4 we obtain

Corollary 11.3.5 *Proposition 11.3.3 also holds with Π^0_1–$AC(\xi uv)$.*

[70] ζ is defined as the composition of Ψ_2 from prop.11.3.1 and ξ.

Arithmetical consequences of Π^0_1–CA(f) and Π^0_1–AC(f)
Using Π^0_1–CA(f) we can prove (relatively to $\mathrm{G_2A}^\omega$) every instance of Δ^0_2–IA with fixed function parameters:

$$\Delta^0_2\text{–IA}(f,g) :\equiv \begin{cases} \bigwedge l^0 \Big(\bigwedge x^0 \big(\bigvee u^0 \bigwedge v^0 (flxuv =_0 0) \leftrightarrow \bigwedge \tilde{u}^0 \bigvee \tilde{v}^0 (glx\tilde{u}\tilde{v} =_0 0) \big) \to \\ \qquad \big[\bigvee u \bigwedge v (fl0uv = 0) \wedge \bigwedge x (\bigvee u \bigwedge v (flxuv = 0) \to \bigvee u \bigwedge v (flx'uv = 0)) \\ \qquad\qquad \to \bigwedge x \bigvee u \bigwedge v (flxuv = 0) \big] \Big). \end{cases}$$

Define $f' := \lambda i^0, v^0. f(\nu^3_1(i), \nu^3_2(i), \nu^3_3(i), v)$ and $g' := \lambda i^0, v^0. \overline{sg}(g(\nu^3_1(i), \nu^3_2(i), \nu^3_3(i), v))$. We now show

Proposition: 11.3.6
$$G_2A^\omega + AC^{0,0}\text{–}qf \vdash \bigwedge f, g \big(\ \Pi^0_1\text{–}CA(f') \wedge \Pi^0_1\text{–}CA(g') \to \Delta^0_2\text{–}IA(f,g) \big).$$

Proof: Π^0_1–CA(f') and Π^0_1–CA(g') imply the existence of functions h_1, h_2 such that

$$h_1 lxu =_0 0 \leftrightarrow \bigwedge v (flxuv =_0 0) \text{ and } h_2 lxu =_0 0 \leftrightarrow \bigvee v (glxuv =_0 0).$$

Assume now that

$$\bigwedge x^0 \big(\bigvee u^0 \bigwedge v^0 (flxuv =_0 0) \leftrightarrow \bigwedge \tilde{u}^0 \bigvee \tilde{v}^0 (glx\tilde{u}\tilde{v} =_0 0) \big).$$

Then

$$\bigwedge x \big(\bigvee u (h_1 lxu = 0) \leftrightarrow \bigwedge \tilde{u} (h_2 lx\tilde{u} = 0) \big).$$

With classical logic this yields

$$\bigwedge x \bigvee z^0 \big(\underbrace{\big[\bigwedge \tilde{u} (h_2 lx\tilde{u} = 0) \to z = 0 \big] \wedge \big[z = 0 \to \bigvee u (h_1 lxu = 0) \big]}_{\in \Sigma^0_1} \big).$$

By $\mathrm{AC}^{0,0}$–qf we obtain a function α such that

$$\bigwedge x (\alpha x = 0 \leftrightarrow \bigvee u (h_1 lxu = 0)).$$

Δ^0_2–IA(f,g) now follows by applying QF–IA to $A_0(x) :\equiv (\alpha x = 0)$.

Next we show that Π^0_1–instances (with fixed function parameters) of the so–called 'collection principle'[71]

$$\mathrm{CP} \ : \ \bigwedge \tilde{x} <_0 x \bigvee y^0 A(x, \tilde{x}, y) \to \bigvee y_0 \bigwedge \tilde{x} <_0 x \bigvee y <_0 y_0 A(x, \tilde{x}, y).$$

are derivable from Π^0_1–AC–instances.

$$\Pi^0_1\text{–CP}(f) :\equiv \bigwedge l^0, x^0 \big(\bigwedge \tilde{x} < x \bigvee y^0 \bigwedge z^0 (flx\tilde{x}yz =_0 0) \to \bigvee y_0 \bigwedge \tilde{x} < x \bigvee y <_0 y_0 \bigwedge z (flx\tilde{x}yz = 0) \big).$$

[71]For a detailed discussion of this principle and its relation to induction see [52].

Proposition: 11.3.7

$G_2A^\omega \vdash \bigwedge f\big(\Pi^0_1\text{–}AC(f') \to \Pi^0_1\text{–}CP(f)\big)$,

where f' *such that* $f'i\tilde{x}yz =_0 0 \leftrightarrow (\tilde{x} < \nu^2_2(i) \to f(\nu^2_1(i), \nu^2_2(i), \tilde{x}, y, z) =_0 0)$.

Proof: Π^0_1–AC(f') yields

$$\bigwedge l^0, x^0\big(\bigwedge \tilde{x} < x \bigvee y \bigwedge z(flx\tilde{x}yz = 0) \to \bigvee h^1 \bigwedge \tilde{x} < x \bigwedge z(flx\tilde{x}(h\tilde{x})z = 0)\big).$$

Define $y_0 := 1 + \Phi_1 hx$ (Recall that $\Phi_1 hx := \max\limits_{i \leq x}(hi)$).

We conclude this paragraph by showing that cor.10.11 is false when $\mathrm{G}_n\mathrm{A}^\omega$, $\mathrm{G}_n\mathrm{R}^\omega$, $\mathrm{G}_n\mathrm{A}^\omega_i$ are replaced by $\mathrm{G}_n\mathrm{A}^\omega + \Sigma^0_1$–IA, $\widehat{PR}^\omega$, $\mathrm{G}_n\mathrm{A}^\omega_i + \Sigma^0_1$–IA or PRA^ω, $\widehat{PR}^\omega$, PRA^ω_i:

It is well–known that there is an (function parameter–free) instance G of Π^0_2–IA such that

$$\mathrm{G}_2\mathrm{A}^\omega + \Sigma^0_1\text{–IA } + G \vdash \bigwedge x^0 \bigvee y^0 A_0(x, y),$$

where $\bigwedge x \bigvee y \leq fx\, A_0(x, y)$ implies that f has the growth of the Ackermann function.

Let $B(x^0) :\equiv \bigwedge u^0 \bigvee v^0 B_0(a^0, u, v, x)$ be the induction formula of G, where $B_0(a, u, v, x)$ contains only a, u, v, x as free variables. By applying Π^0_1–CA(f) to $f := \overline{sg} \circ t_{B_0}$, where t_{B_0} is the characterictic function of B_0, G reduces to an instance of Σ^0_1–IA. Hence

$$\mathrm{G}_2\mathrm{A}^\omega + \Sigma^0_1\text{–IA } \vdash \Pi^0_1\text{–CA}(f) \to \bigwedge x \bigvee y\, A_0(x, y).$$

If cor.10.11 would apply to $\mathrm{G}_2\mathrm{A}^\omega + \Sigma^0_1$–IA and $\widehat{PR}^\omega$ we would obtain (by the proof of prop. 11.3.3) a term $s^1 \in \widehat{PR}^\omega$ such that $\bigwedge x \bigvee y \leq sx\, A_0(x, y)$. This however would contradict the well–known fact that every $s^1 \in \widehat{PR}^\omega$ is primitive recursive.

The same argument applies to PRA^ω since PRA^ω+$\mathrm{AC}^{0,0}$ –qf $\vdash \Sigma^0_1$–IA (see e.g. [32],pp.8–9).

11.4 The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$ (for every fixed d)

We now consider the Bolzano–Weierstraß principle for sequences in $[-1, 1]^d \subset \mathbb{R}^d$. The restriction to the special bound 1 is convenient but not essential: If $(x_n) \subset \mathbb{R}^d$ is bounded by $C > 0$, we define $x'_n := \frac{1}{C} \cdot x_n$ and apply the Bolzano–Weierstraß principle to this sequence. For simplicity we formulate the Bolzano– Weierstraß principle w.r.t. the maximum norm $\|\cdot\|_{\max}$. This of course implies the principle for the Euclidean norm $\|\cdot\|_E$ since $\|\cdot\|_E \leq \sqrt{d} \cdot \|\cdot\|_{\max}$.

We start with the investigation of the following formulation of the Bolzano–Weierstraß principle:

$$BW \ : \ \bigwedge(x_n) \subset [-1, 1]^d \bigvee x \in [-1, 1]^d \bigwedge k^0, m^0 \bigvee n >_0 m\big(\|x - x_n\|_{\max} \leq \frac{1}{k+1}\big),$$

i.e. (x_n) possesses a limit point x.

Later on we discuss a second formulation which (relatively to $\mathrm{G}_n\mathrm{A}^\omega$) is slightly stronger than BW:

$$BW^+ \ : \ \begin{cases} \bigwedge(x_n) \subset [-1, 1]^d \bigvee x \in [-1, 1]^d \bigvee f^1\big(\bigwedge n^0(fn <_0 f(n+1)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \wedge \bigwedge k^0\big(\|x - x_{fk}\|_{\max} \leq \frac{1}{k+1}\big)\big), \end{cases}$$

i.e. (x_n) has a subsequence (x_{fn}) which converges (to x) with the modulus $\frac{1}{k+1}$.
Using our representation of $[-1,1]$ from chapter 3, the principle BW has the following form

$$\bigwedge x_1^{1(0)},\dots,x_d^{1(0)} \underbrace{\bigvee a_1,\dots,a_d \leq_1 M \bigwedge k^0, m^0 \bigvee n >_0 m \bigwedge_{i=1}^{d} \big(|\tilde{a}_i -_{\mathbb{R}} \widetilde{x_i n}| \leq_{\mathbb{R}} \frac{1}{k+1}\big)}_{BW(\underline{x}^{1(0)}):\equiv},$$

where M and $y^1 \mapsto \tilde{y}$ are the constructions from our representation of $[-1,1]$ in chapter 3. We now prove

$$(*)\ \mathrm{G_2A}^\omega + \mathrm{AC}^{1,0}\text{–qf} \ \vdash F^- \to \bigwedge x_1^{1(0)},\dots,x_d^{1(0)} \big(\Pi_1^0\text{–CA}(\chi\underline{x}) \to BW(\underline{x})\big),$$

for a suitable $\chi \in \mathrm{G_2R}^\omega$:

$BW(\underline{x})$ is equivalent to

$$(1)\ \bigvee a_1,\dots,a_d \leq_1 M \bigwedge k^0 \bigvee n >_0 k \bigwedge_{i=1}^{d} \big(|\tilde{a}_i -_{\mathbb{R}} \widetilde{x_i n}| \leq_{\mathbb{R}} \frac{1}{k+1}\big)$$

which in turn is equivalent to

$$(2)\ \bigvee a_1,\dots,a_d \leq_1 M \bigwedge k^0 \bigvee n >_0 k \bigwedge_{i=1}^{d} \big(|\tilde{a}_i k -_{\mathbb{Q}} (\widetilde{x_i n})(k)| \leq_{\mathbb{Q}} \frac{3}{k+1}\big).$$

Assume $\neg(2)$, i.e.

$$(3)\ \bigwedge a_1,\dots,a_d \leq_1 M \bigvee k^0 \bigwedge n >_0 k \bigvee_{i=1}^{d} \big(|\tilde{a}_i k -_{\mathbb{Q}} (\widetilde{x_i n})(k)| >_{\mathbb{Q}} \frac{3}{k+1}\big).$$

Let $\chi \in \mathrm{G_2R}^\omega$ be such that

$$\begin{aligned}\mathrm{G_2A}^\omega \vdash \bigwedge x_1^{1(0)},\dots,x_d^{1(0)} \bigwedge l^0, n^0 \big(\chi \underline{x} l n =_0 0 \leftrightarrow \\ \big[n >_0 \nu_{d+1}^{d+1}(l) \to \bigvee_{i=1}^{d} |\nu_i^{d+1}(l) -_{\mathbb{Q}} (\widetilde{x_i n})(\nu_{d+1}^{d+1}(l))| >_{\mathbb{Q}} \tfrac{3}{\nu_{d+1}^{d+1}(l)+1}\big]\big).\end{aligned}$$

Π_1^0–CA$(\chi\underline{x})$ yields the existence of a function h such that

$$(4)\ \bigwedge l_1^0,\dots. l_d^0, k^0 \big(h l_1 \dots, l_d k =_0 0 \leftrightarrow \bigwedge n >_0 k \bigvee_{i=1}^{d} \big(|l_i -_{\mathbb{Q}} (\widetilde{x_i n})(k)| >_{\mathbb{Q}} \frac{3}{k+1}\big).$$

Using h, (3) has the form

$$(5)\ \bigwedge a_1,\dots,a_d \leq_1 M \bigvee k^0 \big(h(\tilde{a}_1 k,\dots,\tilde{a}_d k, k) =_0 0\big).$$

By Σ_1^0–UB$^-$ (which follows from AC^{1,0}–qf and F^- by prop. 7.2.19) we obtain

$$(6)\ \bigvee k_0 \bigwedge a_1,\dots,a_d \leq_1 M \bigwedge m^0 \bigvee k \leq_0 k_0 \bigwedge n >_0 k \bigvee_{i=1}^{d} \big(|(\widetilde{\overline{a_i}, m})(k) -_{\mathbb{Q}} (\widetilde{x_i n})(k)| >_{\mathbb{Q}} \frac{3}{k+1}\big)$$

and therefore

$$(7)\ \bigvee k_0 \bigwedge a_1,\dots,a_d \le_1 M \bigwedge m^0 \bigwedge n >_0 k_0 \bigvee_{i=1}^{d} \big(|(\widetilde{\overline{a_i,m}}) -_{\mathrm{I\!R}} \widetilde{x_i n}| >_{\mathrm{I\!R}} \frac{1}{k_0+1}\big).$$

Since $|\widetilde{\overline{a_i,3(m+1)}} -_{\mathrm{I\!R}} \tilde{a}_i| <_{\mathrm{I\!R}} \frac{2}{m+1}$ (see the definition of $y \mapsto \tilde{y}$ from chapter 3) it follows

$$(8)\ \bigvee k_0 \bigwedge a_1,\dots,a_d \le_1 M \bigwedge n >_0 k_0 \bigvee_{i=1}^{d} \big(|\tilde{a}_i -_{\mathrm{I\!R}} \widetilde{x_i n}| >_{\mathrm{I\!R}} \frac{1}{2(k_0+1)}\big),\ \text{i.e.}$$

$$(9)\ \bigvee k_0 \bigwedge (a_1,\dots,a_d) \in [-1,1]^d \bigwedge n >_0 k_0 \big(\|\underline{a} - \underline{x}n\|_{\max} > \frac{1}{2(k_0+1)}\big).$$

By applying this to $\underline{a} := \underline{x}(k_0+1)$ yields the contradiction $\|\underline{x}(k_0+1) - \underline{x}(k_0+1)\|_{\max} > \frac{1}{2(k_0+1)}$, which concludes the proof of $(*)$.

Remark 11.4.1 In the proof of $(*)$ we used a combination of Π^0_1–CA(f) and Σ^0_1–UB$^-$ to obtain a restricted form Π^0_1–UB$^-\restriction$ of the extension of Σ^0_1–UB$^-$ to Π^0_1–formulas:

$$\Pi^0_1\text{–UB}^-\restriction\ :\ \begin{cases} \bigwedge f \le_1 s \bigvee n^0 \bigwedge k^0 A_0(t^0[f],n,k) \to \\ \bigvee n_0 \bigwedge f \le_1 s \bigwedge m^0 \bigvee n \le_0 n_0 \bigwedge k^0 A_0(t[\overline{f,m}],n,k), \end{cases}$$

where k does not occur in $t[f]$ and f does not occur in $A_0(0,0,0)$.
Π^0_1–UB$^-\restriction$ follows by applying Π^0_1–CA to $\lambda n,k.t_{A_0}(a^0,n^0,k^0)$, where t_{A_0} is such that $t_{A_0}(a^0,n^0,k^0) =_0 0 \leftrightarrow A_0(a^0,n^0,k^0)$, and subsequent application of Σ^0_1–UB$^-$.
Π^0_1–CA and Σ^0_1–UB$^-$ do not imply the unrestricted form Π^0_1–UB$^-$ of Π^0_1–UB$^-\restriction$:

$$\Pi^0_1\text{–UB}^-\ \begin{cases} \bigwedge f \le_1 s \bigvee n^0 \bigwedge k^0 A_0(f,n,k) \to \\ \bigvee n_0 \bigwedge f \le_1 s \bigwedge m^0 \bigvee n \le_0 n_0 \bigwedge k^0 A_0((\overline{f,m}),n,k) \end{cases}$$

since a reduction of Π^0_1–UB$^-$ to Σ^0_1–UB$^-$ would require a comprehension functional in f:

$$(+)\ \bigvee \Phi \bigwedge f^1, n^0(\Phi f n =_0 0 \leftrightarrow \bigwedge k^0 A_0(f,n,k)).$$

In fact Π^0_1–UB$^-$ can easily be refuted by applying it to $\bigwedge f \le_1 \lambda x.1 \bigvee n^0 \bigwedge k^0 (fk = 0 \to fn = 0)$, which leads to a contradiction. This reflects the fact that we had to use F^- to derive Σ^0_1–UB$^-$, which is incompatible with $(+)$ since Φ+AC^{1,0}–qf produces (see above) a non–majorizable functional, whereas F^- is true only in $\mathcal{M}^\omega$.

