Effective Uniform Bounds from Proofs in Abstract Functional Analysis

Ulrich Kohlenbach

Department of Mathematics
Darmstadt University of Technology
Schlossgartenstraße 7
D-64289 Darmstadt, Germany
kohlenbach@mathematik.tu-darmstadt.de
http://www.mathematik.tu-darmstadt.de/~kohlenbach

1 Introduction

In recent years (though influenced by papers of G. Kreisel going back to the 50's, e.g. [75–77]) as well as subsequent work by H. Luckhardt ([83, 84]) and others an applied form of proof theory systematically evolved which is also called 'Proof Mining' ([73]). It is concerned with transformations of prima facie ineffective proofs into proofs from which certain quantitative computational information as well as new qualitative information can be read off which was not visible beforehand. Applications have been given in the areas of number theory ([83]), combinatorics [2, 36, 100, 101], algebra ([23–26, 21, 22]) and, most systematically, in the area of functional analysis (see the references below). In particular, general logical metatheorems ([55, 35, 65]) have been proved which guarantee a-priorily for large classes of theorems and proofs in analysis the extractability of effective bounds which are independent from parameters in general classes of metric, hyperbolic and normed spaces if certain local boundedness conditions are satisfied. Unless separability assumptions on the spaces involved are used in a given proof, the independence results from parameters only need metric bounds but no compactness ([35, 65]). The theorems treat results involving concrete Polish metric spaces P (such as $\mathbb{R}^n$ or $C[0,1]$) as well as **abstract** structures (metric, hyperbolic, normed spaces etc.) which are axiomatically added to the formal systems as kind of 'Urelements'. It is for the latter structures that we can replace the dependency of the bounds from inputs involving elements of these spaces by hereditary bounds ('majorants') of such elements which in our applications will be either natural numbers or number theoretic functions. So we can apply the usual notions of computability and complexity for type-2 functionals and do not have to restrict ourselves to instances of these structures which are representable in some effective way or would carry a computability structure. The latter is only required for the **concrete** Polish metric spaces where we rely on the usual 'standard (Cauchy) representation'.

Obviously, certain restrictions on the logical form of the theorems to be proved as well as on the axioms to be used in the proofs are necessary (for a large class of semi-constructive proofs the restrictions on the form of the theorems can largely be avoided, see [34]). These restrictions in turn depend on the language of the formal systems used as well as the representation of the relevant mathematical objects such as general function spaces. The correctness of the results, moreover, depends in subtle ways on the amount of extensionality properties used in the proof which has a direct analytic counterpart in terms of uniform continuity conditions.

The applications which we discuss in this survey include a number of new qualitative existence results in the area of nonlinear functional analysis which follow from the metatheorems but so far did not have a functional analytic proof. Applying the extraction algorithm provided by the proofs of the metatheorems to these results yields the explicit quantitative versions stated below and at the same time direct proofs which no longer rely on the logical metatheorems themselves ([10, 33, 62, 64, 69, 70, 66, 81]).

The page limitations of this paper prevent us from formulating precisely the various logical metatheorems and the formal systems involved and we refer to [65, 35]. We will rather give a comprehensive presentation of the effective bounds obtained with the help of this logical approach in analysis (often all the qualitative features of the bounds concerning the (in)dependence from various parameters as well as some crude complexity estimates are guaranteed a-priorily by logical metatheorems) and refer for information on the logical background as well as for the proofs of these bounds to the literature.

Notations: $\mathbb{Q}^*_+$ and $\mathbb{R}^*_+$ denote the sets of strictly positive rational and real numbers respectively. The bounds presented below are all obviously effective if stated for $\varepsilon \in \mathbb{Q}^*_+$. Sometimes it is more convenient to state them (and to formulate the various moduli involved) for $\varepsilon \in \mathbb{R}^*_+$. It will, nevertheless, always be straightforward to make the use of e.g. $\lceil x \rceil$ effective by restricting things to rational ε (and corresponding moduli formulated for rationals).

2 Logical metatheorems

In this section we give an informal presentation of the main metatheorems on which the applications reported in this paper are based (details can be found in [65, 35]).

Definition 1. 1) The set $\mathbf{T}$ of all finite types over 0 is defined inductively by the clauses

$$(i)\ 0 \in \mathbf{T},\ (ii)\ \rho, \tau \in \mathbf{T} \Rightarrow (\rho \to \tau) \in \mathbf{T}.$$

2) The set $\mathbf{T}^X$ of all finite types over the two ground types 0 and X is defined by

$$(i)\ 0, X \in \mathbf{T}^X,\ (ii)\ \rho, \tau \in \mathbf{T}^X \Rightarrow (\rho \to \tau) \in \mathbf{T}^X.$$

3) A type is called small if it is of degree 1 (i.e. $0 \to \ldots \to 0 \to 0$) or the form $\rho_1 \to \ldots \to \rho_k \to X$ with the ρ_i being 0 or X.[1]

The theory $\mathcal{A}^\omega$ for classical analysis is the extension of the weakly extensional Peano arithmetic in all types WE-PA$^\omega$ by the schemata of quantifier-free choice QF-AC and dependent choice DC for all types in $\mathbf{T}$ (formulated for tuples of variables).
The theories $\mathcal{A}^\omega[X, d]_{-b}$ and $\mathcal{A}^\omega[X, d, W]_{-b}$ result[2] by extending $\mathcal{A}^\omega$ to all types in $\mathbf{T}^X$ and adding axioms for an abstract metric (in the case of $\mathcal{A}^\omega[X, d]_{-b}$) resp. hyperbolic (in the case of $\mathcal{A}^\omega[X, d, W]_{-b}$) space. $\mathcal{A}^\omega[X, d, W, \mathrm{CAT}(0)]_{-b}$ is the extension by an abstract CAT(0)-space. Analogously, one has theories $\mathcal{A}^\omega[X, \|\cdot\|]$ with an abstract non-trivial real normed space added (as well as further extensions $\mathcal{A}^\omega[X, \|\cdot\|, C]$ resp. $\mathcal{A}^\omega[X, \|\cdot\|, C]_{-b}$ with bounded resp. general convex subsets $C \subseteq X$ which we will, however, due to lack of space not formulate here). Our theories also contain a constant 0_X of type X which in the normed case represents the zero vector and in the other cases stands for an arbitrary element of the metric space. For details on all this see [65, 35].

Real numbers are represented as Cauchy sequences of rationals with fixed rate 2^{-n} of convergence which in turn are encoded as number theoretic functions f^1, where an equivalence relation $f =_{\mathbb{R}} g$ expresses that f^1, g^1 denote the same real numbers, and $\le_{\mathbb{R}}, <_{\mathbb{R}}, |\cdot|_{\mathbb{R}}$ express the obvious relations

[1] In [35] a somewhat bigger class of types of so-called degree $(1, X)$ is allowed. However, for the applications presented in this paper the small types suffice which simplifies the statement of the metatheorem below.

[2] The index '$-b$' indicates that in contrast to the corresponding theories in [65] we (following [35]) do not require the metric space to be bounded.

and operations on the level of these codes. Here $=_{\mathbb{R}}, \leq_{\mathbb{R}} \in \Pi^0_1$ whereas $<_{\mathbb{R}} \in \Sigma^0_1$. Again details can be found in [65].

'Weakly extensional' means that we only have Spector's quantifier-free extensionality rule. In particular, for the defined equality $x =_X y :\equiv (d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}})$, we do not have

$$x =_X y \to f^{X\to X}(x) =_X f(y)$$

but only from a proof of $s =_X t$ can infer that $f(s) =_X f(t)$. This is of crucial importance for our metatheorems to hold. Fortunately, we can in most cases prove the extensionality of f for those functions we consider, e.g. for nonexpansive functions, so that this only causes some need for extra care in few cases (for an extensive discussion of this point see [65]).

Definition 2. *For $\rho \in \mathbf{T}^X$ we define $\widehat{\rho} \in \mathbf{T}$ inductively as follows*

$$\widehat{0} := 0,\ \widehat{X} := 0,\ \widehat{(\rho \to \tau)} := (\widehat{\rho} \to \widehat{\tau}),$$

i.e. $\widehat{\rho}$ is the result of replacing all occurrences of the type X in ρ by the type 0.

Definition 3 ([35]). *We define a ternary majorization relation $\gtrsim^a_\rho$ between objects x, y and a of type $\widehat{\rho}, \rho$ and X respectively by induction on ρ as follows:*[3]

- $x^0 \gtrsim^a_0 y^0 :\equiv x \geq_{\mathbb{N}} y$,
- $x^0 \gtrsim^a_X y^X :\equiv (x)_{\mathbb{R}} \geq_{\mathbb{R}} d_X(y,a)$,
- $x \gtrsim^a_{\rho\to\tau} y :\equiv \forall z', z(z' \gtrsim^a_\rho z \to xz' \gtrsim^a_\tau yz) \wedge \forall z', z(z' \gtrsim^a_{\widehat{\rho}} z \to xz' \gtrsim^a_{\widehat{\tau}} xz)$.

For normed linear spaces we choose $a = 0_X$.

Definition 4. A formula F in $\mathcal{L}(\mathcal{A}^\omega[X,\ldots]_{-b})$ is called $\forall$-formula (resp. $\exists$-formula) if it has the form $F \equiv \forall \underline{a}^{\underline{\sigma}} F_{qf}(\underline{a})$ (resp. $F \equiv \exists \underline{a}^{\underline{\sigma}} F_{qf}(\underline{a})$) where F_{qf} does not contain any quantifier and the types in $\underline{\sigma}$ are small.

In the following $\mathcal{S}^\omega = \langle S_\rho \rangle_{\rho \in \mathbf{T}}$ refers to the full set-theoretic type structure of all set-theoretic functionals of finite type.

Theorem 1 ([35]).

1) *Let ρ be a small type and let $B_\forall(x,u)$, resp. $C_\exists(x,v)$, be $\forall$- and $\exists$-formulas that contain only x, u free, resp. x, v free. Assume that the constant 0_X does not occur in $B_\forall, C_\exists$ and that*

$$\mathcal{A}^\omega[X,d]_{-b} \vdash \forall x^\rho (\forall u^0 B_\forall(x,u) \to \exists v^0 C_\exists(x,v)).$$

Then there exists a computable functional[4] *$\Phi : S_{\widehat{\rho}} \to \mathbb{N}$ such that the following holds in all nonempty metric spaces (X,d): for all $x \in S_\rho$, $x^* \in S_{\widehat{\rho}}$ if there exists an $a \in X$ s.t. $x^* \gtrsim^a x$ then*

$$\forall u \leq \Phi(x^*) B_\forall(x,u) \to \exists v \leq \Phi(x^*) C_\exists(x,v).$$

If 0_x does occur in $B_\forall$ and/or $C_\exists$, then the bound Φ depends (in addition to x^*) on an upper bound $\mathbb{N} \ni n \geq d(0_X, a)$.

2) *The theorem also holds for nonempty hyperbolic spaces $\mathcal{A}^\omega[X,d,W]_{-b}$, (X,d,W) and for $\mathcal{A}^\omega[X,d,W,\mathrm{CAT}(0)]_{-b}$, where (X,d,W) is a $\mathrm{CAT}(0)$ space.*
3) *The theorem also holds for non-trivial real normed spaces $\mathcal{A}^\omega[X,\|\cdot\|]$, $(X,\|\cdot\|)$, where then 'a' has to be interpreted by the zero vector 0_X in $(X,\|\cdot\|)$ and 0_X is allowed to occur in $B_\forall, C_\exists$.*

[3] Here $(x)_{\mathbb{R}}$ refers to the embedding of $\mathbb{N}$ into $\mathbb{R}$ in the sense of our representation of $\mathbb{R}$.
[4] Note that for small types ρ the type $\widehat{\rho}$ is of degree 1. So Φ essentially is a type-2 functional : $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$.