Next we prove

$$(**)\ \mathrm{G_2A}^\omega + \mathrm{AC}^{0,0}\text{–qf} \vdash \bigwedge x_1^{1(0)},\dots,x_d^{1(0)} \big(\Sigma^0_1\text{–IA}(\chi\underline{x}) \wedge BW(\underline{x}) \to BW^+(\underline{x})\big)$$

for a suitable term $\chi \in \mathrm{G_2R}^\omega$, where

$$\Sigma^0_1\text{–IA}(f) :\equiv \begin{cases} \bigwedge l^0 \big(\bigvee y^0 (fl0y =_0 0) \wedge \bigwedge x^0 (\bigvee y(flxy = 0) \to \bigvee y(flx'y = 0)) \\ \qquad\qquad \to \bigwedge x \bigvee y(flxy = 0)\big). \end{cases}$$

$BW(\underline{x})$ implies the existence of $a_1, \ldots, a_d \leq_1 M$ such that

$$(10)\ \bigwedge k, m \bigvee n > m \bigwedge_{i=1}^{d} \big(|\tilde{a}_i(2(k+1)(k+2)) -_{\mathbb{Q}} (\widetilde{x_i n})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \frac{1}{k+1}\big).$$

Define (for $x_1^{1(0)}, \ldots, x_d^{1(0)}, l_1^0, \ldots, l_d^0$)

$$F(\underline{x}, \underline{l}, k, m, n) :\equiv$$
$$\big(\underline{x}n \text{ is the } m\text{–th element in } (\underline{x}(l))_l \text{ such that } \bigwedge_{i=1}^{d} \big(|l_i -_{\mathbb{Q}} (\widetilde{x_i n})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \tfrac{1}{k+1}\big)\big).$$

One easily verifies that $F(\underline{x}, \underline{l}, k, m, n)$ can be expressed in the form $\bigvee a^0 F_0(\underline{x}, \underline{l}, k, m, n, a)$, where F_0 is a quantifier–free formula in $\mathcal{L}(\mathrm{G_2A}^\omega)$, which contains only $\underline{x}, \underline{l}, k, m, n, a$ as free variables. Let $\tilde{\chi} \in \mathrm{G_2R}^\omega$ such that

$$\tilde{\chi}(\underline{x}, \underline{l}, k, m, n, a) =_0 0 \leftrightarrow F_0(\underline{x}, \underline{l}, k, m, n, a)$$

and define $\chi(\underline{x}, q, m, p) := \tilde{\chi}(\underline{x}, \nu_1^{d+1}(q), \ldots, \nu_{d+1}^{d+1}(q), m, j_1(p), j_2(p))$.
Σ_1^0–IA$(\chi \underline{x})$ yields

$$(11) \begin{cases} \bigwedge l_1, \ldots, l_d, k\big(\bigvee n\, F(\underline{x}, \underline{l}, k, 0, n) \wedge \bigwedge m\big(\bigvee n F(\underline{x}, \underline{l}, k, m, n) \rightarrow \bigvee n F(\underline{x}, \underline{l}, k, m', n)\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \rightarrow \bigwedge m \bigvee n F(\underline{x}, \underline{l}, k, m, n)\big). \end{cases}$$

(10) and (11) imply

$$(12) \begin{cases} \bigwedge k, m \bigvee n\big(\underline{x}n \text{ is the } m\text{–th element of } (\underline{x}(l))_l \text{ such that} \\ \qquad \bigwedge_{i=1}^{d} \big(|\tilde{a}_i(2(k+1)(k+2)) -_{\mathbb{Q}} (\widetilde{x_i n})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \tfrac{1}{k+1}\big)\big). \end{cases}$$

and therefore

$$(13) \begin{cases} \bigwedge k \bigvee n\big(\underline{x}n \text{ is the } k\text{–th element of } (\underline{x}(l))_l \text{ such that} \\ \qquad \bigwedge_{i=1}^{d} \big(|\tilde{a}_i(2(k+1)(k+2)) -_{\mathbb{Q}} (\widetilde{x_i n})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \tfrac{1}{k+1}\big)\big). \end{cases}$$

By $\mathrm{AC}^{0,0}$–qf we obtain a function g^1 such that

$$(14) \begin{cases} \bigwedge k\big(\underline{x}(gk) \text{ is the } k\text{–th element of } (\underline{x}(l))_l \text{ such that} \\ \qquad \bigwedge_{i=1}^{d} \big(|\tilde{a}_i(2(k+1)(k+2)) -_{\mathbb{Q}} (\widetilde{x_i(gk)})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \tfrac{1}{k+1}\big)\big). \end{cases}$$

We show

$$(15)\ \bigwedge k(gk < g(k+1)) :$$

Define $A_0(\underline{x}l, k) :\equiv \bigwedge_{i=1}^{d} \big(|\tilde{a}_i(2(k+1)(k+2)) -_{\mathbb{Q}} (\widetilde{x_i l})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \frac{1}{k+1}\big)$. Let l be such that $A_0(\underline{x}l, k+1)$. Because of

$$|\tilde{a}_i(2(k+1)(k+2)) -_{\mathbb{Q}} (\widetilde{x_i l})(2(k+1)(k+2))| \leq$$
$$|\tilde{a}_i(2(k+2)(k+3)) -_{\mathbb{Q}} (\widetilde{x_i l})(2(k+2)(k+3))| + \tfrac{2}{2(k+1)(k+2)} \overset{A_0(\underline{x}l, k+1)}{\leq}$$
$$\tfrac{1}{k+2} + \tfrac{2}{2(k+1)(k+2)} = \tfrac{1}{k+1},$$

this yields $A_0(\underline{x}l,k)$. Thus the $(k+1)$–th element $\underline{x}l$ such that $A_0(\underline{x}l,k+1)$ is at least the $(k+1)$–th element such that $A_0(\underline{x}l,k)$ and therefore occurs later in the sequence than the k-th element such that $A_0(\underline{x}l,k)$, i.e. $gk < g(k+1)$.
It remains to show

$$(16)\ \bigwedge k \bigwedge_{i=1}^{d} \big(|\tilde{a}_i -_{\mathrm{IR}} \widetilde{x_i(fk)}| \leq_{\mathrm{IR}} \frac{1}{k+1}\big),\ \text{where } fk := g(2(k+1)):$$

This follows since

$$\bigwedge_{i=1}^{d} \big(|\tilde{a}_i(2(k+1)(k+2)) -_{\mathbb{Q}} (\widetilde{x_i(gk)})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \frac{1}{k+1}\big)$$

implies

$$\bigwedge_{i=1}^{d} \big(|\tilde{a}_i -_{\mathrm{IR}} \widetilde{x_i(gk)}| \leq_{\mathrm{IR}} \frac{1}{k+1} + \frac{2}{2(k+1)(k+2)+1} \leq \frac{2}{k+1}\big).$$

(15) and (16) imply $BW^+(\underline{x})$ which concludes the proof of $(**)$.

Remark 11.4.2 One might ask why we did not use the following obvious proof of $BW^+(\underline{x})$ from $BW(\underline{x})$:
Let $\underline{a}$ be such that $\bigwedge k \bigvee n > k \bigwedge_{i=1}^{d} \big(|\tilde{a}_i -_{\mathrm{IR}} \widetilde{x_i n}| <_{\mathrm{IR}} \frac{1}{k+1}\big)$. $\mathrm{AC}^{0,0}$–qf yields the existence of a function g such that

$$\bigwedge k (gk > k \wedge \bigwedge_{i=1}^{d} (|\tilde{a}_i -_{\mathrm{IR}} \widetilde{x_i(gk)}| <_{\mathrm{IR}} \frac{1}{k+1})).$$

Now define $fk := g^{(k+1)}(0)$. It is clear that f fulfils $BW^+(\underline{x})$.
The problem with this proof is that we cannot use our results from chapter 10 in the presence of the iteration functional Φ_{it} (see §3 above) which is needed to define f as a functional in g. To introduce the graph of Φ_{it} by Σ^0_1–IA and AC–qf does not help since this would require an application of Σ^0_1–IA which involves (besides $\underline{x}$) also $\underline{a}$ as genuine function parameters. In contrast to this situation, our proof of $BW(\underline{x}) \to BW^+(\underline{x})$ uses Σ^0_1–IA only for a formula with (besides $\underline{x}$) only $k, \underline{a}k$ as parameters. Since k (as a parameter) remains fixed throughout the induction, $\underline{a}$ only occurs as the **number parameter** $\underline{a}k$ but **not as genuine function parameter**. This is the reason why we are able to construct a term χ such that $\Sigma^0_1\text{–IA}(\chi\underline{x}) \wedge BW(\underline{x}) \to BW^+(\underline{x})$.

Using $(*)$ and $(**)$ we are now able to prove

Proposition: 11.4.3 *Let $n \geq 2$ and $B_0(u^1, v^\tau, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula which contains only u^1, v^τ, w^γ free, where $\gamma \leq 2$. Furthermore let $\underline{\xi}, t \in G_nR^\omega$ and Δ be as in thm.2.2.2.*

Then the following rule holds

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(BW^+(\underline{\xi}uv) \to \bigvee w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.)\chi \in G_nR^\omega \text{ such that} \\ G_{\max(n,3)}A_i^\omega + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \Psi^*\big((\Psi^* \text{ satifies the mon. funct.interpr. of} \\ \qquad \bigwedge a^{1(0)}, k^0, g^1 \bigvee n^0(gn > n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1})) \to \bigvee w \leq_\gamma \chi u \Psi^* B_0(u,v,w)\big) \\ \Rightarrow PRA_i^\omega + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w \leq_\gamma \chi u \Psi B_0(u,v,w), \end{cases}$$

where $\Psi := \lambda a, k, g. \max\limits_{i<C(a)k'} \big(\Phi_{it} i0g\big)$.

If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\bigvee$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta + AC$–qf may be replaced by E–$G_nA^\omega + \Delta + AC^{\alpha,\beta}$–qf, where α, β are as in cor.10.10.

As in prop.11.1.1 we also have a term $\tilde{\chi}$ which needs only a $\tilde{\Psi}^*$ for an instance $a := \zeta uv$ (where ζ is a suitable term in G_2R^ω).

Proof: By $(*)$,$(**)$ and the proof of prop.11.3.6 there are functionals $\varphi_1, \varphi_2 \in \mathrm{G_2R}^\omega$ such that

$$\mathrm{G_2A}^\omega + \mathrm{AC}^{1,0}\text{–qf} \vdash F^- \to \bigwedge \underline{x}\big(\Pi_1^0\text{–CA}(\varphi_1 \underline{x}) \wedge \Pi_1^0\text{–CA}(\varphi_2 \underline{x}) \to BW^+(\underline{x})\big).$$

Furthermore

$$\mathrm{G_2A}^\omega \vdash \Pi_1^0\text{–CA}(\psi f_1 f_2) \to \Pi_1^0\text{–CA}(f_1) \wedge \Pi_1^0\text{–CA}(f_2),$$

where

$$\psi f_1 f_2 x^0 y^0 =_0 \begin{cases} f_1(j_2x, y), \text{ if } j_1x = 0 \\ f_2(j_2x, y), \text{ otherwise.} \end{cases}$$

Hence

$$\mathrm{G_2A}^\omega + \mathrm{AC}^{1,0}\text{–qf} \vdash F^- \to \bigwedge \underline{x}\big(\Pi_1^0\text{–CA}(\varphi_3 \underline{x}) \to BW^+(\underline{x})\big),$$

for a suitable $\varphi_3 \in \mathrm{G_2R}^\omega$ and thus

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC}\text{–qf} \vdash F^- \to \bigwedge u^1 \bigwedge v \leq_\tau tu\big(\Pi_1^0\text{–CA}(\varphi_3(\underline{\xi}uv)) \to \bigvee w\, B_0\big).$$

By the proof of thm.7.2.20 we obtain

$$\mathrm{G}_n\mathrm{A}^\omega + \widetilde{\Delta} + (*) + \mathrm{AC}\text{–qf} \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(\Pi_1^0\text{–CA}(\varphi_3(\underline{\xi}uv)) \to \bigvee w\, B_0\big),$$

where

$$\widetilde{\Delta} := \{\bigvee Y \leq_{\rho\delta} s \bigwedge x^\delta, z^\eta A_0(x, Yx, z) : \bigwedge x \bigvee y \leq sx \bigwedge z^\eta A_0 \in \Delta\},$$

$$(*) :\equiv \bigwedge n_0 \bigvee Y \leq \lambda \Phi^{2(0)}, y^{1(0)}.y \bigwedge \Phi, \tilde{y}^{1(0)}, k^0, \tilde{z}^1 \bigwedge n \leq_0 n_0\big(\bigwedge_{i<n}(\tilde{z}i \leq \tilde{y}ki) \to \Phi k(\overline{\tilde{z}, n}) \leq \Phi k(Y\Phi\tilde{y}k)\big).$$

Prop.11.3.3 (with $\Delta' := \widetilde{\Delta} \cup \{(*)\}$) yields the conclusion of our proposition in $\mathrm{G}_n\mathrm{A}_i^\omega + \Delta + (*) +$ b–AC and so (since, again by the proof of thm.7.2.20, $\mathrm{G_3A}_i^\omega \vdash (*)$) in $\mathrm{G}_{\max(3,n)}\mathrm{A}_i^\omega + \Delta +$ b–AC.

Remark 11.4.4 *Analogously to $(PCM2^*)$ one can generalize $BW^+(\underline{x})$ to $BW^*(\underline{x}_{(\cdot)})$, where $BW^*(\underline{x}_{(\cdot)})$ asserts the existence of a sequence of subsequences for a sequence of bounded sequences.*

11.5 The Arzela–Ascoli lemma

Under the name 'Arzelà–Ascoli' lemma' we understand (as in the literature on 'reverse mathematics') the following proposition:
Let $(f_l) \subset C[0,1]$ be a sequence of functions[72] which are equicontinuous and have a common bound, i.e. there exists a common modulus of uniform continuity ω for all f_l and a bound $C \in \mathbb{N}$ such that $\|f_l\|_\infty \le C$. Then

(i) (f_l) possesses a limit point w.r.t. $\|\cdot\|_\infty$ which also has the modulus ω, i.e.

$$\bigvee f \in C[0,1]\big(\bigwedge k^0 \bigwedge m \bigvee n >_0 m\big(\|f - f_n\|_\infty \le \frac{1}{k+1}\big) \wedge f \text{ has modulus } \omega\big);$$

(ii) there is a subsequence (f_{gl}) of (f_l) which converges with modulus $\frac{1}{k+1}$.

As in the case of the Bolzano–Weierstraß principle we deal first with (i). The sligthly stronger assertion (ii) can then be obtained from (i) using Σ^0_1–IA(f) analogously to our proof of $BW^+(\underline{x})$ from $BW(\underline{x})$. For notational simplicity we may assume that $C = 1$. When formalized in $\mathrm{G}_n\mathrm{A}^\omega$, the version (i) of the Arzelà–Ascoli lemma has the form

$$\begin{aligned}
&\text{A–A}(f^{1(0)(0)}_{(\cdot)}, \omega^1) :\equiv \Big(f_{(\cdot)} \le_{1(0)(0)} \lambda l^0, n^0.M \wedge \\
&\bigwedge l^0, m^0, u^0, v^0 \overbrace{\Big(|qu -_{\mathbb{Q}} qv| \le_{\mathbb{Q}} \frac{1}{\omega(m)+1} \to |\widetilde{f_l u} -_{\mathbb{R}} \widetilde{f_l v}| \le_{\mathbb{R}} \frac{1}{m+1}\Big)}^{\Pi^0_1 \ni F(f_l,m,u,v) :\equiv \bigwedge a^0 F_0(f_l,m,u,v,a) :\equiv} \\
&\to \bigvee g \le_{1(0)} \lambda n.M\big(\bigwedge m,u,v F(g,m,u,v) \wedge \bigwedge k \bigvee n >_0 k(\|\lambda x^1.g(x)_{\mathbb{R}} - \lambda x^1.f_n(x)_{\mathbb{R}}\|_\infty \le \tfrac{1}{k+1})\big)\Big).
\end{aligned}$$

Here M, q and $y^1 \mapsto \tilde{y}$ are the constructions from our representation of $[-1,1]$ in chapter 3. For notational simplicity we omit in the following $\tilde{(\;)}$.
A–A(f,ω) is equivalent to[73]

$$\begin{aligned}
&f_{(\cdot)} \le l^0, n^0.M \wedge \bigwedge l^0, m^0, u^0, v^0 F(f_l,m,u,v) \to \bigvee g \le_{1(0)} \lambda n.M\big(\bigwedge m,u,v F(g,m,u,v) \wedge \\
&\bigwedge k \bigvee n >_0 k \bigwedge_{i=0}^{\omega(k)+1} \big(|g(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k)| \le_{\mathbb{Q}} \tfrac{5}{k+1}\big)\big).
\end{aligned}$$

Assume $\neg$A–A(f,ω), i.e. $f_{(\cdot)} \le \lambda l^0, n^0 M \wedge \bigwedge l,m,u,v F(f_l,m,u,v)$ and

$$(1)\ \begin{cases} \bigwedge g \le_{1(0)} \lambda n.M\Big(\bigwedge m,u,v\, F(g,m,u,v) \to \\ \bigvee k \bigwedge n\big(n >_0 k \to \bigvee_{i=0}^{\omega(k)+1} \big(|g(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \tfrac{5}{k+1}\big)\big)\Big). \end{cases}$$

Let α be such that

$$\alpha(l^0, k^0, n^0) =_0 0 \leftrightarrow \big[n > k \to \bigvee_{i=0}^{\omega(k)+1} \big(|(l)_i -_{\mathbb{Q}} f_n(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \frac{5}{k+1}\big)\big].$$

[72] The restriction to the unit interval $[0,1]$ is convenient for the following proofs but not essential.
[73] For better readability we write $\frac{i}{\omega(k)+1}$ instead of its code.

Π^0_1–CA(α') (where $\alpha' in := \alpha(j_1 i, j_2 i, n)$) yields the existence of a function h such that

$$hlk =_0 0 \leftrightarrow \bigwedge n\big(\alpha(l,k,n)=0\big).$$

Hence

$$(2)\ \begin{cases} h(\overline{\lambda i.g(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k)}(\omega(k)+1),k) =_0 0 \leftrightarrow \\ \bigwedge n >_0 k \bigvee\limits_{i=0}^{\omega(k)+1} \big(|g(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \frac{5}{k+1}\big). \end{cases}$$

(1),(2) and Σ^0_1–UB$^-$ yield (using the fact that g can be coded into a type–1–object by $g'x^0 := g(j_1 x, j_2 x)$)

$$(3)\ \begin{cases} \bigvee k_0 \bigwedge g' \leq_1 \lambda x.M(j_1 x) \bigwedge l^0 \Big(\bigwedge m,u,v,a \leq k_0 F_0(\lambda x,y.(\overline{g',l})(j(x,y))),m,u,v,a) \rightarrow \\ \bigvee k \leq k_0 \bigwedge n > k_0 \bigvee\limits_{i=0}^{\omega(k)+1} \big(|(\lambda x,y.(\overline{g',l})(j(x,y)))(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \frac{5}{k+1})\Big), \end{cases}$$

and therefore using

$$g_l mn := \begin{cases} gmn, \text{ if } m,n \leq l \\ 0^0, \text{ otherwise, } \text{ and } g_l =_{1(0)} \lambda x,y.(\overline{(g_l)',r})(j(x,y)) \text{ for } r > j(x,y) \end{cases}$$

$$(4)\ \begin{cases} \bigvee k_0 \bigwedge g \leq_{1(0)} \lambda n.M \bigwedge l^0 \Big(\bigwedge m,u,v,a \leq k_0 F_0(g_l,m,u,v,a) \rightarrow \\ \bigvee k \leq k_0 \bigwedge n > k_0 \bigvee\limits_{i=0}^{\omega(k)+1} \big(|g_l(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \frac{5}{k+1})\Big), \end{cases}$$

By putting $g := f_{k_0+1}$ and $l^0 := 3(c+1)$, where c is the maximum of k_0+1 and the codes of all $\frac{i}{\omega(k)+1}$ for $i \leq \omega(k)+1$ and $k \leq k_0$, (4) yields the contradiction

$$\bigvee k \leq k_0 \bigvee\limits_{i=0}^{\omega(k)+1} \big(|f_{k_0+1}(\frac{i}{\omega(k)+1})(k) -_{\mathbb{Q}} f_{k_0+1}(\frac{i}{\omega(k)+1})(k)| >_{\mathbb{Q}} \frac{5}{k+1}\big).$$

α' can be defined as a functional ξ in $f_{(\cdot)}, \omega$, where $\xi \in \mathrm{G}_2\mathrm{R}^\omega$. Since the proof above can be carried out in $\mathrm{G}_3\mathrm{A}^\omega$+AC^{1,0}–qf[74] (under the assumption of F^- and Π^0_1–CA($\xi(f,\omega)$) using prop.7.2.19) we have shown that

$$\mathrm{G}_3\mathrm{A}^\omega + \mathrm{AC}^{1,0}\text{–qf} \vdash F^- \rightarrow \bigwedge f^{1(0)(0)}, \omega^1 \big(\Pi^0_1\text{–CA}(\xi(f,\omega)) \rightarrow \text{ A–A}(f,\omega)\big).$$

Analogously to BW^+ one defines a formalization A–A$^+(f,\omega)$ of the version (ii) of the Arzelà–Ascoli lemma. Similarly to the proof of $BW(\underline{x}) \rightarrow BW^+(\underline{x})$ one shows (using Σ^0_1–IA($\chi(f,\omega)$) for a suitable $\chi \in \mathrm{G}_2\mathrm{R}^\omega$) that A–A$(f,\omega) \rightarrow$ A–A$^+(f,\omega)$. Aanalogously to prop.11.4.3 one so obtains

Proposition: 11.5.1 *For $n \geq 3$ proposition 11.4.3 holds with $BW^+(\xi uv)$ replaced by A–A(ξuv) or A–A$^+(\xi uv)$.*

[74]We have to work in $\mathrm{G}_3\mathrm{A}^\omega$ instead of $\mathrm{G}_2\mathrm{A}^\omega$ since we have used the functional $\Phi_{\langle\rangle} fx = \overline{f}x$.

11.6 The existence of $\limsup$ and $\liminf$ for bounded sequences in $\mathbb{R}$

Definition 11.6.1 *$a \in \mathbb{R}$ is the* $\limsup$ *of $(x_n) \subset \mathbb{R}$ iff*

$$(*)\ \bigwedge k^0 \big(\bigwedge m \bigvee n >_0 m(|a - x_n| \leq \frac{1}{k+1}) \wedge \bigvee l \bigwedge j >_0 l(x_j \leq a + \frac{1}{k+1})\big).$$

Remark 11.6.2 *This definition of* $\limsup$ *is equivalent to the following one:*
$(**)$ a is the greatest limit point of (x_n).
The implication $(*) \to (**)$ is trivial and can be proved e.g. in G_2A^ω. The implication $(**) \to (*)$ uses the Bolzano–Weierstraß principle.
In the following we determine the rate of growth caused by the assertion of the existence of $\limsup$ *(for bounded sequences) in the sense of $(*)$ and thus a fortiori in the sense of $(**)$.*

We may restrict ourselves to sequences of rational numbers: Let $x^{1(0)}$ represent a sequence of real numbers with $\bigwedge n(|x_n| \leq_{\mathbb{R}} C)$. Then $y_n := \widehat{x_n}(n)$ represents a sequence of rational numbers which is bounded by $C+1$. Let a^1 be the $\limsup$ of (y_n), then a also is the $\limsup$ of x. Hence the existence of $\limsup x_n$ follows from the existence of $\limsup y_n$. Furthermore we may assume that $C = 1$.

The existence of $\limsup$ for a sequence of rational numbers $\in [-1,1]$ is formalized in $\mathrm{G}_n\mathrm{A}^\omega$ (for $n \geq 2$) as follows:

$$\exists \limsup(x^1) :\equiv \bigvee a^1 \bigwedge k^0 \big(\bigwedge m \bigvee n >_0 m(|a -_{\mathbb{R}} \breve{x}(n)| \leq_{\mathbb{R}} \frac{1}{k+1}) \wedge \bigvee l \bigwedge j >_0 l(\breve{x}(j) \leq_{\mathbb{R}} a + \frac{1}{k+1})\big),$$

where $\breve{x}(n) := \max_{\mathbb{Q}}(-1, \min_{\mathbb{Q}}(xn, 1))$. In the following we use the usual notation $\breve{x}_n$ instead of $\breve{x}(n)$.

We now show that $\exists \limsup(x^1)$ can be reduced to a purely **arithmetical** assertion $L(x^1)$ on x^1 in proofs of $\bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w^\gamma A_0$–sentences:

$$L(x^1) :\equiv \bigwedge k \bigvee l >_0 k \bigwedge K \geq_0 l \bigvee j \bigwedge q, r \geq_0 j \underbrace{\bigwedge m, n(K \geq_0 m, n \geq_0 l \to |x_q^m -_{\mathbb{Q}} x_r^n| \leq_{\mathbb{Q}} \frac{1}{k+1}}_{L_0(x,k,l,K,q,r):\equiv}),$$

where $x_q^m := \max_{\mathbb{Q}}(\breve{x}_m, \ldots, \breve{x}_{m+q})$ (Note that L_0 can be expressed as a quantifier–free formula in $\mathrm{G}_n\mathrm{A}^\omega$).

Lemma: 11.6.3 *1) $G_2A^\omega \vdash Mon(\bigvee k \bigwedge l \bigvee K \bigwedge j \bigvee q, r(l > k \to K \geq l \wedge q, r \geq j \wedge \neg L_0)$.*

2) $G_2A^\omega \vdash \bigwedge x^1 \big(\exists \limsup(x) \to L(x)\big)$.

3) $G_2A^\omega \vdash \bigwedge x^1 \big((L(x)^s \to \exists \limsup(x)\big)$.
(The facts 1)–3) combined with the results of chapter 10 imply that $\exists \limsup(\xi uv)$ can be reduced to $L(\xi uv)$ in proofs of sentences $\bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w^\gamma A_0$, see prop. 11.6.4 below).

4) $G_3A^\omega + \Sigma_2^0$–$IA \vdash \bigwedge x^1 L(x)$.

Proof: **1)** is obvious.
2) By $\exists \limsup(x^1)$ there exists an a^1 such that

$$(1)\ \bigwedge k^0 \bigwedge m \bigvee n >_0 m(|a -_{\mathbb{R}} \breve{x}_m| \leq_{\mathbb{R}} \frac{1}{k+1})$$

and

$$(2)\ \bigwedge k^0 \bigvee l \bigwedge j >_0 l(\breve{x}_j \leq_{\mathbb{R}} a + \frac{1}{k+1}).$$

Assume $\neg L(x)$, i.e. there exists a k_0 such that

$$(3)\ \bigwedge l > k_0 \bigvee K \geq l \bigwedge j \bigvee q, r \geq j \bigvee m, n\big(K \geq m, n \geq l \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big).$$

Applying (2) to $2k_0 + 1$ yields an u_0 such that

$$(4)\ \bigwedge j \geq u_0(\breve{x}_j \leq_{\mathbb{R}} a + \frac{1}{2(k_0+1)}).$$

(3) applied to $l := \max_0(k_0, u_0) + 1$ provides a K_0 with

$$(5)\ K_0 \geq u_0 \wedge \bigwedge j \bigvee q, r \geq j \bigvee m, n\big(K_0 \geq m, n \geq u_0 \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big).$$

(1) applied to $k := 2k_0 + 1$ and $m := K_0$ yields a d_0 such that

$$(6)\ d_0 > K_0 \wedge \big(|a - \breve{x}_{d_0}| \leq \frac{1}{2(k_0+1)}\big).$$

By (5) applied to $j := d_0$ we obtain

$$(7)\ \begin{cases} K_0 \geq u_0 \wedge d_0 > K_0 \wedge \big(|a -_{\mathbb{R}} \breve{x}_{d_0}| \leq \frac{1}{2(k_0+1)}\big) \wedge \\ \bigvee q, r \geq d_0 \bigvee m, n\big(K_0 \geq m, n \geq u_0 \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big). \end{cases}$$

Let q, r, m, n be such that

$$(8)\ q, r \geq d_0 \wedge K_0 \geq m, n \geq u_0 \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}.$$

Then $x_q^m \geq \breve{x}_{d_0} \overset{(6)}{\geq} a - \frac{1}{2(k_0+1)}$ since $m \leq K_0 \leq d_0 \leq m + q$. Analogously: $x_r^n \geq a - \frac{1}{2(k_0+1)}$.
On the other hand, (4) implies $x_q^m, x_r^n \leq a + \frac{1}{2(k_0+1)}$. Thus $|x_q^m -_{\mathbb{Q}} x_r^n| \leq \frac{1}{k_0+1}$ which contradicts (8).
3) Let f, g be such that L^s is fulfilled, i.e.

$$(*)\ \begin{cases} \bigwedge k\big(fk > k \wedge \bigwedge K \geq fk \bigwedge q, r \geq gkK \\ \qquad\qquad \bigwedge m, n(K \geq m, n \geq fk \to |x_q^m -_{\mathbb{Q}} x_r^n| \leq_{\mathbb{Q}} \frac{1}{k+1})\big). \end{cases}$$

We may assume that f, g are monotone for otherwise we could define $f^M k := \max_0(f0, \ldots, fk), g^M kK := \max_0\{gxy : x \leq_0 k \wedge y \leq_0 K\}$ (f^M, g^M can be defined in

G_1R^ω using Φ_1 and λ–abstraction). If f, g satisfy $(*)$, then f^M, g^M also satisfy $(*)$.
Define

$$h(k) :=_0 \begin{cases} \min i[f(k) \leq_0 i \leq_0 f(k) + gk(fk) \wedge \breve{x}_i =_{\mathbb{Q}} x^{fk}_{gk(fk)}], \text{ if existent} \\ 0^0, \text{ otherwise.} \end{cases}$$

h can be defined in G_2A^ω as a functional in f, g. The case 'otherwise' does not occur since

$$\bigwedge m, q \bigvee i(m \leq_0 i \leq_0 m + q \wedge \breve{x}_i =_{\mathbb{Q}} \max_{\mathbb{Q}}(\breve{x}_m, \ldots, \breve{x}_{m+q})).$$

By the definition of h we have $(+)$ $\breve{x}_{hk} =_{\mathbb{Q}} x^{fk}_{gk(fk)}$ for all k. Assume that $m \geq k$. By the monotonicity of f, g we obtain

$$fm \geq_0 fk \wedge gm(fm) \geq_0 gk(fm) \geq_0 gk(fk).$$

Hence $(*)$ implies

$$(1)\ |x^{fk}_{gk(fm)} -_{\mathbb{Q}} x^{fm}_{gm(fm)}| \leq \frac{1}{k+1}$$

and

$$(2)\ |x^{fk}_{gk(fk)} -_{\mathbb{Q}} x^{fk}_{gk(fm)}| \leq \frac{1}{k+1}$$

and therefore

$$(3)\ |x^{fk}_{gk(fk)} -_{\mathbb{Q}} x^{fm}_{gm(fm)}| \leq \frac{2}{k+1}.$$

Thus for $m, \tilde{m} \geq k$ we obtain

$$(4)\ |x^{fm}_{gm(fm)} -_{\mathbb{Q}} x^{f\tilde{m}}_{g\tilde{m}(f\tilde{m})}| \leq \frac{4}{k+1}.$$

For $\tilde{h}(k) := h(4(k+1))$ this yields

$$(5)\ \bigwedge k \bigwedge m, \tilde{m} \geq k\big(\breve{x}_{\tilde{h}m} -_{\mathbb{Q}} \breve{x}_{\tilde{h}\tilde{m}}| \leq \frac{1}{k+1}\big).$$

Hence for $a :=_1 \lambda m^0.\breve{x}_{\tilde{h}m}$ we have $\widehat{a} =_1 a$, i.e. a represents the limit of the Cauchy sequence $(\breve{x}_{\tilde{h}m})$.
Since $\tilde{h}(k) = h(4(k+1)) \geq f(4(k+1)) \overset{(*)}{\geq} 4(k+1) > k$, we obtain

$$(6)\ \bigwedge k\big(\tilde{h}(k) > k \wedge |\breve{x}_{\tilde{h}k} -_{\mathbb{R}} a| \leq_{\mathbb{R}} \frac{1}{k+1}\big),$$

i.e. a is a limit point of x.
It remains to show that

$$(7)\ \bigwedge k \bigvee l \bigwedge j >_0 l\big(\breve{x}_j \leq_{\mathbb{R}} a + \frac{1}{k+1}\big):$$

Define $c(k) := g(4(k+1), f(4(k+1)))$. Then by $(*)$

$$\bigwedge q, r \geq c(k)\big(|x^{f(4(k+1))}_q -_{\mathbb{Q}} x^{f(4(k+1))}_r| \leq \frac{1}{4(k+1)}\big)$$

and by (+)

$$a(k) =_{\mathbb{Q}} x^{f(4(k+1))}_{g(4(k+1),f(4(k+1)))}$$

and therefore

$$\bigwedge j \geq c(k)\big(|x_j^{f(4(k+1))} -_{\mathbb{Q}} a(k)| \leq \frac{1}{4(k+1)}\big).$$

Hence

$$\bigwedge j \geq c(k)\big(\breve{x}_{f(4(k+1))+j} \leq_{\mathbb{Q}} a(k) + \frac{1}{4(k+1)}\big)$$

which implies

$$\bigwedge j \geq c(k) + f(4(k+1))\big(\breve{x}_j \leq_{\mathbb{R}} a + \frac{1}{4(k+1)} + \frac{1}{k+1}\big).$$

Thus (7) is satisfied by $l := c(2(k+1)) + f(4(2k+1)+1)$.

4) Assume $\neg L(x)$, i.e. there exists a k_0 such that

$$(+)\ \bigwedge \tilde{l} > k_0 \bigvee K \geq \tilde{l} \bigwedge j \bigvee q, r \geq j \bigvee m, n\big(K \geq m, n \geq \tilde{l} \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big).$$

We show (using Σ^0_1–IA on l^0):

$$(++)\ \bigwedge l \geq_0 1 \bigvee i^0 \underbrace{\big(lth(i) = l \wedge \bigwedge j < l \dot{-} 1\big((i)_j < (i)_{j+1}\big) \wedge \bigwedge j, j' \leq l \dot{-} 1(j \neq j' \to |\breve{x}_{(i)_j} -_{\mathbb{Q}} \breve{x}_{(i)_{j'}}| > \frac{1}{k_0+1}\big)}_{A_0(i,l):\equiv}.$$

$l = 1$: Obvious.
$l \mapsto l+1$: By the induction hypothesis their exists an i which satisfies $A_0(i, l)$.
Case 1: $\bigwedge j \leq l \dot{-} 1 \bigvee a \bigwedge b > a\big(|\breve{x}_b -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1}\big)$.
Then by Π^0_1–CP there exists an a_0 such that

$$\bigwedge j \leq l \dot{-} 1 \bigwedge b > a_0\big(|\breve{x}_b -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1}\big).$$

Hence
$i' := i * \langle \max_0(a_0, (i)_{l \dot{-} 1}) + 1\rangle$ satisfies $A_0(i', l+1)$.
Case 2: $\neg$ Case 1. Let us assume that $\breve{x}_{(i)_0} < \ldots < \breve{x}_{(i)_{l \dot{-} 1}}$ (If not we use a permutation of $(i)_0, \ldots, (i)_{l \dot{-} 1}$). Let $j_0 \leq_0 l \dot{-} 1$ be maximal such that

$$(1)\ \bigwedge \tilde{m} \bigvee n \geq_0 \tilde{m}\big(|\breve{x}_n -_{\mathbb{Q}} \breve{x}_{(i)_{j_0}}| \leq \frac{1}{k_0+1}\big).$$

(The existence of j_0 follows from the least number principle for Π^0_2–formulas Π^0_2–LNP: Let j_1 be the least number such that $(l \dot{-} 1) \dot{-} j_1$ satisfies (1). Then $j_0 = (l \dot{-} 1) \dot{-} j_1$).
The definition of j_0 implies

$$\bigwedge j \leq l \dot{-} 1\big(j > j_0 \to \bigvee a \bigwedge b > a(|\breve{x}_b -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1})\big).$$

Hence (again by Π^0_1–CP)

$$(2)\ \bigvee a_1 > j_0 \bigwedge j \le l \dot{-} 1 \big(j > j_0 \to \bigwedge b > a_1 (|\breve{x}_b -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1})\big).$$

Let $c \in \mathbb{N}$ be arbitrary. By (+) (applied to $\tilde{l} := \max_0(k_0, c) + 1$) there exists a K_1 such that

$$(3)\ \bigwedge j \bigvee q, r \ge j \bigvee m, n \big(K_1 \ge m, n \ge c, k_0 \wedge |x^m_q -_{\mathbb{Q}} x^n_r| > \frac{1}{k_0+1}\big).$$

By (1) applied to $\tilde{m} := K_1$ there exists a $u \ge K_1$ such that $|\breve{x}_u -_{\mathbb{Q}} \breve{x}_{(i)_{j_0}}| \le \frac{1}{k_0+1}$.
(3) applied to $j := u$ yields q, r, m, n such that

$$(5)\ q, r \ge u \wedge K_1 \ge m, n \ge c, k_0 \wedge |x^m_q -_{\mathbb{Q}} x^n_r| > \frac{1}{k_0+1} \wedge x^m_q, x^n_r \ge_{\mathbb{Q}} \breve{x}_{(i)_{j_0}} - \frac{1}{k_0+1}$$

(since $m, n \le u \le m+q, n+r$).
Because of $m, n \ge c, k_0$ this implies the existence of an $\alpha \ge c, k_0$ such that $\breve{x}_\alpha > \breve{x}_{(i)_{j_0}}$. Thus we have shown

$$(6)\ \bigwedge c \bigvee \alpha \ge_0 c, k_0 (\breve{x}_\alpha > \breve{x}_{(i)_{j_0}}).$$

For $c := \max_0(a_1, (i)_{l \dot{-} 1}) + 1$ this yields the existence of an $\alpha_1 > a_1, (i)_{l \dot{-} 1}, k_0$ such that $\breve{x}_{\alpha_1} > \breve{x}_{(i)_{j_0}}$.
Let K_{α_1} be (by (+)) such that

$$(7)\ \bigwedge j \bigvee q, r \ge j \bigvee m, n \big(K_{\alpha_1} \ge m, n \ge \alpha_1 (\ge a_1, k_0) \wedge |x^m_q -_{\mathbb{Q}} x^n_r| > \frac{1}{k_0+1}\big).$$

(6) applies to $c := K_{\alpha_1}$ provides an $\alpha_2 \ge K_{\alpha_1}$ such that $\breve{x}_{\alpha_2} > \breve{x}_{(i)_{j_0}}$. Hence (7) applied to $j := \alpha_2$ yields q, r, m, n with

$$(8)\ q, r \ge \alpha_2 \wedge K_{\alpha_1} \ge m, n \ge \alpha_1 \wedge |x^m_q -_{\mathbb{Q}} x^n_r| > \frac{1}{k_0+1} \wedge x^m_q, x^n_r \ge_{\mathbb{Q}} \breve{x}_{\alpha_2}.$$

Since $m, n \ge \alpha_1 > a_1, (i)_{l \dot{-} 1}$, (8) implies the existence of an $\alpha_3 > (i)_{l \dot{-} 1}, a_1$ such that

$$(9)\ \breve{x}_{\alpha_3} >_{\mathbb{Q}} \breve{x}_{(i)_{j_0}} + \frac{1}{k_0+1}.$$

Since $\breve{x}_{(i)_j} \le \breve{x}_{(i)_{j_0}}$ for $j \le j_0$, this implies

$$(10)\ \bigwedge j \le j_0 \big(\breve{x}_{\alpha_3} >_{\mathbb{Q}} \breve{x}_{(i)_j} + \frac{1}{k_0+1}\big).$$

Let $j \le l \dot{-} 1$ be $> j_0$. Then by (2) and $\alpha_3 > a_1$: $|\breve{x}_{\alpha_3} -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1}$. Put together we have shown

$$(11)\ \alpha_3 > (i)_{l \dot{-} 1} \wedge \bigwedge j \le l \dot{-} 1 \big(|\breve{x}_{\alpha_3} -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1}\big).$$

Define $i' := i * \langle \alpha_3 \rangle$. Then $A_0(i, l)$ implies $A_0(i', l+1)$, which concludes the proof of (++).
(++) applied to $l := 2(k_0+1)+1$ yields the existence of indices $i_0 < \ldots < i_{2(k_0+1)}$ such that $|\breve{x}_{(i)_j} -_{\mathbb{Q}} \breve{x}_{(i)_{j'}}| > \frac{1}{k_0+1}$ for $j, j' \le 2(k_0+1) \wedge j \ne j'$, which contradicts $\bigwedge j^0(-1 \le_{\mathbb{Q}} \breve{x}_j \le_{\mathbb{Q}} 1)$.
Hence we have proved $L(x)$. This proof has used Σ^0_1–IA, Π^0_1–CP and Π^0_2–LNP. Since Π^0_2–LNP is equivalent to Σ^0_2–IA (see [52]), and Π^0_1–CP follows from Σ^0_2–IA by [51] (where CP is denoted by M), the proof above can be carried out in $\mathrm{G}_3\mathrm{A}^\omega + \Sigma^0_2$–IA.