Instead of single variables x, u, v and single premises $\forall u B_\forall(x,u)$ we may have tuples of variables and finite conjunctions of premises. In the case of a tuple $\underline{x}$ we then have to require that we have a tuple $\underline{x}^*$ of a-majorants for a common $a \in X$ for all the components of the tuple $\underline{x}$.

Remark 1. From the proof of Theorem 1 two further extensions follow:

1) The language may be extended by a-majorizable constants (in particular constants of types 0 and 1, which always are uniformly majorizable) where the extracted bounds then additionally depend on (a-majorants for) the new constants.
2) The theory may be extended by purely universal axioms or, alternatively, axioms which can be reformulated into purely universal axioms using new majorizable constants if the types of the quantifiers are small.

Using these extension, the theorem above can be adapted to other structures such as uniformly convex normed spaces or inner product spaces ([35]) as well as to uniformly convex hyperbolic spaces, δ-hyperbolic spaces (in the sense of Gromov) and $\mathbb{R}$-trees in the sense of Tits (see [81]).

A crucial aspect of theorem 1 is that the bound Φ operates on objects of degree ≤ 1, i.e. natural numbers or n-ary number theoretic functions so that the usual type-2 computability theory as well as well-known subrecursive classes of such functionals apply here irrespectively of whether the metric and normed spaces to which the bounds are applied come with any notion of computability or not. Since we included the axiom of dependent choice (and so also countable choice and hence full comprehension over numbers) in our systems, the functional Φ extracted will be in general a bar recursive functional in the sense of Spector [97]. However, if (as usually is the case) only small fragments of this are used, e.g. if in addition to basic arithmetic only the weak König's lemma WKL is used, then the bound will be primitive recursive in the sense of Gödel's T ([40]) if full induction is used resp. primitive recursive in the ordinary sense of Kleene if only Σ^0_1-induction is used. If not even full Σ^0_1-induction is used then in many cases even polynomial bounds (in the data) can be expected (see [55, 58–60]).
The proof of theorem 1 provides an algorithm (based on (monotone) functional ('Dialectica') interpretation [40, 97, 57, 43]) for the extraction of Φ.

In the concrete applications theorem 1 is used via various applied corollaries of which we give an example now:

Definition 5. *Let (X,d) be a metric space. A mapping $f : X \to X$ is called nonexpansive (short 'n.e.') if*
$$\forall x, y \in X (d(f(x), f(y)) \leq d(x,y)).$$

Corollary 1 ([35]). *Let $C_\exists$ be an $\exists$-formula and P, K Polish resp. compact metric spaces in standard representation by $\mathcal{A}^\omega$-definable terms (see [55] for a precise definition). If $\mathcal{A}^\omega[X,d,W]_{-b}$ proves a sentence*
$$\forall x \in P \forall y \in K \forall z^X, \tilde{z}^X, c^{0\to X}, f^{X\to X}\big(f \text{ nonexpansive} \to \exists v^{\mathbb{N}} C_\exists\big)$$
then there is a computable functional $\Phi(g_x, b, h)$ s.t. for all $x \in P, g_x \in \mathbb{N}^{\mathbb{N}}$ representative of x, $b \in \mathbb{N}, h \in \mathbb{N}^{\mathbb{N}}$
$$\forall y \in K \forall z, \tilde{z} \in X \forall c : \mathbb{N} \to X \forall f : X \to X \big(f \text{ n.e.} \wedge d(z, f(z)), d(z,\tilde{z}) \leq b \wedge \forall n (d(z, c(n)) \leq h(n) \\ \to \exists v \leq \Phi(g_x, b, h)\, C_\exists\big)$$
holds in **any** *nonempty hyperbolic space (X,d,W).*

Proof (sketch): The fact that P, K have a standard representation by $\mathcal{A}^\omega$-terms essentially means that $\forall$-quantification over P resp. K can be expressed as quantification $\forall x^1$ resp. $\forall y \leq_1 N$ where N

is a fixed simple (primitive recursive) function depending on K. Here the number theoretic functions encode Cauchy sequences (with fixed rate of convergence) of elements from the countable dense subset of P resp. K on which the standard representations are based. We now apply theorem 1 with $a := z$. For this we have to construct $\gtrsim^z$-majorants for $x^1, y^1, z^X, \tilde{z}^X, c^{0\to X}$ and $f^{X\to X}$:

$$x^* := x^M := \lambda n. \max\{x(i) : i \leq n\}, y^* := N^M, z^* := 0^0, \tilde{z}^* := b, c^* := h^M,$$
$$f^* := \lambda n^0.n + b.$$

For f^* we use that

$$d(x,z) \leq n \to d(f(x),z) \leq d(f(x),f(z)) + d(f(z),z) \leq d(x,z) + d(f(z),z) \leq n + b.$$

Note that the majorants only depend on x, b, h. □

3 Applications of proof mining in approximation theory

Let $(X, \|\cdot\|)$ be a (real) normed linear space and $E \subseteq X$ a finite dimensional subspace. By a standard (ineffective) compactness argument each $x \in X$ possesses at least one element $y_b \in E$ of best apprxomation, i.e.

$$\|x - y_b\| = \inf_{y\in E} \|x - y\| =: dist(x, E).$$

In some important cases (see further below) y_b is uniquely determined

$$\forall x \in X \forall y_1, y_2 \in E(\|x - y_1\|, \|x - y_2\| = dist(x, E) \to y_1 = y_2)$$

which can be written as follows

$$\forall x \in X \forall y_1, y_1 \in E \forall k \in \mathbb{N} \exists n \in \mathbb{N}(\|x - y_1\|, \|x - y_2\| \leq dist(x, E) + 2^{-n} \to \|y_1 - y_2\| < 2^{-k}),$$

where (using the representation of real numbers mentioned above)

$$\|x - y_1\|, \|x - y_2\| \leq dist(x, E) + 2^{-n} \to \|y_1 - y_2\| < 2^{-k}$$

is equivalent to a Σ^0_1-formula.

Every best approximation $y_b \in E$ clearly satisfies $\|y_b\| \leq 2\|x\|$ (since otherwise $0 \in E$ would be a better appoximation). Hence we can replace above the space E by the compact subset $K_x := \{y \in E : \|y\| \leq 2\|x\|\}$. Now suppose that one has a computable bound $\Phi(x, k)$ (depending on a suitable representation of x) for '$\exists n \in \mathbb{N}$' that is independent of $y_1, y_2 \in K_x$, i.e.

$$\forall x \in X \forall y_1, y_1 \in K_x \forall k \in \mathbb{N}(\|x - y_1\|, \|x - y_2\| \leq dist(x, E) + 2^{-\Phi(x,k)} \to \|y_1 - y_2\| < 2^{-k}).$$

We call such a Φ a modulus of uniqueness. Then any algorithm for computing 2^{-n}-best approximations $y_n \in K_x$, i.e. $\|x - y_n\| \leq dist(x, E) + 2^{-n}$ can be used to compute y_b with any prescribed precision since

$$\forall k \in \mathbb{N}\ (\|y_{\Phi(x,k)} - y_b\| < 2^{-k}).$$

If we use $\tilde{K}_x := \{y \in E : \|y\| \leq \frac{5}{2}\|x\|\}$ instead of K_x, then by an easy argument a modulus of uniqueness on $\tilde{K}_x$ can be extended effectively to the whole space E. So we now always refer to moduli of uniqueness on all of E and – for convenience – use $q \in \mathbb{Q}^*_+$ instead of 2^{-k} with $\Phi(x, q) \in \mathbb{Q}^*_+$. The next proposition further indicates the relevance of this notion:

Proposition 1 ([55]). *Let $(X, \|\cdot\|)$ be a real normed linear space, $E \subseteq X$ a finite dimensional subspace. Assume that every $x \in X$ possesses a uniquely determined best approximation in E and that the operation Φ is a modulus of uniqueness. Then the following holds*

1) $\frac{1}{2}\cdot\Phi$ *is a modulus of pointwise continuity for the projection* $\mathcal{P}: X\to E$ *which maps* $x\in X$ *to its best approximation* $y_b\in E$*, i.e.*

$$\forall x,x_0\in X, q\in\mathbb{Q}^*_+\big(\|x-x_0\|\le\frac{1}{2}\Phi x_0q\to\|\mathcal{P}x-\mathcal{P}x_0\|\le q\big).$$

2) If Φ *is linear in* q*,i.e.* $\Phi xq=q\cdot\gamma(x)$*, then* $\gamma(x)$ *is a 'constant of strong unicity', i.e.*

$$\forall x\in X,y\in E\big(\|x-y\|\ge\|x-y_b\|+\gamma(x)\cdot\|y-y_b\|\big),$$

where y_b *is the best approximation of* x *in* E*,*

3) For $\gamma(x)$ *as in '2)' we get that* $\lambda(x):=\frac{2}{\gamma(x)}$ *is a pointwise Lipschitz constant for* $\mathcal{P}$*, i.e.*

$$\forall x,x_0\in X\big(\|\mathcal{P}x-\mathcal{P}x_0\|\le\lambda(x_0)\cdot\|x-x_0\|\big).$$

In the following, we discuss two specific best approximation problems. Let $C[0,1]$ be the space of all continuous real valued functions on $[0,1]$ and P_n the subspace of all polynomials of degree $\le n$. We consider best approximations of $f\in C[0,1]$ by polynomials in P_n w.r.t. the maximum norm $\|f\|_\infty:=\sup\limits_{x\in[0,1]}|f(x)|$ (called best Chebycheff approximation) as well as w.r.t. the L_1-norm $\|f\|_1:=\int_0^1|f|$ (also called 'approximation in the mean'). Even in the latter case we represent $C[0,1]$ as a Polish space w.r.t. the metric induced by $\|\cdot\|_\infty$ since it is not complete w.r.t. $\|\cdot\|_1$. The usual so-called standard representation of $(C[0,1],\|\cdot\|_\infty)$ is constructively equivalent to the representation of f via its restriction to the rational numbers in $[0,1]$ and a modulus $\omega:\mathbb{Q}^*_+\to\mathbb{Q}^*_+$ of uniform continuity of f, i.e.

$$\forall x,y\in[0,1]\forall\varepsilon\in\mathbb{Q}^*_+(|x-y|<\omega(\varepsilon)\to|f(x)-f(y)|<\varepsilon)$$

so that the bounds will depend on ω.

Since in this section we do not use abstract classes of metric spaces but (in addition to $\mathbb{R}$) only the concrete Polish metric space $(C[0,1],\|\cdot\|_\infty)$ the applications in this section are instances already of the older metatheorems from [55].