Proposition: 11.6.4 *Let $n \geq 2$ and $B_0(u^1, v^\tau, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula which contains only u^1, v^τ, w^γ free, where $\gamma \leq 2$. Furthermore let $\xi, t \in G_nR^\omega$ and Δ be as in thm.2.2.2. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(\exists \limsup(\xi uv) \to \bigvee w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.) \chi \in G_nR^\omega \textit{ such that} \\ G_nA_i^\omega + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \underline{\Psi}^* \big((\underline{\Psi}^* \textit{ satifies the mon. funct.interpr. of} \\ \qquad\qquad \textit{the negative translation } L(\xi uv)' \textit{ of } L(\xi uv)) \ \to \bigvee w \leq_\gamma \chi u \underline{\Psi}^* \, B_0(u,v,w)\big) \\ \Rightarrow \exists \Psi \in \mathrm{T}_1 \textit{ such that} \\ PA_i^\omega + \Delta + b\text{-}AC \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w \leq_\gamma \Psi u \, B_0(u,v,w). \end{cases}$$

where T_1 is the restriction of Gödel's T which contains only the recursor R_ρ for $\rho = 1$ (see chapter 2). The Ackermann function (but no functions having an essentially greater order of growth) can be defined in T_1.
If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\bigvee$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta + AC$–qf may be replaced by E–$G_nA^\omega + \Delta + AC^{\alpha,\beta}$–qf, where α, β are as in cor.10.10.

Proof: Prenexation of $\bigwedge u^1 \bigwedge v \leq_\tau tu\big(L(\xi uv) \to \bigvee w^\gamma B_0(u,v,w)\big)$ yields

$$G :\equiv \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee k \bigwedge l \bigvee K \bigwedge j \bigvee q, r, w\big[(l > k \wedge (K \geq l \wedge q, r \geq j \to L_0)) \to B_0(u,v,w)\big].$$

Lemma 11.6.3.1) implies

(1) $\mathrm{G_2A}^\omega \vdash Mon(G)$.

The assumption of the proposition combined with lemma 11.6.3.3) implies

(2) $\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC\text{–}qf} \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu\big(L(\xi uv)^S \to \bigvee w^\gamma B_0(u,v,w)\big)$

and therefore

(3) $\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC\text{–}qf} \vdash G^H$.

Theorem 10.8 applied to (1) and (3) provides the extractability of a tuple $\underline{\varphi} \in \mathrm{G}_n\mathrm{R}^\omega$ such that

(4) $\mathrm{G}_n\mathrm{A}_i^\omega + \Delta + \mathrm{b\text{-}AC} \vdash \big(\underline{\varphi}$ satisfies the monotone functional interpretation of $G'\big)$.

G' intuitionistically implies

(5) $\bigwedge u^1 \bigwedge v \leq_\tau tu\big(L(\xi uv)' \to \neg\neg \bigvee w^\gamma B_0(u,v,w)\big)$.

Hence from $\underline{\varphi}$ one obtains a term $\tilde{\varphi} \in \mathrm{G}_n\mathrm{R}^\omega$ such that (provably in $\mathrm{G}_n\mathrm{A}_i^\omega + \Delta + \mathrm{b\text{-}AC}$)

(6) $\bigvee \psi\big(\tilde{\varphi}$ s–maj $\psi \wedge \bigwedge u^1 \bigwedge v \leq_\tau tu \bigwedge \underline{a}\big(\bigwedge \underline{b}(L(\xi uv)')_D \to B_0(u,v,\psi uv\underline{a})\big)\big)$,

where $\bigvee \underline{a} \bigwedge \underline{b}\big(L(\xi uv)'\big)_D$ is the usual functional interpretation of $L(\xi uv)'$.
Let $\underline{\Psi}^*$ satisfy the monotone functional interpretation of $L(\xi uv)'$ then

(7) $\bigvee \underline{a}\big(\underline{\Psi}^*$ s–maj $\underline{a} \wedge \bigwedge \underline{b}\big(L(\xi uv)'\big)_D\big)$.

Hence for such a tuple $\underline{a}$ we have

(8) $\lambda u^1.\tilde{\varphi}u(t^*u)\underline{\Psi}^*$ s–maj $\psi uv\underline{a}$ for $v \leq tu$

(Use lemma 1.2.11. t^* in $\mathrm{G}_n\mathrm{R}^\omega$ is a majorant for t).
Since $\gamma \leq 2$ this yields a $\geq_2$ bound $\chi u\underline{\Psi}^*$ for $\psi uv\underline{a}$ (lemma 1.2.11).
The second part of the proposition follows from lemma 11.6.3.4) and the fact that $\mathrm{G}_n\mathrm{A}^\omega + \Sigma^0_2$–IA has a monotone functional interpretation in PA^ω_i by terms $\in \mathrm{T}_1$ (By [52] Σ^0_2–IA has a functional interpretation in T_1. Since every term in T_1 has a majorant in T_1, also the monotone functional interpretation can be satisfied in T_1).

Remark 11.6.5 *By the theorem above the use of the analytical axiom $\exists \limsup(\xi uv)$ in a given proof of $\bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w^\gamma B_0$ can be reduced to the use of the arithmetical principle $L(\xi uv)$. By lemma 11.6.3.2) this reduction is optimal (relatively to G_2A^ω).*

In this chapter we have determined the impact of **sequences of instances** $\bigwedge l^0 B(\xi uvl)$ of the following **analytical** principles $\bigwedge x^{1(0)} B(x)$ on the growth of bounds for sentences

(+) $\bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\rho t\underline{u}\,\underline{k} \bigvee w^0 A_0$

extractable from proofs using such instances:

1) the convergence of bounded monotone sequences in $\mathbb{R}$,
2) the existence of a greatest lower bound for sequences in $\mathbb{R}$ which are bounded from below,
3) the existence of a convergent subsequence for bounded sequences in $\mathbb{R}^d$,
4) the Arzelà–Ascoli lemma,
5) the existence of lim sup and lim inf for bounded sequences in $\mathbb{R}$.

We have shown that the use of **sequences of instances** $\bigwedge l^0 B(\xi uvl)$ of 1)–4) in a proof of (+) (relatively to $G_n\mathrm{A}^\omega$+AC–qf+ the analytical principles discussed in chapters 3–7) can be reduced to a suitable sequence $\bigwedge l^0 PCM1(\tilde{\xi}uvl)$ of instances of the **arithmetical** principle $PCM1$ (i.e. the Cauchy property of bounded monotone sequences in $\mathbb{R}$) studied in chapter 9.[75] So the results on the growth of bounds stated at the end of chapter 9 apply. In particular the contribution of $\bigwedge l^0 PCM1(\tilde{\xi}uvl)$ and even $\bigwedge a^{1(0)} PCM1(a)$ is not stronger then Φ_{it} and hence a primitive recursive bound is always guaranted (this is in contrast to the use of the full universal closure $\bigwedge x^{1(0)} B(x)$ of the principles 1)–4) which are equivalent to CA_{ar} and therefore make all $\alpha(< \varepsilon_0)$–recursive functions provably recursive). However for special $\tilde{\xi}$ and if Φ_{it} is applied only to $g := S$ (see the discussion at the end of chapter 9) one still may obtain polynomial bounds (for $n = 2$ and 1)–3)).
These results also apply to instances of Π^0_1–CA and its arithmetical consequences (relatively to $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf) Δ^0_2–IA and Π^0_1–CP.

Instances of 5) also can be reduced to corresponding instances of a certain arithmetical principle $L \in \Pi^0_5$. L can be proved using Σ^0_2–IA which suffices to prove the totality of the Ackermann function (but not of functions having an essentially greater order of growth). So w.r.t. its impact on the growth of provably recursive functions, 5) is the strongest tool used in standard analysis.

[75] For 1)–3) this works for all $n \geq 2$ and for 4) if $n \geq 3$.

12 False theorems on Π_1^0–CA$^-$ and Σ_2^0–AC$^-$ in the literature

By Π_1^0–CA$^-$ and Σ_2^0–AC$^-$ we denote the schemas of Π_1^0–comprehension and Σ_2^0–choice for formulas **without parameters of type** ≥ 1, i.e.

$$\Pi_1^0\text{–CA}^- \;:\; \bigvee f \bigwedge x^0 (fx =_0 0 \leftrightarrow \bigwedge y^0 A_0(x,y,\underline{a}^0)),\ \text{(only } x,y,\underline{a} \text{ free in } A_0),$$

$$\Sigma_2^0\text{–AC}^- \;:\; \bigwedge x^0 \bigvee y^0 \bigvee z^0 \bigwedge v^0 A_0(x,y,z,v,\underline{a}^0) \to \bigvee f \bigwedge x \bigvee z \bigwedge v\, A_0(x,fx,z,v,\underline{a}),$$

where only $x,y,z,v,\underline{a}$ occur free in $A_0(x,y,z,v,\underline{a})$.

As a special case of cor.11.3.5 we have

Proposition: 12.1 *Let $\gamma \leq 2$ and $A_0(u^1,v^\tau,w^\gamma)$ contains only u,v,w as free variables; $t \in G_nR^\omega$. Then the following rule holds*

$$\begin{cases} G_nA^\omega \oplus AC\text{–}qf \oplus \Pi_1^0\text{–}CA^- \oplus \Sigma_2^0\text{–}AC^- \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w^\gamma B_0(u,v,w) \\ \Rightarrow \exists \Psi \in \widehat{PR}^\omega \text{ such that} \\ PRA_i^\omega \vdash \bigwedge u^1 \bigwedge v \leq_\tau tu \bigvee w \leq_\gamma \Psi u\, B_0(u,v,w). \end{cases}$$

If $\tau \leq 1$, we may replace $G_nA^\omega \oplus AC\text{–}qf \oplus \Pi_1^0\text{–}CA^- \oplus \Sigma_2^0\text{–}AC^-$ by
$E\text{–}G_nA^\omega + AC^{\alpha,\beta}\text{–}qf + \Pi_1^0\text{–}CA^- + \Sigma_2^0\text{–}AC^-$, where $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$.
In particular

$$\begin{cases} E\text{–}G_nA^\omega + AC^{\alpha,\beta}\text{–}qf + \Pi_1^0\text{–}CA^- + \Sigma_2^0\text{–}AC^- \;\vdash \bigwedge u^0 \bigvee v^0 R(u,v) \\ \Rightarrow \exists \text{ primitive recursive function } \varphi : \\ \bigwedge u\, R(u,\varphi u) \text{ is true,} \end{cases}$$

where R is a primitive recursive relation. If in the definition of G_nA^ω the universal axioms 9) are replaced by the schema of quantifier–free induction one has $PRA \vdash R(u,\varphi u)$
(Note that this proposition also holds for $n = \infty$. Since all primitive recursive functions (but not all primitive recursive functionals of type 2!) can be defined in $G_\infty A^\omega$ (see chapter 1) we may assume that $G_\infty A^\omega \supset PRA$).

Proof: The proposition follows from cor.11.3.5 using the fact that Σ_2^0–AC$^-$ can be derived from Π_1^0–AC$^-$ (using pairing) and the fact that there is a $\xi \in$ G$_n$R$^\omega$ such that $\xi(x,y,z,v,\underline{a}) =_0 0 \leftrightarrow A_0(x^0,y^0,z^0,v^0,\underline{a}^0)$. Thus Π_1^0–AC$^-$ follows from Π_1^0–AC(ξ) for a term ξ without parameters of type ≥ 1.

In this final chapter we diccuss the two treatments of Π_1^0–CA$^-$ and Σ_2^0–AC$^-$ in the literature due to Mints [46] and Sieg [57], which are carried out in the context of a second–order fragment BT of PRA$^\omega$ and which state some conservativity results. By constructing counterexamples to these results we show the incorrectness of these treatments. Furthermore a weakening of their results which is correct by our prop.12.1 does not follow by the proofs in [46] and [57].

Let BT denote the extension of primitive recursive arithmetic PRA to the second–order theory which results if function variables and (two–sorted) classical predicate logic with quantifiers for number variables as well as for function variables are added and the schema of quantifier–free induction QF–IA is extended to this language, i.e. instances of QF–IA in BT may contain function variables. Furthermore BT contains (at least[76]) the functionals $\Phi_1 fx = \max(f0,\ldots,fx)$, $\Phi_2 fx = \sum_{i=0}^{x} fi$, $\Phi_{\langle\rangle} fx = \overline{f}x$ and μ_b together with their defining equations. Finally we have the schema of so–called 'explicit definition' in BT:

$$\bigvee f \bigwedge x (fx = t[x]), \text{ where } t \text{ is a term of BT.}$$

(In our theories $\mathrm{G}_n\mathrm{A}^\omega$ these schema is superfluous because of the defininability of λ–operators by means of Π and Σ).

Both Mints ans Sieg are not very explicit on the inclusion of primitive recursive functionals in BT:

'The formalization of PRA being examined by us contains variables for the positive intergers ..., and for unary number–theoretic functions $f, g, h, \ldots$. Functors are constructed from the functional variables and the constants $J_{n,k}$ (projection), Z (function identically equal 0), and s (addition of 1) with the aid of substitution and primitive recursion formulas. Terms are constructed from objective variables and 0 with the aid of the functors' ([46], p.1488)[77] .

'The base theory for subsystems is formulated in $\mathcal{L}^2$ and is called (BT); it includes the axioms of (QF–IA) (but possibly with second–order parameters in the defining equations for primitive recursive functions and the instances of IA) and the schema for explicit definitions of functions ED $(\exists f)(\forall x)\, fx = t_a[x] \ldots$' ([57], p.37).

Since Mints explicitely uses the functional Φ_2 and both Mints and Sieg use the functional $\Phi_{\langle\rangle}$ for the formulation of WKL, it is clear that genuine primitive recursion in function arguments is allowed. Here 'genuine' means primitive recursion which depends on a **variable <u>number</u> of values** of the function arguments as in Φ_i $(i = 1, 2, \ldots)$ or $\Phi_{\langle\rangle}$. Such primitive recursive functionals can not be obtained from primitive recursive functions by substitution of number terms (which may contain function variables) for number variables (an example for the later e.g. is the functional $\Phi fx = x + fx$ which is not genuine in our sense).
The iteration functional $\Phi_{it} xyf = f^x(y)$ is also a (genuine) primitive recursive functional. However it has quite different properties than $\Phi_1, \Phi_2, \ldots$ and $\Phi_{\langle\rangle}$ as we have seen in chapter 2 and chapter 9. Since it is not clear to us whether Mints or Sieg intend to include Φ_{it} in BT, we treat the theories BT and BT+Φ_{it} seperately. It turns out that our refutations apply in an even stronger sense when Φ_{it} is added to BT.

Let Π^0_2–IR$^-$ denote the **rule** of induction for Π^0_2–formulas **without function parameters**. In

[76] If further primitive recursive functionals of type 2 (in the sense of [29]) are added our refutation of the results stated in [46] ,[57] applies a fortiori. Prop.12.3 and prop.12.5 below as well as their corollaries even hold when BT does not contain any of these funcionals at all.

[77] Note that the restriction to unary function variables is no real restriction since coding of finite tuples of numbers is possible in BT.

[46] the following theorem is stated

$$(1)\quad \begin{cases} \text{If BT } + \Pi^0_2\text{–IR}^- + \Pi^0_1\text{–CA}^- \vdash \bigwedge x \bigvee y\, A_0(x,y), \\ \text{then there is a primitive recursive function } \varphi \text{ such that} \\ \text{PRA } \vdash \bigwedge x\, A_0(x, \varphi x). \end{cases}$$

(Here $A_0(x,y) \in \mathcal{L}(\text{PRA})$ contains only x, y free).

Remark 12.2 *Mints uses* (1) *to show the* Π^0_2*–conservativity of* WKL *restricted to primitive recursive trees when added to* BT+Π^0_2–IR$^-$ + Π^0_1–CA$^-$ *over* PRA *(In fact Mints claims to have proved the conservativity of full WKL, which however does not follow from* (1) *since the derivation of* WKL *from* Π^0_1–CA *is possibly only when function parameters are allowed to occur in* Π^0_1–CA*. This has already been noticed in* [57] p.65*).*

In [57] various generalizations of (1) are stated:

(2) ([57], thm.5.8): Let Γ be a set of Σ^0_3–sentences in $\mathcal{L}^2$. Then BT+Σ^0_2–AC$^-$+Π^0_2–IR$^-$+WKL+Γ is conservative over BT+Γ for Π^0_3–sentences.