We first consider the case of best Chebycheff approximation: A well-known theorem in so-called Chebycheff approximation theory states that every $f\in C[0,1]$ possesses a unique polynomial $p_b\in P_n$ of best approximation in the $\|\cdot\|_\infty$-norm, i.e. a polynomial in P_n such that $\|f-p_b\|_\infty=dist_\infty(f,P_n):=\inf\limits_{p\in P_n}\|f-p\|_\infty$. Both the existence as well as the uniqueness of p_b are established by classical arguments which make use of the theorem that continuous real valued functions attain their minimum on compact spaces, i.e. use the ineffective weak König's lemma WKL (see [95]).
By (the algorithm implicit in) our general metatheorems from [55] it is guaranteed that the uniqueness proof, nevertheless, allows one to extract a (primitive recursively) computable modulus of uniqueness (even of relatively low complexity), a concept which – under the name of strong unicity – plays an important role in approximation theory (see [18]). By proposition 1 such a modulus of uniqueness provides a stability rate for the Chebycheff projection which assigns to $f\in C[0,1]$ the unique polynomial p_b of best approximation in P_n. Furthermore, it can be used to compute p_b and to (upper) estimate its computational complexity (see [55] for all this). In [56] the following explicit moduli (also for the case of general Haar spaces) were extracted from the classical uniqueness proof due to [102]:

Theorem 2 ([56]). *Let*

$$\Phi(\omega,n,\varepsilon):=\min\left\{\varepsilon/4,\frac{\lfloor\frac{n}{2}\rfloor!\,\lceil\frac{n}{2}\rceil!}{2(n+1)}\cdot(\omega_n(\varepsilon/2))^n\cdot\varepsilon\right\},$$

with

$$\omega_n(\varepsilon) := \begin{cases} \min\left\{\omega\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{8n^2\lceil\frac{1}{\omega(1)}\rceil}\right\}, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

Then Φ is a common modulus of uniqueness for all $f \in C[0,1]$ which have the modulus of uniform continuity ω, i.e. for all $n \in \mathbb{N}$. More precisely, we have

$$\forall p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}^*_+ \big(\bigwedge_{i=1}^{2} (\|f - p_i\|_\infty - dist_\infty(f, P_n) < \Phi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_\infty \leq \varepsilon \big).$$

Moreover if $dist_\infty(f, P_n) > 0$ and $l \in \mathbb{Q}^*_+$ such that $l \leq dist_\infty(f, P_n)$ and

$$\tilde{\Phi}(\omega, n, l) := \frac{\lfloor\frac{n}{2}\rfloor! \lceil\frac{n}{2}\rceil!}{2(n+1)} \cdot (\omega_n(2l))^n,$$

then $\tilde{\Phi}(\omega, n, l) \cdot \varepsilon$ is a modulus of uniqueness for f which is linear in ε and so $\tilde{\Phi}(\omega, n, l)$ (by proposition 1) is a 'constant of strong unicity'.

Remark 2. 1) The most important aspect of $\Phi, \tilde{\Phi}$ above is that these bounds do not depend on p_1, p_2. This is guaranteed by the metatheorems in [55] since one can – as discussed above – restrict things to the bounded (and hence compact) subset $\tilde{K}_{f,n} := \{p \in P_n : \|p\|_\infty \leq \frac{5}{2}\|f\|_\infty\}$ of the finite dimensional space P_n.

2) Instead of the term $\lceil\frac{1}{\omega(1)}\rceil$ in the definition of ω_n we may use an arbitrary upper bound $M \geq \|f\|_\infty$. Actually the result is proved in this form in [56]. Using the construction $f \mapsto \tilde{f}, \tilde{f}(x) := f(x) - f(0)$ (using that $dist_\infty(f, P_n) = dist_\infty(\tilde{f}, P_n)$) one sees that one may assume without loss of generality that $f(0) = 0$. With this assumption $\lceil\frac{1}{\omega(1)}\rceil$ is an upper bound of $\|f\|_\infty$ which reduces the dependence of the bound on f to just ω.

3) Our constant of strong unicity tends to 0 as $n \to \infty$. Except for the trivial case where $f \in P_n$ this is unavoidable by a deep result in [32].

The modulus of uniqueness in theorem 2 is significantly better than the one implicit in [53, 54] (see [56] for a comparison).

The existence of a unique element of best approximation to $f \in C[0,1]$ extends from $P_{n-1} := Lin_{\mathbb{R}}\{1, x, \ldots, x^{n-1}\}$ to general so-called Haar spaces $H := Lin_{\mathbb{R}}\{\phi_1, \ldots, \phi_n\}$, i.e. n-dimensional subspaces of $C[0,1]$ which have the unique interpolation property, i.e.

$$\forall \phi \in H \forall \underline{x} \in [0,1] \big(\bigwedge_{i=1}^{n-1} (x_i < x_{i+1}) \wedge \bigwedge_{i=1}^{n} (\phi(x_i) = 0) \to \phi \equiv 0 \big).$$

The tuple $(\phi_1, \ldots, \phi_n)$ of functions in $C[0,1]$ is called a Chebycheff system over $[0,1]$.
Let $\phi := (\phi_1, \ldots, \phi_n)$ be a Chebycheff system over $[0,1]$, $\underline{\phi}(x) := \big(\phi_1(x), \ldots, \phi_n(x)\big) \in \mathbb{R}^n$, $\|\underline{\phi}\| := \sup_{x \in [0,1]} \|\underline{\phi}(x)\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$.
$\beta, \gamma, \kappa : (0, \frac{1}{n}] \to \mathbb{R}^*_+$ are defined by

$$\beta(\alpha) := \begin{cases} \inf_{x \in [0,1]} |\phi_1(x)|, \text{ if } n = 1 \\ \inf\left\{ |\det(\phi_j(x_i))| : 0 \leq x_1, \ldots, x_n \leq 1, \bigwedge_{i=1}^{n-1} (x_{i+1} - x_i \geq \alpha) \right\}, \text{ if } n > 1 \end{cases}$$

and

$$\gamma(\alpha) := \min\left\{ \|\underline{\phi}\|, \frac{\beta(\alpha)}{n^{\frac{1}{2}}(n-1)! \prod_{i=1}^{n}(1+\|\phi_i\|_\infty)} \right\},\ \kappa(\alpha) := \gamma(\alpha)^{-1}\cdot\|\underline{\phi}\|$$

for $\alpha \in (0, \frac{1}{n}]$. Since ϕ is a Chebycheff system it follows that $\beta(\alpha) > 0$.
$H := Lin_{\mathbb{R}}(\phi_1, \dots, \phi_n)$; $\omega_{\underline{\phi}}$ denotes a modulus of uniform continuity of $\underline{\phi}$. $E_{H,f} := dist_\infty(f, H)$.

Lemma 1 ([6, 7]).

1) Suppose that $A \subset C[0,1]$ is totally bounded, ω_A is a common modulus of uniform continuity for all $f \in A$ and $M > 0$ is a common bound $M \geq \|f\|_\infty$ for all $f \in A$. Then

$$\omega_{A,H}(\varepsilon) := \min\left\{ \omega_A(\frac{\varepsilon}{2}), \omega_{\underline{\phi}}\left(\frac{\varepsilon\cdot\beta(\frac{1}{n})}{4Mn^{\frac{3}{2}}(n-1)! \prod_{i=1}^{n}(1+\|\phi_i\|_\infty)} \right) \right\}$$

is a common modulus of uniform continuity for all $\psi_b - f$ where $f \in A$ and ψ_b is the best approximation of f in H.

2) Assume $0 < \alpha \leq \frac{1}{n}$ and $\bigwedge_{i=1}^{n-1}(x_{i+1} - x_i \geq \alpha)$ $(x_1, \dots, x_n \in [0,1])$ for $n \geq 2$. Then

$$\forall \psi \in H, \varepsilon > 0 \Big(\bigwedge_{i=1}^{n} |\psi(x_i)| \leq \frac{\gamma(\alpha)}{n\cdot\|\underline{\phi}\|}\cdot\varepsilon \to \|\psi\|_\infty \leq \varepsilon \Big).$$

Theorem 3 ([56]). *Let $A, \omega_{A,H}, \gamma, \kappa$ be as in lemma 1 and $E_{H,A} := \inf_{f\in A} E_{H,f}$. Then*

$$\Phi_A\varepsilon := \min\left\{ \frac{\varepsilon}{4}, \frac{1}{2}\frac{\gamma\big(\min\{\frac{1}{n}, \omega_{A,H}(\frac{\varepsilon}{2})\}\big)}{n\cdot\|\underline{\phi}\|}\cdot\varepsilon \right\} = \min\left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{2n\kappa\big(\min\{\frac{1}{n}, \omega_{A,H}(\frac{\varepsilon}{2})\}\big)} \right\}$$

is a common modulus of uniqueness (and a common modulus of continuity for the Chebycheff projection in f) for all $f \in A$.
For $l_{H,A} \in \mathbb{Q}_+^*$ such that $l_{H,A} < E_{H,A}$ and $0 < \alpha \leq \min\{\frac{1}{n}, \omega_{A,H}(2\cdot l_{H,A})\}$ we have $\frac{\gamma(\alpha)}{n\cdot\|\underline{\phi}\|}$ (resp. $2n\kappa(\alpha)$) as a uniform constant of strong unicity (resp. Lipschitz constant) for all $f \in A$.

The bounds in theorem 3 are significantly better than the ones obtained in [6–8] (see [56] for a detailed comparison). The (ineffective) existence of a constant of strong unicity was proved first in [86]. The existence of a uniform such constant (in the sense above) was established (again ineffectively) first in [42]. The local Lipschitz continuity of the projection is due to [31].

If the Haar space contains the constant-1 function then, using again the transformation $f \mapsto \tilde{f}$, with $\tilde{f}(x) := f(x) - f(0)$, one can even eliminate the dependence of the bounds on $M \geq \|f\|_\infty$ and conclude:

Theorem 4. *Let $\{\phi_1, \dots, \phi_n\}$ be a Chebycheff system such that $1 \in H := Lin_{\mathbb{R}}(\phi_1, \dots, \phi_n)$ and let $\omega : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be any function. Then*

$$\Phi_H(\omega, \varepsilon) := \min\left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{2n\kappa\big(\min\{\frac{1}{n}, \omega^H(\frac{\varepsilon}{2})\}\big)} \right\}$$

with

$$\omega^H(\varepsilon) := \min\left\{\omega(\frac{\varepsilon}{2}), \omega_{\underline{\phi}}\left(\frac{\varepsilon\cdot\beta(\frac{1}{n})}{4\lceil\frac{1}{\omega(1)}\rceil n^{\frac{3}{2}}(n-1)!\prod_{i=1}^{n}(1+\|\phi_i\|_\infty)}\right)\right\}$$

is a common modulus of uniqueness (and a common modulus of continuity for the Chebycheff projection) for all functions $f \in C[0,1]$ which have ω as a modulus of uniform continuity.

As a corollary we obtain that for arbitrary Haar spaces having the constant function 1 the continuity behavior of the Chebycheff projection is uniform for any class of equicontinuous functions which generalizes a result of [79] for the case of (trigonometric) polynomials.

We now move to best approximations of f by polynomials in P_n w.r.t. the L_1-norm $\|f\|_1 := \int_0^1 |f(x)|dx$, so-called best 'approximation in the mean'.

Theorem 5 ([45]). Let $f \in C[0,1]$ and $n \in \mathbb{N}$. There exists a unique polynomial $p_b \in P_n$ of degree $\leq n$ that approximates f best in the L_1-norm, i.e.

$$\|f - p_b\|_1 = \inf_{p\in P_n} \|f - p\|_1 =: dist_1(f, P_n).$$

Since $C[0,1]$ is not complete w.r.t. the norm $\|f\|_1$ we still use the representation w.r.t. $\|f\|_\infty$ in which the norm $\|f\|_1$ can easily be computed. As a result of this we again have to expect our modulus of uniqueness to depend on a modulus ω of uniform continuity of f. Again, both the existence and the uniqueness part are proved using compactness arguments which are equivalent to WKL. Despite of this ineffectivity, using the algorithm implicit in the logical metatheorems from [55] the following result was extracted from the ineffective uniqueness proof due to [17] (the extractability of a primitive recursive modulus of uniqueness again is a-priorily guaranteed by logical metatheorems, see [55]):

Theorem 6 ([72]). Let

$$\Phi(\omega, n, \varepsilon) := \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \frac{c_n\varepsilon}{2}\omega_n(\frac{c_n\varepsilon}{2})\},$$

where

$$c_n := \frac{\lfloor n/2\rfloor!\lceil n/2\rceil!}{2^{4n+3}(n+1)^{3n+1}} \text{ and } \omega_n(\varepsilon) := \min\{\omega(\tfrac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4\lceil\frac{1}{\omega(1)}\rceil}\}.$$

Then $\Phi(\omega, n, \varepsilon)$ is a modulus of uniqueness for the best L_1-approximation of any function f in $C[0,1]$ having modulus of uniform continuity ω from P_n, i.e. for all n and $f \in C[0,1]$:

$$\forall p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}_+^*\, (\bigwedge_{i=1}^{2}(\|f - p_i\|_1 - dist_1(f, P_n) \leq \Phi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_1 \leq \varepsilon),$$

where ω is a modulus of uniform continuity of the function f. Note that again Φ only depends on f only via the modulus ω.

The uniqueness of the best L_1-approximation was proved already in 1921 ([45]). In 1975, Björnestål [3] proved ineffectively the existence of a rate of strong unicity Φ having the form $c_{f,n}\,\varepsilon\,\omega_n(c_{f,n}\,\varepsilon)$, for some constant $c_{f,n}$ depending on f and n. In 1978, Kroó [78] improved Björnestål's results by showing – again ineffectively – that a constant $c_{\omega,n}$, depending only on the modulus of uniform continuity of f and n exists. Moreover, Kroó proved that the ε-dependency established by Björnestal is optimal. Note that the effective rate given above has this optimal dependency.
The effective rate of strong unicity given above allows one for the first time to effectively compute the best approximation. An upper bound on the complexity of that procedure is given in [88].

4 Effective computation of fixed points for functions of contractive type

There is a long history on extensions of Banach's well-known fixed point theorem for contractions to various more liberal notions of contractive type functions. The results usually are of the same shape as Banach's theorem, i.e. they state that the functions under consideration have a unique fixed point and that the Picard iteration $(f^n(x))_{n\in\mathbb{N}}$ of an arbitrary starting point converges to this fixed point. However, in contrast to Banach's theorem, in general no explicit rates of convergence can be read off from the (often ineffective) proofs.
The oldest of these results are due to Edelstein [28] and Rakotch [90].

Definition 6 ([28]). *A self-mapping f of a metric space (X,d) is contractive if*

$$\forall x,y \in X(x \neq y \rightarrow d(f(x),f(y)) < d(x,y)).$$

Theorem 7 ([28]). *Let (X,d) be a complete metric space, let f be a contractive self-mapping on X and suppose that for some $x_0 \in X$ the sequence $(f^n(x_0))$ has a convergent subsequence $(f^{n_i}(x_0))$. Then $\xi = \lim\limits_{n\to\infty} f^n(x_0)$ exists and is a unique fixed point of f.*

Rakotch observed that when contractivity is formulated in the following uniform way (which in the presence of compactness is equivalent to Edelstein's definition but in general is a strictly stronger condition) then it is possible to drop the assumption of the existence of convergent subsequences.

Definition 7 ([90]). [5] *A selfmapping $f: X \to X$ of a metric space is called uniformly contractive with modulus $\alpha : \mathbb{Q}^*_+ \to (0,1) \cap \mathbb{Q}$ if*

$$\forall \varepsilon \in \mathbb{Q}^*_+ \forall x,y \in X(d(x,y) > \varepsilon \rightarrow d(f(x),f(y)) \leq \alpha(\varepsilon)\cdot d(x,y)).$$

Theorem 8 ([90]). *Let (X,d) be a complete metric space and let f be a uniformly contractive self-mapping on X (i.e. f has modulus of contractivity α), then, for all $x \in X$, $\xi = \lim\limits_{n\to\infty} f^n(x)$ exists and is a unique fixed point of f.*

Example 1. The functions $f : [1,\infty) \to [1,\infty), f(x) := x + \frac{1}{x}$ and $f : \mathbb{R} \to \mathbb{R}, f(x) := \ln(1+e^x)$ are both contractive in the sense of Edelstein but not uniformly contractive in the sense of Rakotch. The function $f : [1,\infty) \to [1,\infty), f(x) := 1 + \ln x$ is uniformly contractive in the sense of Rakotch but not a contraction.

From the essentially constructive proof in [90] one obtains (as predicted by a general logical metatheorem established in [34]) the following bound (see also [9] for a related result):

Theorem 9 ([34]). *With the conditions as in the previous theorem we have the following rate of convergence of the Picard iteration from an arbitrary point $x \in X$ towards the unique fixed point ξ of f:*

$$\forall x \in X \forall \varepsilon \in \mathbb{Q}^*_+ \forall n \geq \delta(\alpha,b,\varepsilon)(d(f^n(x),\xi) \leq \varepsilon),$$

where

$$\delta(\alpha,b,\varepsilon) = \left\lceil \frac{\log\varepsilon - \log b'(\alpha,b)}{\log\alpha(\varepsilon)} \right\rceil \text{ for}$$
$$b'(\alpha,b) = \max\left(\rho, \frac{2\cdot b}{1-\alpha(\rho)}\right) \text{ with } \mathbb{N} \ni b \geq d(x,f(x)) \text{ and } \rho > 0 \text{ arbitrary.}$$

Remark 3. 1) Note that the rate of convergence depends on f, x only via α and an upper bound for $d(x,f(x))$.

[5] This definition is taken from [34] and slightly more general than Rakotch's original definition.

2) Instead of the multiplicative modulus of uniform contractivity α one can also consider an additive modulus $\eta : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ s.t.

$$\forall \varepsilon \in \mathbb{Q}^*_+ \forall x, y \in X \big(d(x,y) > \varepsilon \to d(f(x), f(y)) + \eta(\varepsilon) \leq d(x,y) \big)$$

and construct a rate of convergence in terms of η (see [34]).

Instead of starting from a constructive proof one also could take an ineffectice proof of $f^n(x) \to 0$ and first extract an effective bound Φ such that

$$\forall x \in X \forall \varepsilon \in \mathbb{Q}^*_+ \exists n \leq \Phi(\alpha, b, \varepsilon)(d(f^n(x), f^{n+1}(x)) < \varepsilon)$$

using theorem 1 (which is possible since '$\exists n(d(f^n(x), f^{n+1}(x)) < \varepsilon)$' is purely existential). Since the sequence $(d(f^n(x), f^{n+1}(x)))_n$ is nonincreasing this yields

$$\forall x \in X \forall \varepsilon \in \mathbb{Q}^*_+ \forall n \geq \Phi(\alpha, b, \varepsilon)(d(f^n(x), f^{n+1}(x)) < \varepsilon).$$

One then extracts (using again theorem 1) a modulus Ψ of uniqueness from the uniqueness proof. Similarly to our applications in approximation theory, these two moduli Φ, Ψ together then provide a rate of convergence towards the fixed point (for details see [73]).

In the fixed point theorems due to Kincses/Totik ([46]) and Kirk ([49]) which we discuss next, only ineffective proofs were known so that an approach as outlined above had to be anticipated. However, due to the lack monotonocity of $(d(f^n(x), f^{n+1}(x)))_n$ in these cases, this approach would not yield a full rate of convergence. Nevertheless, this problem could be overcome and, in fact, recent work of E.M. Briseid ([15]) shows that under rather general conditions on the class of functions to be considered (satisfied in the two cases at hand for uniformly continuous functions) theorem 1 **can** be used to guarantee effective rates of convergence of $(f^n(x))_n$ towards a unique fixed point from a given ineffective proof of this fact.

In [92, 93], 25 different notions of contractivity are considered starting from Edelstein's definition. The most general one among those is called 'generalized contractivity' in [10, 11]. If only some iterate f^p for $p \in \mathbb{N}$ is required to satisfy this condition, the function is called 'generalized p-contractive':

Definition 8 ([92]). *Let (X, d) be a metric space and $p \in \mathbb{N}$. A function $f : X \to X$ is called generalized p-contractive if*

$$\forall x, y \in X \big(x \neq y \to d(f^p(x), f^p(y)) < \text{ diam}\{x, y, f^p(x), f^p(y)\} \big).$$

Theorem 10 (Kincses/Totik,[46]). *Let (K, d) be a compact metric space and $f : K \to K$ a continuous function which is generalized p-contractive for some $p \in \mathbb{N}$. Then f has a unique fixed point ξ and for every $x \in K$ we have*

$$\lim_{n \to \infty} f^n(x) = \xi.$$

Guided by the logical metatheorems from [55, 63, 65], Briseid ([10]) (i) generalized theorem 10 to the noncompact case (similar to Rakotch's form of Edelstein's theorem) and (ii) provided a fully effective quantitative form of this generalized theorem:

Definition 9 ([10, 11]). *Let (X, d) be a metric space, $p \in \mathbb{N}$. $f : X \to X$ is called uniformly generalized p-contractive with modulus $\eta : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ if*

$$\forall x, y \in X \forall \varepsilon \in \mathbb{Q}^*_+ (d(x,y) > \varepsilon \to d(f^p(x), f^p(y)) + \eta(\varepsilon) < \ diam\{x, y, f^p(x), f^p(y)\}).$$

It is clear that for compact spaces and continuous f the notions 'generalized p-contractive' and 'uniformly generalized p-contractive (with some modulus η)' coincide.

Theorem 11 ([10, 11]). *Let (X, d) be a complete metric space and $p \in \mathbb{N}$. Let $f : X \to X$ be a uniformly continuous and uniformly generalized p-contractive function with moduli of uniform continuity ω and uniform generalized p-contractivity η. Let $x_0 \in X$ be the starting point of the Picard iteration $(f^n(x_0))$ of f and assume that $(f^n(x_0))$ is bounded by $b \in \mathbb{Q}^*_+$. Then f has a unique fixed point ξ and $(f^n(x_0))$ converges to ξ with rate of convergence $\Phi : \mathbb{Q}^*_+ \to \mathbb{N}$, i.e.*

$$\forall \varepsilon \in \mathbb{Q}^*_+ \forall n \geq \Phi(\varepsilon)(d(f^n(x_0), \xi) \leq \varepsilon),$$

where

$$\Phi(\varepsilon) := \begin{cases} p\lceil (b - \varepsilon)/\rho(\varepsilon)\rceil \text{ if } b > \varepsilon, \\ 0, \text{otherwise} \end{cases}$$

with

$$\rho(\varepsilon) := \min \left\{ \eta(\varepsilon), \frac{\varepsilon}{2}, \eta(\frac{1}{2}\omega^p(\frac{\varepsilon}{2})) \right\}.$$

For a discussion of the logical background of this result see [10].

Another notion of contractivity was recently introduced by Kirk and has received quite some interest in the last few years:

Definition 10 ([49]). *Let (X, d) be a metric space. A selfmapping $f : X \to X$ is called an asymptotic contraction with moduli $\Phi, \Phi_n : [0, \infty) \to [0, \infty)$ if Φ, Φ_n are continuous, $\Phi(s) < s$ for all $s > 0$ and*

$$\forall n \in \mathbb{N} \forall x, y \in X (d(f^n(x), f^n(y)) \leq \Phi_n(d(x, y)),$$

and $\Phi_n \to \Phi$ uniformly on the range of d.

Theorem 12 (Kirk,[49]). *Let (X, d) be a complete metric space and $f : X \to X$ a continuous asymptotic contraction. Assume that some orbit of f is bounded. Then f has a unique fixed point $\xi \in X$ and the Picard sequence $(f^n(x))$ converges to ξ for each $x \in X$.*

The following definition is essentially due to [33] (with a small generalization given by [12]) and was prompted by applying the method of monotone functional interpretation on which the logical metatheorems mentioned before are based to Kirk's definition.