(3) ([57], cor.5.9): Let Γ be a set of Σ^0_3–sentences in $\mathcal{L}^2$. Then BT+Π^0_1–CA$^-$+Π^0_2–IR$^-$+WKL+Γ is conservative over BT+Γ for Π^0_3–sentences. Consequently BT+Π^0_1–CA$^-$ + Π^0_2–IR$^-$+WKL is conservative over PRA.

These theorems are also stated in a generalized hierarchy version in [57] (5.13,5.14).

In contrast to these claims we now show:

Proposition: 12.3 BT+Π^0_2–IR$^-$ + Π^0_1–CA$^-$ *proves the totality of the Ackermann function and therefore is not conservative over* PRA*.*

Corollary 12.4 (1), (2) *and* (3) *above (even when* $\Gamma = \emptyset$, WKL *is dropped and conservativity is claimed only for* Π^0_2*–sentences) are wrong. This applies a fortiori to* BT+Φ_{it}*.*

Proposition: 12.5 BT+Π^0_1–CA$^-$ *is not* Π^0_3*–conservative over the first order fragment of* BT+Φ_{it}*.*

Corollary 12.6 *The* Π^0_3*–conservativity assertion in* (2), (3) *is wrong when* $\Gamma = \emptyset$ *and* WKL*,* Π^0_2–IR$^-$ *are dropped (for both theories* BT and BT+Φ_{it} *).*

Proof of prop.12.3: Let $\bigvee y A_0(x,y)$ be a Σ^0_1–formula of BT which does not contain any function variable. By Π^0_1–CA$^-$ there exists a function g such that $\bigwedge x\big(gx = 0 \leftrightarrow \bigvee y A_0(x,y)\big)$. Since function variables are allowed to occur in instances of QF–IA we can apply QF–IA to $F_0(x,g) :\equiv (gx = 0)$ and obtain

$$\bigvee y\, A_0(0,y) \wedge \bigwedge x\big(\bigvee y\, A_0(x,y) \to \bigvee y\, A_0(x',y)\big) \to \bigwedge x \bigvee y\, A_0(x,y).$$

Hence every function variable free instance of Σ^0_1–IA, i.e. every instance of Σ^0_1–IA$^-$ can be proved in BT+Π^0_1–CA$^-$. On the other hand it is known (see [51]) that there is an instance of Σ^0_1–IA$^-$ which together with an application of Π^0_2–IR$^-$ proves the totality of the Ackermann function (This fact is mentioned e.g. in [57](!) note 16).

Proof of prop.12.5: By the proof of prop.12.3 every instance of Σ^0_1–IA$^-$ is provable in BT+Π^0_1–CA$^-$. Since every instance of Σ^0_1–IA$^-$ can be logically transformed into a prenex normal form $\in \Pi^0_3$ it suffices to show that there is such an instance – lets call it A– with BT+$\Phi_{it} \nvdash A$: We first notice that BT+Φ_{it} is conservative over the first order fragment BT$'$ of BT: Every model of BT$'$ can be extended to a model of BT+Φ_{it} by letting range the function variables over all primitive recursive functions. It is known from [40] (see also [51], cor.to thm.1) that there exists an instance A of Σ^0_1–IA$^-$ such that BT$' \nvdash A$.

Corollary 12.7 (to the proofs of prop.12.3 and 12.5) *Prop.12.3 and 12.5 also hold if the functionals* $\Phi_1, \Phi_2, \Phi_{\langle\rangle}$ *are omitted from* BT.

In the proofs of prop.12.3 and prop.12.5 we essentially used the fact that in BT function variables may occur in instances of QF–IA. Let QF–IA$^-$ be the restriction of QF–IA to formulas **without function variable** and BT$^-$ the restriction of BT which results if QF–IA is replaced by QF–IA$^-$. Within BT$^-$ we are not able to derive the usual properties of functionals like Φ_1, Φ_2 or $\Phi_{\langle\rangle}$ from their defining equations. Thus in order to deal with WKL (as formulated in [46] ,[57]) we have to add the axiom

$$(*) \bigwedge f, x, y(y < x \to (\overline{f}x)_y = fy),$$

which is provable in BT.

Proposition: 12.8 *1)* BT$^-$ + $(*)$ + Π^0_2–IR$^-$ + Π^0_1–CA$^-$ *proves the totality of the Ackermann function.*

2) BT$^-$ + $(*)$ + Π^0_1–CA$^-$ *is not* Π^0_3*–conservative over the first order fragment of* BT+Φ_{it}.

Proof: Let $A(x)$ be a Σ^0_1–formula without function variables: By Π^0_1–CA$^-$ there exists a function $f \leq \lambda x.1$ such that $\bigwedge x(fx = 0 \leftrightarrow A(x))$. By $(*)$ there exists a number y –namely $y := \overline{f}a'$ – for each a such that

$$\bigwedge x \leq a\big((y)_x = (\overline{f}a')_x = fx \wedge [\big(fx = 0 \wedge A(x)\big) \vee \big(fx = 1 \wedge \neg A(x)\big)]\big)$$

and therefore

$$(1) \bigwedge x \leq a\Big(\big((y)_x = 0 \wedge A(x)\big) \vee \big((y)_x = 1 \wedge \neg A(x)\big)\Big).$$

By QF–IA$^-$ we have

$$(2)\ (y)_0 = 0 \wedge \bigwedge x < a\big((y)_x = 0 \to (y)_{x'} = 0\big) \to (y)_a = 0.$$

(1) and (2) yield

$$(3)\ A(0) \wedge \bigwedge x < a\big(A(x) \to A(x')\big) \to A(a).$$

Hence BT$^-$ + $(*)$ + Π^0_1–CA$^-$ proves every instance of Σ^0_1–IA$^-$. 1) and 2) now follow from the proofs of prop.12.3,12.5.

As we already have mentioned above there is a further negative result if Φ_{it} is added to BT. Then even without Π^0_2–IR$^-$ the principle Σ^0_2–AC$^-$ is not conservative over PRA:

Proposition: 12.9 $\mathrm{BT}+\Phi_{it}+\Sigma^0_2\text{–AC}^-$ *proves every (function parameter free)* Π^0_2*–consequence of* $\Sigma^0_2\text{–IA}^-$ *and hence is not conservative over* PRA.

Corollary 12.10 *For* $\mathrm{BT}+\Phi_{it}$ *instead of* BT (2) *is false already with respect to* Π^0_2*–conservativity even when* $\Pi^0_2\text{–IR}^-$, WKL *and* Γ *are omitted.*

Proof of prop.12.9: Let us consider an instance

$$(1)\ \begin{cases} \bigvee x \bigwedge y\, A_0(0,x,y) \wedge \bigwedge z\big(\bigvee x \bigwedge y\, A_0(z,x,y) \to \bigvee x \bigwedge y\, A_0(z',x,y)\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to \bigwedge z \bigvee x \bigwedge y\, A_0(x,y,z) \end{cases}$$

of $\Sigma^0_2\text{–IA}^-$. By $\Pi^0_1\text{–CA}^-$ (which follows from $\Sigma^0_2\text{–AC}^-$) there exists a function g such that

$$(2)\ \bigwedge z, x\big(gzx = 0 \leftrightarrow \bigwedge y\, A_0(z,x,y)\big).$$

Using g, (1) reduces to an instance of Σ^0_1–IA. One easily shows that $\mathrm{BT}+\Phi_{it}+\mathrm{AC}^{0,0}\text{–qf} \vdash \Sigma^0_1\text{–IA}$ (see e.g. [32]). Hence

$$(3)\ \mathrm{BT}+\Pi^0_1\text{–CA}^- + \Phi_{it}+\mathrm{AC}^{0,0}\text{–qf} \ \vdash \Sigma^0_2\text{–IA}^-.$$

Since $\Sigma^0_2\text{–IA}^-$ (which is equivalent to $\Pi^0_2\text{–IA}^-$ relatively to BT, see e.g. [57]) proves the totality of the Ackermann function, the theory $\mathrm{BT}+\Pi^0_1\text{–CA}^- + \Phi_{it}+\mathrm{AC}^{0,0}\text{–qf}$ is not Π^0_2–conservative over PRA.
We now show that $\mathrm{BT}+\Phi_{it}+\Sigma^0_2\text{–AC}^-$ proves every Π^0_2– consequence of $\mathrm{BT}+\Pi^0_1\text{–CA}^-+\Phi_{it}+\mathrm{AC}^{0,0}$–qf (together with (3) this concludes the proof of the proposition): Suppose that

$$(4)\ \ \mathrm{BT}+\Pi^0_1\text{–CA}^- + \Phi_{it}+\mathrm{AC}^{0,0}\text{–qf} \ \vdash \bigwedge u \bigvee v\, F_0(u,v),\ \text{where } F_0 \text{ contains only } u,v \text{ free.}$$

For notational simplicity let us assume that only one instance

$$(5)\ \bigvee g \bigwedge x(gx = 0 \leftrightarrow \bigwedge y\, A_0(x,y))$$

of $\Pi^0_1\text{–CA}^-$ is used in the proof of (4). Let h be a new function constant with the axiom

$$(*)\ \bigwedge x, y\big(\neg A_0(x,hx) \vee A_0(x,y)\big).$$

It is clear $\mathrm{BT}+(*) \vdash (5)$. Hence

$$(6)\ \mathrm{BT}+(*)+\Phi_{it}+\mathrm{AC}^{0,0}\text{–qf} \ \vdash \bigwedge u \bigvee v\, F_0(u,v).$$

By functional interpretation there exists a term $t[h]$ in the set of all closed terms of $\mathrm{BT}+(*)+\Phi_{it}$ such that[78]

$$(7)\ \mathrm{BT}+(*)+\Phi_{it} \vdash \bigwedge u\, F_0(u,t[h]u).$$

[78] More precisely one obtains a functional $\Psi[h] \in \widehat{PR}^{\omega}[h]$ such that $\widehat{\mathrm{PA}}^{\omega}{\restriction} + (*)$ proves (7). By normalization one eliminates the higher type levels in $\Psi[h]$ and realizes that $\Psi[h]$ reduces to a functional $t[h]$ which is primitive recursive in h in the sense of [29]. Finally one verifies that $\widehat{\mathrm{PA}}^{\omega}{\restriction} + (*)$ is conservative over the first order part $(\mathrm{BT}+(*)+\Phi_{it})'$ of $\mathrm{BT}+(*)+\Phi_{it}$ (more precisely the first order part of BT plus the defining equations for all functions which are primitive recursive in h) for arithmetical sentences. This follows from the fact that every model of $(\mathrm{BT}+(*)+\Phi_{it})'$ can be extended to a model of $\widehat{\mathrm{PA}}^{\omega}{\restriction} + (*)$ by letting range the variables for functionals over all functionals which are primitive recursive (in the sense of $\widehat{PR}^{\omega}$) in h.

Hence

(8) $\mathrm{BT} + \Phi_{it} + \bigvee h \bigwedge x, y\big(\neg A_0(x, hx) \vee A_0(x, y)\big) \vdash \bigwedge u \bigvee v\, F_0(u, v)$

and therefore

(9) $\mathrm{BT} + \Phi_{it} + \Sigma^0_2\text{–AC}^- \vdash \bigwedge u \bigvee v\, F_0(u, v)$.

We now discuss where the errors in the proofs of (1)–(3) in [46] and [57] occur:

Mints reduces QF–IA to the **rule** of quantifier free induction

$$\text{QF–IR} \;:\; \frac{B_0(0)\;,\; B_0(x) \to B_0(x')}{\bigwedge x\, B_0(x)}.$$

This can be done by applying QF–IR to

$$A_0(x) :\equiv B_0(0) \wedge \bigwedge y < x\big(B_0(y) \to B_0(y')\big) \to B_0(x).$$

In order to express A_0 as a quantifier–free formula one has to eliminate the bounded quantifier $\bigwedge y < x$. Since B_0 may contain function variables (e.g. $B_0(x) :\equiv (fx = 0)$) this elimination requires the use of a primitive recursive function**al** as e.g. Φ_1 or Φ_2 (Mints uses Φ_2 to express bounded quantification in a quantifier free way). If now B_0 involves a function g which results from Π^0_1–CA$^-$ then the corresponding instance of QF–IA is reduced to a g–free instance of Π^0_2–IR by replacing g by its graph (which 'is described in the form of a Π^0_2–formula' ([46] p.1490)). Then Mints applies a previous result that BT is closed under Π^0_2–IR which finishes his proof.
The problem with this argument is that the elimination of g only works in this way if g occurs everywhere in the form $g(t)$ in B_0 but not if g occurs also as a function argument $\Phi_1 g$ or $\Phi_2 g$. In the later case one first has to reduce expressions like $\Phi_1 gx = y$ to $\bigwedge i \le x(y \ge gi) \wedge \bigvee j \le x(y = gj)$ and to eliminate g from the result. However the bounded quantifiers $\bigwedge i \le x$, $\bigvee j \le x$ now stand in front of the Π^0_2–formula which results from the g–elimination. In contrast to bounded quantifiers in front of quantifier–free formulas these bounded quantifiers can not be neglected in BT. In fact to express e.g. $\bigvee y \le x \bigwedge u \bigvee v\, B_0$ as a Π^0_2-formula requires Π^0_1–CP which implies Σ^0_1–IA (and combined with Π^0_2–IR proves the totality of the Ackermann function).

Sieg uses a sort of ε–terms to reduce theories like BT+Π^0_1–CA$^-$ + Π^0_2–IR$^-$ to certain 'operator theories' Π^0_0–OT^2_1+QF–AC$_0$ + Π^0_2–IR. He also does not treat the (genuine) primitive recursive functionals in his definition of the number terms of OT^2: If one adds here the clause 'If f is a function term and t a number term then $\Phi f t$ is a number term' (where e.g. $\Phi = \Phi_1$ or $= \Phi_{\langle\rangle}$) one gets problems with the interpretation of the operator theories: The reduction of OT^2_n to fragments of second order arithmetic via the τ_n–translation no longer works in the way presented in [57] (2.2). It is not even clear how to define the ν–depth of $\Phi(\nu x.(A_0(x)), t)$ anymore. Besides this in Sieg's proof of 5.8 one has to understand Π^0_0 w.r.t. $\mathcal{L}(\mathrm{OT}^2_1)$ and not with respect to $\mathcal{L}(Z)$ (as is claimed by Sieg): Since function parameters are allowed to occur in QF–IA, in particular the functions obtained from Σ^0_2–AC$^-_0$ may occur in QF–IA. Therefore the reduction of Σ^0_2–AC$^-_0$ to OT^2_1+QF–AC$_0$ only works if in the schema of quantifier–free induction of OT^2_1 ν–terms (which are used in the reduction of

Σ^0_2–AC^-_0 to QF–AC_0) are allowed to occur. However the τ_n–translation of such instances of QF–IA requires (according to Sieg's remarks on p.45) already Π^0_2–IA which proves the totality of the Ackermann function.

Finally both arguments by Mints and Sieg do not establish (as they stand) the following weakening of (1) which is a corollary of our prop.12.1:

$$(1)' \begin{cases} \text{If BT } + \Pi^0_1\text{–CA}^- \vdash \bigwedge x \bigvee y\, A_0(x,y), \\ \text{then there is a primitive recursive function } \varphi \text{ such that} \\ \text{PRA } \vdash \bigwedge x\, A_0(x, \varphi x). \end{cases}$$

(Here $A_0(x,y) \in \mathcal{L}(\text{PRA})$ contains only x, y free. This result also follows if all the functionals Φ_i with $i \in \mathbb{N}$ from $G_\infty A^\omega$ are added to BT but not if Φ_{it} is added: compare prop.12.9 .) Since the reduction of Π^0_1–CA^- to Π^0_2–IR by Mints uses Σ^0_1–CP which proves (combined with Π^0_2–IR) the totality of the Ackermann function it is not possible to obtain $(1)'$ using his argument. The failure of Sieg's proof has nothing to do with Π^0_2–IR and its straightforward correction needs Π^0_2–IA (which is not conservative over PRA) already for the treatment of BT+Π^0_1–CA^-. In any case both methods (even if they can be corrected to yield $(1)'$) are not usuable for our results on finite type theories from chapter 11, since they rest on the elimination of function symbols f by their graphs which is not possible if a proof applies for instance variables of type 2 to these function symbols as it is possible in our context (e.g. we may use f as the bounding function of the fan in the axiom F from chapter 7).

13 Summary of results on the growth of uniform bounds

In this paper we have considered proofs of sentences[79]

$$(+)\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w)\ \text{(where } \gamma \leq 2)$$

in various parts of classical analysis, more precisely in $\mathrm{G}_n\mathrm{A}^\omega + \Gamma$+AC–qf, where Γ is a set of analytical theorems. Using proof–theoretic methods we are able to extract **uniform** bounds Φ on $\bigvee w^\gamma$ which do **not depend on** v such that

$$(++)\ \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi \underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w)$$

holds classically[80] , i.e. is true in the full set–theoretic type structure $\mathcal{S}^\omega$.

In chapter 2 (see thm.2.2.2 and the remark on it) we have shown in particular that

Theorem 13.1 *Let Δ be a set of sentences having the form $\bigwedge x^\delta \bigvee y \leq_\rho sx \bigwedge z^\eta B_0(x, y, z)$, where $s \in G_nR^\omega$. Let $A_0(\underline{u}^1, \underline{k}^0, v, w^\gamma)$ contain only $\underline{u} = u_1^1, \ldots, u_j^1, \underline{k} = k_1^0, \ldots k_l^0$ and v, w as free variables, where $\gamma \leq 2$. Then the following rule holds:*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w) \\ \Rightarrow \textit{ one can extract a term } \Phi \in G_nR^\omega \textit{ such that} \\ G_nA_i^\omega + \Delta + \textit{ b-AC} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi \underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w). \end{cases}$$

If $\Delta = \emptyset$, then b-AC is not needed in the conclusion.
For $\gamma \leq 1$ ($\gamma = 2$), Φ has the form $\lambda \underline{u}, \underline{k}.\tilde{\Phi}\underline{u}^M \underline{k}$ ($\lambda \underline{u}, \underline{k}, y^1.\tilde{\Phi}\underline{u}^M \underline{k} y^M$) , where $\tilde{\Phi} \in G_nR^\omega_-$ and by the results from chapter 1 (prop.1.2.21, the cor. to the proof of 1.2.21 and prop.1.2.22) we have:

For $n = 1$: $\tilde{\Phi}\underline{u}^M \underline{k}$ *(* $\tilde{\Phi}\underline{u}^M \underline{k} y^0$ *resp.* $\tilde{\Phi}\underline{u}^M \underline{k} y^M$*) is a linear function in* $\underline{u}^M, \underline{k}$ *(*$\underline{u}^M, \underline{k}, y^0$ *resp.* $\underline{u}^M, \underline{k}, y^M$*)*

For $n = 2$: $\tilde{\Phi}\underline{u}^M \underline{k}$ *(* $\tilde{\Phi}\underline{u}^M \underline{k} y^0$ *resp.* $\tilde{\Phi}\underline{u}^M \underline{k} y^M$*) is a polynomial in* $\underline{u}^M, \underline{k}$ *(*$\underline{u}^M, \underline{k}, y^0$ *resp.* $\underline{u}^M, \underline{k}, y^M$*)*

For $n = 3$: $\tilde{\Phi}\underline{u}^M \underline{k}$ *(* $\tilde{\Phi}\underline{u}^M \underline{k} y^0$ *resp.* $\tilde{\Phi}\underline{u}^M \underline{k} y^M$*) is an elementary recursive function in* $\underline{u}^M, \underline{k}$ *(*$\underline{u}^M, \underline{k}, y^0$ *resp.* $\underline{u}^M, \underline{k}, y^M$*).*

We recall that by definition $t\underline{f}^1\underline{x}^0$ is a linear function (polynomial resp. elementary recursive function) in $\underline{f}, \underline{x}$ if there is a term $\widehat{t}[\underline{f}, \underline{x}]$ of type 0 containing only $\underline{f} := f_1^1, \ldots, f_i^1$ and $\underline{x} := x_1^0, \ldots, x_j^0$ free such that

(i) $\bigwedge \underline{f}, \underline{x}(t\underline{f}\underline{x} =_0 \widehat{t}[\underline{f}, \underline{x}])$ and

[79] Sometimes we have formulated (for notational simplicity) only the case $\bigwedge u^1$ instead of $\bigwedge \underline{u}^1, \underline{k}^0$. However using suitable coding the general case reduces to the special one in $\mathrm{G}_n\mathrm{A}_i^\omega$ for $n \geq 2$ (Also all of our proofs immediately generalize to tuples without the need of any coding).

[80] For the mathematical significance of sentences (+) and of such uniform bounds see [37],[38], [39] and the discussion at the end of chapter 3 of the present paper. We recall that $A_0, B_0, \ldots$ always denote quantifier–free formulas.

(ii) $\widehat{t}[\underline{f},\underline{x}]$ is built up from $0^0, x_1^0, \ldots, x_j^0, f_1^1, \ldots, f_i^1, S^1, +$ (or $0^0, x_1^0, \ldots, x_j^0, f_1^1, \ldots, f_i^1, S^1, +, \cdot$ resp. $0^0, x_1^0, \ldots, x_j^0, f_1^1, \ldots, f_i^1, S^1, +, \cdot, \lambda x^0, y^0.x^y$) only.

In particular if tfx is a polynomial in f^M, x, then for every polynomial $p \in \mathbb{N}[x]$ the function $\lambda x.tpx$ can be written as a polynomial in $\mathbb{N}[x]$. Moreover (by prop.1.2.30) for $t^{1(1)} \in \mathrm{G}_2\mathrm{R}^\omega$ there exists a polynomial $q \in \mathbb{N}[x]$ (depending only on the term structure of t) such that

$$\begin{cases} \text{For every polynomial } p \in \mathbb{N}[x] \\ \text{one can construct a polynomial } r \in \mathbb{N}[x] \text{ such that} \\ \bigwedge f^1 \big(f \leq_1 p \to \bigwedge x^0 (tfx \leq_0 r(x))\big) \text{ and } deg(r) \leq q(deg(p)). \end{cases}$$

The cases $n = 2$ and $n = 3$ are of particular interest since within $\mathrm{G}_2\mathrm{A}^\omega$ many of the fundamental notions of the analysis of continuous functions can be treated but some as e.g. the unrestricted exponential function exp need $\mathrm{G}_3\mathrm{A}^\omega$.

Let us consider the following analytical properties, principles and theorems:

I.
- Basic properties of the operations $+, -, \cdot, (\cdot)^{-1}, |\cdot|, \max, \min$ and the relations $=, \leq, <$ for rational numbers and real numbers (which are given by Cauchy sequences of rationals with fixed Cauchy rate, see chapter 3 §1 for details).
- Basic properties of maximum and sum for sequences of real numbers of variable length (see chapter 3 §3).
- Basic properties of uniformly continuous[81] functions $f : [a,b]^d \to \mathbb{R}$, $\sup\limits_{x\in[a,b]} fx$ and $\int\limits_a^x f(x)dx$ for $f \in C[a,b]$ where $a < b$ (see chapter 3 §2,3).
- The Leibniz criterion, the quotient criterion, the comparison test for series of real numbers. The convergence of the geometric series together with its sum formula. The nonconvergence of the harmonic series. (But not: The Cauchy property of bounded monotone sequences in $\mathbb{R}$ or the Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}$). See chapter 4 for details.
- Characteristic properties of the trigonometric functions sin, cos, tan, arcsin, arccos, arctan and of the restrictions $\exp_k$ and $\ln_k$ of exp, ln to $[-k, k]$ for every **fixed** number k.
- Fundamental theorem of calculus.
- Fejér's theorem on uniform approximation of 2π–periodic uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ by trigonometric polynomials.
- Equivalence (local and global) of sequential continuity and ε–δ–continuity for $f : \mathbb{R} \to \mathbb{R}$.

II.
- Attainment of the maximum of $f \in C([a,b]^d, \mathbb{R})$ on $[a,b]^d$.
- Mean value theorem of differentiation.
- Mean value theorem for integrals.

[81]Uniformly continuous is meant always endowed with a modulus of uniform continuity. In the presence of III. below we can prove the uniform continuity (with a modulus) of pointwise continuous functions $f : [a,b]^d \to \mathbb{R}$. Thus together with III. we have I. also for pointwise continuous functions. Instead of $[a,b]^d$ we may also have $[a_1,b_1] \times \ldots \times [a_d,b_d]$ where $a_i < b_i$ for $i = 1, \ldots, d$.

- Cauchy–Peano existence theorem.
- Brouwer's fixed point theorem for uniformly continuous functions $f : [a,b]^d \to [a,b]^d$.

(See chapter 7 for precise formulations of these principles).

III.
- Uniform continuity (together with the existence of a modulus of uniform continuity) of pointwise continuous functions $f : [a,b]^d \to \mathbb{R}$.
- Sequential form of the Heine–Borel covering property of $[a,b]^d \subset \mathbb{R}^d$.
- Dini's theorem: Every sequence (G_n) of pointwise continuous functions $G_n : [a,b]^d \to \mathbb{R}$ which increases pointwise to a pointwise continuous function $G : [a,b]^d \to \mathbb{R}$ converges uniformly on $[a,b]^d$ to G and there exists a modulus of uniform convergence.
- Every strictly increasing pointwise continuous function $G : [a,b] \to \mathbb{R}$ possesses a uniformly continuous strictly increasing inverse function $G^{-1} : [Ga, Gb] \to [a,b]$.
- König's lemma WKL^2_{seq} for sequences of binary trees.

(See chapter 7 for precise formulations of these principles).

In the chapters 3–6 we have shown that $\mathrm{G_2A}^\omega$+$\mathrm{AC}^{0,1}$–qf proves the analytical facts summerized under I. so that theorem 13.1 applies with $\Delta = \emptyset$.

In chapter 7 §1 we have shown that the principles II. can be expressed in the language of $\mathrm{G_2A}^\omega$ as sentences $(*)$ $\bigwedge x^1 \bigvee y \leq_1 sx \bigwedge z^{0/1} B_0 \in \Delta$ or follow relatively to $\mathrm{G_2A}^\omega$+$\mathrm{AC}^{0,0}$–qf from such sentences. In the following let Δ denote the finite set of these sentences $(*)$ used in chapter 7 §1. One clearly has $\mathcal{S}^\omega \models \Delta$. Hence by thm.13.1 we obtain the following results for $\gamma \leq 2$ and $n \geq 2$:

$$\begin{cases} \text{From a proof} \\ \mathrm{G}_n\mathrm{A}^\omega\text{+AC–qf+I+II} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0(\underline{u},\underline{k},v,w) \\ \text{one can extract a bound } \Phi \in \mathrm{G}_n\mathrm{R}^\omega \text{ such that } \Phi \text{ has the form as in thm.13.1 and} \\ \mathrm{G}_n\mathrm{A}^\omega_i + \Delta\text{+b-AC} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u},\underline{k},v,w) \\ \text{and therefore} \\ \mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u},\underline{k},v,w). \end{cases}$$

In particular (for $\gamma = 0$):

$\Phi\underline{u}\,\underline{k}$ is a polynomial in $\underline{u}^M, \underline{k}$ if $n = 2$ and

$\Phi\underline{u}\,\underline{k}$ is an elementary recursive function in $\underline{u}^M, \underline{k}$ if $n = 3$.

The theorems III. can be proved in $\mathrm{G_2A}^\omega \oplus \mathrm{AC}^{1,0}$–qf $\oplus F^-$ (see chapter 7 §2,3).
Hence for $n \geq 2$

$$\mathrm{G}_n\mathrm{A}^\omega\text{+AC–qf+I+II}\oplus\text{III} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0(\underline{u},\underline{k},v,w)$$

implies

$$\mathrm{G}_n\mathrm{A}^\omega\text{+AC–qf} + \Delta \vdash F^- \to \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0(\underline{u},\underline{k},v,w)$$

Thus combined with the proof of thm.7.2.20 we obtain

Theorem 13.2 *Let $n \geq 2$ and $\gamma \leq 2$. Then the following rule holds:*

$$(+) \begin{cases} \textit{From a proof} \\ \mathrm{G}_n\mathrm{A}^\omega\text{+AC–qf+I+II}\oplus\text{III} \ \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w) \\ \textit{one can extract a bound } \Phi \in G_nR^\omega \textit{ such that } \Phi \textit{ has the form as in thm.13.1 and} \\ \mathrm{G}_{\max(n,3)}\mathrm{A}_i^\omega + \Delta\text{+b-AC} \ \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w) \\ \textit{and therefore} \\ \mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w). \end{cases}$$

In particular (for $\gamma = 0$):

> $\Phi\underline{u}\,\underline{k}$ *is a polynomial in* $\underline{u}^M, \underline{k}$ *if* $n = 2$ *and*
>
> $\Phi\underline{u}\,\underline{k}$ *is an elementary recursive function in* $\underline{u}^M, \underline{k}$ *if* $n = 3$.

(In the case $n \geq 3$ the proof of the assumption may use also e.g. the unrestricted exponential function exp *and the unrestricted logarithm* ln.*)*

For $\tau \leq 2$ the conclusion

$$\mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w)$$

holds even when $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+I+II$\oplus$III *is replaced by* $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+I+II+III.

For $\tau \leq 1$ (which is the most important case for applications) (+) holds also for

> E–$\mathrm{G}_n\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$–qf+$\mathrm{AC}^{0,1}$–qf+I+II+III

instead of

> $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+I+II$\oplus$III.

A result analogous to (+) holds for $PRA^\omega, \widehat{PR}$, PRA_i^ω and PA^ω, T, PA_i^ω instead of G_nA^ω, G_nR^ω, $G_{\max(n,3)}A_i^\omega$.

Proof: In view of the comments above it remains to show the special assertions for $\tau \leq 2$ and $\tau \leq 1$:
For $\tau \leq 2$ the elimination of F^- is not needed for a classical verification of Φ since F^- has the logical form of an axiom Δ in thm.13.1 and

> $\mathcal{M}^\omega \models \mathrm{G}_n\mathrm{A}^\omega + \Delta$+b-AC+$F^-$ (see chapter 7 §2) and thus
>
> $\mathcal{M}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w)$ which implies
>
> $\mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w)$

(since $\tau, \gamma \leq 2$ and $v \leq t\underline{u}\,\underline{k}$ is majorized by $t^*\underline{u}^M\underline{k}$, where t^* is a majorant of t).
For $\tau \leq 1$ we argue as follows:

> E–$\mathrm{G}_n\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$–qf+$\mathrm{AC}^{0,1}$–qf+I+II+III $\vdash (\ldots)$

implies

$$\text{E–G}_n\text{A}^\omega\text{+AC}^{1,0}\text{–qf+AC}^{0,1}\text{–qf+}\Delta\ \vdash F^- \to (\ldots)$$

since E–$\text{G}_n\text{A}^\omega$ satisfies the deduction theorem w.r.t. +. Elimination of extensionality now yields

$$\text{G}_n\text{A}^\omega\text{+AC}^{1,0}\text{–qf+AC}^{0,1}\text{–qf+}\Delta\ \vdash F^- \to (\ldots)$$

(Note that the sentences in Δ used to derive II only have variables of type ≤ 1 and that $(F^-)_e$ is implied by F^-). Now one proceeds as in the proof of (+).

Growth caused by (function parameter–free) applications of the Σ^0_1–induction rule Σ^0_1–IR$^-$:

¿From chapter 9 it follows that a single application of Σ^0_1–IR$^-$ may increase the growth of the bound Φ in thm.13.2 by one level in the hierarchy $(\text{G}_n\text{R}^\omega)_{n\in\mathbb{N}}$. Thus if the proof of

$$\bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w)$$

uses besides $\text{G}_n\text{A}^\omega$+AC–qf and I–III a single application of Σ^0_1–IR$^-$ whose upper formulas are provable in $\text{G}_k\text{A}^\omega$+AC–qf plus I–III ($k \geq 2$), then only a bound $\Phi\ \in \text{G}_{\max(n,k+1)}\text{R}^\omega$ is guaranteed (In chapter 9 we have presented an example from analysis where such a speed up actually happens).

Growth caused by the axiom of Σ^0_1–induction Σ^0_1–IA and the Cauchy property of bounded monotone sequences in $\mathbb{R}$ ($PCM1$):

In chapter 9 we have shown that $\text{G}_3\text{A}^\omega$ proves the equivalence of Σ^0_1–IA and $(PCM1)$. The implication $(PCM1) \to \Sigma^0_1$–IA holds even in $\text{G}_2\text{A}^\omega_i$ and we have constructed a functional $\chi\ \in \text{G}_2\text{R}^\omega$ such that $PCM1(\chi(g)) \to \Sigma^0_1\text{–IA}(g)$ (see prop.9.2).
According to the results in chapter 9 the contribution of $PCM1(x_n)$ (where (x_n) is a decreasing sequence of positive real numbers[82]) to the growth of bounds is given by a functional Ψ such that

$$(1)\ \bigwedge k^0, g^1 \bigvee n \leq_0 \Psi((x_n), k, g)\big(gn >_0 n \to x_n -_{\mathbb{R}} x_{gn} \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

A functional Ψ which satisfies (1) is given by

$$(2)\Psi((x_n), k, g) := \max_{i<C(x_0)(k+1)} (g^i(0)),$$

where $g^i(x)$ is the i–th iteration of g (i.e. $\overbrace{g(\ldots(gx)}^{i \text{ times}}\ldots)$ and $\mathbb{N} \ni C(x_0) \geq x_0$ (e.g. $C(x_0) := x_0(0) + 1$).
Since this functional Ψ satisfies (provably in PRA$^\omega$) the monotone functional interpretation of the negative translation of $PCM1$ we have

[82]The restriction to the special lower bound 0 is convenient but of course not essential. Analogous results hold for increasing sequences (x_n) which are bounded from above.

Theorem 13.3 *Let $n \geq 2$ and $\gamma \leq 2$. Then the following rule holds:*

$$\begin{cases} \textit{From a proof} \\ \mathrm{G}_n\mathrm{A}^\omega\text{+AC–qf+}(PCM1)\text{+I+II}\oplus\text{III} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w) \\ \textit{one can extract a bound } \Phi \in G_nR^\omega[\Psi] \textit{ (where } \Psi \textit{ is defined as in (2) above) such that} \\ \mathrm{PRA}^\omega_i + \Delta\text{+b-AC} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w) \\ \textit{and therefore} \\ \mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w). \end{cases}$$

In particular Φ is a primitive recursive functional in the sense of [30].
The special assertions for $\tau \leq 2$ and $\tau \leq 1$ from thm.13.2 hold analogously.
This result is valid also for $PRA^\omega, \widehat{PR}$, PRA^ω_i and PA^ω, T, PA^ω_i instead of G_nA^ω, $G_nR^\omega[\Psi]$, PRA^ω_i.

Since Σ^0_1–IA (and so $PCM1$) suffices to introduce every primitive recursive function relative to $\mathrm{G}_n\mathrm{A}^\omega$ (for $n \geq 2$) in general only a primitive recursive bound is guaranteed. However in concrete proofs in analysis usually $PCM1$ is not used iterated and so Φ will have only 'Ψ–depth' 1. In this case $\lambda\underline{k}.\Phi\underline{u}\,\underline{k}$ has a growth of type $\mathrm{G}_{\max(n,k)+1}\mathrm{R}^\omega$ for input functions $\underline{u}$ having growth of type $\mathrm{G}_k\mathrm{R}^\omega$. A particular important special case is where (1) is applied only to $g := S$. Then $\Psi((x_n), k, g) \leq C(x_0)\cdot(k+1)$ contributes only polynomial (in fact quadratic) growth and thus for $n = 2$ one obtains a bound $\Phi\underline{u}\,\underline{k}$ which is polynomial in $\underline{u}^M, \underline{k}$ in this situation.

Growth caused by single (sequences of) instances of analytical principles involving arithmetical comprehension:

In chapter 11 we have studied the following principles:

1) The Cauchy property **together with the existence of a Cauchy modulus** (which implies the convergence) for bounded monotone sequences (x_n) in $\mathbb{R}$ (short: $PCM2(x_n)$)[83]

2) The existence of a greatest lower bound for sequences $(x_n) \subset \mathbb{R}_+$ (short: $GLB(x_n)$)

3) Comprehension for Π^0_1–formulas

$$\Pi^0_1\text{–CA}(f^{1(0)}) :\equiv \bigvee g^1 \bigwedge x^0 \big(gx =_0 0 \leftrightarrow \bigwedge y^0 (fxy =_0 0)\big)$$

4) Choice for Π^0_1–formulas

$$\Pi^0_1\text{–AC}(f^{1(0)(0)}) :\equiv \bigwedge x^0 \bigvee y^0 \bigwedge z^0 (fxyz =_0 0) \rightarrow \bigvee g^1 \bigwedge x, z (fx(gx)z =_0 0)$$

5) The Bolzano–Weierstraß principle for bounded sequences $(x_n) \subset \mathbb{R}^d$ for every fixed number d (short: $BW(x_n)$)[84]

[83] For simplicity we may consider only decreasing sequences in $\mathbb{R}_+$.
[84] In chapter 11 we have distinguished between two versions of this principle (called BW and BW^+). BW asserts the existence of a limit point whereas BW^+ asserts the existence of a convergent subsequence of (x_n). Since both principles have the same impact on the growth of bounds (which however is more difficult to prove for BW^+) we now denote both versions by BW. Similarly for A–A in 6) below.