Definition 11 ([33, 12]). *A selfmapping $f : X \to X$ of a metric space (X, d) is called an asymptotic contraction in the sense of Gerhardy and Briseid if for each $b > 0$ there exist moduli $\eta^b : (0, b] \to (0, 1)$ and $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ such that the following hold*

1) There exists a sequence of functions $\phi^b_n : (0, \infty) \to (0, \infty)$ such that for each $0 < l \leq b$ the function $\beta^b_l := \beta^b(l, \cdot)$ is a modulus of uniform convergence for $(\phi^b_n)_n$ on $[l, b]$, i.e.

$$\forall \varepsilon > 0 \forall s \in [l, b] \forall m, n \geq \beta^b_l(\varepsilon)(|\phi^b_m(s) - \phi^b_n(s)| \leq \varepsilon).$$

Furthermore, if $\varepsilon < \varepsilon'$ then $\beta^b_l(\varepsilon) \geq \beta^b_l(\varepsilon')$.

2) For all $x, y \in X$, for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$ with $\beta^b_\varepsilon(1) \leq n$ we have that

$$b \geq d(x, y) \geq \varepsilon \to d(f^n(x), f^n(y)) \leq \phi^b_n(\varepsilon) d(x, y).$$

3) For $\phi^b := \lim_{n\to\infty} \phi^b_n$ we have

$$\forall \varepsilon \in (0, b] \forall s \in [\varepsilon, b](\phi^b(s) + \eta^b(\varepsilon) \leq 1).$$

As shown in [33] (see also [12]) every asymptotic contraction in the sense of Kirk is also an asymptotic contraction in the sense of Gerhardy and Briseid (for suitable moduli). Moreover, as shown in [12], in the case of bounded and complete metric spaces, both notions coincide and are equivalent to the existence of a rate of convergence of the Picard iterations which is uniform in the starting point (as the one presented below).

Guided by logical metatheorems Gerhardy [33] not only developed the above explicit form of asymptotic contractivity but also extracted from Kirk's proof an effective so-called rate of proximity $\Psi(\eta, \beta, b, \varepsilon)$ such that

$$(f^n(x))_n \text{ bounded by } b \ \rightarrow \forall \varepsilon > 0 \exists n \leq \Psi(\eta, \beta, b, \varepsilon)(d(f^n(x), \xi))$$

for the unique fixed point ξ of f. For functions f which in addition to being continuous asymptotic contractions (with moduli η, β) are quasi-nonexpansive (see the final section of this paper) this already yields a rate of convergence towards the fixed point since $(d(f^n(x), \xi))_n$ is non-increasing in this case. Building upon Gerhardy's result Briseid [12] gave an effective rate of convergence in the general case:

Theorem 13 ([12]). *Let (X, d) be a complete metric space and f a continuous asymptotic contraction (in the sense of Gerhardy and Briseid) with moduli η, β. Let, furthermore, $b > 0$. If for some $x_0 \in X$ the Picard iteration sequence $f^n(x_0)$ is bounded by b, then f has a unique fixed point ξ and*

$$\forall \varepsilon > 0 \forall n \geq \Phi(\eta, \beta, b, \varepsilon)\big(d(f^n(x_0), \xi) \leq \varepsilon\big),$$

where

$$\Phi(\eta, \beta, b, \varepsilon) :=$$

$$\max\{k(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1), (k-1) \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1) + M_\gamma + 1\},$$

with $k := \left\lceil \frac{\ln \varepsilon - \ln b}{\ln(1 - \frac{\eta(\gamma)}{2})} \right\rceil, \ M_\gamma := K_\gamma \cdot \left\lceil \frac{\ln \gamma - \ln b}{\ln(1 - \frac{\eta(\gamma)}{2})} \right\rceil, \ K_\gamma := \beta_\gamma\big(\frac{\eta(\gamma)}{2}\big),$

$\delta := \min\{\frac{\varepsilon}{2}, \frac{\eta(\frac{\varepsilon}{2})}{2}\}, \ \gamma := \min\{\delta, \frac{\delta\varepsilon}{4}\}.$

Using results from [33] it is shown in [13] that Picard iteration sequences of asymptotic contractions always are bounded so that the corresponding assumption in Kirk's theorem 12 is superfluous (see also [98, 14]). Moreover, [13] gives an effective rate of convergence which does not depend on a bound b on (x_n) but instead on (strictly positive) lower and upper bounds on $d(x_0, f(x_0))$.

5 Fixed points and approximate fixed points of nonexpansive functions in hyperbolic spaces

Already for bounded metric spaces we cannot even hope that nonexpansive functions have approximate fixed points. This is due to the fact that (in contrast to functions of contractive type treated above) we can always change a given metric d to a bounded one by defining the truncated metric $D(x, y) := \max\{d(x, y), 1\}$ without destroying the property of nonexpansiveness: e.g. consider the bounded metric space $(\mathbb{R}, D)$ where $D(x, y) := \max\{|x - y|, 1\}$ and the nonexpansive function $f(x) := x + 1$. Then $\inf\{D(x, f(x)) : x \in \mathbb{R}\} = 1$. In the case of bounded, closed and convex subsets C of Banach spaces, nonexpansive mappings always have approximate fixed points (see e.g. [61] for an easy proof of this fact) but in general they have no fixed points (see [96]). Moreover, as the example $f = id_X$ shows, if a fixed point exists it will in general no longer be unique and even in cases where a unique fixed point exists, the Picard iteration will not necessarily converge to the fixed point and may even fail to produce approximate fixed points: consider e.g. $f : [0,1] \to [0,1], f(x) := 1-x$. Then for each $x \in [0,1] \setminus \{\frac{1}{2}\}$ the iteration sequence $f^n(x)$ oscillates between x and $1-x$ and so stays bounded away from the unique fixed point $\frac{1}{2}$. This is the reason why one considers so-called Krasnoselski-Mann iterations $(x_n)_{n\in\mathbb{N}}$ (see below) which make use of a concept of convex combination which exists in normed spaces but also in so-called hyperbolic spaces. Even in cases where (x_n) converges to a fixed point one can no longer hope for an effective rate of convergence. In fact it has been shown that already in almost trivial contexts such effective rates do not exist (see [66]). This failure of effectivity is largely due to the non-uniqueness of the fixed point (and hence the absence of a modulus of uniqueness in the sense of section 3). However, in many cases one can extract from the proofs effective rates on the so-called asymptotic regularity

$$d(x_n, f(x_n)) \to 0,$$

which holds under much more general conditions than the ones needed to guarantee the existence of fixed points. As mentioned above, we need somewhat more structure than just a metric space to define the Krasnoselski-Mann iteration:

Definition 12 ([47, 38, 91, 65]). *(X, d, W) is called a hyperbolic space if (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ a function satisfying*

(i) $\forall x, y, z \in X \forall \lambda \in [0,1]\big(d(z, W(x,y,\lambda)) \le (1-\lambda)d(z,x) + \lambda d(z,y)\big)$,
(ii) $\forall x, y \in X \forall \lambda_1, \lambda_2 \in [0,1]\big(d(W(x,y,\lambda_1), W(x,y,\lambda_2)) = |\lambda_1 - \lambda_2| \cdot d(x,y)\big)$,
(iii) $\forall x, y \in X \forall \lambda \in [0,1]\big(W(x,y,\lambda) = W(y,x,1-\lambda)\big)$,
(iv) $\begin{cases} \forall x, y, z, w \in X, \lambda \in [0,1] \\ \big(d(W(x,z,\lambda), W(y,w,\lambda)) \le (1-\lambda)d(x,y) + \lambda d(z,w)\big). \end{cases}$

Remark 4. The definition (introduced in [65]) is slightly more restrictive than the notion of 'space of hyperbolic type' as defined in [38] (which results if (iv) is dropped) but somewhat more general than the concept of 'space of hyperbolic type' as defined in [47] and – under the name of 'hyperbolic space' – in [91]. Our definition was prompted by the general logical metatheorems developed in [65] and appears to be most useful in the context of proof mining (see [65, 35] for detailed discussions). Moreover, our notion comprises the important class of CAT(0)-spaces (in the sense of Gromov) whereas the concept from [47, 91] only covers CAT(0)-spaces having the so-called geodesic line extension property. With axiom (i) alone the above notion coincides with the concept of 'convex metric space' as introduced in [99].

In the following we denote $W(x, y, \lambda)$ by $(1-\lambda)x \oplus \lambda y$.
In this section (X, d, W) always denotes a hyperbolic space and (λ_n) a sequence in $[0, 1)$ which is bounded away from 1 (i.e. $\limsup \lambda_n < 1$) and divergent in sum (i.e. $\sum\limits_{i=0}^{\infty} \lambda_i = \infty$). $f : X \to X$ is a selfmapping of X. Furthermore, given an $x \in X$, the sequence (x_n) refers (unless stated otherwise) to the so-called Krasnoselski-Mann iteration of f, i.e.

$$x_0 := x,\ \ x_{n+1} := (1-\lambda_n)x_n \oplus \lambda_n f(x_n).$$

Theorem 14 ([44, 38]). *Let (X, d, W) be a hyperbolic space and $f : X \to X$ nonexpansive. Then for all $x \in X$ the following holds:*

$$\textit{If } (x_n) \textit{ is bounded, then } d(x_n, f(x_n)) \to 0.$$

Theorem 15 ([4]). *Let (X, d, W) be a hyperbolic space and $f : X \to X$ be a nonexpansive function. Then for all $x \in X$ the following holds:*

$$d(x_n, f(x_n)) \to r_X(f) := \inf_{y \in X} d(y, f(y)).$$

The quantity $r_X(f)$ is often called 'minimal displacement of f on X.

As shown in [35], corollary 1 a-priorily guarantees that the proofs of the previous two results allow one to extract effective bounds on both theorems depending only on those parameters the concrete bounds in theorems 16 and 19 below depend which are extracted in this way. We start with theorem 15: Since $(d(x_n, f(x_n)))$ is non-increasing, theorem 15 formalizes as either

$$(a)\ \forall \varepsilon > 0 \exists n \in \mathbb{N} \forall x^* \in X (d(x_n, f(x_n)) < d(x^*, f(x^*)) + \varepsilon)$$

or

$$(b)\ \forall \varepsilon > 0 \forall x^* \in X\ \exists n \in \mathbb{N}(d(x_n, f(x_n)) < d(x^*, f(x^*)) + \varepsilon).$$

Trivially, (a) implies (b) but, ineffectively (using the existence of $r_X(f)$) also the implication in the other direction holds. Only (b) meets the specification in the metatheorem.