6) The Arcelà–Ascoli lemma for bounded equicontinuous sequences $(f_n) \subset C[0,1]$ (short: A–$A(f_n)$).

Whereas the universal closure of these principles (i.e. $\bigwedge(x_n)\big(PCM2(x_n)\big)$ in the case of $PCM2$) implies full arithmetical comprehension and thus makes every $\alpha(<\varepsilon_0)$–recursive function provably recursive when added to $\mathrm{G}_n\mathrm{A}^\omega$ for $n \geq 2$ (see chapter 11) this does not happen if only **single sequences** of instances (which may depend on the parameters $\underline{u}, \underline{k}, v$) of these principles are used in a proof of $\bigwedge\underline{u}, \underline{k}\bigwedge v \leq_\tau t\underline{u}\,\underline{k}\bigvee w^\gamma A_0$, i.e.

$$\bigwedge\underline{u}, \underline{k}\bigwedge v \leq_\tau t\underline{u}\,\underline{k}\big(\bigwedge l^0\big(PCM2(\xi\underline{u}\,\underline{k}vl)\big) \to \bigvee w^\gamma A_0\big),$$

where ξ is a closed term of $\mathrm{G}_n\mathrm{A}^\omega$.
More precisely we have the following theorem

Theorem 13.4 *Let $n \geq 2$ and $\tau, \gamma \leq 2$, $\xi \in G_nR^\omega$ (of suitable type). Then the following rule holds*

$$\begin{cases} \textit{From a proof} \\ \mathrm{G}_n\mathrm{A}^\omega\text{+AC–qf+I+II+III} \ \vdash \bigwedge\underline{u}^1, \underline{k}^0\bigwedge v \leq_\tau t\underline{u}\,\underline{k}\big(\bigwedge l^0\big(PCM2(\xi\underline{u}\,\underline{k}vl)\big) \to \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w)\big) \\ \textit{one can extract a bound } \Phi \in G_nR^\omega \textit{ such that} \\ \mathrm{G}_{\max(n,3)}\mathrm{A}_i^\omega + \Delta + F^- \text{+b-AC} \ \vdash \bigwedge\underline{u}^1, \underline{k}^0\bigwedge v \leq_\tau t\underline{u}\,\underline{k}\bigwedge\Psi^*\big((\Psi^* \textit{ satisfies the mon.funct. interp. of} \\ \quad \bigwedge a^{1(0)(0)}, k^0, g^1\bigvee n^0(gn > n \to \bigwedge l \leq k\big(\widetilde{(a)_l}(n) -_{\mathbb{R}} \widetilde{(a)_l}(gn) \leq_{\mathbb{R}} \frac{1}{k+1}\big))\big) \to \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\Psi^*A_0\big) \\ \textit{and} \\ \mathcal{S}^\omega \models \bigwedge\underline{u}^1, \underline{k}^0\bigwedge v \leq_\tau t\underline{u}\,\underline{k}\bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\Psi A_0(\underline{u}, \underline{k}, v, w), \end{cases}$$

where $\Psi := \lambda a, k, g. \max\limits_{i<C(a,k)(k+1)^2}(g^i(0))$, $\mathbb{N} \ni C(a,k) \geq \max_{\mathbb{R}}((a_0)(0), \ldots, (a_k)(0))$ and $\tilde{a}(n) := \max_{\mathbb{R}}(0, \min\limits_{i\leq n}{}_{\mathbb{R}}(a(i)))$.
In fact Φ (more precisely a slight variant of Φ) only needs (instead of Ψ^* as input) a functional $\tilde{\Psi}^*$ which satisfies the monotone functional interpretation of the instance $\lambda l^0.\xi\underline{u}\,\underline{k}vl$ of '$\bigwedge a^{1(0)(0)}(\ldots)$'.
If only a single instance $PCM2(\xi\underline{u}\,\underline{k}v)$ is used then even a functional $\tilde{\Psi}^*$ which satisfies the monotone functional interpretation of $\bigwedge k, g\bigvee n(gn > n \to (\widetilde{\xi\underline{u}\,\underline{k}v})(n) -_{\mathbb{R}} (\widetilde{\xi\underline{u}\,\underline{k}v})(gn) \leq_{\mathbb{R}} \frac{1}{k+1})$ is sufficient.

This result also holds for $\bigwedge l^0\big(GLB(\xi\underline{u}\,\underline{k}vl)\big)$, $\bigwedge l^0\big(\Pi^0_1\text{–}CA(\xi\underline{u}\,\underline{k}vl)\big)$, $\bigwedge l^0\big(\Pi^0_1\text{–}AC(\xi\underline{u}\,\underline{k}vl)\big)$, $\bigwedge l^0\big(BW(\xi\underline{u}\,\underline{k}vl)\big)$ and (for $n \geq 3$) also for $\bigwedge l^0\big(A\text{–}A(\xi\underline{u}\,\underline{k}vl)\big)$ instead of $\bigwedge l^0\big(PCM2(\xi\underline{u}\,\underline{k}vl)\big)$ ($\xi \in G_nR^\omega$ of suitable type).[85]

For $\tau \leq 1$: $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+I+II+III *may be replaced by* E–$\mathrm{G}_n\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$–qf+$\mathrm{AC}^{0,1}$–qf+I +II+III.

Proof: As in the proof of thm.13.2 the assumption yields that

$$\mathrm{G}_n\mathrm{A}^\omega\text{+AC–qf+}\Delta + F^- \ \vdash \bigwedge\underline{u}^1, \underline{k}^0\bigwedge v \leq_\tau t\underline{u}\,\underline{k}\big(\bigwedge l^0\big(PCM2(\xi\underline{u}\,\underline{k}vl)\big) \to \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w)\big).$$

[85] Then $\tilde{\Psi}^*$ has to satisfy the monotone functional interpretation of the instance $\lambda l.\xi'\underline{u}\,\underline{k}vl$ of '$\bigwedge a^{1(0)(0)}(\ldots)$'for a suitable $\xi' \in \mathrm{G}_n\mathrm{R}^\omega$.

The conclusion (for $\bigwedge l^0\big(PCM2(\xi\underline{u}\,\underline{k}vl)\big)$ now follows from prop.11.1.3, the fact that $\mathcal{M}^\omega \models \mathrm{G}_n\mathrm{A}^\omega +$ $\Delta + F^-$+b-AC (see chapter 7) and the fact that (as in the proof of thm.13.2)

$$\mathcal{M}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u},\underline{k},v,w) \text{ implies}$$

$$\mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi\underline{u}\,\underline{k}\, A_0(\underline{u},\underline{k},v,w).$$

The assertions for the other principles follow from 11.2 and the propositions 11.3.3,11.3.4,11.4.3 and 11.5.1 (More precisely from the proofs of these propositions some of which are formulated only for single instances ξuv instead of sequences $\lambda l^0.\xi uvl$. However the proof e.g. for BW reduces every instance $BW(f)$ to an instance $PCM2(tf)$ and hence $\bigwedge l^0\big(BW(\xi\underline{u}\,\underline{k}vl)\big)$ reduces to $\bigwedge l^0\big(PCM2(\xi'\underline{u}\,\underline{k}vl)\big)$ and so to $PCM2^*(\xi''\underline{u}\,\underline{k}v)$ for a suitable $\xi', \xi'' \in \mathrm{G}_n\mathrm{R}^\omega$ so that prop.11.1.3 applies. Similar for the other principles).

Remark 13.5 *1) Instead of $\bigwedge l^0\big(PCM2(\xi\underline{u}\,\underline{k}vl)\big)$ we may also use a strengthened version $PCM2^*$ which asserts the existence of a sequence of Cauchy moduli for the sequence $\lambda l^0.\widetilde{\xi\underline{u}\underline{k}vl}$ of monotone sequences (see prop. 11.1.3).*

2) For $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+I+II$\oplus$III instead of $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf+I+II+III we can eliminate F^- from the conclusion and may have an arbitrary type τ as in thm.13.2.

3) In the theorem above we may also have the conjunction $\bigwedge l^0\big(PCM2(\xi_1\underline{u}\,\underline{k}vl)\big) \wedge \bigwedge l^0\big(GLB(\xi_2\underline{u}\,\underline{k}vl)\big) \wedge \ldots$ of sequences of instances of the principles treated in this theorem (for $\xi_1, \xi_2, \ldots \in G_nR^\omega$) since a (fixed) finite number of sequences of instances of $PCM2$ can be coded into a single sequence of such instances.

4) In thm.13.4 we may add also single sequences of instances of Δ^0_2–IA and Π^0_1–CP since they follow from suitable sequences of instance of Π^0_1–CA and Π^0_1–AC (see chapter 11). But note that the theorem becomes false if the full axiom Σ^0_1–IA is added: Using suitable instances of Π^0_1–CA one can prove (in the presence of Σ^0_1–IA) Σ^0_2–IA^- which suffices to establish the totality of the Ackermann function. In particular the theorem does not hold for $PRA^\omega, \widehat{PR}$, PRA^ω_i instead of G_nA^ω, G_nR^ω, $G_nA^\omega_i$ since PRA^ω+AC–qf proves Σ^0_1–IA.

Finally we have investigated the following principle w.r.t. its impact on the growth of bounds

7) For every bounded sequence $(x_n) \subset \mathbb{R}$ there exists the lim sup (short: $\exists \limsup(x_n)$).
For simplicity we restrict ourselves to sequences in $\mathbb{Q} \cap [-1,1]$ (In chapter 11 we have seen that the general case can be reduced to this).

Theorem 13.6 *Let $n \geq 2$ and $\tau, \gamma \leq 2$, $\xi \in G_n R^\omega$ (of suitable type). Then the following rule holds:*

$$\begin{cases} \textit{From a proof} \\ \mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC\text{–}qf} + \mathrm{I+II+III} \ \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k}\big(\exists \limsup(\xi \underline{u}\,\underline{k}v) \to \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w)\big) \\ \textit{one can extract a bound } \chi \in \mathrm{G}_n\mathrm{R}^\omega \textit{ such that} \\ \mathrm{G}_n\mathrm{A}_i^\omega + \Delta + F^- + b\text{-}AC \ \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigwedge \underline{\Psi}^* \big((\underline{\Psi}^* \textit{ satifies the mon. funct.interpr. of} \\ \qquad\qquad \textit{the negative translation } L(\xi \underline{u}\,\underline{k}v)' \textit{ of } L(\xi \underline{u}\,\underline{k}v)) \ \to \bigvee w \leq_\gamma \chi \underline{u}\,\underline{k}\,\underline{\Psi}^* A_0(\underline{u}, \underline{k}, v, w)\big) \\ \textit{and in particular one can construct a bound } \Phi \in \mathrm{T}_1 \textit{ such that} \\ \mathrm{PA}_i^\omega + \Delta + F^- + \text{b-AC} \ \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi \underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w) \\ \textit{and} \\ \mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi \underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w), \end{cases}$$

where

$$L(x^1) :\equiv \bigwedge k \bigvee l >_0 k \bigwedge K \geq_0 l \bigvee j \bigwedge q, r \geq_0 j \bigwedge m, n (K \geq_0 m, n \geq_0 l \to |x_q^m -_{\mathbb{Q}} x_r^n| \leq_{\mathbb{Q}} \frac{1}{k+1}),$$

with $x_q^m := \max_{\mathbb{Q}}(x_m, \ldots, x_{m+q})$ and
T_1 is the restriction of Gödel's T which contains only the recursor R_ρ for $\rho = 1$ (see chapter 2). The Ackermann function (but no functions of essentially greater order of growth) can be defined in T_1.

Proof: As in the proof of thm.13.2 the assumption implies that

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC\text{–}qf} + \Delta + F^- \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k}\big(\exists \limsup(\xi \underline{u}\,\underline{k}v) \to \bigvee w^\gamma A_0(\underline{u}, \underline{k}, v, w)\big).$$

The theorem now follows from prop.11.6.4 using that

$\mathcal{M}^\omega \models \ \mathrm{PA}^\omega + \Delta + F^-$+b-AC and the fact that

$\mathcal{M}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi \underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w)$ implies

$\mathcal{S}^\omega \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_\tau t\underline{u}\,\underline{k} \bigvee w \leq_\gamma \Phi \underline{u}\,\underline{k}\, A_0(\underline{u}, \underline{k}, v, w).$

By lemma 11.6.3.2) the reduction of $\exists \limsup(\xi \underline{u}\,\underline{k}v)$ to $L(\xi \underline{u}\,\underline{k}v)$ is sharp. Since it is very unlikely that $L(x_n)$ has a monotone functional interpretation without R_1, the principle $\exists \limsup$ seems to be the strongest principle (w.r.t. its impact on growth) used in the standard parts of classical analysis of continuous functions.

Growth of functional dependencies for logically complex formulas in (non–constructive) analytical proofs relatively to the intuitionistic theories E–$\mathbf{G}_n\mathbf{A}_i^\omega$:

Let $\mathcal{A}$ be the set of the following theorems and principles:[86]

[86] In 1)–4) continuous functions on $[a, b]^d$ are always understood to be endowed with a modulus of uniform continuity.

1) Attainment of the maximum of $f \in C([a,b]^d, \mathbb{R})$

2) Mean value theorem for integrals

3) Cauchy–Peano existence theorem

4) Brouwer's fixed point theorem for uniformly continuous functions $f : [a,b]^d \to [a,b]^d$

5) The generalization WKL^2_{seq} of the binary König's lemma WKL

6) The 'double negation shift' DNS : $\bigwedge x \neg\neg A \to \neg\neg \bigwedge x\, A$

7) The 'lesser limited principle of omniscience'

$$\mathrm{LLPO}\ :\ \bigwedge x^1, y^1 \bigvee k \leq_0 1([k=0 \to x \leq_{\mathbb{R}} y] \wedge [k=1 \to y \leq_{\mathbb{R}} x])$$

8) Comprehension for negated formulas:

$$CA^{\rho}_{\neg} :\ \bigvee \Phi \leq_{0\rho} \lambda x^{\rho}.1^0 \bigwedge y^{\rho} \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big),\ \text{where } A \text{ is arbitrary.}$$

Theorem 13.7 *Let $\gamma \leq 2$, $n \geq 2$, $t \in G_nR^{\omega}$ and C, D arbitrary formulas of $E\text{–}G_nA^{\omega}$ such that $\bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_{\tau} t\underline{u}\,\underline{k}\big(\neg C \to \bigvee w^{\gamma} D(\underline{u}, \underline{k}, v, w)\big)$ is closed. Then the following rule holds*

$$\begin{cases} \textit{From a proof} \\ \mathrm{E\text{–}G}_n\mathrm{A}^{\omega}_i\mathrm{+AC}\ + IP_{\neg} + \mathcal{A} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_{\tau} t\underline{u}\,\underline{k}\big(\neg C \to \bigvee w^{\gamma} D(\underline{u}, \underline{k}, v, w)\big) \\ \textit{one can extract a bound } \Phi \in G_nR^{\omega} \textit{ such that} \\ \mathrm{E\text{–}G}_n\mathrm{A}^{\omega}\mathrm{+AC}\ + \mathcal{A} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_{\tau} t\underline{u}\,\underline{k} \bigvee w \leq_{\gamma} \Phi \underline{u}\,\underline{k}\big(\neg C \to D(\underline{u}, \underline{k}, v, w)\big) \\ \textit{and therefore} \\ \mathcal{S}^{\omega} \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_{\tau} t\underline{u}\,\underline{k} \bigvee w \leq_{\gamma} \Phi \underline{u}\,\underline{k}\big(\neg C \to D(\underline{u}, \underline{k}, v, w)\big). \end{cases}$$

An analogous result holds $E\text{–}PRA^{\omega}_i, \widehat{PR}$, $E\text{–}PRA^{\omega}$ and $E\text{–}PA^{\omega}_i, T$, $E\text{–}PA^{\omega}$ instead of $E\text{–}G_nA^{\omega}_i$, G_nR^{ω}, $E\text{–}G_nA^{\omega}$.

Proof: The theorem follows immediately from thm.8.3 and the fact that the sentences in $\mathcal{A}$ can be expressed in the logical form $\bigwedge x(A \to \bigvee y \leq sx \neg B)$ as we have seen in chapter 8.

Let $\mathcal{B}$ consist of the following theorems and principles:

1) Attainment of the maximum of $f \in C([a,b]^d, \mathbb{R})$

2) Mean value theorem for integrals

3) Cauchy–Peano existence theorem

4) Brouwer's fixed point theorem for uniformly continuous functions $f : [a,b]^d \to [a,b]^d$

5) The generalization WKL^2_{seq} of the binary König's lemma WKL

6) The 'double negation shift' DNS : $\bigwedge x\neg\neg A \to \neg\neg\bigwedge x\, A$

7) The 'lesser limited principle of omniscience'

$$\text{LLPO} \ : \ \bigwedge x^1, y^1 \bigvee k \leq_0 1([k=0 \to x \leq_{\mathbb{R}} y] \wedge [k=1 \to y \leq_{\mathbb{R}} x])$$

8) Comprehension for $\bigvee$–free formulas:

$$CA^{\rho}_{\vee f} : \ \bigvee \Phi \leq_{0\rho} \lambda x^{\rho}.1^0 \bigwedge y^{\rho}\big(\Phi y =_0 0 \leftrightarrow A(y)\big), \text{ where } A \text{ is } \bigvee\text{–free}$$

9) The generalization of the axiom F to arbitrary types ρ:

$$F_{\rho} :\equiv \bigwedge \Phi^{0\rho 0}, y^{\rho 0} \bigvee y_0 \leq_{\rho 0} y \bigwedge k^0 \bigwedge z \leq_{\rho} yk\big(\Phi k z \leq_0 \Phi k(y_0 k)\big)$$

10) Every pointwise continuous function $F : [a,b]^d \to \mathbb{R}$ is uniformly continuous (together with a modulus of uniform continuity)

11) Every sequence of continuous functions $F_n : [a,b]^d \to \mathbb{R}$ which converges pointwise to a continuous function $F : [a,b]^d \to \mathbb{R}$ converges uniformly on $[a,b]^d$ (together with a modulus of convergence)

12) Every sequence of balls (not necessarily open ones) which cover $[a,b]^d$ contains a finite subcovering.