In the following, let $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be such that[6]

$$\forall i, n \in \mathbb{N}\big(\alpha(i, n) \le \alpha(i + 1, n)\big) \text{ and}$$

$$\forall i, n \in \mathbb{N}\big(n \le \sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s\big).$$

Let $k \in \mathbb{N}$ be such that $\lambda_n \le 1 - \frac{1}{k}$ for all $n \in \mathbb{N}$.
Corollary 1 predicts a uniform bound depending on x, x^*, f only via $b \ge d(x, x^*)$, $d(x, f(x))$ and on (λ_k) only via k, α (see [35]):

Theorem 16 ([70]). *Let (X, d, W) be a hyperbolic space and $(\lambda_n)_{n\in\mathbb{N}}, k, \alpha$ as above. Let $f : X \to X$ be nonexpansive and $b > 0, x, x^* \in X$ with $d(x, x^*), d(x, f(x)) \le b$. Then for the Krasnoselski-Mann iteration (x_n) of f starting from x the following holds:*

$$\forall \varepsilon \in \mathbb{Q}^*_+ \forall n \ge \Psi(k, \alpha, b, \varepsilon)\ (d(x_n, f(x_n)) < d(x^*, f(x^*)) + \varepsilon),$$

where

$$\begin{aligned}&\Psi(k, \alpha, b, \varepsilon) := \widehat{\alpha}(\lceil 2b \cdot \exp(k(M + 1))\rceil \dot- 1, M),\\ &\textit{with } M := \left\lceil \tfrac{1+2b}{\varepsilon} \right\rceil \textit{ and}\\ &\widehat{\alpha}(0, M) := \tilde{\alpha}(0, M),\ \widehat{\alpha}(m + 1, M) := \tilde{\alpha}(\widehat{\alpha}(m, M), M) \textit{ with}\\ &\tilde{\alpha}(m, M) := m + \alpha(m, M) \quad (m \in \mathbb{N}).\end{aligned}$$

Definition 13 ([48, 70]). *If (X, d, W) is a hyperbolic space, then $f : X \to X$ is called directionally nonexpansive (short 'f d.n.e') if*

$$\forall x \in X \forall y \in seg(x, f(x))\big(d(f(x), f(y)) \le d(x, y)\big),$$

where

$$seg(x, y) := \{\ W(x, y, \lambda) : \lambda \in [0, 1]\ \}.$$

Example 2. Consider the convex subset $[0, 1]^2$ of the normed space $(\mathbb{R}^2, \|\cdot\|_{\max})$ and the function

$$f : [0, 1]^2 \to [0, 1]^2,\ f(x, y) := \begin{cases} (1, y), & \text{if } y > 0 \\ (0, y), & \text{if } y = 0. \end{cases}$$

f is directionally nonexpansive but discontinuous at $(0, 0)$ and so, in particular, not nonexpansive.

[6] One easily verifies that one could start with any function $\beta : \mathbb{N} \to \mathbb{N}$ satisfying $n \le \sum_{s=0}^{\beta(n)} \lambda_s$ and then define $\alpha(i, n) := \max_{j\le i}(\beta(n + j) - j + 1)$ to get an α satisfying these conditions. However, this would in general give less good bounds than when working with α directly. See [62, 70] for more information in this point.

Theorem 16 generalizes to directionally nonexpansive mappings. The additional assumption needed is redundant in the case of nonexpansive mappings:

Theorem 17 ([70]). *The previous theorem (and bound) also holds for directionally nonexpansive mappings if* $d(x,x^*) \le b$ *is strengthened to* $d(x_n,x^*_n) \le b$ *for all* n.

The next result is proved in [64] for the case of convex subsets of normed spaces but the proof immediately extends to hyperbolic spaces. We include the proof for completeness. It applies corollary 1 to a formalization of theorem 15 which corresponds to the Herbrand normal form of (a) and constructively has a strength in between (a) and (b). Here x^* is replaced by a sequence (y_n) and we search for an n such that

$$d(x_n,f(x_n)) < d(y_n,f(y_n)) + \varepsilon,$$

i.e. (b) is just the special case with the constant sequence $y_n := x^*$. As predicted by corollary 1 we get a quantitative version of the following form:

Theorem 18. *Under the same assumptions as in theorem 16 the following holds: Let* (b_n) *be a sequence of strictly positive real numbers. Then for all* $x \in X$, $(y_n)_{n\in\mathbb{N}} \subset X$ *with*

$$\forall n \in \mathbb{N}(d(x,f(x)), d(x,y_n) \le b_n)$$

and all $\varepsilon > 0$ *there exists an* $i \le j(k,\alpha,(b_n)_{n\in\mathbb{N}},\varepsilon)$ *s.t.*[7]

$$d(x_i,f(x_i)) < d(y_i,f(y_i)) + \varepsilon,$$

where (omitting the arguments k,α *for better readability)*

$$j((b_n)_{n\in\mathbb{N}},\varepsilon) := \max_{i\le h((b_n)_{n\in\mathbb{N}},\varepsilon)} \Psi(k,\alpha,b_i,\varepsilon/2)$$

with

$$h((b_n)_{n\in\mathbb{N}},\varepsilon) := \max_{i<N} g^i(0),\ \ g(n) := \Psi(k,\alpha,b_n,\varepsilon/2), N := \left\lceil \frac{6b_0}{\varepsilon} \right\rceil.$$

Here Ψ *is the bound from theorem 16 and* $g^n(0)$ *is defined primitive recursively:*
$g^0(0) := 0,\ g^{n+1}(0) := g(g^n(0))$.
Instead of N, *we can take any integer upper bound for* $6b_0/\varepsilon$.

Proof: By theorem 16 we have that

$$(1)\ \forall n \in \mathbb{N}\big(d(x_{g(n)},f(x_{g(n)})) < d(y_n,f(y_n)) + \frac{\varepsilon}{2}\big),$$

where $g(n) := \Psi(k,\alpha,b_n,\varepsilon/2)$. Let $N := \lceil \frac{6b_0}{\varepsilon} \rceil$ and $l := \max_{i<N} g^i(0)$. Using that

$$(2)\ d(y_0,f(y_0)) \le d(y_0,x) + d(x,f(x)) + d(f(x),f(y_0)) \le 2d(y_0,x) + d(x,f(x)) \le 3b_0$$

we now show that

$$(3)\ \exists i < N\big(d(y_{(g^i(0))},f(y_{(g^i(0))})) \le d(y_{(g^{i+1}(0))},f(y_{(g^{i+1}(0))})) + \frac{\varepsilon}{2}\big):$$

Suppose not, then for all $i < N$

$$d(y_{(g^{i+1}(0))},f(y_{(g^{i+1}(0))})) < d(y_{(g^i(0))},f(y_{(g^i(0))})) - \frac{\varepsilon}{2}$$

[7] Recall that whereas (y_n) is an arbitrary sequence of points in X, (x_n) denotes the Krasnoselski-Mann iteration of f starting from x.

and, therefore,

$$d(y_{(g^N(0))}, f(y_{(g^N(0))})) < d(y_0, f(y_0)) - N\frac{\varepsilon}{2} \overset{(2)}{\leq} 3b_0 - N\frac{\varepsilon}{2} \leq 0,$$

which is a contradiction and finishes the proof of (3).
Let i be as in (3). Then by (1) we get for $p := g^i(0)$

$$(4)\ \forall n \in \mathbb{N}\big(d(x_{g(p)}, f(x_{g(p)})) < d(y_{g(p)}, f(y_{g(p)})) + \varepsilon\big),$$

where $p \leq l$. Hence the theorem is satisfied with $j((b_n)_n, \varepsilon) := \max_{i \leq l} g(i)$. $\dashv$

The next theorem gives a uniform quantitative version of the theorem of Ishikawa [44] as generalized by Goebel and Kirk [38] to hyperbolic spaces.

Theorem 19 ([70, 68]). *Let (X, d, W) be a nonempty hyperbolic space and $f : X \to X$ a nonexpansive mapping, $(\lambda_n)_{n\in\mathbb{N}}, \alpha$ and k be as before. Let $b > 0, x, x^* \in X$ be such that*

$$d(x, x^*) \leq b \wedge \forall n, m \in \mathbb{N}(d(x^*_n, x^*_m) \leq b),$$

where (x^*_n) is the Krasnoselski-Mann iteration starting from x^*. Then the following holds

$$\forall \varepsilon > 0 \forall n \geq h(k, \alpha, b, \varepsilon)\big(d(x_n, f(x_n)) \leq \varepsilon\big),$$

where

$$h(k, \alpha, b, \varepsilon) := \widehat{\alpha}(\lceil 10b \cdot \exp(k(M + 1))\rceil - 1, M)), \text{ with } M := \left\lceil \tfrac{1+4b}{\varepsilon} \right\rceil \text{ and } \widehat{\alpha} \text{ as before.}$$

Next we generalize the previous theorem (for $x^* := x$) to directionally nonexpansive functions.

Theorem 20 ([70]). *Let (X, d, W) be a nonempty hyperbolic space and $f : X \to X$ a directionally nonexpansive mapping. Let $(\lambda_n)_{n\in\mathbb{N}}, \alpha, k$ be as before.*
Let $b > 0$ and $x \in X$ such that

$$\forall n, k, m \in \mathbb{N}\big(d(x_n, (x_k)_m) \leq b\big),$$

where

$$(x_k)_0 = x_k, \qquad (x_k)_{m+1} = (1 - \lambda_m)(x_k)_m \oplus \lambda_k f((x_k)_m).$$

Then the following holds

$$\forall \varepsilon > 0 \forall n \geq h(k, \alpha, b, \varepsilon)\big(d(x_n, f(x_n)) \leq \varepsilon\big),$$

where

$$h(k, \alpha, b, \varepsilon) := \alpha(0, 1) + \widehat{\alpha^*}(\lceil 2b \cdot \alpha(0, 1) \cdot \exp(k(M + 1))\rceil - 1, M), \text{ with}$$
$$M := \left\lceil \tfrac{1+2b}{\varepsilon} \right\rceil \text{ and } \widehat{\alpha^*}(0, n) := \tilde{\alpha^*}(0, n),\ \widehat{\alpha^*}(i + 1, n) := \tilde{\alpha^*}(\widehat{\alpha^*}(i, n), n) \text{ with}$$
$$\tilde{\alpha^*}(i, n) := i + \alpha^*(i, n),$$
$$\alpha^*(i, n) := \alpha(i + \alpha(0, 1), n)\ (i, n \in \mathbb{N}).$$

Remark 5. Note that for constant $\lambda_k := \lambda$ we have $(x_k)_m = x_{k+m}$ so that the assumption $d(x_n, x_m) \leq b$ for all m, n suffices.

Previously known existence and uniformity results in the bounded case:[8]

[8] I.e. the case of bounded convex subsets in the normed case resp. bounded hyperbolic spaces.

- Krasnoselski(1955,[74]): Uniformly convex normed spaces X and special constant $\lambda_k = \frac{1}{2}$, no uniformity.
- Browder/Petryshyn(1966,[16]): Uniformly convex normed spaces X and constant $\lambda_k = \lambda \in (0,1)$, no uniformity.
- Groetsch (1972,[41]): X uniformly convex, general (λ_k), no uniformity (see also below).
- Ishikawa (1976,[44]): General normed space X and general (λ_k), no uniformity.
- Edelstein/O'Brien (1978,[29]): General normed space X and constant $\lambda_k := \lambda \in (0,1)$. Uniformity w.r.t. $x_0 \in C$ (and implicitly, though not stated, w.r.t. f).
- Goebel/Kirk (1983,[38]): General hyperbolic X and general (λ_k). Uniformity w.r.t. x_0 and f.
- Kirk/Martinez (1990,[51]): Uniformity w.r.t. x_0, f for uniformly convex normed spaces X and special constant $\lambda_k := 1/2$.
- Goebel/Kirk (1990,[39]): Conjecture: no uniformity w.r.t. C.[9]
- Baillon/Bruck (1996,[1]): Uniformity w.r.t. x_0, f, C for general normed spaces X and constant $\lambda_k := \lambda \in (0,1)$.
- Kirk (2000,[48]): Uniformity w.r.t. x_0, f for constant $\lambda_k := \lambda \in (0,1)$ for directionally nonexpansive functions in normed spaces.
- Kohlenbach (2001,[62]): Uniformity w.r.t. x_0, f, C for general (λ_k) for nonexpansive functions in the normed case.
- K./Leustean (2003,[70]): Uniformity w.r.t. x_0, f, C for general (λ_k) for directionally nonexpansive functions in the hyperbolic case.