Theorem 13.8 *Let $n \geq 2$, $\gamma, \tau \leq 2$, C be $\bigvee$–free and $D \in \Gamma_1$ such that $\bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_{\tau} t\underline{u}\,\underline{k}\big(C \to \bigvee w^{\gamma} D\big)$ is closed, where $t \in G_n R^{\omega}$. Suppose that all positively occuring $\bigwedge x^{\rho}$ (resp. negatively occuring $\bigvee x^{\rho}$) in $C \to \bigvee w\, D$ have types ≤ 1 and all other quantifiers have types ≤ 2. Then the following rule holds:*

$$\begin{cases} \textit{From a proof} \\ \text{E–G}_n\text{A}^{\omega}_i\text{+AC } + IP_{\vee f} + \mathcal{B} \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_{\tau} t\underline{u}\,\underline{k}\big(C \to \bigvee w^{\gamma} D(\underline{u}, \underline{k}, v, w)\big) \\ \textit{one can extract a bound } \Phi \in G_n R^{\omega} \textit{ such that} \\ \text{E–G}_n\text{A}^{\omega}\text{+b–AC } + \mathcal{B}^- \vdash \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_{\tau} t\underline{u}\,\underline{k} \bigvee w \leq_{\gamma} \Phi\underline{u}\,\underline{k}\big(C \to D(\underline{u}, \underline{k}, v, w)\big) \\ \textit{and} \\ \mathcal{S}^{\omega} \models \bigwedge \underline{u}^1, \underline{k}^0 \bigwedge v \leq_{\tau} t\underline{u}\,\underline{k} \bigvee w \leq_{\gamma} \Phi\underline{u}\,\underline{k}\big(C \to D(\underline{u}, \underline{k}, v, w)\big), \end{cases}$$

where $\mathcal{B}^- := \mathcal{B} \setminus \{10), 11), 12)\}$.
An analogous result holds E–PRA$^{\omega}_i$, $\widehat{PR}$, E–PRA$^{\omega}$ and E–PA$^{\omega}_i$, T, E–PA$^{\omega}$ instead of E–G$_n$A$^{\omega}_i$, G$_n$R$^{\omega}$, E–G$_n$A$^{\omega}$.

Proof: The first part of the theorem follows from thm.8.8 , the fact that the principles 1)–9) from $\mathcal{B}$ have the logical form $\bigwedge x\big(G \to \bigvee y \leq sxH\big)$ (where $G \in \Gamma_1$ and H is $\bigvee$–free) and the fact that principles 10)–12) follow from AC and F relatively to E–G$_2$A$^{\omega}_i$ (see chapter 8).
Since $\mathcal{M}^{\omega} \models$ E–G$_n$A$^{\omega}$+b–AC+B^- the conclusion holds in $\mathcal{M}^{\omega}$ and so (since $\tau, \gamma \leq 2$) in $\mathcal{S}^{\omega}$.

Remark 13.9 *As a special corollary of thm.13.8 one obtains the consistency of* $E\text{–}G_nA^\omega_i + AC + IP_{\vee f} + \mathcal{B}$ *which is not obvious since (due to 10)–12)*$\in \mathcal{B}$*) the corresponding classical theory is inconsistent.*

In this paper we have studied the impact of many analytical theorems on the growth of extractable bounds. Moreover we have developed general methods to determine this impact. These methods can be applied to many further analytical tools. In practice one will try to apply them directly to the analytical lemmas G which are used in a concrete given proof (even if these lemmas can be proved e.g. in $\mathrm{G}_n\mathrm{A}^\omega$ plus analytical theorems whose contribution to growth has already been determined) because this may avoid the need of analyzing the whole proof of G (e.g. if G can be reduced to a sentence $\bigwedge x^\delta \bigvee y \leq_\rho sx \bigwedge z^\eta A_0$ then the proof of G is not relevant for the construction of the bound but only for its verification) and will in general provide better bounds.

References

[1] Ackermann, W., Zum Aufbau der rellen Zahlen. Math. Annalen **99** pp.118–133 (1928).

[2] Beeson, M.J., Principles of continuous choice and continuity of functions in formal systems for constructive mathematics. Annals of Math. Logic **12**, pp.249–322 (1977).

[3] Beeson, M.J., Foundations of Constructive Mathematics. Springer Ergebnisse der Mathematik und ihrer Grenzgebiete 3.Folge, Bd.6., Berlin Heidelberg New York Tokyo 1985.

[4] Bezem, M.A., Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals. J. Symb. Logic **50** pp. 652–660 (1985).

[5] Bishop, E., Foundations of constructive analysis. McGraw–Hill, New–York (1967).

[6] Bishop, E.– Bridges, D. , Constructive analysis. Springer Grundlehren der mathematischen Wissenschaften vol.279, Berlin 1985.

[7] Bridges, D.–Richman, F., Varieties of constructive mathematics. London Math. Soc. Lecture Note Series **97**, Cambridge University Press (1987).

[8] Brown, D.K.– Simpson, S.G., Which set existence axioms are needed to prove the separable Hahn–Banach theorem? Ann. Pure Appl. Logic **31**, pp. 123–144 (1986).

[9] Coddington, E.A.–Levinson, N., Theory of ordinary differential equations. McGraw–Hill, New–York,Toronto,London (1955).

[10] Feferman, S., Systems of predicative analysis. J. Symbolic Logic **29**, pp. 1–30 (1964).

[11] Feferman, S., Theories of finite type. In: Barwise, J. (ed.), Handbook of Mathematical Logic, North–Holland, Amsterdam, pp. 913–972 (1977).

[12] Felgner, U., Models of ZF–Set Theory. Lecture Notes in Mathematics **223**, Springer, Berlin 1971.

[13] Felscher, W., Berechenbarkeit: Rekursive und programmierbare Funktionen. Springer–Lehrbuch, Berlin Heidelberg New York 1993.

[14] Ferreira, F., A feasible theory for analysis. J. Smybolic Logic **59**, pp. 1001–1011 (1994).

[15] Forster, O., Analysis 1. Vieweg, Braunschweig/Wiesbaden (1976).

[16] Friedman, H., Some systems of second order arithmetic and their use. In: Proc. 1974 International Congress of Mathematicians, Vancouver 1974, vol. 1 (Canadian Mathematical Congress, 1975), pp.235–242 (1974).

[17] Friedman, H., Systems of second order arithmetic with restricted induction (abstract), J. Symbolic Logic **41**, pp. 558-559 (1976).

[18] Friedman, H., A strong conservative extension of Peano arithmetic. In: Barwise, J., Keisler, H.J., Kunen, K. (eds.), The Kleene Symposium, pp. 113–122, North–Holland Publishing Company (1980).

[19] Friedman, H., Beyond Kruskal's theorem I–III. Unpublished manuscripts, Ohio State University (1982).

[20] Gödel, K., Zur intuitionistischen Arithmetik und Zahlentheorie. Ergebnisse eines Mathematischen Kolloquiums, vol. 4 pp. 34–38 (1933).

[21] Gödel, K., Über eine bisher noch nicht benutzte Erweiterung des finiten Standpunktes. Dialectica **12**, pp. 280–287 (1958).

[22] Grzegorczyk, A., Some classes of recursive functions. Rozprawny Matematyczne, 46 pp., Warsaw (1953).

[23] Halpern, J.D.–Lvy, A., The Boolean Prime Ideal theorem does not imply the axiom of choice. In: Proc. of Symp. Pure Math. vol.XIII, pp.83–134, AMS, Providence (1971).

[24] Heuser, H., Lehrbuch der Analysis: Teil 1. Teubner, Stuttgart (1980).

[25] Hilbert, D. – Bernays, P., Grundlagen der Mathematik, vol. II. Springer Berlin (1939).

[26] Howard, W.A., Hereditarily majorizable functionals of finite type. In: Troelstra (1973).

[27] Jägermann, M., The axiom of choice and two definitions of continuity. Bull. Acad. Polon. Sci. **13**, pp.699–704 (1965).

[28] Kirby, L. – Paris, J., Accessible independence results for Peano arithmetic. Bull. London Math. Soc. **14**, pp. 285–293 (1982).

[29] Kleene, S.C.,Introduction to Metamathematics. North– Holland (Amsterdam), Noordhoff (Groningen), New–York (Van Nostrand) (1952).

[30] Kleene, S.C., Recursive functionals and quantifiers of finite types, I. Trans. A.M.S. **91**, pp.1–52 (1959).

[31] Ko, K.–I., Complexity theory of real functions. Birkhäuser; Boston, Basel, Berlin (1991).

[32] Kohlenbach, U., Theorie der majorisierbaren und stetigen Funktionale und ihre Anwendung bei der Extraktion von Schranken aus inkonstruktiven Beweisen: Effektive Eindeutigkeitsmodule bei besten Approximationen aus ineffektiven Eindeutigkeitsbeweisen. Dissertation, Frankfurt/Main, pp. xxii+278 (1990).

[33] Kohlenbach, U., Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. J. Symbolic Logic **57**, pp. 1239–1273 (1992).

[34] Kohlenbach, U., Pointwise hereditary majorization and some applications. Arch. Math. Logic **31**, pp.227–241 (1992).

[35] Kohlenbach, U., Remarks on Herbrand normal forms and Herbrand realizations. Arch. Math. Logic **31**, pp.305–317 (1992).

[36] Kohlenbach, U., Constructions in classical analysis by unwinding proofs using majorizability (abstract). J. Symbolic Logic **57**, p.307 (1992).

[37] Kohlenbach, U., Effective moduli from ineffective uniqueness proofs. An unwinding of de La Vallée Poussin's proof for Chebycheff approximation. Ann. Pure Appl. Logic **64**, pp. 27–94 (1993).

[38] Kohlenbach, U., New effective moduli of uniqueness and uniform a–priori estimates for constants of strong unicity by logical analysis of known proofs in best approximation theory. Numer. Funct. Anal. and Optimiz. **14**, pp. 581–606 (1993).

[39] Kohlenbach, U., Analyzing proofs in analysis. Preprint 25 p. To appear in: Proceedings Logic Colloquium'93 (Keele), Oxford University Press (A).

[40] Kreisel, G., On the concepts of completeness and interpretation of formal systems. Fund.Math. **39**, pp.103–127 (1952).

[41] Kreisel, G., On weak completeness of intuitionistic predicate logic. J. Symbolic Logic **27**, pp.139–158 (1962).

[42] Kreisel, G.–MacIntyre, A., Constructive logic versus algebraization. In: Troelstra, A.S.–van Dalen, D. (eds.) L.E.J. Brouwer Centenary Symposium, pp. 217–260. North–Holland (1982).

[43] Luckhardt, H., Extensional Gödel functional interpretation. A consistency proof of classical analysis. Springer Lecture Notes in Mathematics **306** (1973).

[44] Luckhardt, H., Beweistheorie. Mitschrift einer 90/91 gehaltenen Vorlesung.

[45] Mandelkern, M., Limited omniscience and the Bolzano–Weierstraß principle. Bull. London Math. Soc. **20**,pp. 319–320 (1988).

[46] Mints, G.E., What can be done with PRA. J.Soviet Math. **14**, pp.1487–1492, 1980 (Translation from: Zapiski Nauchuyh Seminarov, LOMI,vol. 60 (1976),pp. 93–102).

[47] Orevkov, V.P., Certain types of continuity of constructive operators (russian). Trudy Mat.Inst.Steklov.93(1967),164–186. English translation in: Orevkov, V.P.–Sanin, M.A. (eds.) Problems in the constructive trends in mathematics IV. Proc. of the Steklov Inst. of Math. no. 93 (1967), AMS, Rhode Island (1970).

[48] Orevkov, V.P., Equivalence of two definitions of continuity (russian). Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskoyo Instituta im V.A. Steklova Akademii Nauk SSSR (LOMI)vol. 20 (1971),145-159. English translation in: Journal of Soviet Math. vol.1,pp. 92 –99 (1973).

[49] Parikh, R.J. Existence and feasibility in arithmetic. J. Symbolic Logic **36**, pp.494–508 (1971).

[50] Paris, J.– Harington, L., A mathematical incompleteness in Peano arithmetic. In: Barwise, J. (ed.), Handbook of Mathematical Logic, pp. 1133–1142, North–Holland Publishing Company, Amsterdam 1977.

[51] Parsons, C., On a number theoretic choice schema and its relation to induction. In: Intuitionism and proof theory,pp. 459–473. North–Holland, Amsterdam 1970.

[52] Parsons, C., On n-quantifier induction. J. Symbolic Logic **37**, pp. 466–482 (1972).

[53] Rice, J.R., The Approximation of Functions, vol. 1, Addison–wesley, Reading, 1964.

[54] Ritchie, R.W., Classes of recursive functions based on Ackermann's function. Pacific J. Math. **15**, pp.1027– 1044 (1965).

[55] Schwichtenberg, H., Complexity of normalization in the pure lambda–calculus. In: Troelstra, A.S.–van Dalen, D. (eds.) L.E.J. Brouwer Centenary Symposium, pp. 453–458. North–Holland (1982).

[56] Shioji, N.–Tanaka, K., Fixed point theory in weak second–order arithmetic. Ann. Pure Appl. Logic **47**,pp. 167–188 (1990).

[57] Sieg, W., Fragments of arithmetic. Ann. Pure Appl. Logic **28**,pp. 33–71 (1985).

[58] Sieg, W., Reduction theories for analysis. In: Dorn, Weingartner (eds.), Foundations of Logic and Linguistics, pp. 199–231, New York, Plenum Press (1985).

[59] Sieg, W., Herbrand analyses. Arch. Math. Logic **30**, pp. 409–441 (1991).

[60] Simpson, S.G., Which set existence axioms are needed to prove the Cauchy/Peano theorem for ordinary differential equations. J. Symbolic Logic **49**,pp. 783–801 (1984).

[61] Simpson, S.G., Reverse Mathematics. Proc. Symposia Pure Math. **42**, pp. 461–471, AMS, Providence (1985).

[62] Simpson, S.G., Nonprovability of certain combinatorial properties of finite trees. In: Harrington et al. (eds.), H. Friedman's Research on the Foundations of Mathematics, pp. 87–117, North–Holland Amsterdam, New York, Oxford (1985).

[63] Smorynski, C. Logical number theory I. Springer Universitext, Berlin Heidelberg (1991).

[64] Spector, C., Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics. In: Dekker ,J.C.E. (ed.) Recursive function theory, Proc. Symp. Pure Math. AMS V, pp. 1–27 (1962).

[65] Takeuti, G., A conservative extension of Peano arithmetic, Part II of 'Two applications of logic to mathematics,' Publ. Math. Soc. Japan **13** (1978).

[66] Troelstra, A.S , Intuitionistic continuity. Nieuw Arch. Wisk. **15**,pp. 2–6 (1967).

[67] Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes in Mathematics **344** (1973).

[68] Troelstra, A.S., Note on the fan theorem. J. Symbolic Logic **39**, pp. 584–596 (1974).

[69] Troelstra, A.S., Realizability. ILLC Prepublication Series for Mathematical Logic and Foundations ML–92–09, 60 pp., Amsterdam (1992).

[70] Troelstra, A.S. – van Dalen, D., Constructivism in mathematics: An introduction. Vol. I,II. North–Holland, Amsterdam (1988).

[71] Weyl, H., Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis. Veit, Leipzig 1918.

[72] Zhang, W., Cut elimination and automatic proof procedures. Theoretical Computer Sccience **91**, pp. 265–284 (1991).

Errata (1998):

P.ii: delete footnote 2.
P.2, l.14: 'addition of 0^0 and of' instead of 'addition of'
P.2, l.23: add '$x \leq_0 y \wedge y \leq_0 x \leftrightarrow x =_0 y$'
P.8, Prop.1.2.16 must be modified into '$f^* \geq 1 \wedge f^*$ s-maj $f \wedge x^* \geq_0 x \to \Phi_j f^* x^* \geq_0 \Phi_j f x$'. In the proof of 1.2.16, the case 2.1 (whose treatment is incorrect) now falls away. The proof of 1.2.18 (which is the only application of 1.2.16) remains unchanged.
p.19, l.-9: '$\vdash A$' instead of '$\vdash (A')$'
P.21, l.-12: 'strengthen' instead of 'strenghten'
P.25, l.7: '$\max(u_i 0, \ldots, u_i x)$' instead of '$\max(u0, \ldots, ux)$'
P.29, l.-5: '$<_{\mathbb{R}}$' instead of '$<_{\mathbb{Q}}$'
P.30, l.-11: '$|\hat{x}_{2(m+1)k} \cdot \hat{\hat{\tilde{x}}}_{2(m+1)k} -_{\mathbb{Q}} \hat{x}_{2(\tilde{m}+1)k} \cdot \hat{\hat{\tilde{x}}}_{2(m+1)k}|$'
P.30, l.-4: '$\lambda n.\hat{x}_k$' instead of '$\lambda n.x_k$'
P.43, Def.3.3.1: add 'k even' and 'k odd' to the 1st and 2nd case resp.
P.76, l.11,13: '$\bigwedge x^1 \bigwedge y \leq_1 sx$' instead of '$\bigwedge x^1 \bigvee y \leq_1 sx$'
P.76, l.-12: 'boundedness' instead of 'boundednes'
P.77, l.10: '$\mathrm{G}_n\mathrm{A}^\omega \oplus \mathrm{AC}^{1,0}$-qf' instead of '$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{1,0}$-qf'
P.81, Prop.7.3.1: '$\mathrm{AC}^{1,0}$-qf' instead of '$\mathrm{AC}^{1,0}$'
P.88, l.3: 'E-$\mathrm{G}_n\mathrm{A}^\omega$' instead of 'E-$\mathrm{G}_n\mathrm{A}^\omega_i$'
P.105, l.13, '$\mathcal{T}^\omega$' instead of '$\mathcal{T}^\omega_i$'
P.106, footnote 5: 'Herbrand' instead of 'Hebrand'
P.117, l.9: 'upper index' instead of 'lower index'
P.117, l.-14: 'under S, definition ...' instead of 'under definition...'
P.117, l.-8: '$\Psi^*[x^0, \underline{h}^1]$' instead of '$\Psi^*[x^0, h^1]$'
P.118, l.-11: '$\bigvee w^\gamma A_0^H$' instead of 'A_0^H'
P.121, l.2: 'interpretation of the negative translation of' instead of 'interpretation of'
P.124, l.6: '$\mathrm{G}_n\mathrm{A}^\omega$' instead of '$\mathrm{G}_n\mathrm{A}^\omega_i$'
P.126, last line: 'Π^0_1-CA(f)' instead of 'Π^0_1-CA(g)'
P.127, l.10: '$\mathrm{G}_n\mathrm{R}^\omega$' instead of '$\mathrm{G}_2\mathrm{R}^\omega$'
P.134, l.12: '$\mathrm{G}_n\mathrm{R}^\omega$' instead of '$\mathrm{G}_2\mathrm{R}^\omega$'
P.138, l.3: '$\breve{x}_n$' instead of '$\breve{x}_m$'
P.150, l.11: 'Π^0_1-CP' instead of 'Σ^0_1-CP'
P.158, l.1: 'PRA^ω' instead of '$\mathrm{G}_n\mathrm{A}^\omega$'
P.158, l.3,4: '$\Phi \underline{u}\, \underline{k}\, \psi$' instead of '$\Phi \underline{u}\, \underline{k}$'
P.161, l.1: add 'for $\bigvee$-free A