Theorem 14 by Ishikawa [44] and Goebel and Kirk [38] has the following consequence in the compact case:

Theorem 21 ([44, 38]). *Let (X, d, W) be a compact hyperbolic space and $(\lambda_n), f, (x_n)$ as in theorem 14. Then $(x_n)_n$ converges towards a fixed point of f for any starting point $x_0 := x \in X$ of the Krasnoselski-Mann iteration (x_n).*

By theorem 14, the completeness of the space and the continuity of f, the conclusion of theorem 21 is equivalent to the property of (x_n) being a Cauchy sequence. That property is Π^0_3 and so of too complicated a logical form to allow for an effective bound in general. In fact, as shown in [66] there is no effective bound (uniformly in the parameters) even in the most simple cases. However, we can extract an effective bound on the Herbrand normal form

$$(H)\ \forall k \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n; n + g(n)](d(x_i, x_j) < 2^{-k})$$

of the Cauchy property which classically is equivalent to the latter. Here $[n; n + g(n)]$ denotes the set of all natural numbers j with $n \le j \le n + g(n)$. Note that '$\forall i, j \in [n; n + g(n)](d(x_i, x_j) < 2^{-k})$' is equivalent to a purely existential formula.

Since

$$\lambda_n d(x_n, f(x_n)) = d(x_n, x_{n+1})$$

the asympotic regularity $d(x_n, f(x_n)) \to 0$ property is equivalent to the special case of (H) with $g \equiv 1$ (for seqences (λ_n) which are bounded away from 0). So (H) is a generalization of asymptotic regularity which for general g fails in the absence of compactness whereas asymptotic regularity only needs the boundedness of X (or rather of the sequence (x_n)). Our effective bound on (H), therefore, will depend on a modulus of total boundedness of the space (see [68] for a detailed discussion).

Definition 14. *Let (M, d) be a totally bounded metric space. We call $\gamma : \mathbb{N} \to \mathbb{N}$ a modulus of total boundedness for M if for any $k \in \mathbb{N}$ there exist elements $a_0, \ldots, a_{\gamma(k)} \in M$ such that*

$$\forall x \in M \exists i \le \gamma(k)\big(d(x, a_i) \le 2^{-k}\big).$$

[9] By uniformity w.r.t. C it is meant that the bound depends on C only via an upper bound on the diameter of C.

Definition 15. *Let (M,d) be a metric space, $f : M \to M$ a selfmapping of M and (x_n) an arbitrary sequence in M. A function $\delta : \mathbb{N} \to \mathbb{N}$ is called an approximate fixed point bound for (x_n) if*

$$\forall k \in \mathbb{N} \exists m \le \delta(k) \big(d(x_m, f(x_m)) \le 2^{-k}\big).$$

Of course, an approximate fixed point bound only exists is (x_n) contains arbitrarily good approximate fixed points.

Theorem 22 ([66]). *Let $(X,d,W), (\lambda_n), f, (x_n)$ be as in theorem 14 and $k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}, \delta : \mathbb{N} \to \mathbb{N}$ and $\gamma : \mathbb{N} \to \mathbb{N}$. We define a function $\Omega(k,g,\delta,\gamma)$ (primitive) recursively as follows:*

$$\Omega(k,g,\delta,\gamma) := \max_{i \le \gamma(k+3)} \Psi_0(i,k,g,\delta),$$

where

$$\begin{cases} \Psi_0(0,k,g,\delta) := 0 \\ \Psi_0(n+1,k,g,\delta) := \delta\left(k+2+\lceil \log_2(\max_{l \le n} g(\Psi_0(l,k,g,\delta))+1)\rceil\right). \end{cases}$$

If δ is an approximate fixed point bound for the Krasnoselski-Mann iteration (x_n) starting from $x \in X$ and γ a modulus of total boundedness for X, then

$$\forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists n \le \Omega(k,g,\delta,\gamma) \forall i,j \in [n; n+g(n)] \big(d(x_i,x_j) \le 2^{-k}\big).$$

We now extend the previous theorem to asymptotically nonexpansive functions (though only in the context of convex subsets C of normed linear spaces $(X, \|\cdot\|)$:

Definition 16 ([37]). *Let $(X,\|\cdot\|)$ be normed space and $C \subset X$ a nonempty convex subset. $f : C \to C$ is said to be asymptotically nonexpansive with sequence $(k_n) \in [0,\infty)^{\mathbb{N}}$ if $\lim_{n\to\infty} k_n = 0$ and*

$$\forall n \in \mathbb{N} \forall x,y \in X \big(\|f^n(x) - f^n(y)\| \le (1+k_n)\|x-y\|\big).$$

In the context of asymptotically nonexpansive mappings $f : C \to C$, the Krasnoselski-Mann iteration starting from $x \in C$ is defined in a slightly different form as

$$(+)\ x_0 := x, \quad x_{n+1} := (1-\lambda_n)x_n + \lambda_n f^n(x_n).$$

Definition 17. *An approximate fixed point bound $\Phi : \mathbb{Q}^*_+ \to \mathbb{N}$ is called monotone if*

$$q_1 \le q_2 \to \Phi(q_1) \ge \Phi(q_2), \quad q_1, q_2 \in \mathbb{Q}^*_+.$$

Remark 6. Any approximate fixed point bound Φ for a sequence (x_n) can effectively be converted into a monotone approximate fixed point bound for (x_n) by

$$\Phi_M(q) := \Phi_m(\min k[2^{-k} \le q]), \text{ where } \Phi_m(k) := \max_{i \le k} \Phi(2^{-i}).$$

We now assume that C is totally bounded.

Theorem 23 ([66]). *Let $k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}, \Phi : \mathbb{Q}^*_+ \to \mathbb{N}$ and $\gamma : \mathbb{N} \to \mathbb{N}$. Let $f : C \to C$ be asymptotically nonexpansive with a sequence (k_n) such that $\mathbb{N} \ni K \ge \sum_{n=0}^{\infty} k_n$ and $N \in \mathbb{N}$ be such that $N \ge e^K$. We define a function $\Psi(k,g,\Phi,\gamma)$ (primitive) recursively as follows:*

$$\Psi(k,g,\Phi,\gamma) := \max_{i \le \gamma(k+\log_2(N)+3)} \Psi_0(i,k,g,\Phi),$$

where (writing $\Psi_0(l)$ for $\Psi_0(l,k,g,\Phi)$)

$$\begin{cases} \Psi_0(0) := 0 \\ \Psi_0(n+1) := \\ \quad \Phi\left(2^{-k-\log_2(N)-2}/(\max_{l\le n}[g^M(\Psi_0(l))(\Psi_0(l)+g^M(\Psi_0(l))+\log_2(N))+1])\right) \end{cases}$$

with $g^M(n) := \max_{i\le n} g(i)$.
If Φ is a monotone approximate fixed point bound for the Krasnoselski-Mann iteration (x_n) (defined by (+)) and γ a modulus of total boundedness for C then

$$\forall k\in\mathbb{N}\forall g:\mathbb{N}\to\mathbb{N}\exists n\le\Psi(k,g,\Phi,\gamma)\forall i,j\in[n;n+g(n)]\big(\|x_i-x_j\|\le 2^{-k}\big).$$

Remark 7. The previous two theorems even hold for arbitrary sequences (λ_n) in $[0,1]$. However, in order to construct approximate fixed point bounds one will need extra conditions.
For uniformly convex spaces and (λ_n) bounded away from both 0 and 1 an approximate fixed point bound Φ for asymptotically nonexpansive mappings will be presented in the last section.

We will now show that the qualitative features of the bounds in theorem 16 and 19 can be used to obtain new information on the approximate fixed point property (AFPP) for product spaces. A metric space (M,ρ) is said to have the AFPP for nonexpansive mappings if every nonexpansive mapping $f:M\to M$ has arbitrarily good approximate fixed points, i.e. if $\inf_{u\in M}\rho(u,f(u))=0$.

Let (X,d,W) be a hyperbolic space and (M,ρ) a metric space with AFPP for nonexpansive mappings. Let $\{C_u\}_{u\in M}\subseteq X$ be a family of convex sets such that there exists a nonexpansive *selection* function $\delta: M\to\bigcup_{u\in M}C_u$ with

$$\forall u\in M\big(\delta(u)\in C_u\big).$$

Consider subsets of $(X\times M)_\infty$ (with the metric $d_\infty((x,u),(y,v)) := \max\{d(x,y),\rho(u,v)\}$)

$$H := \{(x,u): u\in M, x\in C_u\}.$$

If $P_1: H\to\bigcup_{u\in M}C_u, P_2: H\to M$ are the projections, then for any nonexpansive function $T:H\to H$ w.r.t. d_∞ satisfying

$$(*)\ \forall(x,u)\in H\ \big((P_1\circ T)(x,u)\in C_u\big)$$

we can define for each $u\in M$ the nonexpansive function

$$T_u: C_u\to C_u,\quad T_u(x) := (P_1\circ T)(x,u).$$

We denote the Krasnoselski-Mann iteration starting from $x\in C_u$ and associated with T_u by (x_n^u) ((λ_n) as in theorem 14).

$r_S(F)$ always denotes the minimal displacement of F on S.

Theorem 24 ([71]). *Assume that $T:H\to H$ is nonexpansive with $(*)$ and $\sup_{u\in M}r_{C_u}(T_u)<\infty$. Suppose there exists $\varphi:\mathbb{R}_+^*\to\mathbb{R}_+^*$ s.t.*

$$\forall\varepsilon>0\,\forall v\in M\,\exists x^*\in C_v\ \big(d(\delta(v),x^*)\le\varphi(\varepsilon)\wedge \\ \wedge\, d(x^*,T_v(x^*))\le\sup_{u\in M}r_{C_u}(T_u)+\varepsilon\big).$$

Then

$$r_H(T)\le\sup_{u\in M}r_{C_u}(T_u).$$

Theorem 25 ([71]). *Assume that there is* $b > 0$ *s.t.*

$$\forall u \in M \exists x \in C_u \big(d(\delta(u), x) \le b \wedge \forall n, m \in \mathbb{N} (d(x_n^u, x_m^u) \le b).$$

Then $r_H(T) = 0$.

Corollary 2 ([71]). *Assume that there is a* $b > 0$ *with the property that*

$$\forall u \in M \big(diam(C_u) \le b \big).$$

Then H *has AFPP for nonexpansive mappings* $T : H \to H$ *satisfying* $(*)$.

As a special case of the previous corollary we obtain a recent result of Kirk (note that for $C_u := C$ being constant, we can take as δ any constant function : $M \to C$):

Corollary 3 ([50]). *If* $C_u := C$ *constant and* C *bounded, then* H *has the approximate fixed point property.*

6 Bounds on asymptotic regularity in the uniformly convex case

Prior to Ishikawa's paper [44] the fixed point theory of nonexpansive mappings was essentially restricted to the case of uniformly convex normed spaces ([74, 16]). Although Ishikawa showed that the fundamental theorem 14 holds without uniform convexity the case of uniformly convex spaces is still of interest for the following reasons (among others):

- As shown by Groetsch [41] (see below) in the uniformly convex case the conditions on (λ_n) in theorem 14 can be weakened to
$$\sum_{i=0}^{\infty} \lambda_i (1 - \lambda_i) = \infty$$
which is known to be optimal even for the case of Hilbert spaces (for general normed spaces it is still open whether this condition is sufficient).
- The bounds extracted from proofs using uniform convexity are often better than the ones known for the general case (see below; a notably exception is the optimal quadratic bound from [1] for the case of general normed spaces and constant $\lambda_k = \lambda \in (0, 1)$).
- Only in the uniformly convex case corresponding results for more general classes of functions such as asymptotically nonexpansive functions and (weakly) quasi-nonexpansive functions are known (see below).

Definition 18 ([19]). *A normed linear space* $(X, \|\cdot\|)$ *is uniformly convex if for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\forall x, y \in X \big(\|x\|, \|y\| \le 1 \ \wedge \|x - y\| \ge \varepsilon \to \|\frac{1}{2}(x + y)\| \le 1 - \delta \big).$$

A mapping $\eta : (0, 2] \to (0, 1]$ *providing such a* $\delta := \eta(\varepsilon) > 0$ *for given* $\varepsilon \in (0, 2]$ *is called a modulus of uniform convexity.*

Theorem 26 ([41]). *Let* C *be a convex subset of uniformly convex Banach space* $(X, \|\cdot\|)$ *and let* (λ_n) *be a sequence in* $[0, 1]$ *with* $\sum_{i=0}^{\infty} \lambda_i (1 - \lambda_i) = \infty$. *If* $f : C \to C$ *is nonexpansive and has at least one fixed point, then for the Krasnoselski-Mann iteration* (x_n) *of* f *starting at any point* $x_0 \in C$ *the following holds:*

$$\|x_k - f(x_k)\| \stackrel{k \to \infty}{\to} 0.$$

We now give a quantitative version of a strengthening of Groetsch's theorem which only assumes the existence of approximate fixed points in some neighborhood of x (see [65, 81] for a discussion on how this fits under the logical metatheorems):

Theorem 27 ([64]).
Let $(X, \|\cdot\|)$ be a uniformly convex normed linear space with modulus of uniform convexity η, $d > 0$, $C \subseteq X$ a (non-empty) convex subset, $f : C \to C$ nonexpansive and $(\lambda_k) \subset [0,1]$ and $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N}\big(\sum_{k=0}^{\gamma(n)} \lambda_k(1-\lambda_k) \geq n\big).$$

Then for all $x \in C$ which satisfy that for all $\varepsilon > 0$ there is a $y \in C$ with

$$\|x-y\| \leq d \text{ and } \|y - f(y)\| < \varepsilon,$$

one has

$$\forall \varepsilon > 0 \forall n \geq h(\varepsilon, d, \gamma, \eta)\big(\|x_n - f(x_n)\| \leq \varepsilon\big),$$

where $h(\varepsilon, d, \gamma, \eta) := \gamma\left(\left\lceil \frac{3(d+1)}{2\varepsilon \cdot \eta(\frac{\varepsilon}{d+1})}\right\rceil\right)$ for $\varepsilon < 2d$ and $h(\varepsilon, d, \gamma, \eta) := 0$ otherwise.
Moreover, if $\eta(\varepsilon)$ can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with

$$\varepsilon_1 \geq \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \geq \tilde{\eta}(\varepsilon_2), \text{ for all } \varepsilon_1, \varepsilon_2 \in (0,2], \tag{1}$$

then the bound $h(\varepsilon, d, \gamma, \eta)$ can be replaced (for $\varepsilon < 2d$) by

$$\tilde{h}(\varepsilon, d, \gamma, \tilde{\eta}) := \gamma\left(\left\lceil \frac{d+1}{2\varepsilon \cdot \tilde{\eta}(\frac{\varepsilon}{d+1})}\right\rceil\right).$$

For a Hilbert space one can take as modulus of uniform convexity $\eta(\varepsilon) := \varepsilon^2/8$ and hence the bound in theorem 27 applies with $\tilde{\eta}(\varepsilon) := \varepsilon/8$. If, moreover, $\lambda_n := \lambda \in (0,1)$ for all n then we can take $\gamma(n) := \lceil n/(\lambda(1-\lambda))\rceil$. So for the case of Hilbert spaces and constant λ we obtain a quadratic bound in ε.

In [82], Groetsch's theorem and its quantitative analysis from [64] is extended to uniformly convex hyperbolic spaces. The bounds obtained are roughly the same as in theorem 27 but now also apply e.g. to the important class of CAT(0)-spaces which are uniformly convex with the same modulus as in the Hilbertian case. Hence as a corollary the following quadratic bound follows:

Theorem 28 ([82]). *Let (X, d) be a* CAT(0)*-space, $C \subseteq X$ a nonempty convex subset whose diameter is bounded by $d \in \mathbb{Q}^*_+$. Let $f : C \to C$ be nonexpansive and $\lambda \in (0,1)$. Then*

$$\forall \varepsilon \in \mathbb{Q}^*_+ \forall n \geq g(\varepsilon, d, \lambda)(d(x_n, f(x_n)) < \varepsilon),$$

where (x_n) is the Krasnoselski-Mann iteration starting from $x_0 := x \in C$ and

$$g(\varepsilon, d, \lambda) := \begin{cases} \frac{1}{\lambda(1-\lambda)}\left\lceil \frac{4(d+1)^2}{\varepsilon^2}\right\rceil, & \text{for } \varepsilon < 2d \\ 0, & \text{otherwise.}\end{cases}$$

In the following, $C \subseteq X$ is a convex subset of a normed linear space $(X, \|\cdot\|)$.

Definition 19 ([94]). *$f : C \to C$ is said to be uniformly λ-Lipschitzian ($\lambda > 0$) if*

$$\forall n \in \mathbb{N} \forall x, y \in C\big(\|f^n(x) - f^n(y)\| \leq \lambda\|x-y\|\big).$$

Definition 20 ([27]). *$f : C \to C$ is quasi-nonexpansive if*

$$\forall x \in C \forall p \in Fix(f)\big(\|f(x) - p\| \le \|x - p\|\big),$$

where $Fix(f)$ is the set of fixed points of f.

Example 3. $f : [0, 1) \to [0, 1), f(x) := x^2$ is quasi-nonexpansive but not nonexpansive.

Definition 21 ([89]). *$f : C \to C$ is asymptotically quasi-nonexpansive with $k_n \in [0, \infty)^{\mathbb{N}}$ if $\lim\limits_{n\to\infty} k_n = 0$ and*

$$\forall n \in \mathbb{N} \forall x \in X \forall p \in Fix(f)\big(\|f^n(x) - p\| \le (1 + k_n)\|x - p\|\big).$$

Definition 22 ([69, 35]).

1) $f : C \to C$ is weakly quasi-nonexpansive[10] *if*

$$\exists p \in Fix(f)\forall x \in C\big(\|f(x) - f(p)\| \le \|x - p\|\big)$$

or – equivalently –

$$\exists p \in C \forall x \in X\big(\|f(x) - p\| \le \|x - p\|\big).$$

2) $f : C \to C$ is asymptotically weakly quasi-nonexpansive if

$$\exists p \in Fix(f)\forall x \in C \forall n \in \mathbb{N}\big(\|f^n(x) - f^n(p)\| \le (1 + k_n)\|x - p\|\big).$$

Example 4. $f : [0, 1] \to [0, 1], f(x) := x^2$ is weakly quasi-nonexpansive but not quasi-nonexpansive.

For asymptotically (weakly) quasi-nonexpansive mappings $f : C \to C$ the Krasnoselski-Mann iteration with errors is

$$(++)\ x_0 := x \in C,\ x_{n+1} := \alpha_n x_n + \beta_n f^n(x_n) + \gamma_n u_n,$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $u_n \in C$.

Relying on previous results of Opial, Dotson, Schu, Rhoades, Tan, Xu and – most recently – Qihou we have

Theorem 29 ([69]). *Let $(X, \|\cdot\|)$ be a uniformly convex normed space and $C \subseteq X$ convex. $(k_n) \subset \mathbb{R}_+$ with $\sum k_n < \infty$. Let $k \in \mathbb{N}$ and $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ such that $1/k \le \beta_n \le 1-1/k$, $\alpha_n+\beta_n+\gamma_n = 1$ and $\sum \gamma_n < \infty$. $f : C \to C$ uniformly Lipschitzian and asymptotically weakly quasi-nonexpansive and (u_n) be a bounded sequence in C. Then the following holds for (x_n) as defined in $(++)$ for an arbitrary starting point $x \in X$:*

$$\|x_n - f(x_n)\| \to 0.$$

Unless f is nonexpansive we in general cannot conclude (in contrast to the situation in theorems 14 and 15) that $(\|x_n - f(x_n)\|)_{n\in\mathbb{N}}$ is non-increasing which is needed to reduce the logical complexity of the convergence statement from Π^0_3 to Π^0_2. That's why we can apply our metatheorems only to the Herbrand normal form to get the following result (see [69]) for an extended discussion on how the metatheorems apply here and to a large extent predict the general form of the result):

[10] The same class of mappings has recently been introduced also in [30] under the name of J-type mappings.

Theorem 30 ([69]). *Let* $(X, \|\cdot\|)$ *be uniformly convex with modulus of convexity* η, $C \subseteq X$ *convex,* $x \in C, f : C \to C, k, \alpha_n, \beta_n, \gamma_n, k_n, u_n$ *as before with* $\sum \gamma_n \leq E$, $\sum k_n \leq K, \forall n \in \mathbb{N}(\|u_n - x\| \leq u)$ *and* $E, K, u \in \mathbb{Q}_+$. *Let* $d \in \mathbb{Q}^*_+$ *and* (x_n) *as in theorem 29.*
If f *is* λ*-uniformly Lipschitzian and*

$$\forall \varepsilon > 0 \exists p_\varepsilon \in C \begin{pmatrix} \|f(p_\varepsilon) - p_\varepsilon\| \leq \varepsilon \wedge \|p_\varepsilon - x\| \leq d \wedge \\ \forall y \in C \forall n \in \mathbb{N}\big(\|f^n(y) - f^n(p_\varepsilon)\| \leq (1 + k_n)\|y - p_\varepsilon\|\big) \end{pmatrix},$$

then

$$\forall \varepsilon \in (0, 1] \forall g : \mathbb{N} \to \mathbb{N} \exists n \leq \Phi \forall m \in [n, n + g(n)] \left(\|x_m - f(x_m)\| \leq \varepsilon\right),$$

where

$$\begin{aligned} &\Phi := \Phi(K, E, u, k, d, \lambda, \eta, \varepsilon, g) := h^i(0), \ \textit{where} \\ &h(n) := g(n + 1) + n + 2, \\ &i = \left\lceil \tfrac{3(5KD+6E(U+D)+D)k^2}{\tilde{\varepsilon}\eta(\tilde{\varepsilon}/(D(1+K)))} \right\rceil, \\ &D := e^K(d + EU), U := u + d, \\ &\tilde{\varepsilon} := \varepsilon/(2(1 + \lambda(\lambda + 1)(\lambda + 2))). \end{aligned}$$

Remark 8. 1) Specializing theorem 30 to $g \equiv 0$ yields

$$\forall \varepsilon \in (0, 1] \exists n \leq \Psi \left(\|x_n - f(x_n)\| \leq \varepsilon\right),$$

where

$$\begin{aligned} &\Psi := \Psi(K, E, u, k, d, \lambda, \eta, \varepsilon) := 2 \left\lceil \tfrac{3(5KD+6E(U+D)+D)k^2}{\tilde{\varepsilon}\eta(\tilde{\varepsilon}/(D(1+K)))} \right\rceil, \\ &D := e^K(d + EU), U := u + d, \\ &\tilde{\varepsilon} := \varepsilon/(2(1 + \lambda(\lambda + 1)(\lambda + 2))). \end{aligned}$$

2) As in the quantitative analysis of Groetsch's theorem above one can replace in the bound in theorem 30 η by $\tilde{\eta}$ if η can be written in the form $\eta(\varepsilon) = \varepsilon\tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ satisfying

$$0 < \varepsilon_1 \leq \varepsilon_2 \leq 2 \to \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2).$$

3) For asymptotically nonexpansive mappings with sequence (k_n) in $\mathbb{R}_+$ such that $\sum k_n \leq K$ the assumption 'uniformly Lipschitzian' is automatically satisfied by $\lambda := 1 + K$ since $K \geq k_n$ for all n.

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